CHAPTER 04

Work and Energy

4.1 Introduction

4.1.1 Work

A body undergoes some displacement under the application of applied force, then some work is said to be done on the body work is equal to the force multiply by distance travelled in the direction of force.

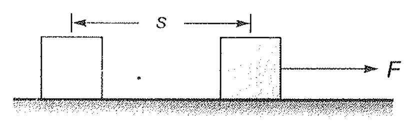


Fig. (a)

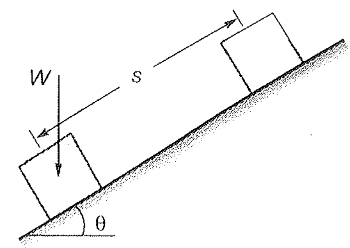


Fig. (b)

Fig. 4.1

Consider the force F acting on the block as shown in figure 4.1 (a). Block moves by distance s under this force, then the work done is

$$U = F \times s$$

In second case figure 4.1 (b) block moves by distance along the inclined plane due to its weight, then work done by its weight is

$$U = W(s \sin\theta)$$

Here the displacement in the direction of force is $s \cdot \sin\theta$.

The unit of work is Nm as unit of force Newton is multiplied by unit of displacement meter. Unit of work is Joule also.

$$1 \text{ J} = 1 \text{ Nm}$$

4.1.2 Energy

Energy of a body is the capacity of doing the work. We always say, a healthy man has high capacity for doing work. A machine having higher amount of energy can perform more work.

The unit of energy is same as work and that is Nm or Joule.

4.1.3 Potential Energy

The energy possessed by an object because of its position is called Potential Energy. If a block of mass m is lifted through a height h above ground, then the work done on the block is mgh. The work remains stored in the block. It can perform the same amount of work again in coming down to its original position. The potential energy of mass m lifted through a height h above ground is given by

$$\text{P.E.} = mgh \text{ (Nm)}$$

4.1.4 Kinetic Energy

The amount of work required to bring a moving body into rest is known as kinetic energy.

A body of mass m is moving with a velocity v and it is brought to rest by applying a force F which causes retardation *'a'* and stops the mass in the distance *'s'*, then

$$\text{Kinetic energy of the body} = \text{work used to stop the body}$$

$$K.E. = F \cdot s = ma \cdot s \quad \dots (i)$$

In this equation, we have assumed that F is constant.

From third equation of motion we can write

$$0 = v^2 - 2as$$

or

$$as = \frac{1}{2}v^2$$

Substituting this in equation (i) we get

$$K.E. = \frac{1}{2}mv^2$$

4.1.5 Spring Energy

Spring also stores work in the form of potential energy and also known as spring energy.

Consider a mass connected to spring as shown in figure 4.2 at equilibrium position $x = 0$. Now spring is stretched by distance x. A force F acts on mass due to stiffness k of the spring that will try to bring it in original position and this force is equal to

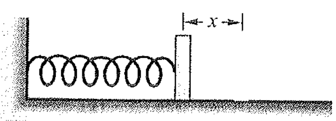


Fig. 4.2

$$F_s = -kx$$

Negative sign shows that this force is always in the opposite direction of displacement. Now consider a small displacement dx is given to mass m. Work done to stretch spring by dx is

$$dU = -F_s\, dx$$

or

$$dU = -(-kx)\, dx$$

or

$$U = k\int_0^{x_0} x\,dx = \frac{1}{2}kx^2$$

4.1.6 Law of conservation of Energy

This law states that Energy can neither be created nor destroyed, though it can be transformed from one form to another forms, in which the energy can exist.

Assume a block of mass m falls from rest from a height H as shown in figure 4.3.

The total energy of the ball at position A is its potential energy,

$$P.E. = mgH$$

Let the ball drop a height h and its velocity at the point B is v_B.

Using third equation of motion

$$v_B^2 = 0 + 2gh = 2gh$$

Total energy at B is potential energy and kinetic energy,

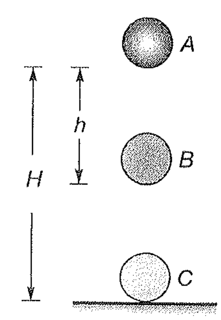


Fig. 4.3

$$P.E. + K.E. = mg(H-h) + \frac{1}{2}mv_B^2$$

$$= mg(H-h) + \frac{m}{2}(2gh) = mgH$$

Consider now the motion of ball from A to its position at C

$$v_c^2 = 0 + 2gH = 2gH$$

Total energy at C is potential energy and kinetic energy,

$$P.E. + K.E. = 0 + \frac{1}{2}mv_c^2 = \frac{1}{2}m(2gH) = mgH$$

This shows that the total energy of the Ball remains constant anywhere in between *A* to *C*. Thus K.E. and P.E. individually change throughout the motion from *A* to *C* but their sum always remains constant.

4.1.7 Work - Energy Relation

According to the work-energy theorem, the work done by an external force on a moving body is equal to the change in kinetic energy of the body.

Suppose an external constant force *F* is applied on a body of mass m and due to this force velocity of mass changes from *u* to *v*.

$$F = ma = m\frac{dv}{dt}$$

Work done for small displacement dx,

$$dU = F\,dx = m\frac{dv}{dt}dx$$

or $$dU = mv\,dv \qquad \text{as } \frac{dx}{dt} = v$$

Total work $$U = \int_u^v mvdv$$

or $$U = m\left[\frac{v^2}{2}\right]_u^v$$

or $$U = \frac{m}{2}[v^2 - u^2] = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

or $$U = K_2 - K_1 = \Delta K.E$$

According to this theorem, if work is done on the body by applying an external force, then it increases the kinetic energy of the body and if a body does work against retarding force (frictional force), then its kinetic energy decrease. If there is no change in the velocity of the body with application of force, then there is no change in kinetic energy. So work done by the force will be zero.

4.2 Rolling Motion under Gravity

Consider a cylinder of radius *r* and mass *m* placed over a inclined surface and released from the position of rest at position 1 and covering a distance *S* along the inclined plane and coming to position 2 (Fig. 4.4). Weight of the cylinder is *mg* and force on the cylinder along the plane is $mg\sin\theta$. Force of friction *F* will act upwards and provides a moment $F \times r$ to rotate the cylinder in the anticlockwise direction.

Normal reaction of plane on cylinder

$$= N = mg\cos\theta$$

Normal reaction *N* does no work as there is no motion of the cylinder in the direction perpendicular to the plane.

Motion is rolling without slipping, i.e., there is no relative motion between the point of contact on the wheel and the surface. Instantaneously the point of contact *C* is at rest as it becomes the instantaneous centre of rotation. *Therefore frictional force acting at contact point C does no work.*

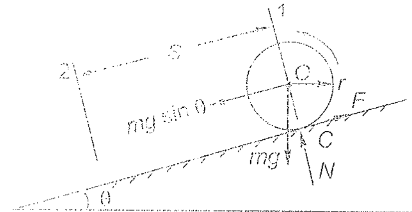


Fig. 4.4

Initial velocity of centre *O* of cylinder $= 0$

Initial angular velocity of cylinder $= 0$

Say in position (2) final velocity of mass centre of cylinder is V and final angular velocity of cylinder is ω. Then work done by force $mg\sin\theta$ along the plane is

$$U_{1-2} = mg.\sin\theta.S$$

Change in kinetic energy $= \frac{1}{2}mV^2 + \frac{1}{2}I\omega^2$

where I = mass moment of inertia of cylinder about axis of rotation $= \frac{mr^2}{2}$

Using the work-energy principle

$$mg\sin\theta.S = \frac{1}{2}mV^2 + \frac{I}{2}\omega^2, \text{ where } I = \frac{mr^2}{2}$$

Therefore $mg.\sin\theta.S = \frac{1}{2}mV^2 + \frac{1}{2}\times\frac{mr^2}{2}\times\omega^2$

but $V = \omega r$

So $S.mg\sin\theta = \frac{1}{2}mV^2 + \frac{1}{4}mV^2 = \frac{3}{4}mV^2$ or velocity, $V = \sqrt{\frac{4S}{3}g\sin\theta}$.

Example 4.1 Determine the power of an engine required to lift 3×10^3 kg water per minutes through a height of 20 m through a part. If the water is coming out with velocity of 5 m/sec, then what is the power.

Solution:

The energy required per second to lift water

$$= \frac{300}{60}gh$$

where m is mass of water lifted per second.

Thus $P = \frac{300}{60}\times 9.8\times 20 = 9800$ N/sec

If the water is coming out with velocity u then engine supplies this extra kinetic energy also. In this case,

$$\text{Energy required} = mgh + \frac{1}{2}mv^2 = m\left(gh + \frac{1}{2}v^2\right)$$

$$= \frac{300}{60}(9.8\times 20 + 0.5\times 5^2)$$

$$= 10425 \text{ Nm/sec}$$

Example 4.2 The position vector of a particle is given by $r = 8ti + 1.2t^2 j - 0.5(t^3 - 1)k$, where t is the time in seconds from the start of the motion and r is expressed in meters. For the condition when $t = 4$ sec, determine the power P developed by the force $F = 40i - 20j - 36k$ kN which acts on the particle.

Solution:

The power is given by force acting and multiplied by the velocity.

$$P = F \cdot v$$

or

$$P = F \cdot \frac{dr}{dt}$$

Here

$$F = 40i - 20j - 36k$$

$$r = 8ti + 1.2t^2 j - 0.5 (t^3 - 1)k$$

$$\frac{dr}{dt} = 8i + 2.4tj - 1.5t^2 k$$

$$P = (40i - 20j - 36k) \cdot (8i + 2.4tj - 1.5 t^2k)$$

or

$$P = 40 \times 8 - 20 \times 2.4t + 36 \times 15t^2$$

or

$$P = 320 - 48t + 54t^2$$

$$P_{t=4} = 320 - 48 \times 4 + 54 \times 4^2$$

$$= 320 - 192 + 864 = 992 \text{ W}$$

Example 4.3 A railways engine draws a train of mass 400 tonnes, including its own mass, on a level growing with a uniform acceleration, until it acquire a velocity of 54 kmph in 5 minutes. The air resistance varies with square of velocity and 500 N at 18 kmph and frictional resistance is 40 N/t. Determine the power of engine.

Solution:

Mass of train $m = 400 \text{ tonne} = 4 \times 10^5 \text{ kg}$

Velocity acquired $v = 54 \text{ kmph} = 15 \text{ m/sec}$

Time $t = 5 \text{ min} = 300 \text{ sec}$

Total frictional resistance, $R_f = 40 \times 400 = 16 \text{ kN}$

Let a be the acceleration of locomotive after 300 second, then

$$15 = 0 + a \times 300$$

or

$$a = \frac{15}{300} = 0.05 \text{ m/sec}^2$$

Required force for this acceleration is

$$F = ma = 400 \times 10^3 \times 0.05 = 20 \text{ kN}$$

As the air resistance varies with square of velocity, therefore air resistance at 54 kmph is

$$R_a = 500\left(\frac{54}{18}\right)^2 = 4.5 \text{ kN}$$

Total required force

$$F_{net} = R_f + F + R_a = 16 + 20 + 4.5 = 40.5 \text{ kN}$$

The power is

$$P = F_{net} \cdot v = 40.5 \times 15 = 607.5 \text{ kN-m/sec}$$

$$= 607.5 \text{ kJ/s}$$

Example 4.4 A train of mass 150 tonnes move on a level track with a speed of 20 m/sec. The tractive resistance is 100 N per tonne. Determine the power of the engine to maintain this speed. Also determine the power of the engine, when the train is to move with an acceleration of 0.3 m/sec^2 on a level track.

Solution:

When speed is constant, the engine supplier power for just tractive resistance. Thus

Tractive resistant	$R_t = 150 \times 100 = 15$ kN
Power	$P = R_t \cdot v = 15 \times 20 = 300$ kN-m/sec

If train is to move with an acceleration of 0.2 m/sec^2, then engine supplies extra force for acceleration also.

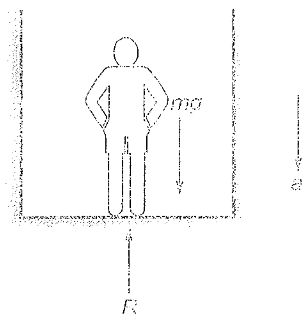
$$F = ma = 150 \times 0.3 = 45 \text{ kN}$$

Total required force	$F_{net} = R_t + F = 15 + 45 = 60$ kN
Power	$P = F_{net} \cdot v = 60 \times 20 = 1200$ kN-m/sec

Example 4.5 A woman having a mass 70 kg stands in an elevator which has a downward acceleration 4 m/sec^2 starting from rest. Determine the work done by her weight and the work of the normal force which the floor exerts of her when the elevator descends a distance 6 m.

Solution:

As per problem statement configuration is shown in figure. Let R be reaction force exerted by floor on woman. Then using Newton second law



$$mg - R = ma$$

$$R = m(g - a)$$

Here $m = 70$ kg

$a = 4$ m/sec^2

Thus $R = 70\,(9.8 - 4) = 406$ N

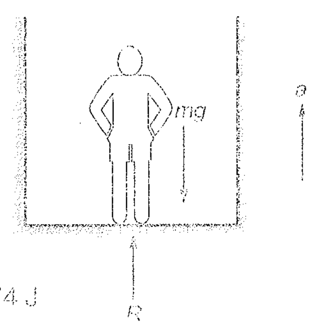
Work done by her weight, $W = mgs = 70 \times 9.8 \times 6 = 4.116$ kJ

Work of the normal force, $W_R = -sR = -6 \times 406 = -2.436$ kJ

Example 4.6 A woman having a mass of 65 kg stands in an elevator which has a upward acceleration of 2 m/sec^2 starting from rest. Determine the work done by her weight and the work of the normal force which the floor exerts of her when the elevator descends a distance 2 m.

Solution:

As per problem statement configuration is shown in figure. Let R be the force exerted by floor on man. Using Newton second law



$$R - mg = ma$$

or $R = m(a + g)$

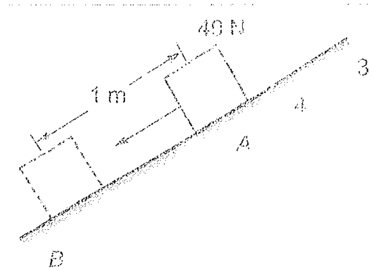
Here $m = 65$ kg

$R = 65\,(9.8 + 2) = 767$ N

Work done by his weight $U_w = -mgs = -65 \times 9.8 \times 2 = -1274$ J

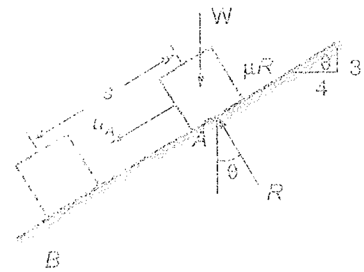
Work done by the normal force, $U_R = R \cdot s = 767 \times 2 = 1534$ J

Example 4.7 A block of weight 49 N has a velocity 2 m/sec^2 when it is at A as shown in figure. Determine its velocity at B after it slides down through 1 m along the plane. The coefficient of kinetic friction between the block and the plane is 0.25. Use the conservation of energy method.



Solution:

Figure shows all the force acting on the block. When it move from A to B work is done by all forces on it. The component of weight W which along plane act in the direction of displacement does positive work where the friction force μR does negative work does negative work since it acts in the opposite direction of that of displacement. The normal reaction, the component of W normal to plane don't displace hence do not work.



Using energy conservation

$$\frac{1}{2}mv_b^2 = \frac{1}{2}mv_a^2 + W\sin\theta \cdot s - \mu W\cos\theta s$$

Here

$$v_a = 2 \text{ m/sec}$$

$$m = \frac{W}{g}$$

$$\sin\theta = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{5}$$

$$\mu = 0.25$$

$$s = 1 \text{ m}$$

Substituting the values

or $$\frac{1}{2}\frac{W}{g}v_b^2 = \frac{1}{2}\frac{W}{g}\cdot 2^2 + W\cdot\frac{3}{5}\cdot 1 - 0.25\cdot W\cdot\frac{4}{5}\cdot 1$$

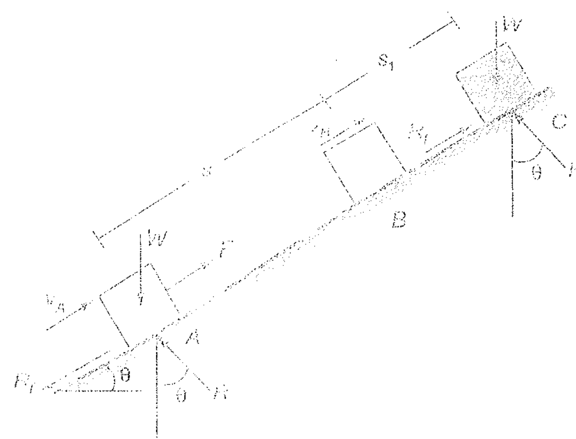
or $$v_b^2 = 4 + 0.4\times 9.8\times 2$$

or $$v_b = \sqrt{17.84} = 3.44 \text{ m/sec}$$

Example 4.8 A block of 1 kN weight is moved up from rest on a rough ($\mu = 0.2$) inclined plane which makes an angle 35°C from rest by a constant force 1.2 kN. When the distance moved by block is 10 m, the external force is removed. Find the velocity of the block when it returns to its original position.

Solution:

As per problem statement the configuration is shown in figure. Here the block moves from A to B under applied force F. Work done on the block under all acting force by moving the block from A to B will increase the kinetics energy of the block.



$$\frac{1}{2}m(v_b^2 - u^2) = (F - W\sin\theta - R_f)\cdot s$$

$$\frac{1}{2}m(v_b^2 - 0) = (F - W\sin\theta - \mu W\cos\theta)\cdot s \quad \text{...(i)}$$

or $$\frac{1}{2}\times\frac{1}{9.8}v_b^2 = (1.2 - 1\sin 35° - 0.2\cos 35°)\times 10$$

$$v_b^2 = 19.62\times 10\,(1.2 - 0.57 - 0.164) = 91.43$$

or $$v_b = 9.56 \text{ m/sec}$$

When the body reaches at B after travelling distance 10 m, the external force is removed. At this point it has velocity 9.56 m/sec and it moves upward till it comes to rest.
Thus substituting $F = 0$, $v = 0$ and $u = 9.56$ in equation
Substituting the values in equation (i) we get

$$\frac{1}{2}m(0 - 9.56^2) = (0 - W\sin\theta - \mu W\cos\theta)\cdot s_1$$

or
$$\frac{1}{2}\frac{W}{g}\times 9.56^2 = (W\sin\theta + \mu W\cos\theta)\cdot s_1$$

or
$$\frac{1}{2}\frac{9.56^2}{g} = (\sin\theta + \mu W\cos\theta)\cdot s_1$$

or
$$\frac{1}{2}\frac{9.56^2}{9.8} = (\sin 35° + 0.2\cos 35°)\cdot s_1$$

$$s_1 = \frac{1}{2}\frac{9.56^2}{9.8(\sin 35° + 0.2\cos 35°)} = 6.35\text{ m}$$

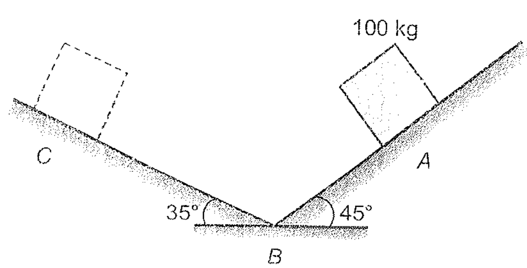
When the body reaches at point C, then it, will start moving downward due to its own weight and when it travels from C to A, it travels distance $s_2 = 10 + 6.35 = 16.35$ m. The initial velocity in this case is zero and final velocity at A is v_a. We can write down the energy equation as

$$\frac{1}{2}m(v_a^2 - u^2) = (W\sin\theta - \mu W\cos\theta)\cdot s_2$$

or
$$\frac{1}{2}\frac{W}{g}v_a^2 = (W\sin\theta - \mu W\cos\theta)\cdot s_2$$

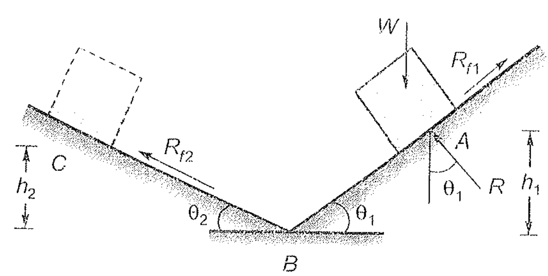
or
$$v_a^2 = 2g(\sin\theta - \mu\cos\theta)\times 16.35$$
$$= 2\times 9.81\times(0.57 - 0.2\times 0.82)\times 16.35 = 130.10\text{ m/sec}$$
$$v_a = 11.4\text{ m/sec}$$

Example 4.9 **A block of 100 kg is at rest at point A. It first slides along $AB = 100$ m and then moves along BC as shown in figure till it comes to the rest. Determine the distance travelled by block on plane BC. Take $\mu = 0.2$ for both the planes.**



Solution:

All force acting on block is shown in figure. At point A block has potential energy mgh due to height h_1. When it comes at foot of the inclined plane some energy is lost in friction force and rest is transferred to its kinetics energy.



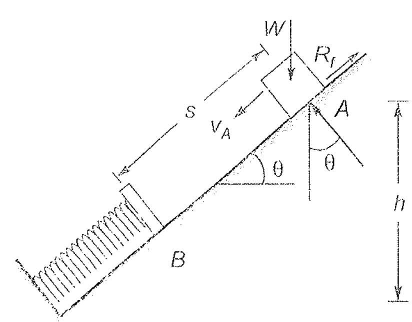
Thus,
$$\frac{1}{2}mv_b^2 = mgh_1 - R_{f1} \cdot AB$$
$$\frac{1}{2}mv_b^2 = mg \cdot AB \sin \theta_1 - \mu W \cos \theta_1 \cdot AB$$
$$\frac{1}{2}mv_b^2 = mg \cdot AB \sin \theta_1 - \mu mg \cos\theta_1 \cdot AB$$
or
$$v_b^2 = 2g \cdot AB(\sin\theta_1 - \mu \cos\theta_1) \quad ...(i)$$
or
$$v_b^2 = 2 \times 9.8 \times 100 (\sin 45^\circ - 0.2 \times \cos 45^\circ)$$
$$= 2 \times 980 \times \frac{1}{\sqrt{2}} \times 0.8 = 1108.7$$
or
$$v_b = 33.3 \text{ m/sec}$$
At this point block has kinetic energy due to velocity u_b and due to this kinetic energy block move on plane BC. Let it travel distance BC along the plane then its height from horizontal plane is h_2. Some of its kinetic energy is lost in work done against friction and rest is transferred to potential energy.

Thus
$$mgh_2 = \frac{1}{2}mv_b^2 - R_{f2} \cdot BC$$
or
$$\frac{1}{2}mv_b^2 = mgh_2 + R_{f2} \cdot BC$$
$$\frac{1}{2}mv_b^2 = mg \cdot BC \sin \theta_2 + \mu W \cos \theta_2 \cdot BC$$
$$\frac{1}{2}mv_b^2 = mg \cdot BC \sin \theta_2 + \mu mg \cos \theta_2 \cdot BC$$
or
$$v_b^2 = 2g \cdot BC (\sin \theta_2 + \mu \cos \theta_2) \quad ...(ii)$$
From equation (i) and (ii) we get
$$2g \cdot AB (\sin \theta_1 + \mu \cos \theta_1) = 2g \cdot BC (\sin \theta_2 + \mu \cos \theta_2)$$
or
$$\frac{BC}{AC} = \frac{(\sin\theta_1 - \mu \cos\theta_1)}{(\sin\theta_2 + \mu \cos\theta_2)}$$
Substituting the values we get
$$\frac{BC}{100} = \frac{(\sin 45^\circ - 0.2\cos 45^\circ)}{(\sin 35^\circ + 0.2\cos 45^\circ)} = \frac{0.565}{0.409}$$
or
$$BC = 76.7 \text{ m}$$

Example 4.10 A block of weight 400 N slides along an inclined plane making an angle 30° with horizontal having initial velocity of 2 m/sec. The distance travelled by the body along the plane is 2 meter and after that it strikes the spring whose stiffness is 100 N/mm. Taking $\mu = 0.2$, find the compression of the spring.

Solution:

As per problem statement the configuration is shown in figure. At point A block has kinetic energy due to its velocity and potential energy due to its height. When it strikes at spring after travelling AB, some part of its total energy is lost in work done against friction and rest is transferred to spring to compress it.



Let x be the compression of spring.

Thus $$mgh + \frac{1}{2}mv_a^2 = R_f \cdot s + \frac{1}{2}kx^2$$

Here $$h = s\sin\theta$$

and $$R_f = \mu W\cos\theta \text{ and } m = \frac{W}{g}$$

Thus $$W \cdot s\sin\theta + \frac{1}{2}\frac{W}{g}v_a^2 = \mu W\cos\theta \cdot s + \frac{1}{2}kx^2$$

or $$s\sin\theta + \frac{v_a^2}{2g} = \mu\cos\theta \cdot s + \frac{k}{2W}x^2$$

Substituting the values we get

$$2\sin 30^\circ + \frac{2^2}{2\times 9.8} = 2\times 0.2\times\cos 30^\circ + \frac{100\times 10^3}{2\times 400}x^2$$

$$2\times 0.5 + \frac{2^2}{9.8} = 0.4\times 0.866 + 125x^2$$

or $$1 + 0.204 = 0.35 + 125x^2$$

or $$x^2 = \frac{0.854}{125} = 6.832\times 10^{-3}$$

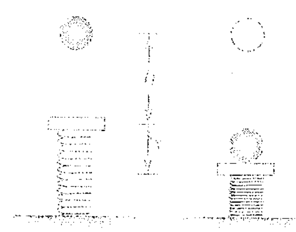
or $$x = 0.0828 \text{ m}$$

or $$x = 82.8 \text{ mm}$$

Example 4.11 A ball of 2 kg, is dropped from a height of 15 cm on a spring of stiffness $k = 980$ N/m. Find the maximum deflection of the spring.

Solution:

When the ball falls through a height h on spring as shown in figure, then its potential energy is transferred to compress the spring. If compression in spring is x then total potential energy is due to height $h + x$



Thus $$mg(h + x) = \frac{1}{2}kx^2$$

Substituting the values

$$2\times 9.8\,(0.15 + x) = \frac{1}{2}\times 980\times x^2$$

or $$0.15 + x = 25x^2$$

or $$x^2 - 0.04x - 0.006 = 0$$

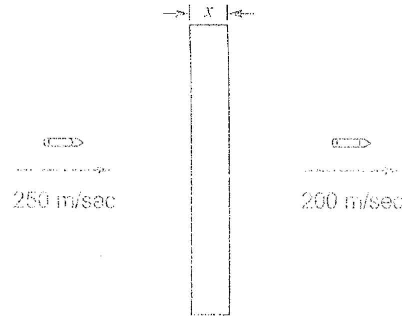
$$x = 0.1 \text{ m}$$

or $$x = 10 \text{ cm}$$

Example 4.12 A bullet with a velocity of 250 m/sec strikes a wooden plank and [illegible] out with a velocity of 200 m/sec. Determine the number of planks required in series when it comes to rest.

Solution:

As per problem statement the configuration is shown in figure.



When bullet strike with plank it losses some of its kinetic energy due to work against resistance force R_f of plank. Let the thickness of each plank be x.

Then $$-R_f x = \frac{1}{2}m(v^2 - u^2)$$

or $$-R_f x = \frac{1}{2}m(200^2 - 250^2)$$

or $$-R_f x = \frac{1}{2}m(-50) \times (450) = -11250\text{ m}$$

or $$R_f x = 11250\text{ m} \quad \text{...(i)}$$

Consider there are n planks in series and bullet comes to rest after passing through n planks, then we can write

$$-n(R_f x) = \frac{1}{2}m(v^2 - u^2) = \frac{1}{2}m(0 - u^2)$$

or $$n(R_f x) = \frac{u^2}{2}m$$

Substituting the value of u and $R_f x$ from equation (i) we get

$$n(11250m) = \frac{250^2}{2}m$$

or $$n = \frac{62500}{2 \times 11250} = 2.78 \approx 3$$

Thus to stop the bullet three plank is required.

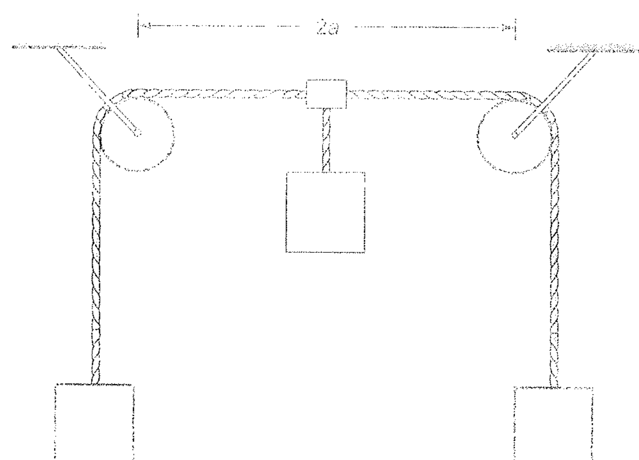
Example 4.13 A weightless string passing over a small smooth weight less pulley located a distance $2a$ and carries masses m_1 at each end of the string. A mass m_2 ($< 2m_1$) initially is at the center of two pulleys. The system is released from rest, prove that the system will come to rest when m_2 will fall through a distance

$$\left(\frac{4am_1m_2}{4m_1^2 - m_2^2}\right)$$

Solution:

As per problem statement the initial condition is shown in figure (a) and condition of the system when comes to the rest after releasing the system are shown in figure (b).

When the system is released, the mass m_2 will start coming down and mass m_1 will start going up and this will continue till the system comes to rest. Assume mass m_2 has fallen down through a height $H = BD$ and the masses m_1 lifted up through an height h.



(a)

Here both mass m_1 will gain potential energy m_1gh and mass m_2 will loose potential energy m_2gH.

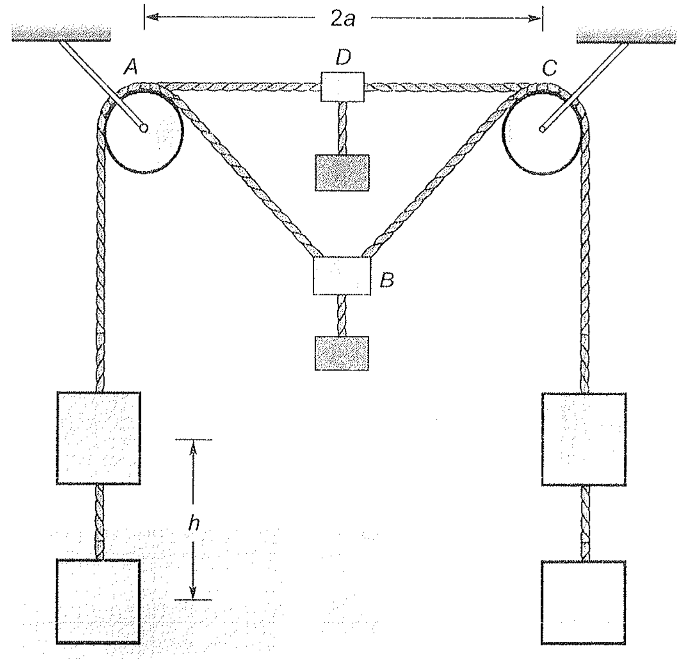
As per energy conservation law

$$2m_1\, gh = m_2\, gH$$

or $$2m_1\, h = m_2\, H \quad ...(i)$$

we have to find out the value of H.

From the initial and final figure of system we can write



(b)

$$h = AB - AD$$

but $$AB^2 = AD^2 + BD^2$$

or $$AB^2 = a^2 + H^2$$

Thus $$h = \sqrt{a^2 + H^2} - a$$

Substituting this value in equation (i) we get

$$2m_1(\sqrt{a^2 + H^2} - a) = m_2\, H$$

or $$2m_1\sqrt{a^2 + H^2} = m_2\, H + 2m_1\, a$$

Taking square both side

$$4m_1^2\,(a^2 + H^2) = m_2^2\, H^2 + 4m_1\, m_2\, aH + 4m_1^2\, a^2$$

$$4\, m_1^2\, H^2 = m_2^2\, H^2 + 4m_1\, m_2\, aH$$

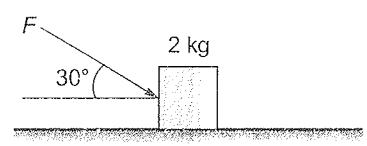
$$4m_1^2\, H = m_2^2\, H + 4m_1\, m_2\, a$$

$$H\,(4m_1^2 - m_2^2) = 4m_1\, m_2\, a$$

$$H = \frac{4am_1m_2}{(4m_1^2 - m_2^2)} \qquad \text{Hence Proved}$$

Example 4.14 The block of mass 2 kg is subjected to a force as shown in figure having a constant direction and a magnitude $F = 300\,/\,(1 + x)$.

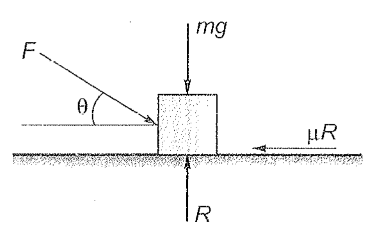
When $x = 4$ m, the block is moving to the right with a speed 8 m/sec. Determine its speed when $x = 12$ m. The coefficient of kinetic friction between the block and the ground is 0.25.



Solution:

Figure shows the all force acting on the block. Here block move in horizontal direction so work is done by the force in horizontal directional only.

Resolving forces in vertical direction.



$$F\sin\theta + mg = R$$

Net force in the direction of motion is $F\cos\theta - \mu R$.

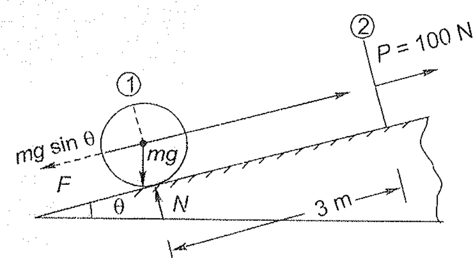
So work done is

$$U = \int_4^{12}(F\cos\theta - \mu R)dx$$

Substituting value of R we get
$$U = \int_4^{12}(F\cos\theta - \mu(F\sin\theta + mg))dx$$
$$U = (\cos\theta - \mu\sin\theta)\int_4^{12} F dx - \mu mg\int_4^{12} dx$$
Substituting all values we get
$$U = (\cos 30^\circ - 0.25\sin 30^\circ)\int_4^{12}\frac{300}{1+x}dx - 2\times 0.25\times 9.8\int_4^{12} dx$$
or
$$U = (0.741)\,300\,[\ln(1+x)]_4^{12} - 4.9\times 8$$
or
$$U = 222.3\ln\frac{(1+12)}{(1+4)} - 39.2$$
or
$$U = 173.2 \text{ J}$$
Now from energy conservation law we can write
$$\frac{1}{2}mu^2 + U = \frac{1}{2}mv^2$$
or
$$v = \sqrt{u^2 + \frac{2U}{m}} = \sqrt{8^2 + \frac{2\times 173.2}{2}} = 15.40 \text{ m/sec}$$

Example 4.15 Find the work done in moving a 15 kg wheel by 2 m up an incline plane with angle of inclination equal to 30°.

Coefficient of friction between wheel and plane is 0.25. A force of 100 N is applied at the centre of wheel as shown in figure. What will be angular velocity of wheel after the wheel has travelled 3 m up the plane? Wheel radius is 0.1 m.



Solution:

Mass of wheel $m = 15$ kg

Weight of wheel, $mg = 15 \times 9.81 = 147.15$ N

Component along the plane, $mg\sin\theta = 147.15 \times \sin 30^\circ = 73.57$ N

Normal reaction, $N = mg\cos\theta = 147.15 \times 0.866 = 127.43$ N

Net force along the plane $= 100 - 73.57$

or $F_{net} = 26.43$ N ...(i)

Please note that in this case force of friction will be equal to F_{net} and provides turning moment to wheel. This is the case of pure rolling without slipping. Force of friction will not be equal to, $\mu N = 0.25 \times 127.43 = 31.86$ N but it will be only 26.43 N.

Work done in rolling the wheel up the plane by 3 m
$$U_{1-2} = (100 - mg\sin\theta)\,S = 26.43 \times 3 = 79.29 \text{ Nm}$$
Say the angular velocity is ω and linear velocity of wheel centre is V at the position 2.
$$U_{1-2} = \frac{1}{2}mV^2 + \frac{I}{2}\omega^2 \quad ...(ii)$$
where
$$V = \omega r = 0.1 \times \omega \text{ m/s}$$
$$I = \frac{mr^2}{2} = \frac{15\times 0.1^2}{2} = 0.075 \text{ kg-m}^2$$

Substituting these values in Equation (ii)

$$79.29 = \frac{1}{2}\times 15\times(0.1\omega)^2 + \frac{0.075}{2}\times\omega^2$$

$$= 0.075\omega^2 + 0.0375\omega^2 = 0.1125\omega^2$$

Angular velocity, $$\omega = \sqrt{\frac{79.29}{0.1125}} = \sqrt{704.8} = 26.54 \text{ rad/s.}$$

Objective Brain Teasers

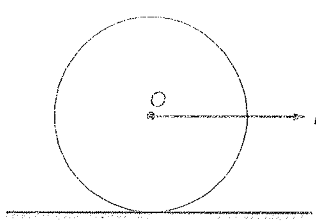
Q.1 A simple pendulum is constructed by attaching a bob of mass m to a string of length l. Fixed at its upper end. The bob oscillates in a vertical circle. It is found that the speed of bob is v when the string makes an angle θ with vertical. The tension in the string at this instant is

(a) $mg\left(g\cos\theta + \frac{v^2}{L}\right)$ (b) $m\left(g\sin\theta + \frac{v^2}{L}\right)$

(c) $m\left(g\cos\theta - \frac{v^2}{L}\right)$ (d) $m\left(g\sin\theta - \frac{v^2}{L}\right)$

Q.2 A particle of mass 50 g moves on a straight line with velocity varying with distance $v = 5\sqrt{x}$ m/s. The total work done by all forces during displacement $x = 0$ to $x = 4$ m is ....Joules.

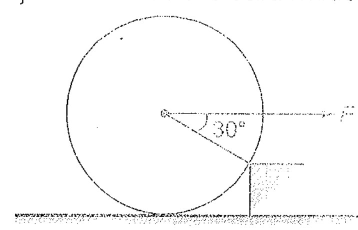
Q.3 A solid sphere of mass m with radius r is subjected to a force F at its centre and it starts rolling without slipping on a rough horizontal surface as shown in figure. The acceleration of the body is



(a) $\frac{5F}{7m}$ (b) $\frac{F}{m}$

(c) $\frac{3F}{2m}$ (d) $\frac{7F}{3m}$

Q.4 A roller of weight W is rolled over the wooden block shown in the given figure. The pull required just to cause the said motion is



(a) W/2 (b) $\sqrt{3}$ W

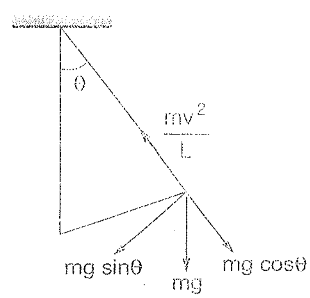
(c) W (d) 2 W

ANSWERS

1. (a) 3. (a) 4. (b)

Hints & Explanation

1. (a)



$$\Rightarrow T - mg\cos\theta = \frac{mv^2}{L}$$

$$\Rightarrow \quad T = m\left(g\cos\theta + \frac{v^2}{L}\right)$$

2. (2.5)

$$V = 5\sqrt{x}$$

$$s = 4 - 0 = 4$$

$$v_1 = 0 \text{ m/s}, \; v_2 = 5 \times \sqrt{4} = 10 \text{ m/s}$$

$$a = \frac{v_2^2 - v_1^2}{2s} = \frac{10^2 - 0^2}{2 \times 4} = 12.5$$

$$F = ma = 12.5 \times 0.05 = 0.625 \text{ N}$$

$$W = F \times s = 0.625 \times s$$

$$= 0.625 \times 4 = 2.5 \text{ J}$$

3. (a)

$$\Sigma M = F \cdot r$$

$$I \cdot \alpha = F \cdot r$$

$$I \cdot a = F r^2$$

$$I = I_0 + mr^2$$

$$= \frac{2}{5} mr^2 + mr^2 = \frac{7}{5} mr^2$$

So, $$a = \frac{Fr^2}{I_P} = \frac{5F}{7m}$$

4. (b)

Taking moment about the edge point, we have

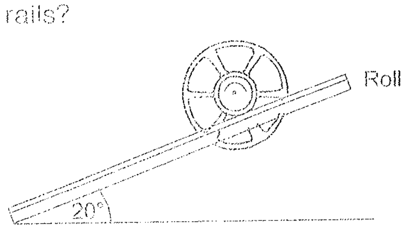
$$W \times r\cos 30° = F \times r \sin 30°$$

[Normal reaction will be zero when the roller is just lifted]

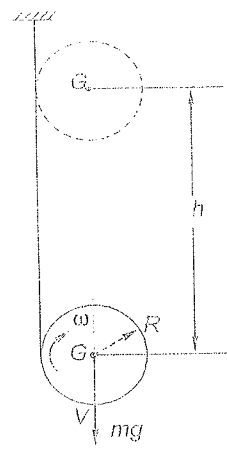
$$F = \frac{W \cos 30°}{\sin 30°} = \sqrt{3}\, W$$

Student's Assignments

Q.1 A flywheel of centroidal radius of gyration 0.5 m is rigidly attached to a shaft of radius 25 mm which may roll along parallel rails inclined at an angle of 20° as shown in figure. The system is released from rest. What is the velocity of the centre of the shaft after it has moved 2 m along the rails?



Q.2 A string is wound over a cylinder of mass m and radius R. At a particular instant the cylinder is released and string unwinds over the cylinder, allowing the cylinder to fall through a height h as shown in figure. What is the angular velocity of the cylinder? What is the linear velocity of the mass centre G of the cylinder?



Q.3 A slender rod OA, hinged at O, of length 1 m is of mass 4 kg and a coller of mass 2 kg is attached to the centre of the rod. Horizontal position shown is at rest. Rod is released and reaches vertical position, what is angular speed of rod in vertical position?