Long Answer Questions-I (PYQ)

[4 Mark]

Q.1. Evaluate: $\int \frac{2x}{(x^2+1)(x^2+3)} dx$

Ans.

Let $x^2 = z \Rightarrow 2x\,dx = dz$

$$\therefore \int \frac{2x}{(x^2+1)(x^2+3)} dx = \int \frac{dz}{(z+1)(z+3)}$$

Now, $\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3} \quad \ldots(i)$

$$\frac{1}{(z+1)(z+3)} = \frac{A(z+3)+B(z+1)}{(z+1)(z+3)}$$

$$\Rightarrow \quad 1 = A(z+3) + B(z+1) \quad \Rightarrow \quad 1 = (A+B)z + (3A+B)$$

Equating the coefficient of z and constant, we get

$$A + B = 0 \quad \ldots(ii)$$

and $\quad 3A + B = 1 \quad \ldots(iii)$

Subtracting (ii) from (iii), we get

$$2A = 1 \quad \Rightarrow \quad A = \frac{1}{2}$$

$$\therefore \quad B = -\frac{1}{2}$$

Putting the values of A and B in (i), we get

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$\therefore \int \frac{2x \; dx}{(x^2+1)(x^2+3)} = \int \frac{dz}{(z+1)(z+3)}$$

$$= \int \left(\frac{1}{2(z+1)} - \frac{1}{2(z+3)} \right) dz = \frac{1}{2} \int \frac{dz}{z+1} - \frac{1}{2} \int \frac{dz}{z+3}$$

$$= \frac{1}{2} \log |z+1| - \frac{1}{2} \log |z+3| + C = \frac{1}{2} \log |x^2+1| - \frac{1}{2} \log |x^2+3|$$

$$= \frac{1}{2} \log \left| \frac{x^2+1}{x^2+3} \right| + C$$

$$\begin{bmatrix} \text{Note: } \log m + \log n = \log m \,.\, n \\ \text{and } \quad \log m - \log n = \log m/n \end{bmatrix}$$

$$= \log \sqrt{\frac{x^2+1}{x^2+3}} + C$$

Q.2. **Evaluate :** $-\int e^{2x} \sin x \, dx$

Ans.

$$\text{Let } I = \int e^{2x} \sin x \; dx$$

$$= -e^{2x} \cos x - \int 2e^{2x} (-\cos x) dx = -e^{2x} \cos x + 2 \int e^{2x} \cos x \; dx$$

$$= -e^{2x} \cos x + 2[e^{2x} \sin x - \int 2e^{2x} \sin x \; dx]$$

$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x \; dx + C'$$

$$= e^{2x} (2 \sin x - \cos x) - 4I + C'$$

$$\Rightarrow \; I = \frac{e^{2x}}{5} [2 \sin x - \cos x] + C \qquad \left[\text{where } C = \frac{C'}{5} \right]$$

Q.3. **Evaluate:** $\int sin\, x\, sin^2\, x sin^3\, x\, dx$

Ans.

Let $I = \int \sin x \sin 2x \sin 3x \ \mathrm{dx}$

$= \frac{1}{2} \int 2 \sin x . \sin 2x . \sin 3x \, \mathrm{dx} = \frac{1}{2} \int \sin x . (2 \sin 2x . \sin 3x) \, \mathrm{dx}$

$= \frac{1}{2} \int \sin x . (\cos x - \cos 5x) \, \mathrm{dx} \qquad [\because 2\sin A \sin B = \cos (A - B) - \cos (A + B)]$

$= \frac{1}{2\times2} \int 2 \sin x . \cos x \, \mathrm{dx} - \frac{1}{2\times2} \int 2 \sin x . \cos 5x \, \mathrm{dx}$

$= \frac{1}{4} \int \sin 2x \, \mathrm{dx} - \frac{1}{4} \int (\sin 6x - \sin 4x) \mathrm{dx}$
$\left[\because \sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}\right]$

$= - \frac{\cos 2x}{8} + \frac{\cos 6x}{24} - \frac{\cos 4x}{16} + C$

Q.4. **Evaluate:** $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x . \cos^2 x} \mathrm{dx}$

Ans.

Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x . \cos^2 x} \mathrm{dx} \quad \Rightarrow \quad I = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x . \cos^2 x} \mathrm{dx}$

$\Rightarrow \quad I = \int \frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x . \cos^2 x + \cos^4 x)}{\sin^2 x . \cos^2 x} \mathrm{dx}$

$\Rightarrow \quad I = \int \frac{\sin^4 x - \sin^2 x . \cos^2 x + \cos^4 x}{\sin^2 x . \cos^2 x} \mathrm{dx} = \int \tan^2 x \, \mathrm{dx} - \int \mathrm{dx} + \int \cot^2 \mathrm{xdx}$

$\Rightarrow \quad I = \int (\sec^2 x - 1) \mathrm{dx} - x + \int (\mathrm{cosec}^2 x - 1) \mathrm{dx}$

$\Rightarrow I = \int \sec^2 x \, \mathrm{dx} + \int \mathrm{cosec}^2 x \, \mathrm{dx} - x - x - x + C = \tan x - \cot x - 3x + C$

Q.5. **Evaluate:** $\int (x - 3)\sqrt{x^2 + 3x - 18} \, \mathrm{dx}$

Ans.

Let $\quad I = \int (x - 3)\sqrt{x^2 + 3x - 18} \, \mathrm{dx} \quad \ldots(i)$

$x - 3 = A\frac{d}{dx}(x^2 + 3x - 18) + B \qquad \Rightarrow \qquad x - 3 = A(2x + 3) + B \ \ldots(ii)$

$\Rightarrow \ x - 3 = 2Ax + (3A + B)$

Equating the co-efficient, we get

$2A = 1$ and $3A + B = -3 \quad \Rightarrow \; A = \frac{1}{2}$ and $3 \times \frac{1}{2} + B = -3$

$\Rightarrow \; A = \frac{1}{2}$ and $B = -3 - \frac{3}{2} = -\frac{9}{2}$

$\therefore \; I \int \left(\frac{1}{2}(2x+3) - \frac{9}{2}\right) \int \sqrt{x^2+3x-18}\, dx$ [From (*i*) and (*ii*)]

$I = \frac{1}{2} \int (2x+3)\sqrt{x^2+3x-18}\, dx - \frac{9}{2} \int \sqrt{x^2+3x-18}\, dx$

$\Rightarrow \; I = \frac{1}{2} I_1 - \frac{9}{2} I_2 \; \ldots(iii)$

$$\begin{bmatrix} \text{where } I_1 = \int (2x+3)\sqrt{x^2+3x-18}\, dx \\ \text{and} \quad I_2 = \int \sqrt{x^2+3x-18}\, dx \end{bmatrix}$$

Now, $I_1 = \int (2x+3)\sqrt{x^2+3x-18}\, dx$

Put $x^2 + 3x - 18 = z \quad \Rightarrow \quad (2x+3)\, dx = dz$

$I_1 = \int \sqrt{z}\, dz = \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C_1 = \frac{2}{3}(z)^{\frac{3}{2}} + C_1 \quad \Rightarrow \quad I_1 = \frac{2}{3}(x^2+3x-18)^{\frac{3}{2}} + C_1 \ldots (iv)$

Again, $I_2 = \int \sqrt{x^2+3x-18}\, dx$

$= \int \sqrt{x^2 + 2.x.\frac{3}{2} + \left(\frac{3}{2}\right)^2 - \frac{9}{4} - 18}\, dx = \int \sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}$

$I_2 = \frac{1}{2}\left(x+\frac{3}{2}\right)\sqrt{x^2+3x-18} - \frac{81}{4\times 2}\log\left|\left(x+\frac{3}{2}\right) + \sqrt{x^2+3x-18}\right|$

$\Rightarrow I_2 = \frac{1}{2}\left(x+\frac{3}{2}\right)\sqrt{x^2+3x-18} - \frac{81}{8}\log\left|\left(x+\frac{3}{2}\right) + \sqrt{x^2+3x-18}\right| + C_2 \ldots (v)$

Putting the value of I_1 and I_2 in (*iii*), we get

$I = \frac{1}{3}(x^2+3x-18)^{\frac{3}{2}} - \frac{9}{4}\left(x+\frac{3}{2}\right)\sqrt{x^2+3x-18} + \frac{729}{16}\log\left|\left(x+\frac{3}{2}\right) + \sqrt{x^2+3x-\right.$

$\left[\text{where } C = \frac{C_1}{2} - \frac{9}{2}C_2\right]$

Q.6. **Evaluate:** $\int \frac{2}{(1-x)(1+x^2)}dx$

Ans.

Here $I = \int \frac{2}{(1-x)(1+x^2)}dx$

Now, $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2} = \frac{A(1+x^2)+(Bx+C)(1-x)}{(1-x)(1+x^2)}$

$\Rightarrow 2 = A(1+x^2) + (Bx+C)(1-x) = A + Ax^2 + Bx - Bx^2 + C - Cx$

$\Rightarrow 2 = (A+C) + (A-B)x^2 + (B-C)x$

Equating co-efficient both sides, we get

$A + C = 2 \quad ...(i)$

$A - B = 0 \quad ...(ii)$

$B - C = 0 \quad ...(iii)$

From (ii) and (iii) $A = B = C$

Putting $C = A$ in (i), we get

$A + A = 2 \quad \Rightarrow \quad 2A = 2 \quad \Rightarrow \quad A = 1$

i.e., $A = B = C = 1$

$\therefore \quad \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$

$\therefore \quad \int \frac{2}{(1-x)(1+x^2)} = \int \frac{1}{1-x}dx + \int \frac{x+1}{1+x^2}dx = -\log|1-x| + \int \frac{x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$

$= -\log|1-x| + \frac{1}{2}\log|1+x^2| + \tan^{-1}x + C_1$

Q.7. **Find:** $\int \frac{(5x-2)}{(3x^2+2x+1)}dx$

Ans.

The given integral is in the form of $\int \frac{(px+q)}{ax^2+bx+c} dx$

Therefore, we express

$$(5x-2) = A\frac{d}{dx}(1+2x+3x^2) + B = A(2+6x) + B$$

Equating the coefficients of x and the constant term on both the sides, we get

$$6A = 5 \text{ and } 2A + B = -2 \quad \text{or} \quad A = \frac{5}{6} \text{ and } B = -\frac{11}{3}$$

$$\therefore \int \frac{(5x-2)}{(1+2x+3x^2)} dx = \frac{5}{6}\int \frac{2+6x}{(1+2x+3x^2)} dx - \frac{11}{3}\int \frac{dx}{3x^2+2x+1}$$

$$= \frac{5}{6} I_1 - \frac{11}{3} I_2 \quad (\text{say}) \qquad \ldots(i)$$

In I_1 putting $1 + 2x + 3x^2 = t$, so that $(2+6x)\,dx = dt$

$$\therefore I_1 = \int \frac{dt}{t} = \log|t| + C_1 = \log|3x^2+2x+1| + C_1 \quad \ldots(ii)$$

and $I_2 = \int \frac{dx}{3x^2+2x+1} = \int \frac{dx}{3\left(x^2+\frac{2x}{3}+\frac{1}{3}\right)} = \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}$

Putting , $\left(x+\frac{1}{3}\right) = t$ so that $dx = dt$, we get

$$I_2 = \frac{1}{3}\int \frac{dt}{t^2+\left(\frac{\sqrt{2}}{3}\right)^2} = \frac{1}{3.\frac{\sqrt{2}}{3}}\tan^{-1}\frac{t}{\frac{\sqrt{2}}{3}} + C_2$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{3t}{\sqrt{2}} + C_2 = \frac{1}{\sqrt{2}}\tan^{-1}\frac{3\left(x+\frac{1}{3}\right)}{\sqrt{2}} + C_2$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{(3x+1)}{\sqrt{2}} + C_2 \qquad \ldots(iii)$$

Using (ii) and (iii) in (i), we get

$$\int \frac{(5x-2)}{(1+2x+3x^2)}.dx = \frac{5}{6}\log|3x^2+2x+1| - \frac{11}{3\sqrt{2}}\tan^{-1}\frac{(3x+1)}{\sqrt{2}} + C$$

where, $C = C_1 + C_2$

Q.8. Evaluate: $\int \frac{3x-1}{(x+2)^2} dx$

Ans.

Let $\frac{3x-1}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} \quad \Rightarrow \quad \frac{3x-1}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2}$

$\Rightarrow 3x - 1 = A(x+2) + B \quad \Rightarrow \quad 3x - 1 = Ax + (2A + B)$

Equating the coefficient of x and constant term both side, we get

$A = 3,\ 2A + B = -1$

$\Rightarrow 2 \times 3 + B = -1 \quad \Rightarrow \quad B = -7$

$\therefore \frac{3x-1}{(x+2)^2} = \frac{3}{x+2} - \frac{7}{(x+2)^2}$

$\Rightarrow \int \frac{3x-1}{(x+2)^2} dx = \int \frac{3}{x+2} dx - \int \frac{7}{(x+2)^2} dx$

$= 3 \log |x+2| - 7 \frac{(x+2)^{-1}}{-1} + C = 3 \log |x+2| + \frac{7}{(x+2)} + C$

Q.9. Evaluate: $\int \frac{\sin (x-a)}{\sin (x+a)} dx$

Ans.

Let $I = \int \frac{\sin (x-a)}{\sin (x+a)} dx$

Let $x + a = t \quad \Rightarrow x = t - a \quad \Rightarrow dx = dt$

$\therefore \quad I = \int \frac{\sin (t-2a)}{\sin t} dt = \int \frac{\sin t . \cos 2a - \cos t . \sin 2a}{\sin t} dt$

$= \cos 2a \int dt - \int \sin 2a . \cot t \ dt = \cos 2a . t - \sin 2a . \log |\sin t| + C$

$= \cos 2a . (x + a) - \sin 2a . \log |\sin (x+a)| + C$

$= x \cos 2a + a \cos 2a - (\sin 2a) \log |\sin (x+a)| + C$

Q.10. Evaluate: $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} dx$

Ans.

Let $I = \int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} dx$

Put $e^x = t \Rightarrow e^x\, dx = dt$, we get

$$\therefore\ I = \int \frac{dt}{\sqrt{5-4t-t^2}} = \int \frac{dt}{\sqrt{-(t^2+4t-5)}} = \int \frac{dt}{\sqrt{-(t^2+2.t.2+2^2-9)}}$$

$$= \int \frac{dt}{\sqrt{3^2-(t+2)^2}} = \sin^{-1}\frac{t+2}{3} + C = \sin^{-1}\left(\frac{e^x+2}{3}\right) + C$$

Q.11. Evaluate: $\int x \sin^{-1} x\, dx$

Ans.

Let $I = \int x \sin^{-1} x\, dx$

$$= \sin^{-1} x.\frac{x^2}{2} - \int \frac{x^2}{2\sqrt{1-x^2}} dx \qquad \text{[By using integration by part]}$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}} dx = \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \sqrt{1-x^2} dx - \frac{1}{2}\int \frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\sin^{-1} x + \frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right] + C$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + \frac{1}{4}\sin^{-1} x + C$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + C$$

Q.12. Evaluate: $\int e^x \left(\frac{\sin 4x-4}{1-\cos 4x}\right) dx$

Ans.

Let $I = \int e^x \left(\frac{\sin 4x - 4}{1 - \cos 4x}\right) dx$

$= \int e^x \left(\frac{2 \sin 2x . \cos 2x - 4}{2 \sin^2 2x}\right) dx$ $\quad [\because \sin 2x = 2 \sin x . \cos x$ and $\cos 2x = 1 - \sin^2 x]$

$= \int e^x (\cot 2x - 2 \operatorname{cosec}^2 2x) dx$

Let $f(x) = \cot 2x \therefore f'(x) = -2\operatorname{cosec}^2 2x$

$\therefore I = \int e^x (f(x) + f'(x)) dx$

$\Rightarrow I = e^x . f(x) + C = e^x . \cot 2x + C$ $\quad [\because \int e^x (f(x) + f'(x)) dx = e^x f(x) + C]$

Q.13. **Evaluate:** $\int \frac{x+2}{\sqrt{x^2+5x+6}} dx$

Ans.

Let $I = \int \frac{x+2}{\sqrt{x^2+5x+6}} dx$

Now, we can express as

$x + 2 = A \frac{d}{dx}(x^2 + 5x + 6) + B$

$\Rightarrow x + 2 = A(2x + 5) + B \quad \Rightarrow \quad x + 2 = 2Ax + (5A + B)$

Equating coefficients both sides, we get

$2A = 1, \; 5A + B = 2 \quad \Rightarrow \quad A = \frac{1}{2}, B = 2 - \frac{5}{2} = -\frac{1}{2}$

$\therefore \; x + 2 = \frac{1}{2}(2x + 5) - \frac{1}{2}$

Hence, $I = \int \frac{\frac{1}{2}(2x+5) - \frac{1}{2}}{\sqrt{x^2+5x+6}} dx = \frac{1}{2} \int \frac{2x+5}{\sqrt{x^2+5x+6}} dx - \frac{1}{2} \int \frac{dx}{\sqrt{x^2+5x+6}}$

$I=\frac{1}{2}.\,I_1-\frac{1}{2}I_2 \quad \dots (i)$

where, $I_1=\int\frac{2x+5}{\sqrt{x^2+5x+6}}\mathrm{dx}\,, I_2=\int\frac{\mathrm{dx}}{\sqrt{x^2+5x+6}}$

Now, $I_1=\int\frac{2x+5}{\sqrt{x^2+5x+6}}\mathrm{dx}\,,$

Let $x^2+5x+6=z \qquad \Rightarrow \qquad (2x+5)\,dx=dz$

$\therefore\ I_1=\int\frac{\mathrm{dz}}{\sqrt{z}}=\int z^{\frac{-1}{2}}\mathrm{dz}=\frac{z^{-\frac{1}{2}+1}}{\frac{-1}{2}+1}+C_1=2\sqrt{z}+C_1=2\sqrt{x^2+5x+6}+C_1$

Again $I_2=\int\frac{\mathrm{dx}}{\sqrt{x^2+5x+6}}=\int\frac{\mathrm{dx}}{\sqrt{x^2+2\times x\times\frac{5}{2}+\left(\frac{5}{2}\right)^2-\frac{25}{4}+6}}$

$=\int\frac{\mathrm{dx}}{\left(x+\frac{5}{2}\right)^2-\frac{1}{4}}=\int\frac{\mathrm{dx}}{\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$

$=\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right|+C_2$

Putting the value of I_1 and I_2 in (i), we get

$I=\frac{1}{2}\left\{2\sqrt{x^2+5x+6}+C_1\right\}-\frac{1}{2}\left\{\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right|+C_2\right\}$

$\Rightarrow\ I=\sqrt{x^2+5x+6}-\frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right|+\frac{1}{2}C_1-\frac{1}{2}C_2$

$=\sqrt{x^2+5x+6}-\frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right|+C$

[where $C=\frac{1}{2}C_1-\frac{1}{2}C_2$]

Q.14. **Evaluate:** $\int\frac{(x^2-3x)}{(x-1)(x-2)}\mathrm{dx}$

Ans.

Let $I=\int\frac{(x^2-3x)}{(x-1)(x-2)}dx=\int\frac{(x^2-3x)}{x^2-3x+2}dx$

$=\int\frac{x^2-3x+2-2}{x^2-3x+2}dx=\int dx-\int\frac{2\,dx}{x^2-3x+2}$

$=x-2\int\frac{dx}{x^2-2.x.\frac{3}{2}+\frac{9}{4}-\frac{9}{4}+2}=x-2\int\frac{dx}{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}$

$=x-2\log\left|\frac{x-\frac{3}{2}-\frac{1}{2}}{x-\frac{3}{2}+\frac{1}{2}}\right|+C$

$=x-2\log\left|\frac{x-2}{x-1}\right|+C \qquad \left[\because\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$

Q.15. Evaluate: $\int x^2\tan^{-1}x\,dx$

Ans.

$\int x^2\tan^{-1}x\,dx=\tan^{-1}x.\frac{x^3}{3}-\int\frac{1}{1+x^2}.\frac{x^3}{3}dx$ [By using integration by parts]

$=\frac{x^3\tan^{-1}x}{3}-\frac{1}{3}\int\left(x-\frac{x}{x^2+1}\right)dx$

$$\left[\begin{array}{l}1+x^2\overset{x}{\overline{\big)x^3}}\\ \quad\underline{-x^3\pm x}\\ \qquad -x\end{array}\right]$$

$=\frac{x^3\tan^{-1}x}{3}-\frac{1}{3}\left[\int x\,dx-\int\frac{x}{x^2+1}dx\right]$

$=\frac{x^3\tan^{-1}x}{3}-\frac{1}{3}\frac{x^2}{2}+\frac{1}{3}\int\frac{dz}{2z}$

$$\left[\begin{array}{l}Let\ x^2+1=z\\ \Rightarrow\ 2xdx=dz\\ \Rightarrow\ xdx=\frac{dz}{2}\end{array}\right]$$

$=\frac{x^3\tan^{-1}x}{3}-\frac{x^2}{6}+\frac{1}{6}\log|z|+C$

$=\frac{x^3\tan^{-1}x}{3}-\frac{x^2}{6}+\frac{1}{6}\log\left|x^2+1\right|+C$

Q.16. Evaluate: $\int \frac{x^2+1}{(x-1)^2\,(x+3)}\text{dx}$

Ans.

Let $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$

$\Rightarrow \quad \frac{x^2+1}{(x-1)^2(x+3)} = \frac{A(x-1)\,(x+3)+B(x+3)+C(x-1)^2}{(x-1)^2(x+3)}$

$\Rightarrow \; x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2 \qquad \ldots(i)$

Putting $x = 1$ in (i), we get

$\Rightarrow \; 2 = 4B \quad \Rightarrow \quad B = \frac{1}{2}$

Putting $x = -3$ in (i), we get

$\Rightarrow 10 = 16C \quad \Rightarrow \quad C = \frac{10}{16} = \frac{5}{8}$

Putting $x = 0,\, B = \frac{1}{2}, C = \frac{5}{8}$ in (i), we get

$1 = A(-1)\,.\,(3) + \frac{1}{2} \times 3 + \frac{5}{8}(-1)^2 \quad \Rightarrow \quad 1 = -3A + \frac{3}{2} + \frac{5}{8}$

$\Rightarrow \; 3A = \frac{12+5}{8} - 1 = \frac{17}{8} - 1 = \frac{9}{8} \quad \Rightarrow \quad A = \frac{3}{8}$

$\therefore \; \frac{x^2+1}{(x-1)^2(x+3)} = \frac{3}{8(x-1)} + \frac{1}{2(x-1)^2} + \frac{5}{8(x+3)}$

$\therefore \; \int \frac{x^2+1}{(x-1)^2(x+3)}\text{dx} = \int \left(\frac{3}{8(x-1)} + \frac{1}{2(x-1)^2} + \frac{5}{8(x+3)} \right) \text{dx}$

$= \frac{3}{8} \int \frac{\text{dx}}{x-1} + \frac{1}{2} \int (x-1)^{-2}\text{dx} + \frac{5}{8} \int \frac{\text{dx}}{x+3}$

$= \frac{3}{8}\log\left|x-1\right| - \frac{1}{2(x-1)} + \frac{5}{8}\log\left|x+3\right| + C$

Q.17. Find : $\int \frac{dx}{\sin x + \sin 2x}$

Ans.

Here, $I = \int \frac{1}{\sin x + \sin 2x} dx$

$\Rightarrow \quad I = \int \frac{1}{\sin x + 2\sin x \cos x} dx \qquad \Rightarrow I = \int \frac{1}{\sin x(1+2\cos x)} dx$

$\Rightarrow \quad I = \int \frac{\sin x}{\sin^2 x(1+2\cos x)} dx \qquad \Rightarrow I = \int \frac{\sin x}{(1-\cos^2 x)(1+2\cos x)} dx$

Let $\cos x = z \quad \Rightarrow \ -\sin x\, dx = dz$

$\Rightarrow \ I = \int \frac{-dz}{(1-z^2)(1+2z)} \qquad \Rightarrow \qquad I = -\int \frac{dz}{(1+z)(1-z)(1+2z)}$

Here, integrand is proper rational function. Therefore, by the form of partial function, we can write

$\frac{1}{(1+z)(1-z)(1+2z)} = \frac{A}{1+z} + \frac{B}{1-z} + \frac{C}{1+2z} \qquad ...(i)$

$\Rightarrow \quad \frac{1}{(1+z)(1-z)(1+2z)} = \frac{A(1-z)(1+2z)+B(1+z)(1+2z)+C(1+z)(1-z)}{(1+z)(1-z)(1+2z)}$

$\Rightarrow \ 1 = A(1-z)(1+2z)+B(1+z)(1+z)(1+2z) + C(1+z)(1-z) \qquad ....(ii)$

Putting the value of $z = -1$ in (ii), we get

$\Rightarrow 1 = -2A + 0 + 0 \qquad \Rightarrow \qquad A = -1/2$

Again, putting the value of $z = 1$ in (ii), we get

$\Rightarrow \ 1 = 0 + B.2.(1+2) + 0 \Rightarrow 1 = 6B \Rightarrow B = \frac{1}{2}$

Similarly, putting the value of $z = -\frac{1}{2}$ in (ii), we get

$\Rightarrow \ 1 = 0 + 0 + C \ \left(\frac{1}{2}\right)\left(\frac{3}{2}\right) \qquad \Rightarrow \qquad 1 = \frac{3}{4}C \qquad \Rightarrow \qquad C = \frac{4}{3}$

Putting the value of A, B, C in (i), we get

$$\frac{1}{(1+z)(1-z)(1-2z)} = \frac{1}{2(1+z)} + \frac{1}{6(1-z)} + \frac{4}{3(1+2z)}$$

$$\therefore\ I = -\int \left[-\frac{1}{2(1+z)} + \frac{1}{6(1-z)} + \frac{4}{3(1+2z)} \right] dz$$

$$\therefore\ I = \int \left[\frac{1}{2(1+z)} - \frac{1}{6(1-z)} - \frac{4}{3(1+2z)} \right] dz$$

$$\Rightarrow\ I = \frac{1}{2}\log|1+z| + \frac{1}{6}\log|1-z| - \frac{4}{3\times 2}\log|1+2z| + C$$

Putting the value of z, we get

$$\Rightarrow\ I = \frac{1}{2}\log|1+\cos x| + \frac{1}{6}\log|1-\cos x| - \frac{2}{3}\log|1+2\cos x| + C$$

Q.18. Integrate the following w.r.t. *x*.

$$\frac{x^2-3x+1}{\sqrt{1-x^2}}$$

Ans.

$$\text{Let } I = \int \frac{x^2-3x+1}{\sqrt{1-x^2}} dx = \int \frac{x^2-1+2-3x}{\sqrt{1-x^2}} dx$$

$$= \int \frac{-(1-x^2)}{\sqrt{1-x^2}} dx + 2\int \frac{dx}{\sqrt{1-x^2}} - 3\int \frac{xdx}{\sqrt{1-x^2}}$$

$$= -\int \sqrt{1-x^2}\, dx + 2\int \frac{dx}{\sqrt{1-x^2}} - 3\int \frac{xdx}{\sqrt{1-x^2}}$$

$$= -\frac{1}{2}x\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x + 2\sin^{-1}x + 3\sqrt{1-x^2} + C$$

$$= \frac{3}{2}\sin^{-1}x - \frac{1}{2}x\sqrt{1-x^2} + 3\sqrt{1-x^2} + C$$

Evaluate: $\int \frac{x^2+1}{(x^2+4)(x^2+25)} dx$

Q.19.

Ans.

Let $I = \int \frac{x^2+1}{(x^2+4)(x^2+25)} dx$

Put $x^2 = y \quad \Rightarrow \quad \frac{x^2+1}{(x^2+4)(x^2+25)} = \frac{y+1}{(y+4)(y+25)}$

Now, $\frac{y+1}{(y+4)(y+25)} = \frac{A}{y+4} + \frac{B}{y+25}$

$\Rightarrow \quad \frac{y+1}{(y+4)(y+5)} = \frac{A(y+25)+B(y+4)}{(y+4)(y+5)}$

$\Rightarrow y + 1 = (A + B)y + (25A + 4B)$

Equating coefficients, we get

$A + B = 1$ and $25A + 4B = 1$

$\Rightarrow \ A = \frac{-1}{7}, B = \frac{8}{7}$

$\therefore \ \frac{x^2+1}{(x^2+4)(x^2+25)} = \frac{-1}{7(x^2+4)} + \frac{8}{7(x^2+25)}$

$\therefore I = \int \left[-\frac{1}{7(x^2+4)} + \frac{8}{7(x^2+25)}\right] dx = -\frac{1}{7} \int \frac{dx}{x^2+2^2} + \frac{8}{7} \int \frac{dx}{x^2+5^2}$

$= -\frac{1}{7} \times \frac{1}{2} \tan^{-1} \frac{x}{2} + \frac{8}{7} \times \frac{1}{5} \tan^{-1} \frac{x}{5} + C$

$= -\frac{1}{14} \tan^{-1} \frac{x}{2} + \frac{8}{35} \tan^{-1} \frac{x}{5} + C$

Q.20. **Evaluate:** $\int \frac{dx}{x(x^5+3)}$

Ans.

Let $I=\int \frac{dx}{x(x^5+3)}=\int \frac{x^4\,dx}{x^5(x^5+3)}=\frac{1}{5}\int \frac{5x^4\,dx}{x^5(x^5+3)}$

Put $x^5=z \quad \Rightarrow \quad 5x^4\,dx=dz$

$\therefore\ I=\frac{1}{5}\int \frac{dz}{z(z+3)}=\frac{1}{5\times 3}\int \frac{z+3-z}{z(z+3)}dz$

$=\frac{1}{15}\int \frac{z+3}{z(z+3)}dz-\frac{1}{15}\int \frac{z}{z(z+3)}dz$

$=\frac{1}{15}\int .\frac{dz}{z}-\frac{1}{15}\int \frac{dz}{z+3}=\frac{1}{15}\{\log z-\log|z+3|\}+C$

$=\frac{1}{15}\log\left|\frac{z}{z+3}\right|+C=\frac{1}{15}\log\left|\frac{x^5}{x^5+3}\right|+C$

Q.21. Evaluate: $\int \frac{2x^2+3}{x^2+5x+6}dx$

Ans.

Let $I=\int \frac{2x^2+3}{x^2+5x+6}dx$

$=\int\left(2-\frac{10x+9}{x^2+5x+6}\right)dx=2\int dx-\int \frac{10x+9}{x^2+5x+6}dx$

$$[x^2+5x+6\overline{\big)2x^2+3}^{\;2}]$$

$$\frac{-2x^2\pm 10x\pm 12}{-10x-9}$$

$=2x-\int \frac{10x+9}{x^2+3x+2x+6}dx$

$=2x-\int \frac{10x+9}{x(x+3)+2(x+3)}dx=2x-\int \frac{10x+9}{(x+3)(x+2)}dx$

$$= 2x - \int \left(\frac{-11}{x+2} + \frac{21}{x+3} \right) dx$$

$$\left[\begin{array}{l} \frac{10x+9}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3} \\ \Rightarrow \quad 10x + 9 = A(x+3) + B(x+2) \\ \text{Putting } x = -3, \text{ we get } B = 21 \\ \text{Putting } x = -2, \text{ we get } A = -11 \end{array} \right]$$

$$= 2x + 11 \int \frac{dx}{x+2} - 21 \int \frac{dx}{x+3}$$

$$= 2x + 11 \log|x+2| - 21 \log|x+3| + C$$

Q.22. Evaluate : $\int (3x - 2)\sqrt{x^2 + x + 1}\, dx$

Ans.

Let $I = \int (3x-2)\sqrt{x^2+x+1}\, dx$

Let $3x - 2 = A\frac{d}{dx}(x^2 + x + 1) + B$

$\Rightarrow 3x - 2 = A(2x+1) + B$

$\Rightarrow 3x - 2 = 2Ax + (A + B)$

Equating we get

$2A = 3$ and $A + B = -2$

$A = \frac{3}{2}$ and $B = -2 - \frac{3}{2} = -\frac{7}{2}$

Now, $I = \int \left\{ \frac{3}{2}(2x+1) - \frac{7}{2} \right\} \sqrt{x^2+x+1}\, dx$

$= \frac{3}{2}\int (2x+1)\sqrt{x^2+x+1}\, dx - \frac{7}{2}\int \sqrt{x^2+x+1}\, dx$

$\Rightarrow I = \frac{3}{2} I_1 - \frac{7}{2} I_2 \quad \ldots (i)$

Where, $I_1 = \int (2x+1)\sqrt{x^2+x+1}\ \mathrm{dx}$ and $I_2 = \sqrt{x^2+x+1}\ \mathrm{dx}$

Now, $I_1 = \int (2x+1)\sqrt{x^2+x+1}\ \mathrm{dx}$

Let $x^2 + x + 1 = z \quad \Rightarrow (2x+1)\ dx = dz$

$\Rightarrow I_1 = \int \sqrt{z}\ \mathrm{dz}$

$= \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C_1 = \frac{2}{3}z^{3/2} + C_1$

$I_1 = \frac{2}{3}(x^2+x+1)^{3/2} + C_1$ ... (ii)

Again $I_2 = \int \sqrt{x^2+x+1}\ \mathrm{dx}$

$= \int \sqrt{x^2 + 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \frac{1}{4} + 1}\,\mathrm{dx}$

$= \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{dx}$

$I_2 = \int \sqrt{x^2+x+1}\ dx$

$I_2 = \frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 .\ \log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right| + C_2$ ... (iii)

Putting value of I_1 and I_2 from (ii), (iii) in (i), we get

$I = (x^2+x+1)^{3/2} - \frac{7}{4}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} - \frac{21}{16}\log\ \left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right| +$

[where $C = \frac{3}{2}C_1 - \frac{7}{2}C_2$]

Q.23. **Evaluate :** $\int \frac{x^2}{(x^2+4)(x^2+9)}\mathrm{dx}$

Ans.

Let $I = \int \frac{x^2}{(x^2+4)(x^2+9)} \mathrm{dx}$

Put $x^2 = t$, we get

$$\therefore \frac{x^2}{(x^2+4)(x^2+9)} = \frac{t}{(t+4)(t+9)}$$

Now, $\frac{t}{(t+4)(t+9)} = \frac{A}{t+4} + \frac{B}{t+9} = \frac{A(t+9)+B(t+4)}{(t+4)(t+9)}$

$\Rightarrow t = (A + B)t + (9A + 4B)$

Equating the coefficients, we get

$$A + B = 1, 9A + 4B = 0$$

Solving above two equations, we get

$A = -\frac{4}{5}, B = \frac{9}{5}$

$$\therefore \frac{x^2}{(x^2+4)(x^2+9)} = -\frac{4}{5(x^2+4)} + \frac{9}{5(x^2+9)}$$

$$=-\frac{4}{5} \times \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{9}{5} \times \frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

$$I = -\frac{4}{5}\int \frac{\mathrm{dx}}{x^2+2^2} + \frac{9}{5}\int \frac{\mathrm{dx}}{x^2+3^2} = \frac{4}{5} \times \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{9}{5} \times \frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

$$= -\frac{2}{5}\tan^{-1}\frac{x}{2} + \frac{3}{5}\tan^{-1}\frac{x}{3} + C$$

Q.24. **Find:** $\int \frac{(3\sin\theta - 2)\cos\theta}{5 - \cos^2\theta - 4\sin\theta} d\theta$

Ans.

We have

$$I = \int \frac{(3\ \sin\theta - 2)\ \cos\theta}{5 - \cos^2\theta - 4\ \sin\theta} d\theta$$

Let sin θ = z

⇒ cos θdθ = dz

$$\therefore\ I = \int \frac{(3z-2)\ \mathrm{dz}}{5-(1-z^2)-4z}$$

$$= \int \frac{(3z-2)\ \mathrm{dz}}{5-1+z^2-4z} = \int \frac{(3z-2)}{4-4z+z^2} \mathrm{dz}$$

$$= \int \frac{3z-2}{(z-2)^2} \mathrm{dz} = \int \frac{3z}{(z-2)^2} \mathrm{dz} - 2\int \frac{\mathrm{dz}}{(z-2)^2}$$

Let $z - 2 = t$ ⇒ $dz = dt$

$$= \int \frac{3(t+2)\,\mathrm{dt}}{t^2} - 2\int \frac{\mathrm{dt}}{t^2} = 3\int \frac{t\,.\,\mathrm{dt}}{t^2} + 6\int \frac{\mathrm{dt}}{t^2} - 2\int \frac{\mathrm{dt}}{t^2}$$

$$= 3\int \frac{\mathrm{dt}}{t} + 4\int \frac{\mathrm{dt}}{t^2} = 3\log|t| + 4\frac{t^{-2+1}}{-2+1} + C$$

$$= 3\log\left|t\right| - 4.\frac{1}{t} + C$$

Putting value of t in terms of z then z in terms of q, we get

$$= 3\ \log\ \left|\sin\theta - \ 2\right| - \frac{4}{\sin\theta - 2} + C$$

Q.25. Find: $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} \mathrm{dx}$

Ans.

We have

$$I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} \mathrm{dx}$$

$$= \int \frac{x^{1/2}\ \mathrm{dx}}{\sqrt{a^3 - x^3}}$$

Let $x^{3/2} = t \Rightarrow \frac{3}{2}x^{1/2}dx = dt \Rightarrow x^{1/2}dx = \frac{2}{3}dt$

$I = \frac{2}{3}\int \frac{\mathrm{dt}}{\sqrt{(a^{3/2})^2 - t^2}}$ [$\because$ x 3/2 =t $\Rightarrow$ x 3 = t 2]

$= \frac{2}{3}\sin^{-1}\left(\frac{t}{a^{3/2}}\right) + C$

$\Rightarrow \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C$

$= \frac{2}{3}\sin^{-1}\sqrt{\frac{x^3}{a^3}} + C$

Q.26. **Find:** $\int \frac{(2x-5)e^{2x}}{(2x-3)^3}\mathrm{dx}$

Ans.

We have,

$I = \int \frac{(2x-5)e^{2x}}{(2x-3)^3}\mathrm{dx}$

$= \int e^{2x}\left[\frac{(2x-3)-2}{(2x-3)^3}\right]\mathrm{dx}$

$\Rightarrow \; e^3 \int e^{2x-3}\left[\frac{1}{(2x-3)^2} - \frac{2}{(2x-3)^3}\right] \mathrm{dx}$

Let $2x - 3 = t$

$\Rightarrow \; 2dx = dt \;\Rightarrow\; dx = \frac{dt}{2}$

$\Rightarrow \; I = \frac{e^3}{2} \int e^t \left[\frac{1}{t^2} - \frac{2}{t^3}\right] \mathrm{dt}$

$\Rightarrow \; I = \frac{e^3}{2} e^t . \frac{1}{t^2} + C$

Putting $t = 2x - 3$

$I = \frac{e^3}{2} e^{2x-3} \frac{1}{(2x-3)^2} + C$

$\Rightarrow \; I = \frac{e^{2x}}{2(2x-3)^2} + C$

Q.27. **Find :** $\int (2x+5)\sqrt{10-4x-3x^2}\,\mathrm{dx}$

Ans.

Let, $I = \int (2x+5)\sqrt{10-4x-3x^2}\,\mathrm{dx}$

Let $(2x+5) = A\frac{d}{\mathrm{dx}}(10-4x-3x^2) + B$

$\Rightarrow 2x + 5 = A(-4 - 6x) + B$

$\Rightarrow 2x + 5 = -4A - 6Ax + B$

Equating, we get

$-4A + B = 5 \qquad ...(1) \quad \text{and} \qquad -6A = 2 \qquad ...(2)$

(2) $\Rightarrow A = -\frac{1}{3}$

Now, from (1) $\frac{4}{3} + B = 5 \Rightarrow B = 5 - \frac{4}{3} = \frac{11}{3}$

$\therefore 2x + 5 = -\frac{1}{3}(-4 - 6x) + \frac{11}{3}$

Now, $I = \int \left\{-\frac{1}{3}(-4 - 6x) + \frac{11}{3}\right\}\sqrt{10 - 4x - 3x^2}\,\mathrm{dx}$

$= -\frac{1}{3}\int (-4 - 6x)\sqrt{10 - 4x - 3x^2}\,\mathrm{dx} + \frac{11}{3}\int \sqrt{10 - 4x - 3x^2}\,\mathrm{dx}$

$I = -\frac{1}{3}I_1 + \frac{11}{3}I_2$ ...(*i*)

where $I_1 = \int (-4 - 6x)\sqrt{10 - 4x - 3x^2}\,\mathrm{dx}$ and $I_2 = \int (10 - 4x - 3x^2)\,\mathrm{dx}$

Now, $I_1 = \int (-4 - 6x)\sqrt{10 - 4x - 3x^2}\,\mathrm{dx}$

Let $10 - 4x - 3x^2 = z$

$\Rightarrow (-4 - 6x)\,dx = dz$

$\therefore I_1 = \int \sqrt{z}\,\mathrm{dz} = \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C_1$

$= \frac{2}{3}(10 - 4x - 3x^2)^{3/2} + C_1$ ...(*ii*)

Again $I_2 = \int \sqrt{10 - 4x - 3x^2}\,\mathrm{dx} = \int \sqrt{-3\left(x^2 + \frac{4}{3}x - \frac{10}{3}\right)}\,\mathrm{dx}$

$= \sqrt{3}\int \sqrt{-\left\{x^2 + 2x.\frac{2}{3} + \frac{4}{9} - \frac{4}{9} - \frac{10}{3}\right\}}\mathrm{dx} = \sqrt{3}\int \sqrt{-\left\{\left(x + \frac{2}{3}\right)^2 - \frac{34}{9}\right\}}\,\mathrm{dx}$

$$= \sqrt{3} \int \sqrt{\frac{34}{9} - \left(x + \frac{2}{3}\right)^2} \, dx = \sqrt{3} \int \sqrt{\left(\frac{\sqrt{34}}{3}\right)^2 - \left(x + \frac{2}{3}\right)^2} \, dx$$

$$= \frac{\sqrt{3}}{2}\left(x + \frac{2}{3}\right)\sqrt{10 - 4x - 3x^2} + \frac{\sqrt{3}}{2} \times \frac{34}{9} \sin^{-1}\left(\frac{x + \frac{2}{3}}{\sqrt{\frac{34}{3}}}\right) + C_2 \quad ...(iii)$$

Putting the value of I_1 and I_2 in (i), we get

$$I = -\frac{1}{3} \times \frac{2}{3}(10 - 4x - 3x^2)^{3/2} + \frac{11}{2\sqrt{3}}\left(x + \frac{2}{3}\right)\sqrt{10 - 4x - 3x^2} + \frac{17}{9}\sin^{-1}\left(\frac{3}{\sqrt{34}}\left(x + \frac{2}{3}\right)\right) + C$$

Q.28. **Evaluate:** $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$

Ans.

Let $I = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx = \int_0^1 \log\left(\frac{1-x}{x}\right) dx \quad \ldots(i)$

$I = \int_0^1 \log\left(\frac{1-(1-x)}{1-x}\right) dx \quad \left[\because \int_0^a f(x) dx = \int_0^a f(a-x)\, dx\right]$

$I = \int_0^1 \log\left(\frac{x}{1-x}\right) dx \quad \ldots(ii)$

Adding (i) and (ii), we get

$2I = \int_0^1 \log\left(\frac{1-x}{x}\right) dx + \int_0^1 \log\left(\frac{x}{1-x}\right) dx$

$= \int_0^1 \log\left(\frac{1-x}{x} \cdot \frac{x}{1-x}\right) dx \quad [\because \log A + \log B = \log(A \times B)]$

$= \int_0^1 \log 1 \, dx$

$2I = 0$

$\therefore \ I = 0$

Q.29. **Evaluate:** $\int_0^{\pi} \frac{4x \sin x}{1 + \cos^2 x} dx$

Ans.

Let $I=\int_0^{\pi}\frac{4x\ \sin\ x}{1+\cos^2 x}dx\ \ldots(i)$

$$=\int_0^{\pi}\frac{4(\pi-x).\ \sin\ (\pi-x)}{1+\cos^2\ (\pi-x)}dx$$

$$I=\int_0^{\pi}\frac{4(\pi-x).\ \sin\ x}{1+\cos^2 x}dx\ \ldots(ii)$$

Adding (i) and (ii), we get

$$2I=\int_0^{\pi}\frac{4\ (x+\pi-x)\ \sin\ x}{1+\cos^2 x}dx \quad\Rightarrow\quad 2I=4\int_0^{\pi}\frac{\pi\ \sin\ x}{1+\cos^2 x}dx$$

$$I=2\pi\int_0^{\pi}\frac{\sin\ x}{1+\cos^2 x}dx$$

Let $\cos x = z \quad\Rightarrow\quad -\sin x\ dx = dz \quad\Rightarrow\quad \sin x\ dx = -\ dz$

The limits are, $x = 0 \quad\Rightarrow\quad z = 1$

$x = \pi \quad\Rightarrow\quad z = -1$

$$\therefore\ I=2\pi\int_1^{-1}\frac{-\ dz}{1+z^2}=2\pi[\tan^{-1} z]_{-1}^{1}$$

$$=2\pi[\tan^{-1}1-\tan^{-1}(-1)]=2\pi\left[\frac{\pi}{4}+\frac{\pi}{4}\right]=2\pi\times\frac{\pi}{2}$$

$\Rightarrow I = \pi^2$.

Q.30. **Evaluate:** $\int_{-\pi}^{\pi}(\cos\ ax-\sin\ bx)^2\,dx$

Ans.

Here, $I = \int_{-\pi}^{\pi} (\cos\ ax - \sin\ bx)^2\, dx$

$\Rightarrow\ I = \int_{-\pi}^{\pi} (\cos^2\ ax + \sin^2\ bx - 2\ \cos\ ax\ \sin\ bx)\, dx$

$\Rightarrow\ I = \int_{-\pi}^{\pi} \cos^2\ ax\ dx + \int_{-\pi}^{\pi} \sin^2 bx\ dx - \int_{-\pi}^{\pi} 2\cos\ ax\ \sin bx\ dx$

$\Rightarrow\ I = 2\int_0^{\pi} \cos^2\ ax\ dx + 2\int_0^{\pi} \sin^2 bx\ dx - 0$ [first two integrands are even function while third is odd function.]

$\Rightarrow I = \int_0^{\pi} 2\cos^2\ ax\ dx + \int_0^{\pi} 2\ \sin^2\ bx\ dx$

$\Rightarrow\ I = \int_0^{\pi} (1 + \cos\ 2ax)\, dx + \int_0^{\pi} (1 - \cos 2bx)\, dx$

$\Rightarrow I = \int_0^{\pi} dx + \int_0^{\pi} \cos 2ax\ dx + \int_0^{\pi} dx - \int_0^{\pi} \cos 2bx\ dx$

$\Rightarrow\ I = 2\int_0^{\pi} dx + \int_0^{\pi} \cos 2\ ax\ dx - \int_0^{\pi} \cos\ 2\ bx\ dx$

$\Rightarrow I = 2[x]_0^{\pi} + \left[\frac{\sin 2\ ax}{2a}\right]_0^{\pi} - \left[\frac{\sin 2\ bx}{2b}\right]_0^{\pi}$

$\Rightarrow I = 2\pi + \frac{\sin 2\ a\pi}{2a} - \frac{\sin 2b\pi}{2b}$

Q.31. **Evaluate:** $\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

Ans.

Let $I = \int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

$= \int_0^{\pi/2} \left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\right) dx = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\, dx$

$= \sqrt{2} \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}}\, dx$ [Multiplying and dividing by $\sqrt{2}$]

$$\Rightarrow\ I=\sqrt{2}\int_0^{\pi/2}\frac{(\sin x+\cos x)}{\sqrt{1-(\sin x-\cos x)^2}}dx$$

Let $\sin x-\cos x=t \Rightarrow (\cos x+\sin x)\ dx=dt$

Now, when $x=0 \Rightarrow t=-1$ and $x=\frac{\pi}{2} \Rightarrow t=1$

$$\therefore I=\sqrt{2}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}=\sqrt{2}\,[\sin^{-1}t]_{-1}^{1}=\sqrt{2}\,[\sin^{-1}(1)-\sin^{-1}(-1)]$$

$$=\sqrt{2}\,[sin^{-1}(1)+\sin^{-1}(1)]=\sqrt{2}.2\sin^{-1}(1)=\sqrt{2}.2.\frac{\pi}{2}=\sqrt{2}\pi.$$

Q.32. **Evaluate:** $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$

Ans.

Let $I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx \Rightarrow I=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)\,dx}{1+\cos^2(\pi-x)}$

$=\int_0^{\pi}\frac{(\pi-x)\sin x\ dx}{1+\cos^2 x}=\pi\int_0^{\pi}\frac{\sin x\ dx}{1+\cos^2 x}-I$

or $2I=\pi\int_0^{\pi}\frac{\sin x\ dx}{1+\cos^2 x}$ or $I=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x\ dx}{1+\cos^2 x}$

Put $\cos x=t$ so that $-\sin x\ dx=dt$.

The limits are, when $x=0$, $t=1$ and $x=$ p, $t=-1$, we get

$I=\frac{-\pi}{2}\int_1^{-1}\frac{dt}{1+t^2}=\pi\int_0^1\frac{dt}{1+t^2}$ $\left[\because \int_a^{-a}f(x)dx=-\int_{-a}^{a}f(x)\,dx \text{ and } \int_0^{2a}f(x)dx=2\int_0\right.$

$=\pi[\tan^{-1}t]_0^1=\pi[\tan^{-1}1-\tan^{-1}0]=\pi\left[\frac{\pi}{4}-0\right]=\frac{\pi^2}{4}$

Q.33. **Find:** $\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$

Ans.

Let $I = \int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$

$= \int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2.2\sin x.\cos x}}$

$= \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{\frac{\sin x}{\cos x}.\cos^2 x}} = \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^4 x\sqrt{\tan x}}$

$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^4 x \ dx}{\sqrt{\tan x}} = \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2 x.\ \sec^2 x \ dx}{\sqrt{\tan x}}$

Let $\tan x = t,\quad x = 0 \quad \Rightarrow \quad t = 0$ and $x = \frac{\pi}{4} \quad \Rightarrow \quad t = 1$

$\sec^2 x\, dx = dt$

$\therefore I = \frac{1}{2}\int_0^1 \frac{(1+t^2)\,dt}{\sqrt{t}}$

$= = \frac{1}{2}\int_0^1 (t^{-1/2} + t^{3/2})dt = \frac{1}{2}\left[\frac{t^{-1/2+1}}{-1/2+1}\right]_0^1 + \frac{1}{2}\left[\frac{t^{3/2+1}}{3/2+1}\right]_0^1$

$\frac{1}{2} \times \frac{2}{1}[\sqrt{t}]_0^1 + \frac{1}{2} \times \frac{2}{5}\left[t^{5/2}\right]_0^1 = 1 + \frac{1}{5} = \frac{6}{5}$

Q.34. **Evaluate:** $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}dx$

Ans.

Let $I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx$

$$= \int_{-\pi/2}^{0} \frac{\cos x}{1+e^x} dx + \int_{0}^{\pi/2} \frac{\cos x}{1+e^x} dx$$

$$= \int_{\pi/2}^{0} \frac{\cos t}{1+e^{-t}} (-dt) + \int_{0}^{\pi/2} \frac{\cos x}{1+e^x} dx$$

In 1st Integrand
Let $x = -t$
$dx = -dt$
$x = -\pi/2 \Rightarrow t = \pi/2$
$x = 0 \quad \Rightarrow t = 0$

$$= \int_{0}^{\pi/2} \frac{\cos t}{1+\frac{1}{e^t}} dt + \int_{0}^{\pi/2} \frac{\cos x}{1+e^x} dx = \int_{0}^{\pi/2} \frac{e^t.\cos t}{1+e^t} dt + \int_{0}^{\pi/2} \frac{\cos x}{1+e^x} dx$$

$$= \int_{0}^{\pi/2} \frac{e^x.\cos x}{1+e^x} dx + \int_{0}^{\pi/2} \frac{\cos x}{1+e^x} dx$$ [By property $\int_a^b f(x)dx = \int_a^b f(t)dt$]

$$= \int_{0}^{\pi/2} \frac{(e^x+1).\cos x}{1+e^x} dx = \int_{0}^{\pi/2} \cos x \, dx = [\sin x]_0^{\frac{\pi}{2}}$$

$= \sin \pi/2 - \sin 0$

$= 1.$

Q.35. Evaluate: $\int_0^{\pi} \frac{x}{1+\sin x} dx$

Ans.

Let $I = \int_0^{\pi} \frac{x \, dx}{1+\sin x}$ $\quad [\because \int_0^a f(x)dx = \int_0^a f(a-x)\,dx] \quad ...(i)$

$\therefore I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)} dx = \pi \int_0^{\pi} \frac{dx}{1+\sin x} - \int_0^{\pi} \frac{xdx}{1+\sin x} = \pi \int_0^{\pi} \frac{dx}{1+\sin x} - I \quad ...(ii)$

$\Rightarrow$

$2I = \pi \int_0^{\pi} \frac{dx}{1+\sin x} = 2\pi \int_0^{\pi/2} \frac{dx}{1+\sin x} \quad [\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ as $\sin(\pi - x) = \sin x$

$\Rightarrow I = \pi \int_0^{\pi/2} \frac{dx}{1+\sin x} = \pi \int_0^{\pi/2} \frac{dx}{1+\sin\left(\frac{\pi}{2} - x\right)} \quad [\because \int_0^a f(x)dx = \int_0^a f(a-x)dx]$

$$= \pi \int_0^{\pi/2} \frac{dx}{1+\cos x} = \pi \int_0^{\pi/2} \frac{dx}{2\cos^2 x/2}$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \sec^2 \frac{x}{2} dx = \frac{\pi}{2} \left[\frac{\tan x/2}{1/2}\right]_0^{\pi/2}$$

$$= \pi \left[\tan \frac{\pi}{4} - \tan 0\right] = \pi [1 - 0] = \pi$$

Q.36. Evaluate: $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

Ans.

Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}$

$[\text{By } u \sin g \text{ property } \int_a^b f(x)dx = \int_a^b f(a+b-x)dx]$ ...(i)

$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}$

$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\frac{1}{\sqrt{\tan x}}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$

$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}} dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_{\pi/6}^{\pi/3} \frac{(1+\sqrt{\tan x})}{(1+\sqrt{\tan x})} dx$

$= \int_{\pi/6}^{\pi/3} dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

$2I = \frac{\pi}{6}$ or $I = \frac{\pi}{12}$

Q.37. Evaluate: $\int_1^3 [|x-1| + |x-2| + |x-3|] \, dx$

Ans.

$$\text{Let } I=\int_1^3[|x-1|+|x-2|+|x-3|]dx=\int_1^3|x-1|dx+\int_1^3|x-2|dx+\int_1^3|x-3|dx$$

$$=\int_1^3|x-1|dx+\int_1^2|x-2|dx+\int_2^3|x-2|dx+\int_1^3|x-3|dx \quad \text{[By property of definite integral]}$$

$$=\int_1^3|x-1|dx+\int_1^2-(x-2)dx+\int_2^3(x-2)dx+\int_1^3-(x-3)dx \quad \left\{\begin{array}{l}x-1\geq 0, \text{ if } 1\leq x\leq 3\\ x-2\leq 0, \text{ if } 1\leq x\leq 2\\ x-2\geq 0, \text{ if } 2\leq x\leq 3\\ x-3\leq 0, \text{ if } 1\leq x\leq 3\end{array}\right\}$$

$$=\left[\frac{(x-1)^2}{2}\right]_1^3-\left[\frac{(x-2)^2}{2}\right]_1^2+\left[\frac{(x-2)^2}{2}\right]_2^3-\left[\frac{(x-3)^2}{2}\right]_1^3$$

$$=\left(\frac{4}{2}-0\right)-\left(0-\frac{1}{2}\right)+\left(\frac{1}{2}-0\right)-\left(-0-\frac{4}{2}\right)=2+\frac{1}{2}+\frac{1}{2}+2=5$$

Q.38. **Evaluate:** $\int_0^{\pi}\frac{x\tan x}{\sec x.\operatorname{cosec} x}\mathrm{dx}$

$$\text{Let } I=\int_0^{\pi}\frac{x\tan x}{\sec x.\operatorname{cos ec} x}\mathrm{dx}=\int_0^{\pi}\frac{x.\frac{\sin x}{\cos x}}{\frac{1}{\cos x}.\frac{1}{\sin x}}\mathrm{dx}$$

$$I=\int_0^{\pi}x\ \sin^2\ x$$

$$=\int_0^{\pi}(\pi-x)\sin^2(\pi-x)\mathrm{dx} \qquad [\because \int_0^a f(x)\mathrm{dx}=\int_0^a f(a-x)\mathrm{dx}]$$

$$I=\int_0^{\pi}\pi\sin^2 x\,\mathrm{dx}-\int_0^{\pi}x\sin^2 x\,\mathrm{dx} \quad\Rightarrow\quad 2I=\frac{\pi}{2}\int_0^{\pi}2\sin^2\ x\ \mathrm{dx}$$

$$=\frac{\pi}{2}\int_0^{\pi}(1-\cos 2x)\mathrm{dx}=\frac{\pi}{2}\int_0^{\pi}\mathrm{dx}-\frac{\pi}{2}\int_0^{\pi}\cos 2x\,\mathrm{dx}=\frac{\pi}{2}[x]_0^{\pi}-\frac{\pi}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi}$$

$$\Rightarrow 2I=\frac{\pi}{2}(\pi-0)-\frac{\pi}{4}(\sin 2\pi-\sin 0)$$

$$\Rightarrow 2I=\frac{\pi^2}{2}-0 \quad\Rightarrow\quad I=\frac{\pi^2}{2}$$

Q.39. Evaluate: $\int_0^{\pi/2}\log(\sin x)\mathrm{dx}$

Ans.

Let $I = \int_0^{\pi/2} \log(\sin x)dx = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right)dx$ ...(i)

$$I = \int_0^{\pi/2} \log \cos x \, dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x)dx = \int_0^{\pi/2} \log \sin x . \cos x \, dx$$

$$= \int_0^{\pi/2} \log \frac{2 \sin x . \cos x}{2} dx = \int_0^{\pi/2} (\log \sin 2x - \log 2)dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - \log 2 \int_0^{\pi/2} dx = I_1 - \log 2 [x]_0^{\pi/2}$$

$\left[\because \text{Here } \log \sin\left(\frac{2\pi}{2} - t\right) = \log \sin t \text{ and } \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a - x) = f(x) \right]$

$$2I = I_1 - \frac{\pi}{2} \log 2 \quad ...(iii)$$

where $I_1 = \int_0^{\pi/2} \log \sin 2x \, dx$

Put $2x = t \Rightarrow dx = \frac{dt}{2}$, If $x = 0, t = 0; x = \frac{\pi}{2}, t = \pi$

$\therefore \quad I_1 = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt$

$\Rightarrow \quad I_1 = \frac{1}{2}\int_0^{2.\pi/2} \log \sin t \, dt \quad \Rightarrow \quad I_1 = \frac{1}{2} \times 2\int_0^{\pi/2} \log \sin t \, dt$

$\Rightarrow \quad I_1 = \int_0^{\pi/2} \log \sin x \, dx \quad \left[\because \int_a^b f(x)dx = \int_a^b f(t)dt\right]$

$\Rightarrow \quad I_1 = I$

Putting it in (iii), we get

$$2I = I - \frac{\pi}{2} \log 2 \quad \Rightarrow \quad I = -\frac{\pi}{2} \log 2$$

Q.40. Evaluate: $\int_0^1 \cot^{-1}(1 - x + x^2)\,dx$

Ans.

Let $I = \int_0^1 \cot^{-1}(1-x+x^2)dx$

$$= \int_0^1 \tan^{-1} \frac{1}{1-x+x^2} dx \qquad \left[\because \cot^{-1} x = \tan^{-1} \frac{1}{x}\right]$$

$$= \int_0^1 \tan^{-1} \frac{x+(1-x)}{1-x(1-x)} dx$$

$$= \int_0^1 \{\tan^{-1} x + \tan^{-1}(1-x)\} dx \qquad \left[\because \tan^{-1}(x+y) = \tan^{-1} \frac{x+y}{1-xy}\right]$$

$$= \int_0^1 \tan^{-1} x \, dx + \int_0^1 \tan^{-1}(1-x) dx$$

$$= \int_0^1 \tan^{-1} x \, dx + \int_0^1 \tan^{-1}(1-(1-x)) dx \quad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^1 \tan^{-1} x \, dx + \int_0^1 \tan^{-1} x \, dx = 2\int_0^1 \tan^{-1} x \, dx = 2\int_0^1 1.\tan^{-1} x \, dx$$

$$= 2\left\{[\tan^{-1} x.x]_0^1 - \int_0^1 \frac{1}{1+x^2}.x.\,dx\right\} \quad \Rightarrow \quad 2\frac{\pi}{4} - \int_0^1 \frac{2x\,dx}{1+x^2} = \frac{\pi}{2} - [\log|1+x^2|]_0^1$$

$$= \frac{\pi}{2} - [\log 2 - \log 1] = \frac{\pi}{2} - \log 2$$

Q.41. Evaluate: $\int_0^1 x^2(1-x)^n \, dx$

Ans.

Let $I = \int_0^1 x^2(1-x)^n \, dx$

$$\Rightarrow \quad I = \int_0^1 (1-x)^2[1-(1-x)]^n dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^1 (1-2x+x^2)x^n dx = \int_0^1 (x^n - 2x^{n+1} + x^{n+2}) dx$$

$$= \left[\frac{x^{n+1}}{n+1} - 2.\frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{n+3}\right]_0^1 = \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}$$

$$= \frac{(n+2)(n+3) - 2(n+1)(n+3) + (n+1)(n+2)}{(n+1)(n+2)(n+3)}$$

$$= \frac{n^2+5n+6-2n^2-8n-6+n^2+3n+2}{(n+1)(n+2)(n+3)} = \frac{2}{(n+1)(n+2)(n+3)}$$

Q.42. Evaluate: $\int_1^3 (2x^2+5x)\,dx$ as a limit of a sum.

Ans.

Let $f(x) = 2x^2 + 5x$

Here $a = 1,\ b = 3$ $\quad\therefore\quad h=\frac{b-a}{n}=\frac{3-1}{n}=\frac{2}{n}$

$\Rightarrow nh = 2$

Also, $n \to \infty \Leftrightarrow h \to 0.$

$$\therefore\ \int_a^b f(x)dx=\lim_{h\to 0} h[f(a)+f(a+h)+\dots+f\{a+(n-1)h\}]$$

$$\therefore\ \int_1^3 (2x^2+5x)dx=\lim_{h\to 0} h[f(1)+f(1+h)+\dots+f\{1+(n-1)h\}]$$

$$=\lim_{h\to 0} h[\{2\times 1^2+5\times 1\}+\{2(1+h)^2+5(1+h)\}+\dots+\{2(1-\ (n-1)h)^2+5((1+(n-1)h\}]$$

$$=\lim_{h\to 0} h[(2+5)+\{2+4h+2h^2+5+5h\}+\dots+\{2+4(n-1)h+2(n-1)^2h^2+5+5(n-1)h\}]$$

$$=\lim_{h\to 0} h[7+\{7+9h+2h^2\}+\dots+\{7+9(n-1)h+2(n-1)^2h^2\}]$$

$$=\lim_{h\to 0} h[7n+9h\{1+2+\dots+(n-1)\}+2h^2\{1^2+2^2+\dots+(n-1)^2\}]$$

$$=\lim_{h\to 0}\left[7nh+9h^2\frac{(n-1).n}{2}+2h^3\frac{(n-1).n\,(2n-1)}{6}\right]$$

$$=\lim_{h\to 0}\left[7(nh)+\frac{9(nh)^2.\left(1-\frac{1}{n}\right)}{2}+\frac{2(nh)^3.\left(1-\frac{1}{n}\right).\left(2-\frac{1}{n}\right)}{6}\right]$$

$$=\lim_{n\to\infty}\left[14+\frac{36\left(1-\frac{1}{n}\right)}{2}+\frac{16\left(1-\frac{1}{n}\right).\left(2-\frac{1}{n}\right)}{6}\right]\quad [\because nh=2]$$

$$=\lim_{n\to\infty}\left[14+18\left(1-\frac{1}{n}\right)+\frac{8}{3}\left(1-\frac{1}{n}\right).\left(2-\frac{1}{n}\right)\right]$$

$$=14+18+\frac{8}{3}\times 1\times 2=32+\frac{16}{3}=\frac{96+16}{3}=\frac{112}{3}$$

Q.43. Evaluate: $\int_0^{\pi/2}\frac{x+\sin x}{1+\cos x}dx$

Ans.

$$\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x} dx = \int_0^{\pi/2} \frac{x}{1+\cos x} dx + \int_0^{\pi/2} \frac{\sin x}{1+\cos x} dx$$

$$= \int_0^{\pi/2} \frac{x}{2\cos^2 x/2} dx + \int_0^{\pi/2} \frac{2\sin x/2.\cos x/2}{2\cos^2 x/2} dx$$

$$= \frac{1}{2}\int_0^{\pi/2} x \sec^2 \frac{x}{2} dx + \int_0^{\pi/2} \tan \frac{x}{2} dx$$

$$= \frac{1}{2}\left[x.2\tan\frac{x}{2}\right]_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2} 1.2\tan\frac{x}{2} dx + \int_0^{\pi/2} \tan\frac{x}{2} dx$$

$$= \frac{1}{2}\left[\frac{\pi}{2}.2\tan\frac{\pi}{4} - 0\right] = \frac{\pi}{2}$$

Q.44. Evaluate : $\int_{-1}^{2} |x^3 - x| dx$

Ans.

If $x^3 - x = 0$

$\Rightarrow x(x^2 - 1) = 0$

$\Rightarrow x = 0$ or $x^2 = 1$

$\Rightarrow x = 0$ or $x = \pm 1$

$\Rightarrow x = 0, -1, 1$

Hence $[-1, 2]$ is divided into three sub intervals $[-1, 0]$, $[0, 1]$ and $[1, 2]$ such that

$x^3 - x \geq 0$ on $[1, 0]$

$x^3 - x \leq 0$ on $[0, 1]$

and $x^3 - x \geq 0$ on $[1, 2]$

Now, $\int_{-1}^{2} |x^3 - x| dx = \int_{-1}^{0} |x^3 - x| dx + \int_{0}^{1} |x^3 - x| dx + \int_{1}^{2} |x^3 - x| dx$

$= \int_{-1}^{0} (x^3 - x) dx + \int_{0}^{1} -(x^3 - x) dx + \int_{1}^{2} (x^3 - x) dx$

$= \left[\frac{x^4}{4} - \frac{x^2}{x}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$

$= \left\{0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right\} - \left\{\left(\frac{1}{4} - \frac{1}{2}\right) - 0\right\} + \left\{(4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right)\right\}$

$= -\frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2} + 2 - \frac{1}{4} + \frac{1}{2} = \frac{3}{2} - \frac{3}{4} + 2 = \frac{11}{4}$

Q.45. **Evaluate :** $\int_{0}^{2\pi} \frac{1}{1+e^{\sin x}} dx$

Ans.

Let $I = \int_{0}^{2\pi} \frac{1}{1+e^{\sin x}} dx$ ...(*i*)

Applying property $\int_{0}^{a} f(x) dx = \int_{0}^{a} f(a - x) dx$, we get

$I = \int_{0}^{2\pi} \frac{dx}{1+e^{\sin(2\pi - x)}} = \int_{0}^{2\pi} \frac{dx}{1+e^{-\sin x}} = \int_{0}^{2\pi} \frac{dx}{1+\frac{1}{e^{\sin x}}}$

$I = \int_{0}^{2\pi} \frac{e^{\sin x} dx}{e^{\sin x}+1}$ ...(*ii*)

Adding (*i*) and (*ii*), we get

$2I = \int_{0}^{2\pi} \frac{dx}{1+e^{\sin x}} + \int_{0}^{2\pi} \frac{e^{\sin x}}{1+e^{\sin x}} = \int_{0}^{2\pi} \frac{1+e^{\sin x}}{1+e^{\sin x}} dx = \int_{0}^{2\pi} dx = [x]_{0}^{2\pi}$

$\Rightarrow 2I = 2\pi \quad \Rightarrow \quad I = \pi.$

Q.46. Evaluate $\int_0^{\pi} e^{2x} . \sin\left(\frac{\pi}{4}+x\right) dx$.

Ans.

We have $I = \int_0^{\pi} e^{2x} . \sin\left(\frac{\pi}{4}+x\right) dx$

Integrating by part, we get.

$$I = \left[\sin\left(\frac{\pi}{4}+x\right) . \frac{e^{2x}}{2}\right]_0^{\pi} - \int_0^{\pi} \cos\left(\frac{\pi}{4}+x\right) . \frac{e^{2x}}{2} dx$$

$$= \frac{1}{2}\left[\sin\frac{5\pi}{4} . e^{2\pi} - \sin\frac{\pi}{4}\right] - \frac{1}{2}\int_0^{\pi} e^{2x} . \cos\left(\frac{\pi}{4}+x\right) dx$$

$$= \frac{1}{2}\left(-\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left[\left\{\cos\left(\frac{\pi}{4}+x\right) . \frac{e^{2x}}{2}\right\}_0^{\pi} + \int_0^{\pi} \sin\left(\frac{\pi}{4}+x\right) . \frac{e^{2x}}{2} dx\right]$$

$$= -\frac{e^{2\pi}+1}{2\sqrt{2}} - \frac{1}{2}\left[\cos\frac{5x}{4} . \frac{2^{2\pi}}{2} - \frac{1}{2}\cos\frac{\pi}{4}\right] - \frac{1}{4}\int e^{2x} . \sin\left(\frac{\pi}{4}+x\right) dx$$

$$I = -\frac{e^{2\pi}+1}{2\sqrt{2}} - \frac{1}{4} . e^{2\pi} . \cos\frac{5\pi}{4} + \frac{1}{4\sqrt{2}} - \frac{1}{4} I$$

$$\frac{5I}{4} = -\frac{e^{2\pi}+1}{2\sqrt{2}} + \frac{e^{2\pi}}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}$$

$$= -\frac{e^{2\pi}+1}{2\sqrt{2}} + \frac{e^{2\pi}+1}{4\sqrt{2}} \quad \Rightarrow \quad \frac{e^{2\pi}+1}{4\sqrt{2}}(-2+1)$$

$$\frac{5I}{4} = -\frac{e^{2\pi}+1}{4\sqrt{2}}$$

$$I = -\frac{e^{2\pi}+1}{5\sqrt{2}}$$

Q.47. Evaluate : $\int_{-2}^{2} \frac{x^2}{1+5^x} dx$ f

Ans.

Let $I=\int_{-2}^{2}\frac{x^2}{1+5^x}\mathrm{dx}$ ... (i)

$=\int_{-2}^{2}\frac{(2+(-2)-x)^2}{1+5^{(2+(-2)-x)}}\mathrm{dx}$ $\left[\int_a^b f(x)\mathrm{dx}=\int f(a+b-x)\mathrm{dx}\right]$

$=\int_{-2}^{2}\frac{(-x)^2}{1+5^{-x}}\mathrm{dx}=\int_{-2}^{2}\frac{x^2}{1+\frac{1}{5^x}}\mathrm{dx}$

$I=\int_{-2}^{2}\frac{5^x x^2}{1+5^x}\mathrm{dx}$ ...(ii)

Adding (i) and (ii), we get

$2I=\int_{-2}^{2}\frac{(1+5^x)x^2}{1+5^x}\mathrm{dx}$

$=\int_{-2}^{2}x^2\mathrm{dx}=\left[\frac{x^3}{3}\right]_{-2}^{2}$

$\Rightarrow\quad 2I=\frac{1}{3}[8-(-8)]$

$\Rightarrow\quad I=\frac{16}{3\times 2}=\frac{8}{3}$

Q.48. **Find :** $\int\left[\log(\log x)+\frac{1}{(\log x)^2}\right]\mathrm{dx}$

Ans.

Let $I=\int[\log(\log x)+\frac{1}{(\log x)^2}]\,dx$

Let $\log x=t \quad \Rightarrow \quad x=e^t \quad \Rightarrow \quad dx=e^t\,dt$

$\therefore I=\int\left\{\log t+\frac{1}{t^2}\right\}e^t\,dt$

$=\int\left\{\log t+\frac{1}{t}-\frac{1}{t}+\frac{1}{t^2}\right\}e^t\,dt$

$=\int(\log t+\frac{1}{t})e^t+\left(-\frac{1}{t}+\frac{1}{t^2}\right)e^t$

$=e^t.\log t-\frac{1}{t}.e^t+C \qquad [\because \int(f(x)+f'(x))e^x\,dx=f(x)e^x+C]$

Put $t=\log x$

$e^{\log x}\log(\log x)-\frac{1}{\log x}e^{\log x}+C$

$\Rightarrow\ x.\log(\log x)-\frac{x}{\log x}+C$

Long Answer Questions-I (OIQ)

[4 Mark]

Q.1. Evaluate : $\int\frac{\tan\theta+\tan^3\theta}{1+\tan^3\theta}d\theta$

Ans.

Let $I=\int\frac{\tan\theta+\tan^3\theta}{1+\tan^3\theta}d\theta=\int\frac{\tan\theta(1+\tan^2\theta)}{1+\tan^3\theta}d\theta=\int\frac{\tan\theta\sec^2\theta}{1+\tan^3\theta}d\theta$

Let $\tan\theta=z \quad \Rightarrow \quad \sec^2\theta\,d\theta=dz$

$\therefore I=\int\frac{zdz}{1+z^3}=\int\frac{zdz}{(1+z)(z^2-z+1)}$

Now $\frac{z}{(1+z)(z^2-z+1)}=\frac{A}{1+z}+\frac{Bz+C}{z^2-z+1} \qquad \Rightarrow \qquad z=A(z^2-z+1)+(Bz+C)(1+z)$

Putting $z = -1$, we get, $A =- \frac{1}{3}$

Putting $z = 0$, we get, $C = \frac{1}{3}$

Putting $z = 1$, we get, $B = \frac{1}{3}$

$\therefore \frac{z}{(1+z)(z^2-z+1)} = \frac{-1}{3(1+z)} + \frac{\frac{1}{3}z+\frac{1}{3}}{z^2-z+1}$

$$\Rightarrow I = -\frac{1}{3}\int \frac{dz}{1+z} + \frac{1}{3}\int \frac{z+1}{z^2-z+1}dz = -\frac{1}{3}\log\left|1+z\right| + \frac{1}{3\times 2}\int \frac{2z-1+3}{z^2-z+1}dz$$

$$= -\frac{1}{3}\log\left|1+z\right| + \frac{1}{6}\int \frac{2z-1}{z^2-z+1}dz + \frac{1}{2}\int \frac{dz}{z^2-z+1}$$

$$I = -\frac{1}{3}\log\left|1+z\right| + \frac{1}{6}\log\left|z^2-z+1\right| + I_1 + C \quad ...(i)$$

Where

$$I_1 = \frac{1}{2}\int \frac{dz}{z^2-z+1} = \frac{1}{2}\int \frac{dz}{\left(z-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2}.\frac{2}{\sqrt{3}} + \tan^{-1}\left(\frac{z-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$\therefore\ I = -\frac{1}{3}\log\left|1+z\right| + \frac{1}{6}\log\left|z^2-z+1\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2z-1}{\sqrt{3}}\right) + C$$

Putting $z = \tan\theta$

$$\therefore\ I = -\frac{1}{3}\log\left|1+\tan\theta\right| + \frac{1}{6}\log\left|\tan^2\theta - \tan\theta + 1\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta-1}{\sqrt{3}}\right) + C$$

Evaluate: $\int e^x \frac{x^2+1}{(x+1)^2}dx$

Q.2.

Ans.

$$\text{Let } I=\int e^{x}\frac{x^{2}+1}{(x+1)^{2}}dx=\int e^{x}\left(1-\frac{2x}{(x+1)^{2}}\right)dx=\int e^{x}-2\left(\frac{e^{x}x.\,dx}{(x+1)^{2}}\right)$$

$$=e^{x}-2\int e^{x}\frac{x+1-1}{(x+1)^{2}}dx=e^{x}-2\int e^{x}\left[\frac{1}{x+1}+\frac{-1}{(x+1)^{2}}\right]dx$$

$$=e^{x}-2e^{x}.\tfrac{1}{x+1}+C \qquad \text{[Note: } \int e^{x}\{f(x)+f'(x)\}dx=e^{x}.f(x)+C]$$

Q.3. **Evaluate:** $\displaystyle\int_{0}^{\pi/2}\frac{\sin 2\theta}{\sin^{4}\theta+\cos^{4}\theta}d\theta$

Ans.

Let $t=\sin^{2}$ q, then dt = 2sin q cos q dq = sin 2q . dq

and $\sin^{4}$ q + $\cos^{4}$ q = $t^{2}+(1-t)^{2}$

$$=2t^{2}-2t+1=2\left(t^{2}-t+\tfrac{1}{2}\right)=2\left[\left(t-\tfrac{1}{2}\right)^{2}+\left(\tfrac{1}{2}\right)^{2}\right]$$

Now, limits are, when θ = 0, t = 0 and when $\theta=\frac{\pi}{2},t=1$

$$\text{Therefore } \int_{0}^{\pi/2}\frac{\sin 2\theta\,.\,d\theta}{\sin^{4}\theta+\cos^{4}\theta}=\int_{0}^{1}\frac{dt}{2\left[\left(t-\frac{1}{2}\right)^{2}+\left(\frac{1}{2}\right)^{2}\right]}$$

$$=\tfrac{1}{2}\cdot\tfrac{1}{\frac{1}{2}}\left[\tan^{-1}\frac{\left(t-\frac{1}{2}\right)}{\frac{1}{2}}\right]_{0}^{1}=[\tan^{-1}(2t-1)]_{2}^{t}$$

$$=\tan^{-1}(1)-\tan^{-1}(-1)=\tfrac{\pi}{4}-\left(-\tfrac{\pi}{4}\right)=\tfrac{\pi}{2}$$

$$\textbf{Evaluate: } \int_0^{\pi/2} \sin 2x \log \tan x \, dx$$

Q.4.

Ans.

$$\text{Let } I = \int_0^{\pi/2} \sin 2x \log \tan x \, dx$$

$$\therefore \; I = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) dx \qquad \left[\text{since } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi/2} \sin(\pi - 2x) \log \cot x \, dx = \int_0^{\pi/2} \sin 2x \log (\tan x)^{-1} dx$$

$$= -\int_0^{\pi/2} \sin 2x \log \tan x \, dx = -I$$

$$\therefore \qquad 2I = 0 \;\Rightarrow\; I = 0$$

$$\textbf{Find: } \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}} dx$$

Q.5.

Ans.

$$\text{Let } I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}} dx$$

$$\text{Putting } x + 1 = t^2 \Rightarrow dx = 2t \, dt$$

$$I = 2\int \frac{t^2+1}{t^4+t^2+1} dt = 2\int \frac{1+\left(\frac{1}{t}\right)^2}{t^2+\frac{1}{t^2}+1} dt \quad [\text{Dividing } N^r \text{ and } D^r \text{ by } t^2]$$

$$= 2\int \frac{1+\left(\frac{1}{t}\right)^2}{\left(t-\frac{1}{t}\right)^2+3}dt$$

Now, Put $t-\frac{1}{t}=z \quad \Rightarrow \quad \left(1+\frac{1}{t^2}\right)dt = dz$, we get

$$I = 2\int \frac{dz}{z^2+3} = \frac{2}{\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{3}t}\right) + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right) + C$$

Q.6. **Find:** $\int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}dx;\ \alpha \neq n\pi,\ n \in Z$

Ans.

Let $I = \int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}dx$

$\sin^3 x \sin(x+a) = \sin^3 x\,(\sin x \,.\, \cos a + \cos x \,.\, \sin a)$

$= \sin^4 x\,(\cos a + \cot x \,.\, \sin a)$

$$\therefore\ I = \int \frac{1}{\sqrt{\sin^3 x\,\sin(x+\alpha)}}.dx = \int \frac{1}{\sin^2 x\sqrt{\cos\alpha + \cot x\,\sin\alpha}}.dx$$

Putting $\cos a + \cot x \,.\, \sin a = t$ so that $-\operatorname{cosec}^2 x \,.\, \sin a \,.\, dx = dt$

$$\therefore\ I = \int -\frac{1}{\sin\alpha\sqrt{t}}.dt = -\frac{1}{\sin\alpha}\int t^{-1/2}dt = -\frac{1}{\sin\alpha}\left(\frac{t^{1/2}}{1/2}\right) + C$$

$$= -2\operatorname{cosec}\alpha.\ \sqrt{t} + C = -2\operatorname{cosec}\alpha\,(\cos\alpha + \cot x\,\sin\alpha)^{1/2} + C$$

Q.7. Find: $-\int 5^{5^{5^x}} . 5^{5^x} . 5^x \, dx$

Ans.

Let $I = \int 5^{5^{5^x}} . 5^{5^x} . 5^x \, dx$

Putting $5^x = t \Rightarrow 5^x . \log 5 \, dx = dt$ or $5^x . dx = \frac{dt}{(\log 5)}$

Therefore, $I = \int 5^{5^{5^x}} . 5^{5^x} . 5^x . dx = \int 5^{5^t} . 5^t . \frac{dt}{(\log 5)} = \frac{1}{(\log 5)} \int 5^{5^t} . 5^t . dt$

Again, putting $5^t = u, \quad 5^t dt = \frac{du}{(\log 5)}$

Therefore, $I = \frac{1}{(\log 5)} \int 5^u . \frac{du}{(\log 5)}$

$$= \frac{1}{(\log 5)^2} \int 5^u \text{du} = \frac{5^u}{(\log 5)^2 . (\log 5)} + C$$

$$= \frac{5^u}{(\log 5)^3} + C = \frac{5^{5^t}}{(\log 5)^3} + C = \frac{5^{5^{5^x}}}{(\log 5)^3} + C$$

Q.8. Show that: $\int_0^{\pi/2} \log(\tan\theta + \cot\theta) \, d\theta = \pi \log 2$

Ans.

Let $I = \int_0^{\pi/2} \log\,(\tan\,\theta + \cot\,\theta)\,d\theta$

$$= \int_0^{\pi/2} \log\,\left(\frac{\sin\,\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right) d\theta \qquad \Rightarrow \qquad \int_0^{\pi/2} \log\,\left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\,\cos\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \log\,\left(\frac{1}{\cos\theta\;\sin\theta}\right) d\theta$$

$$= -\int_0^{\pi/2} \log\,(\cos\,\theta\,\sin\,\theta) d\theta \quad \left[\because\ \log\,\frac{1}{m} = \log\,m^{-1} = -\log m\right]$$

$$= -\int_0^{\pi/2} (\log\,\cos\theta + \log\,\sin\,\theta) d\theta \qquad \Rightarrow \qquad -\int_0^{\pi/2} \log\,\cos\,\theta\,d\theta - \int_0^{\pi/2} \log\,\sin\,\theta\,d\theta$$

$$= -\int_0^{\pi/2} \log\,\cos\left(\frac{\pi}{2} - \theta\right)\,d\theta - \int_0^{\pi/2} \log\,\sin\,\theta\,d\theta$$

$$= -\int_0^{\pi/2} \log\,\sin\,\theta\,d\theta - \int_0^{\pi/2} \log\,\sin\,\theta\,d\theta$$

$$= -2\int_0^{\pi/2} \log\,\sin\theta d\theta = -2\left(-\frac{\pi}{2}\log 2\right) = \pi\log\,2$$

$$\left[\because \int_0^{\pi/2} \log\,\sin\,x dx = \frac{-\pi}{2}\log\,2\right]$$

[**Note:** If this question comes in exam, students are advised to give complete solution of $\int_0^{\pi/2} \log\,\sin\,\theta\,d\theta$.]