Successive Differentiation

Q.1. If y = e^ m cos $^{-1}$ x , prove that (1 – x^2) d2y/dx^2 – x dy/dx = m^2 y.

Solution : 1

We have , y = e^ m cos $^{-1}$ x

Taking log of both sides, we get

log e y = m cos $^{-1}$ x log e e

Or, log e y = m cos $^{-1}$ x [log e e = 1]

Differentiating both sides with respect to x , we get

(1/y) dy/dx = m. (– 1)/√(1 – x^2)

Or, dy/dx = – m y /√(1 – x^2)

Or, dy/dx = (– m ×e^ m cos $^{-1}$ x)/√(1 – x^2)

Differentiating both sides again w. r. t. x , we get d^2y/dx^2

= – m [– √(1 – x^2) . {m e^ m cos –1 x/√(1 – x^2)}+ (2x e^ mcos $^{-1}$ x)/{2√(1 – x2)}]/(1 – x^2)

Or, (1 – x^2) d2y/dx^2 = m^2 . e^m cos $^{-1}$ x – m x e^ m cos $^{-1}$ x /√(1 – x^2)

Or, (1 – x^2) d^2y/dx^2 – x dy/dx = m^2 y . **[Proved.]**

Q.2. If y = e^x (sin x + cos x) , prove that d^2y/dx^2 – 2 dy/dx + 2y = 0 .

Solution : 2

We have y , = e^x (sin x + cos x)

Then differentiating w. r. t. x , we get

dy/dx = e^x . (sin x + cos x) + e^x (cos x – sin x)

Or, $dy/dx = 2\ e^{x} \cos x$.

Differentiating again w. r. t. x , we get

$d2y/dx2 = 2\ e^{x} \cos x + 2\ e^{x} (- \sin x)$

Or, $d2y/dx2 = 2\ e^{x} (\cos x - \sin x)$.

L. H. S. = $d2y/dx2 - 2\ dy/dx + 2y$

$= 2\ e^{x} (\cos x - \sin x) - 2 \times 2e^{x} \cos x + 2\ e^{x} (\sin x + \cos x)$

$= 2\ e^{x} [\cos x - \sin x - 2 \cos x + \sin x + \cos x]$

= 0 . **[Proved.]**

Q.3. If $y = A \cos nx + B \sin nx$, show that $d^{2}y/dx^{2} + n^{2} y = 0$.

Solution : 3

We have , $y = A \cos nx + B \sin nx$

Differentiating w. r. t. x , we get

$dy/dx = - A\ n \sin nx + B\ n \cos nx$.

Differentiating again w. r. t. x , we get

$d^{2}y/dx^{2} = - (An)\ n \cos nx - (Bn)\ n \sin nx$

Or , $d^{2}y/dx^{2} = - n^{2} (A \cos nx + B \sin nx) = - n^{2}y$.

Hence , $d^{2}y/dx^{2} + n^{2} y = 0$. **[Proved.]**

Q.4. Find $d^{2}y/dx^{2}$, when $y = \log (x^{2}/ex)$.

Solution : 4

$y = \log (x^{2}/ex) = \log x^{2} - \log ex$

$= 2 \log x - x \log e = 2 \log x - x$ [As, log e = 1]

Or, $y = 2 \log x - x$

Differentiating w. r. t. x , we get

$dy/dx = 2/x - 1$.

Differentiating again w. r. t. x , we get

$d^2y/dx^2 = -2/x^2$.

Q.5. If $y = \text{cosec } x + \cot x$, show that $\sin x \cdot d^2y/dx^2 = y^2$.

Solution : 5

We have , $y = \text{cosec } x + \cot x$ --- --- --- (1)

Differentiating w. r. t. x , we get

$dy/dx = -\text{cosec } x \cot x - \text{cosec}^2 x$

$= -\text{cosec } x (\cot x + \text{cosec } x)$

$= -y \text{ cosec } x$ --- --- (2) [Using (1)]

differentiating again w. r. t. x , we get

$d^2y/dx^2 = -[dy/dx \cdot \text{cosec } x + y \cdot (-\text{cosec } x \cot x)]$

$= -[(-y \text{ cosec } x) \cdot \text{cosec } x - y \text{ cosec } x \cot x]$ [By (2)]

$= y \text{ cosec}^2 x + y \text{ cosec } x \cot x = y \text{ cosec } x (\text{cosec } x + \cot x)$

$= y \cdot \text{cosec } x \cdot y$ [By (1)] $= y^2 \text{ cosec } x = y^2/\sin x$

Or, $\sin x \, d^2y/dx^2 = y^2$. **[Proved.]**

Q.6. If $y = \tan^{-1} x$, show that $(1 + x^2) d^2y/dx^2 + 2x \, dy/dx = 0$.

Solution : 6

We have , y = tan – 1 x

Differentiating w. r. t. x , we get

$dy/dx = 1/(1 + x^2)$

Or, $(1 + x^2)\ dy/dx = 1$

Differentiating again w. r. t. x , we get

$(1 + x^2)\ d^2y/dx^2 + 2x\ .\ dy/dx = 0$. **[Proved.]**

Q.7. If $y = x + \tan x$, prove that $\cos^2 x\ d^2y/dx^2 - 2y + 2x = 0$.

Solution : 7

We have , $y = x + \tan x$

Differentiating w. r. t. x , we get

$dy/dx = 1 + \sec^2 x$

Differentiating again w. r. t. x, we get

$d^2y/dx^2 = 0 + 2 \sec x\ .\ \sec x \tan x$

$= 2 \sec^2 x \tan x$

$= 2 \tan x /\cos^2 x$.

Or, $\cos^2 x\ d^2y/dx^2 = 2 \tan x = 2(y - x)$

Or, $\cos^2 x\ d^2y/dx^2 - 2\ (y - x) = 0$. **[Proved.]**

Q.8. Find d^2y/dx^2 ; if $x = at^2$, $y = 2\ at$.

Solution : 8

We have , $x = at^2 \Rightarrow dx/dt = 2at$,

Also , $y = 2\ at \Rightarrow dy/dt = 2\ a$.

Therefore , $dy/dx = (dy/dt)/(dx/dt) = (2\ a)/(2\ at) = 1/t$,

Hence , $d^2y/dx^2 = d/dx\ (1/t)$

$= (-1/t^2).dt/dx$

$= (-1/t^2)/(dx/dt) = (-1/t^2)/(2\ at) = (-1/2)\ a\ t^3$.

Q.9. If $y = e^{3x}(A + Bx)$, then prove that : $d^2y/dx^2 - 6\ dy/dx + 9\ y = 0$.

Solution : 9

We have , $y = e^{3x}(A + Bx)$ ---- ---- ---- (1)

Differentiating w. r. t. x , we get

$dy/dx = e^{3x} . 3 .(A + Bx) + e3x .B$

$= 3y + B\ e^{3x}$ ---- ---- (2) [Using (1)]

Differentiating again w. r. t. x , we get

$d^2y/dx^2 = 3\ dy/dx + 3B\ e^{3x}$

$= 3\ dy/dx + 3(dy/dx - 3y)$, [Using (2)]

$= 3\ dy/dx + 3\ dy/dx - 9\ y$,

Or, $d^2y/dx^2 - 6dy/dx + 9\ y = 0$. **[Proved.]**

Q.10. If $y = b\ ex + c\ e^{2x}$, then prove that $d^2y/dx^2 - 3\ dy/dx + 2y = 0$.

Solution : 10

We have , $y = b\ ex + c\ e^{2x}$ ---- ---- ---- (1)

Differentiating both sides w. r. t. x , we get

$dy/dx = b\ ex + 2c\ e^{2x}$ ---- ---- ---- (2)

Subtracting (1) from (2) we get

$dy/dx - y = c\ e^{2x}$ ---- ---- ---- (3)

Differentiating again w. r. t. x , we get

d^2y/dx^2 – dy/dx = 2c e^{2x}---- ---- ---- (4)

Multiplying (3) by 2 and then subtracting from (4) we get

d^2y/dx^2 – dy/dx – 2 (dy/dx – y) = 0

Or, d^2y/dx^2 – 3 dy/dx + 2 y = 0 **[Proved.]**

Q.11. If y = a e^{2x} + b e –x , then prove that d^2y/dx^2 – dy/dx – 2y = 0 .

Solution : 11

We have , y = a e^{2x} + b e –x ------------ (1)

Differentiating both sides w. r. t. x , we get

dy/dx = 2a e^{2x} – b e –x ------------- (2)

Adding (1) and (2) we get

dy/dx + y = 3a e^{2x} -------------- (3)

Differentiating both sides of (3) w. r. t. x , we get

d^2y/dx^2 + dy/dx = 3a e2x . 2

= 2. (dy/dx + y) [Using (3)]

Or, d^2y/dx^2 – dy/dx – 2y = 0 . **[Proved.]**