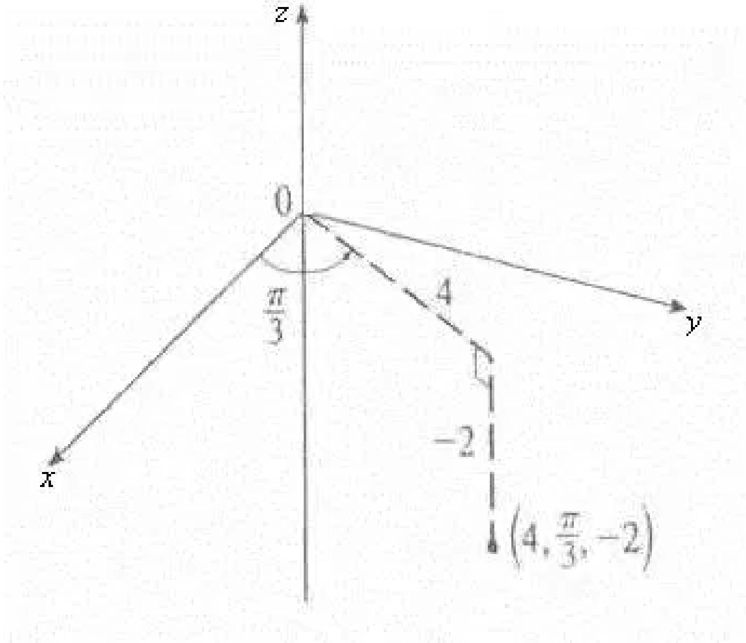
Exercise 15.8

Chapter 15 Multiple Integrals 15.8 1E

(a) First, let us plot the point with the given cylindrical coordinates.



In cylindrical coordinates, we have $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$.

Substitute the known values in $x = r\cos\theta$ and find x.

$$\begin{aligned} x &= 4\cos\frac{\pi}{3} \\ &= 2 \end{aligned}$$

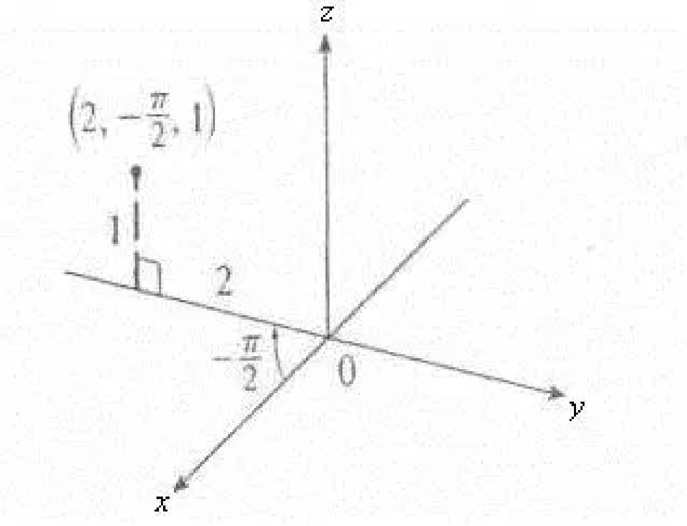
Replace r with 4 and z with $\frac{\pi}{3}$ in $y = r\sin\theta$.

$$\begin{aligned} y &= 4\sin\frac{\pi}{3} \\ &= 2\sqrt{3} \end{aligned}$$

We have $z = -2$.

Therefore, we get the corresponding rectangular coordinates as $\boxed{\left(2,\ 2\sqrt{3},\ -2\right)}$.

(b) Plot the point with the given cylindrical coordinates.



Substitute the known values in $x = r\cos\theta$ and find x.

$$x = 2\cos\left(-\frac{\pi}{2}\right)$$
$$= 0$$

Replace r with 2 and z with $-\frac{\pi}{2}$ in $y = r\sin\theta$.

$$y = 2\sin\left(-\frac{\pi}{2}\right)$$
$$= -2$$

We have $z = 1$.

Therefore, we get the corresponding rectangular coordinates as $\boxed{(0, -2, 1)}$.

Chapter 15 Multiple Integrals 15.8 2E

(a)

Cylindrical coordinates are $\left(\sqrt{2}, \frac{3\pi}{4}, 2\right)$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Here, $r = \sqrt{2},\ \theta = \frac{3\pi}{4},\ \ z = 2$

We have

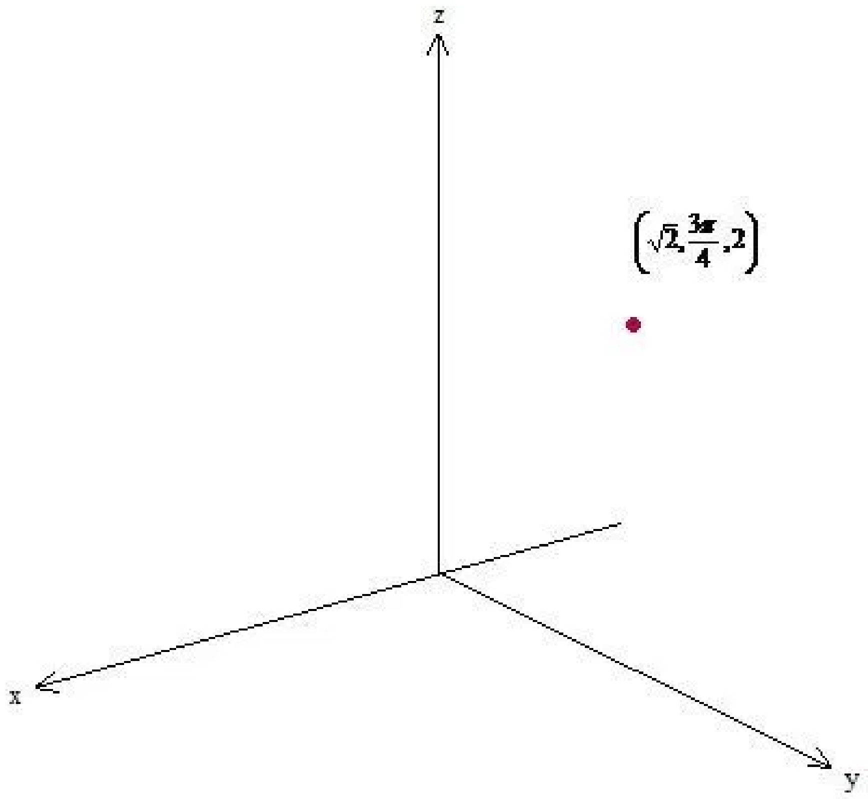
$$x = \sqrt{2}\cos\frac{3\pi}{4}$$
$$= \sqrt{2} \cdot -\frac{1}{\sqrt{2}}$$
$$= -1$$

$$y = \sqrt{2}\sin\frac{3\pi}{4}$$
$$= \sqrt{2} \cdot \frac{1}{\sqrt{2}}$$
$$= 1$$

$$z = 2$$

Thus rectangular coordinates are $\boxed{(-1,1,2)}$

The plotted point is shown as:



(b)

Cylindrical coordinates are $(1,1,1)$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Here, $r = 1,\ \theta = 1,\ \ z = 1$

We have

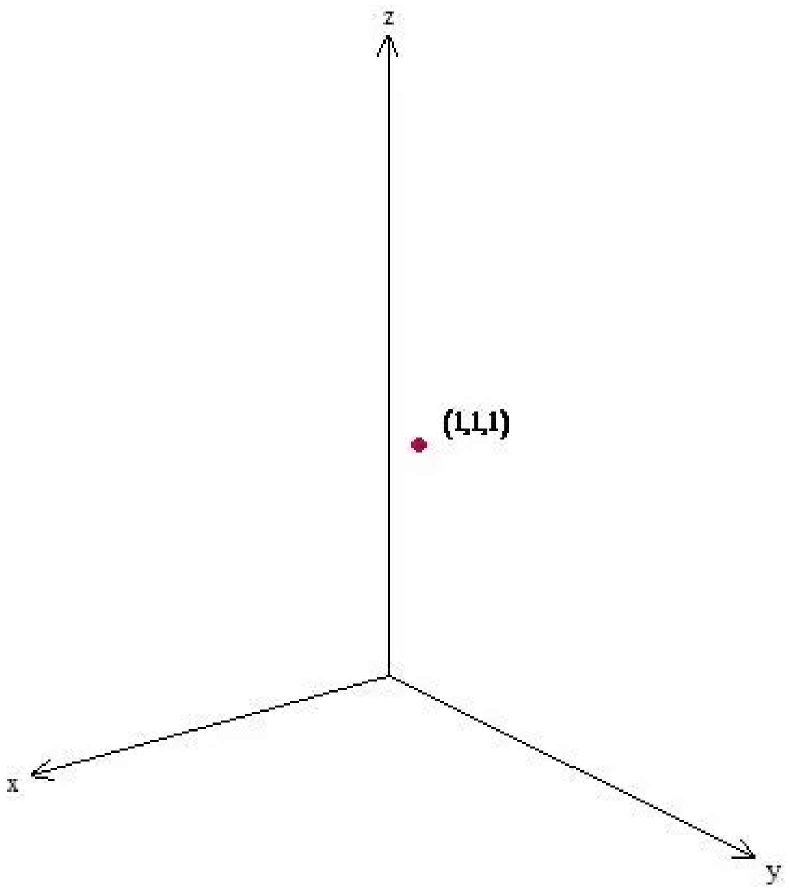
$$\begin{aligned} x &= \cos 1 \\ &= 1\cdot(0.54) \\ &= 0.54 \end{aligned}$$

$$\begin{aligned} y &= \sin 1 \\ &= 0.84 \end{aligned}$$

$z = 1$

Thus rectangular coordinates are $\boxed{(0.54, 0.84, 1)}$

The plotted point is shown as:



Chapter 15 Multiple Integrals 15.8 3E

a)

The point in rectangular coordinates is $(-1,1,1)$.

The relation between rectangular coordinates (x,y,z) and the cylindrical coordinates (r,θ,z) are

$$r=\sqrt{x^2+y^2}$$

$$\theta=\tan^{-1}\frac{y}{x}$$

$$z=z$$

Let $(x,y,z)=(-1,1,1)$.

So,

$$\begin{aligned} r&=\sqrt{x^2+y^2}\\ &=\sqrt{(-1)^2+1^2}\\ &=\boxed{\sqrt{2}} \end{aligned}$$

Now

$$\begin{aligned} \theta&=\tan^{-1}\frac{y}{x}\\ &=\tan^{-1}\left(\frac{1}{-1}\right)\\ &=\tan^{-1}(-1)\\ &=\boxed{\frac{3\pi}{4}} \end{aligned}$$

And

$$\begin{aligned} z&=z\\ &=\boxed{1} \end{aligned}$$

Therefore, the cylindrical coordinates suitable to $(-1,1,1)$ are $\boxed{\left(\sqrt{2},\frac{3\pi}{4},1\right)}$.

(b)

Consider the point in rectangular coordinates is $\left(-2,2\sqrt{3},3\right)$.

Let $(x,y,z)=\left(-2,2\sqrt{3},3\right)$.

So,

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{(-2)^2+\left(2\sqrt{3}\right)^2} \\ &= \sqrt{4+12} \\ &= \sqrt{16} \\ &= \boxed{4} \end{aligned}$$

Now

$$\begin{aligned} \theta &= \tan^{-1}\frac{y}{x} \\ &= \tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) \\ &= \tan^{-1}\left(-\sqrt{3}\right) \\ &= \boxed{\frac{2\pi}{3}} \end{aligned}$$

And

$$\begin{aligned} z &= z \\ &= \boxed{3} \end{aligned}$$

Therefore, the cylindrical coordinates suitable to $\left(-2,2\sqrt{3},3\right)$ are $\boxed{\left(4,\frac{2\pi}{3},3\right)}$.

Chapter 15 Multiple Integrals 15.8 4E

Consider the following point on rectangular coordinates.

$$(x,y,z)=\left(2\sqrt{3},2,-1\right)$$

The objective is to convert the point into rectangular coordinates to cylindrical coordinates.

Use the following conversion formulas to convert the rectangular coordinates (x,y,z) to cylindrical coordinates (r,θ,z).

$$\begin{aligned} r^2 &= x^2+y^2 \\ \tan\theta &= \frac{y}{x} \\ z &= z \qquad \text{......(1)} \end{aligned}$$

Here, $x = 2\sqrt{3}, y = 2,$ and $z = -1$.

Substitute $x = 2\sqrt{3}, y = 2,$ and $z = -1$ in equation (1)

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= \left(2\sqrt{3}\right)^2 + (2)^2 && \text{Since } x = 2\sqrt{3} \text{ and } y = 2 \\ &= 12 + 4 \\ &= 16 \end{aligned}$$

Thus, $r = 4$.

Find the component θ.

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{2}{2\sqrt{3}} \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

Then,

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &= \frac{\pi}{6} && \text{Since } \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \end{aligned}$$

Thus, $\theta = \frac{\pi}{6}$.

Therefore, the cylindrical coordinates of $\left(2\sqrt{3}, 2, -1\right)$ are $\boxed{(r, \theta, z) = \left(4, \frac{\pi}{6}, -1\right)}$

(b)

Consider the following point in rectangular coordinate system.

The objective is to convert the point into rectangular coordinates to cylindrical coordinates.

Convert the rectangular coordinate $(4, -3, 2)$

Here, $x = 4, y = -3,$ and $z = 2$

Substitute $x = 4, y = -3,$ and $z = 2$ in equation (1).

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (4)^2 + (-3)^2 && \text{Since } x = 4 \text{ and } y = -3 \\ &= 16 + 9 \\ &= 25 \end{aligned}$$

Thus, $r = 5$

Find the component θ.

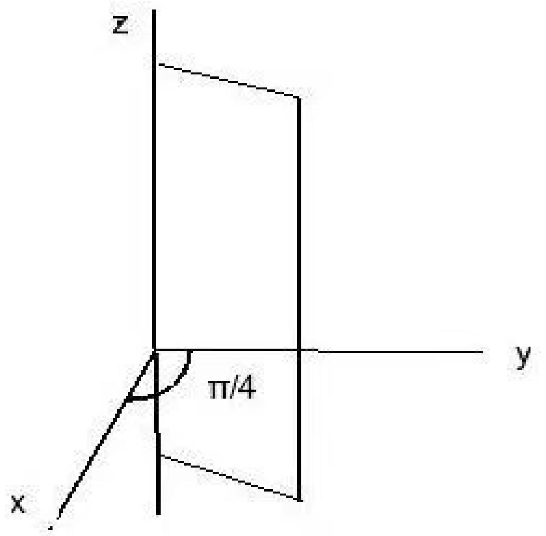
$$\tan\theta = \frac{y}{x} = \frac{-3}{4} = -\frac{3}{4}$$

Then, $\theta = \tan^{-1}\left(-\frac{3}{4}\right)$

Therefore, the cylindrical coordinates of $(-1,1,1)$ is $\boxed{(r,\theta,z) = \left(5, \tan^{-1}\left(-\frac{3}{4}\right), 2\right)}$.

Chapter 15 Multiple Integrals 15.8 5E

Given surface is θ = π/4



The surface is a vertical half-plane through the z-axis that comes out diagonally.

Chapter 15 Multiple Integrals 15.8 6E

Since r=5, but z and θ may vary, the surface is a cylinder with radius 5 around the z-axis.

Chapter 15 Multiple Integrals 15.8 7E

Consider the surface

$z = 4 - r^2$

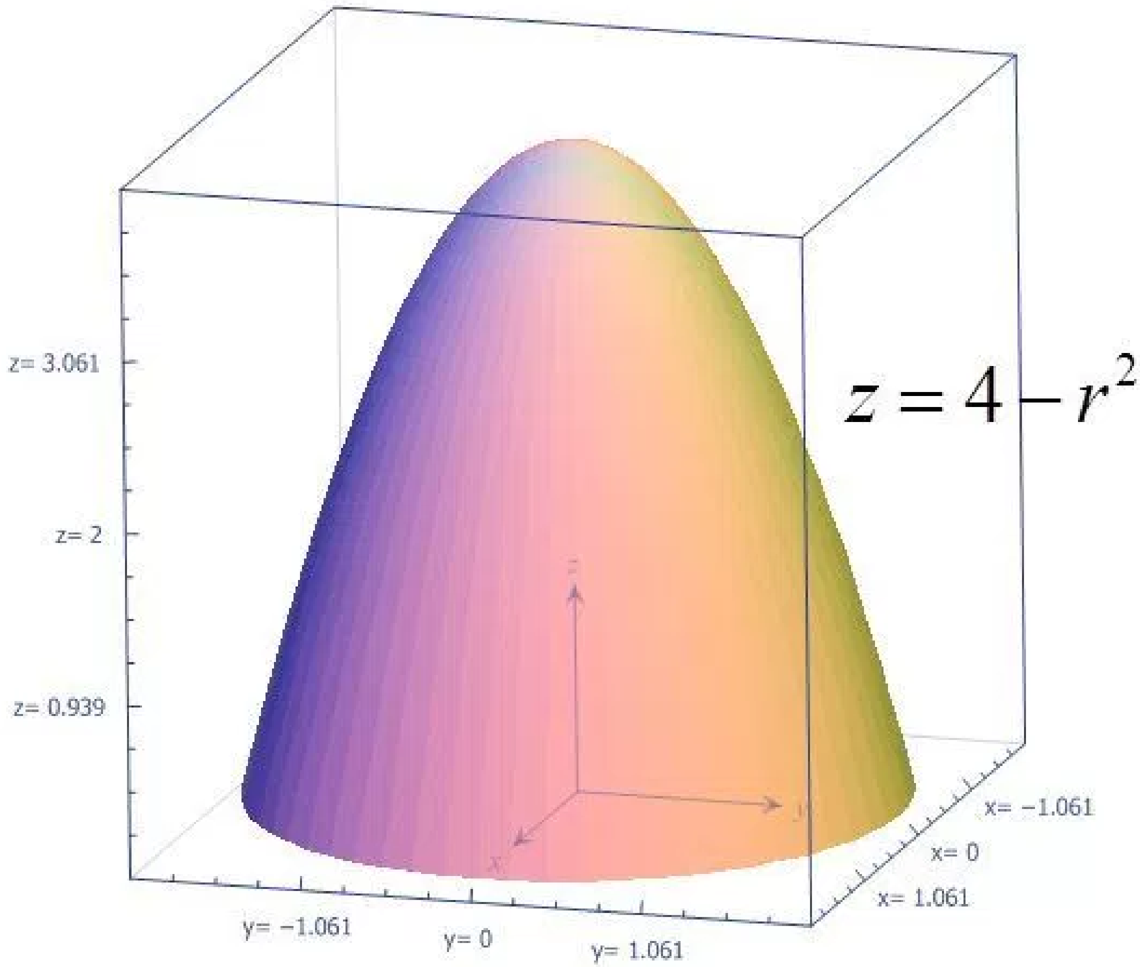
The objective is to identify the surface whose equation is $z = 4 - r^2$.

Here, the equation $z = 4 - r^2$ says the height or z-value, of each point on the surface is the same as the value $4 - r^2$, here z is not given, so it can vary.

So any horizontal trace in the plane $z = k, k < 4$ is a circle of radius $4 - k$.

Hence, these traces suggest that the surface of the equation $z = 4 - r^2$ is circular paraboloid.

The sketch of the surface $z = 4 - r^2$ is,



Chapter 15 Multiple Integrals 15.8 8E

Consider the equation $2r^2+z^2=1.$

Identify the surface of the given equation.

In cylindrical coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$x^2+y^2=r^2$$
$$dzdydx = rdzdrd\theta$$

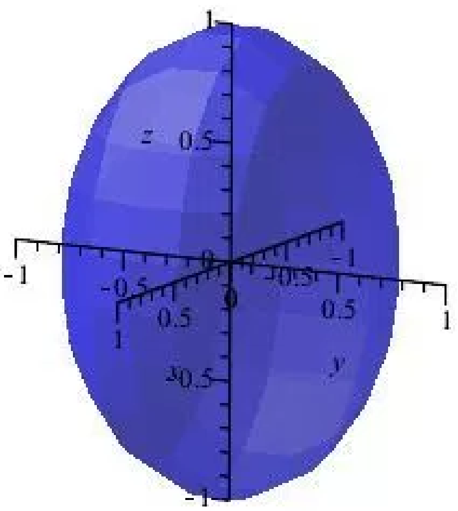
Substitute $r^2=x^2+y^2$ in the given equation.

$$2r^2+z^2=1$$
$$2\left(x^2+y^2\right)+z^2=1$$
$$\frac{x^2}{0.5}+\frac{y^2}{0.5}+\frac{z^2}{1}=1$$
$$\frac{x^2}{\left(\sqrt{0.5}\right)^2}+\frac{y^2}{\left(\sqrt{0.5}\right)^2}+\frac{z^2}{1^2}=1$$

The equation of an ellipsoid centered at origin is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Compare the equation $\frac{x^2}{\left(\sqrt{0.5}\right)^2}+\frac{y^2}{\left(\sqrt{0.5}\right)^2}+\frac{z^2}{1^2}=1$ with ellipsoid equation and conclude that this equation represents an ellipsoid.

Surface of the ellipsoid $2\left(x^2+y^2\right)+z^2=1$ is shown below:



Chapter 15 Multiple Integrals 15.8 9E

(a) We have $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$.

Substitute the known values in $x^2 - x + y^2 + z^2 = 1$ and simplify.

$$(r\cos\theta)^2 - r\cos\theta + (r\sin\theta)^2 + z^2 = 1$$
$$r^2\cos^2\theta - r\cos\theta + r^2\sin^2\theta + z^2 = 1$$
$$r(r - \cos\theta) + z^2 = 1$$

Therefore, the equation in cylindrical coordinates is $\boxed{r(r - \cos\theta) + z^2 = 1}$.

(b) Replace x with $r\cos\theta$ and y with $r\sin\theta$ in $z = x^2 - y^2$.

$$z = (r\cos\theta)^2 - (r\sin\theta)^2$$
$$z = r^2(\cos^2\theta - \sin^2\theta)$$
$$z = r^2\cos 2\theta$$

We have $z = 3$.

Therefore, the equation in cylindrical coordinates is $\boxed{z = r^2\cos 2\theta}$.

Chapter 15 Multiple Integrals 15.8 10E

(a) Given equation is 3x+2y+z=6.

Putting cylindrical coordinates $x = \mathrm{r}\cos\theta,\ y = \mathrm{r}\sin\theta,\ z = z$, equation becomes

$$3\mathrm{r}\cos\theta + 2\mathrm{r}\sin\theta + z = 6$$

(b) Given equation is $-x^2 - y^2 + z^2 = 1$

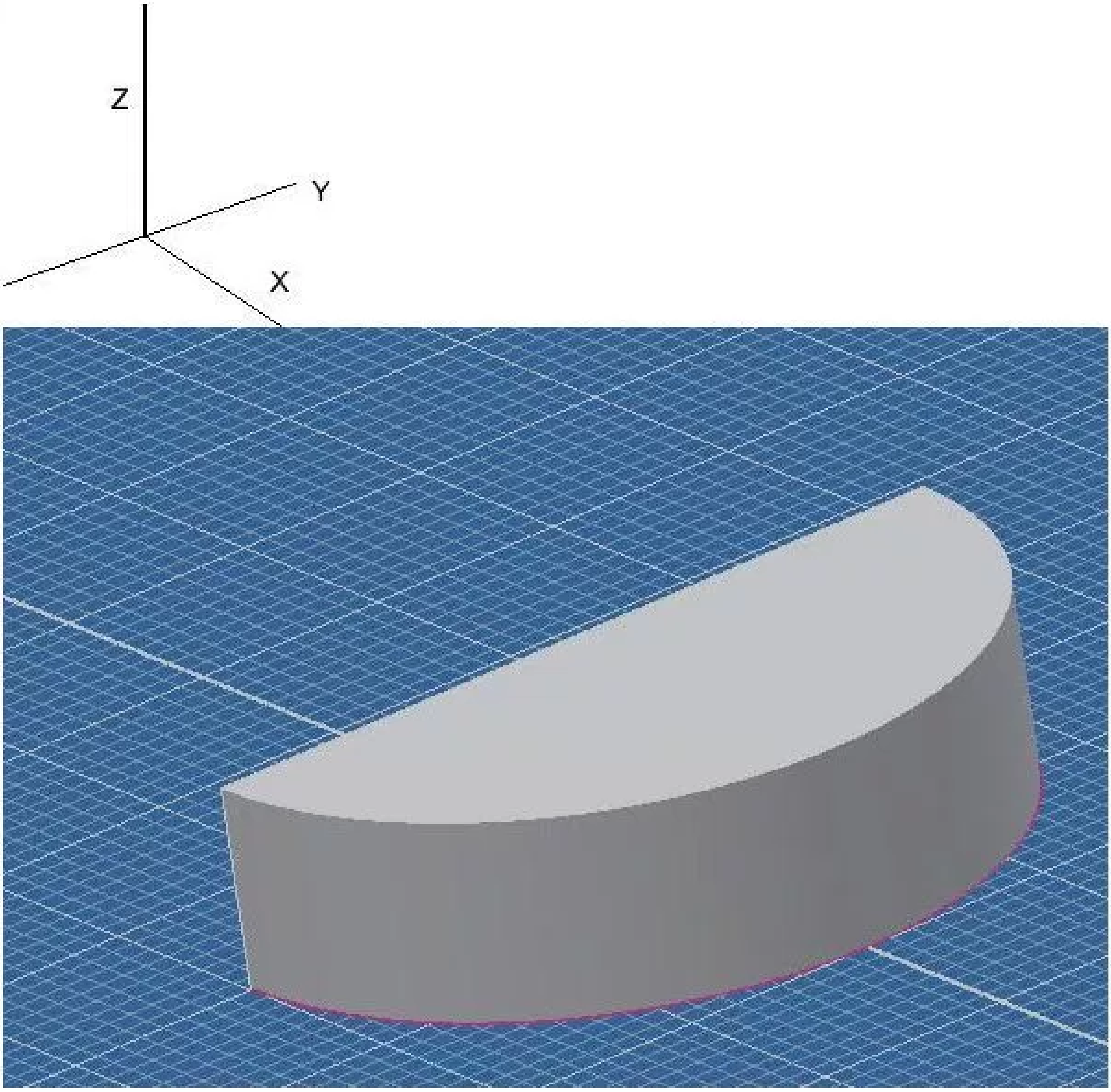
Putting cylindrical coordinates, equation becomes $-(\mathrm{r}\cos\theta)^2 - (\mathrm{r}\sin\theta)^2 + z^2 = 1$

or $-r^2\left(\cos^2\theta + \sin^2\theta\right) + z^2 = 1$

or $-r^2.1 + z^2 = 1$

$$z^2 - r^2 = 1$$

Chapter 15 Multiple Integrals 15.8 11E



Radius=2

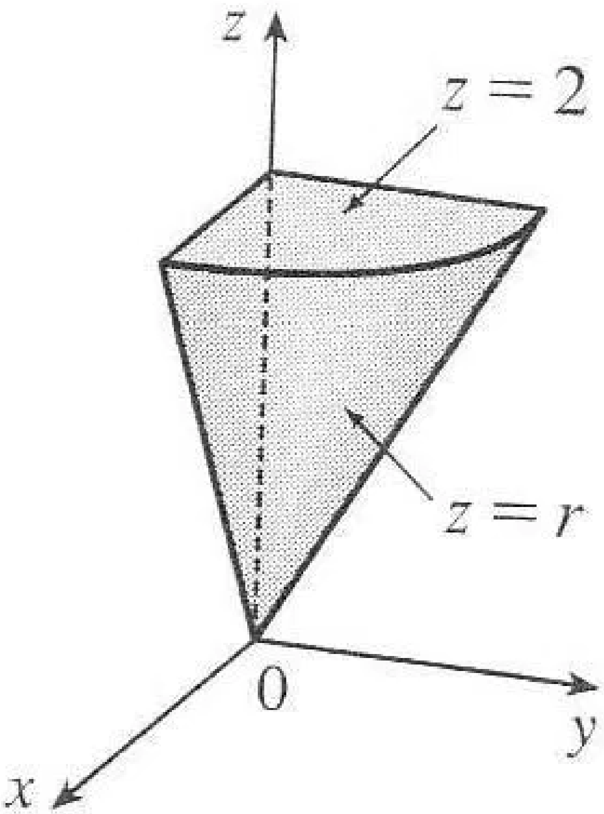
Height=1

Chapter 15 Multiple Integrals 15.8 12E

$z = r = \sqrt{x^2 + y^2}$ is a cone that opens upward.

Therefore, $r \leq z \leq 2$ is the region above this cone and beneath the horizontal plane z = 2.

$0 \leq \theta \leq \frac{\pi}{2}$ restricts the solid to that part of this region in the first octant.



Chapter 15 Multiple Integrals 15.8 13E

Consider a cylindrical shell of 20cm long, with inner radius 6cm, and outer radius 7cm.

It is required to write the inequalities that describe the shell.

Use cylindrical coordinates to describe the shell.

Let z-axis be the central axis of the cylinder. Then $0 \leq z \leq 20.$

The inner and outer radii can be treated as the limits for r . So $6 \leq r \leq 7$

To make a full circle, the angle θ must be in the interval $0 \leq \theta \leq 2\pi$.

Therefore, the following inequalities will describe the shell:

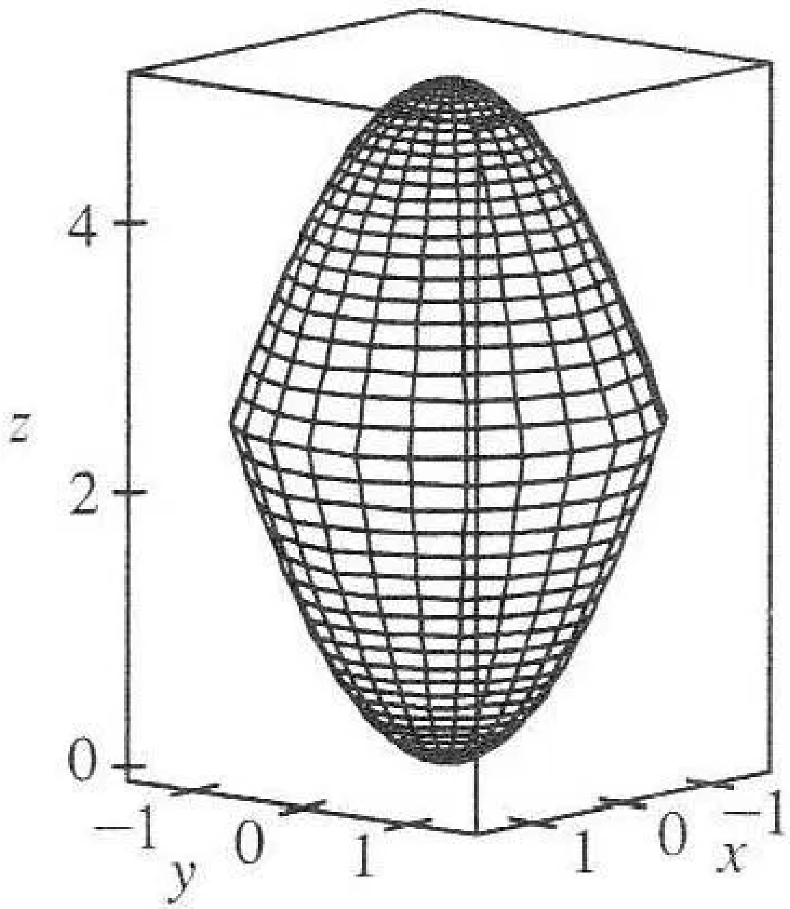
$$\boxed{\begin{cases} 6 \leq r \leq 7 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq 20 \end{cases}}$$

Chapter 15 Multiple Integrals 15.8 14E

In cylindrical coordinates, the equations are z = r2 and z = 5 - r2. The curve of intersection is r2 = 5 - r2 or r = √(5/2).

So we graph the surfaces in cylindrical coordinates, with 0 ≤ r ≤√(5/2).

In Maple, we can use the coords=cylindrical option in a regular plot3d command. In Mathematica, we can use ParametricPlot3D.

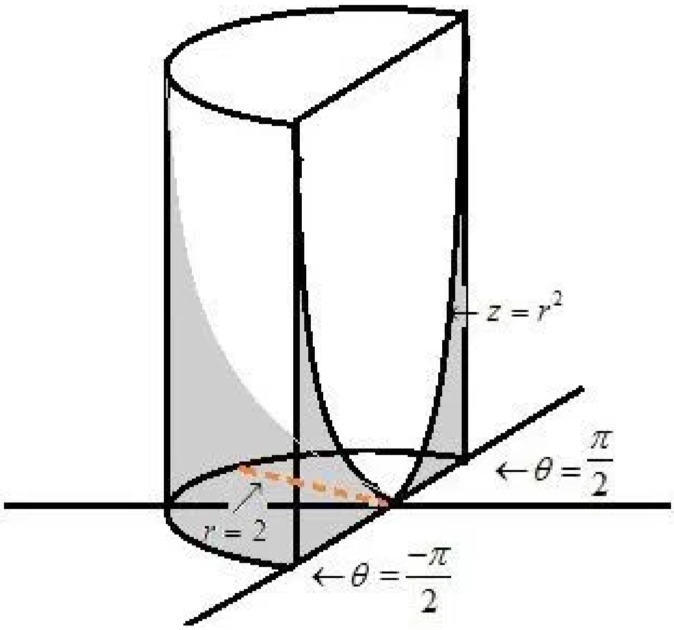


Chapter 15 Multiple Integrals 15.8 15E

Consider the volume integral

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{2}\int_{0}^{r^2} r\,dz\,dr\,d\theta$$

Sketch the solid whose volume is given by the given iterated integral is



Now, evaluate the integral.

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}\int_{0}^{2}\int_{0}^{r^2} r\,dz\,dr\,d\theta &= \int_{-\pi/2}^{\pi/2}\int_{0}^{2} r(z)_0^{r^2}\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2}\int_{0}^{2} r^3\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2}\left(\frac{r^4}{4}\right)_0^2 d\theta \\ &= \int_{-\pi/2}^{\pi/2}\frac{1}{4}\left(2^4-0\right)_0^2 d\theta\end{aligned}$$

$$= \int_{-\pi/2}^{\pi/2} 4\,d\theta$$

$$= 4(\theta)_{-\pi/2}^{\pi/2}$$

$$= 4\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= 4\pi$$

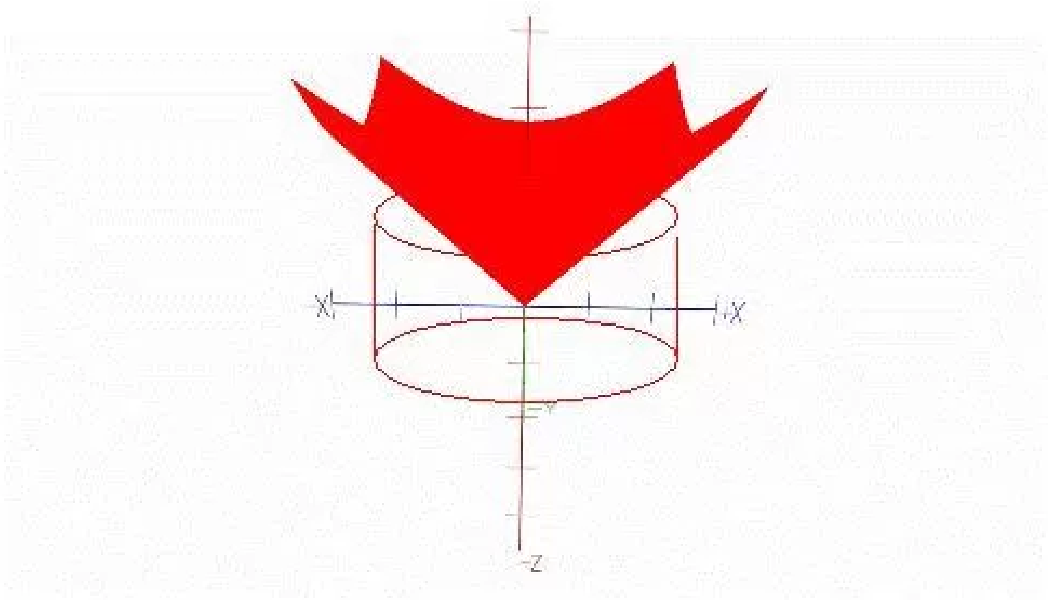
Therefore, the volume of the solid is 4π.

Chapter 15 Multiple Integrals 15.8 16E

Stewart-Calculus-7e-Solutions-Chapter-15.8-Multiple-Integrals-16E

First, the z ranges between 0 and r, which means that the z-values range from the xy-plane to the lateral surface of a right circular cone that has its apex at the origin and opens upwards at an exact 45 degree angle, as on the surface of such a cone is the only way the z-value will equal the r-value (on the surface of the cone the z- and r-values make two legs of an isosceles right triangle, with the lateral surface of the cone forming the hypotenuse). Next, θ ranges between 0 and 2π, so goes around the entire z-axis and does not limit the volume at all. Finally, r goes between 0 and 2, bounding the volume inside a cylinder of radius 2. In other words, this solid is inside a circular cylinder of radius 2 with axis the z-axis, above the xy-plane, and below the right circular cone that rises at 45 degrees from the origin.

Here is a sketch:



The solid is the volume inside the red-outlined cylinder but below the solid red cone surface.

Calculate the integral.

$$\int_0^2 \int_0^{2\pi} \int_0^r r\,dz\,d\theta\,dr$$

Integrate in terms of z:

$$\int_0^2 \int_0^{2\pi} (rz)\Big|_0^r d\theta\,dr$$

$$= \int_0^2 \int_0^{2\pi} \left(r^2 - 0\right) d\theta\,dr$$

$$= \int_0^2 \int_0^{2\pi} \left(r^2\right) d\theta\,dr$$

Integrate in terms of θ:

$$\int_0^2 \int_0^{2\pi} \left(r^2\right) d\theta dr$$

$$= \int_0^2 \left(\theta r^2\right)\Big|_0^{2\pi} dr$$

$$= \int_0^2 \left(2\pi r^2 - 0\right) dr$$

$$= \int_0^2 \left(2\pi r^2\right) dr$$

Integrate in terms of r:

$$\int_0^2 \left(2\pi r^2\right) dr$$

$$= \left(\frac{2\pi r^3}{3}\right)\Bigg|_0^2$$

$$= \frac{2\pi(2)^3}{3} - \frac{2\pi(0)^3}{3}$$

$$= \boxed{\frac{16\pi}{3}}$$

Chapter 15 Multiple Integrals 15.8 17E

Consider the triple integral

$$\iiint_E \sqrt{x^2 + y^2}\, dV$$

Here E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$

Cylindrical Coordinates:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

$$\begin{aligned} x^2 + y^2 &= r^2\cos^2(\theta) + r^2\sin^2(\theta) \\ &= r^2\left(\cos^2(\theta) + \sin^2(\theta)\right) \\ &= r^2(1) \\ &= r^2 \end{aligned}$$

$$dV = r dz dr d\theta$$

The region

$$x^2 + y^2 = 16$$

$$r^2 = 16$$

$r = 4$ Since radius is non-negative

$$E = \{(r, \theta, z) : 0 \le \theta \le 2\pi, 0 \le r \le 4, -5 \le z \le 4\}$$

Now, to find the integral

$$\iiint_E \sqrt{x^2 + y^2}\, dV = \int_0^{2\pi}\int_0^4\int_{-5}^4 \sqrt{r^2} \cdot r dz dr d\theta$$

$$= \int_0^{2\pi}\int_0^4\int_{-5}^4 r.r.dz\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_{-5}^4 dz \int_0^4 r^2\, dr$$

$$= \theta\Big]_0^{2\pi} z\Big]_{-5}^4 \frac{r^3}{3}\Big]_0^4$$

$$= (2\pi - 0)\cdot(4 - (-5))\cdot\frac{1}{3}\left((4)^3 - 0\right)$$

$$= (2\pi)(9)\left(\frac{64}{3}\right)$$

$$= 6\pi(64)$$

$$= \boxed{384\pi}$$

Chapter 15 Multiple Integrals 15.8 18E

Consider the volume of a solid region G bounded by the surfaces

$z = g_1(r, \theta)$ and $z = g_2(r, \theta)$ in cylindrical coordinates. Now, if the projection of the solid on the x-plane is a simple polar region R, and if $f(r, \theta, z)$ is continuous on G, then

$$\iiint_G f(r, \theta, z)\,dV = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r, \theta, z)\,r\,dz\,dr\,d\theta.$$

Let us determine the limits of integration of z by solving the given equation for z. In cylindrical coordinates, $x = r\cos\theta$ and $y = r\sin\theta$.

$$z = (r\cos\theta)^2 + (r\sin\theta)^2$$
$$z = r^2\cos^2\theta + r^2\sin^2\theta$$
$$z = r^2$$

Thus, we get the limits as (r2, 4). We get the limits for r as (0, 2).

Since the region is entirely covered by the given equation, the limits of integration of r is (0, a) and that for θ is $(0, 2\pi)$.

Substitute the known values in

$$V = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r, \theta, z)\,r\,dz\,dr\,d\theta.$$

$$V = \int_0^{2\pi}\int_0^2\int_{r^2}^4 zr\,dz\,dr\,d\theta$$

Evaluate the above integral.

First integrate with respect to z, where z varies from r^2 4 to 4.

$$V = \int_0^{2\pi}\int_0^2 r\left[\frac{z^2}{2}\right]_{r^2}^4 dr\,d\theta$$
$$V = \int_0^{2\pi}\int_0^2 r\left[\frac{4^2 - r^2}{2}\right] dr\,d\theta$$
$$V = \int_0^{2\pi}\int_0^2 \frac{1}{2}r\left(16 - r^4\right)dr\,d\theta$$
$$V = \frac{1}{2}\int_0^{2\pi}\int_0^2 r\left(16 - r^4\right)dr\,d\theta$$
$$V = \frac{1}{2}\int_0^{2\pi}\int_0^2 \left(16r - r^5\right)dr\,d\theta$$

Integrate with respect to r, where r varies from 0 to 2.

$$V=\frac{1}{2}\int_0^{2\pi}\left[\frac{16r^2}{2}-\frac{r^6}{6}\right]_0^2 d\theta$$

$$V=\frac{1}{2}\int_0^{2\pi}\left[\frac{16(4)}{2}-\frac{64}{6}\right]d\theta$$

$$V=\frac{1}{2}\int_0^{2\pi}\left[\frac{16(4)}{2}-\frac{64}{6}\right]d\theta$$

$$V=\frac{1}{2}\int_0^{2\pi}\frac{64}{3}d\theta$$

And integrate with respect to θ, where θ varies from 0 to 2π.

$$V=\frac{1}{2}\cdot\frac{64}{3}\int_0^{2\pi}d\theta$$

$$V=\frac{1}{2}\cdot\frac{64}{3}[\theta]_0^{2\pi}$$

$$V=\frac{1}{\not 2}\cdot\frac{64}{3}\cdot\not 2\pi$$

$$V=\frac{64\pi}{3}$$

The volume of the region is $\frac{64\pi}{3}$.

Chapter 15 Multiple Integrals 15.8 19E

Consider the triple integral

$$\iiint_E (x+y+z)\,dV$$

Where E is the solid in the first octant that lies under the paraboloid $z=4-x^2-y^2$.

Use cylindrical coordinates,

$$x=r\cos\theta,\quad y=r\sin\theta \quad \text{and} \quad z=z$$

Now

$$z=4-x^2-y^2$$
$$\Rightarrow z=4-r^2$$

Now, set $z = 0$, obtain that

$$4-r^2=0$$
$$\Rightarrow \quad r=2$$

Note that the solid lies in the first octant. So, the lower limit of all the three integrals is 0.

So, this region has a much simpler description in cylindrical coordinates:

$$E=\left\{(r,\theta,z)\,|\,0\le\theta\le\frac{\pi}{2},0\le r\le 2,o\le z\le 4-r^2\right\}$$

Therefore,

$$\iiint_E (x+y+z)\,dV = \int_0^{\pi/2}\int_0^{2}\int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{2}\left[(r\cos\theta + r\sin\theta)z + \frac{z^2}{2}\right]_0^{4-r^2} r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{2}\left[r(\cos\theta + \sin\theta)(4-r^2) + \frac{(4-r^2)^2}{2}\right] r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{2}\left[(\cos\theta + \sin\theta)(4r^2-r^4) + \frac{(16r+r^5-8r^3)}{2}\right] dr\, d\theta$$

$$= \int_0^{\pi/2}\left[(\cos\theta+\sin\theta)\left(\frac{4r^3}{3}-\frac{r^5}{5}\right)+\frac{1}{2}\left(8r^2+\frac{r^6}{6}-2r^4\right)\right]_0^2 d\theta$$

$$= \int_0^{\pi/2}\left[(\cos\theta+\sin\theta)\left(\frac{32}{3}-\frac{32}{5}\right)+\frac{1}{2}\left(32+\frac{64}{6}-32\right)\right] d\theta$$

$$= \int_0^{\pi/2}\left[\frac{64}{15}(\cos\theta+\sin\theta)+\frac{16}{3}\right] d\theta$$

Continuation to the above step,

$$\iiint_E (x+y+z)\,dV = \left[\frac{64}{15}(\sin\theta-\cos\theta)+\frac{16}{3}\theta\right]_0^{\pi/2}$$

$$= \left[\left(\frac{64}{15}(1-0)+\frac{16}{3}\cdot\frac{\pi}{2}\right)-\left(\frac{64}{15}(0-1)+\frac{16}{3}\cdot 0\right)\right]$$

$$= \frac{64}{15}+\frac{8\pi}{3}+\frac{64}{15}$$

$$= \frac{8\pi}{3}+\frac{128}{15}$$

Therefore, the volume of the solid is $\frac{8\pi}{3}+\frac{128}{15}$.

Chapter 15 Multiple Integrals 15.8 20E

In cylindrical co – ordinates the region E is bounded by planes $z=0$, $z = r\cos\theta + r\sin\theta + 5$ and the cylinder $r=2$ and $r=3$ therefore E is given by

$$E = \{(r,\theta,z):\ 0\le\theta\le 2\pi,\ 2\le r\le 3,\ \ 0\le z\le r(\sin\theta+\cos\theta)+5\}$$

Then $\iiint\limits_E x\,dV = \int_0^{2\pi}\int_2^3\int_0^{r\cos\theta + r\sin\theta + 5} (r\cos\theta)\, r\, dz\, dr\, d\theta$

$$= \int_0^{2\pi}\int_2^3 r^2\cos\theta\,[z]_0^{r\cos\theta + r\sin\theta + 5}\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_2^3 r^2\cos\theta\,(r\cos\theta + r\sin\theta + 5)\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^4}{4}\cos^2\theta + \frac{r^4}{4}\cos\theta\sin\theta + \frac{5}{3}r^3\cos\theta\right]_{r=2}^{r=3} d\theta$$

$$= \int_0^{2\pi}\left[\frac{65}{4}\cos^2\theta + \frac{65}{4}\cos\theta\sin\theta + 19\times\frac{5}{3}\cos\theta\right] d\theta$$

i.e. $\iiint\limits_E x\,dV = \int_0^{2\pi} \frac{65}{8}(1+\cos 2\theta) + \frac{65}{8}\sin 2\theta + 19\times\frac{5}{3}\cos\theta \Big] d\theta$

$$= \left[\frac{65}{8}\left(\theta + \frac{\sin 2\theta}{2} - \frac{\cos 2\theta}{2}\right) + 19\times\frac{5}{3}\sin\theta\right]_0^{2\pi}$$

$$= \frac{65}{8}\left(2\pi + 0 - \frac{1}{2} + 0 - 0 + \frac{1}{2}\right) + 19\times\frac{5}{3}(0)$$

$$= \boxed{\frac{65\pi}{4}}$$

Chapter 15 Multiple Integrals 15.8 21E

Consider the triple integral $\iiint_E x^2 \, dV$

Here *E* is the solid that lies within the cylinder $x^2+y^2=1$, above the plane $z=0$ and below the cone $z^2=4x^2+4y^2$.

The objective is to evaluate the integral $\iiint_E x^2 \, dV$

In cylindrical coordinates $x=r\cos\theta, y=r\sin\theta, z=z, r^2=x^2+y^2$.

To find the *z* limits,

Given plane is $z=0$

$$z^2=4x^2+4y^2$$
$$z^2=4\left(x^2+y^2\right)$$
$$z^2=4r^2$$
$$\Rightarrow z=2r$$

So, *z* varies from 0 to $2r$.

And

$$x^2+y^2=1$$
$$r^2\sin^2\theta+r^2\cos^2\theta=1$$
$$r^2=1$$
$$r=1$$

Then *r* lies from 0 to 1 and θ lies from 0 to 2π.

The region of integration can be described as

$$E=\{(r,\theta,z)\,|\,0\le z\le 2r, 0\le r\le 1, 0\le\theta\le 2\pi\}.$$

So, solve the integral as follows:

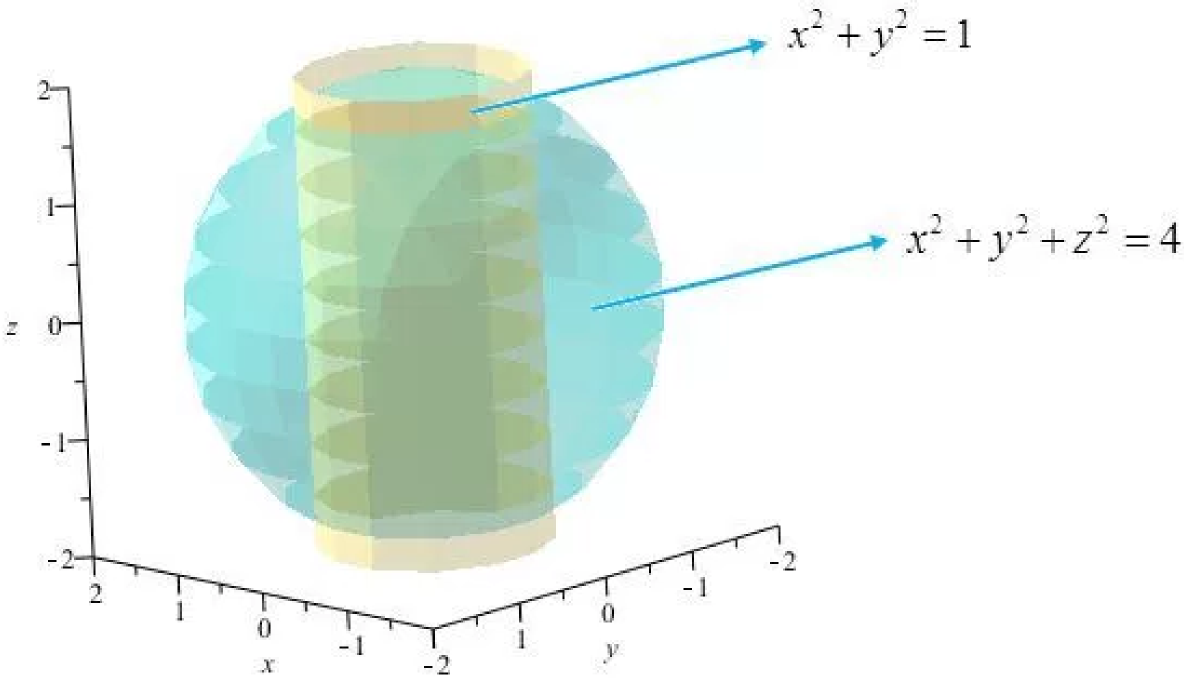
$$\iiint_E x^2\,dV = \int_0^{2\pi}\int_0^1\int_0^{2r} (r\cos\theta)^2\ r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1\int_0^{2r} r^3\cos^2\theta\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 \left[r^3\cos^2\theta\left[z\right]_0^{2r}\,dr\,d\theta\right]$$
$$= \int_0^{2\pi}\int_0^1 \left[r^3\cos^2\theta\left[2r\right]dr\,d\theta\right]$$
$$= 2\left(\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta\right)\left(\int_0^1 r^4\,dr\right)$$
$$= 2\left(\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right)_0^{2\pi}\left(\frac{r^5}{5}\right)_0^1$$
$$= (2\pi + 0 - 0)\left(\frac{1}{5} - 0\right)$$
$$= \frac{2\pi}{5}$$

Therefore, $\boxed{\iiint_E x^2\ dV = \frac{2\pi}{5}}$.

Chapter 15 Multiple Integrals 15.8 22E

Consider the following:

The volume of the solid lies within both the cylinders $x^2 + y^2 = 1$, and the sphere $x^2 + y^2 + z^2 = 4$.



Change the coordinates of the region of solid in terms of cylindrical polar coordinates.

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Consider the volume element.

$$dx\,dy\,dz = \det\frac{\partial(x,y,z)}{\partial(r,\theta,z)}dr\,d\theta\,dz$$
$$= r\,dr\,d\theta\,dz$$

Then, the volume of the solid is as follows:

$$V = \iiint_E dv$$

Here, in polar co – ordinates are as follows:

$x = r\cos\theta$ and $y = r\sin\theta$.

$$x^2 + y^2 = r^2$$
$$1 = r^2 \text{ Since } x^2 + y^2 = 1$$
$$r = 1$$

and,

$$x^2 + y^2 + z^2 = 4$$
$$r^2 + z^2 = 4$$
$$z = \pm\sqrt{4-r^2}$$

Therefore, the region is stated as follows:

$$E = \left\{(r,\theta,z): 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ -\sqrt{4-r^2} \le z \le \sqrt{4-r^2}\right\}$$

Thus ,

$$V = \iiint_E dv$$

$$= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r.dz\, dr\, d\theta$$ Perform the integration on *z*,

$$= \int_0^{2\pi} \int_0^1 r(z)_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}}\, dr\, d\theta$$ Apply the limits for *z*.

$$= \int_0^{2\pi} \int_0^1 r\left(\sqrt{4-r^2} - \left(-\sqrt{4-r^2}\right)\right) dr\, d\theta$$

Simplify,

$$= \int_0^{2\pi} \int_0^1 2r\sqrt{4-r^2}\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 2r\sqrt{4-r^2}\, dr$$ Perform the integration on *r* and θ .

Continue the above steps.

Consider the integral.

$$\int_0^1 2r\sqrt{4-r^2}\, dr$$

Let, $t = 4 - r^2$ then $dt = -2rdr$

and the limits are as follows:

If $r = 0$ then $t = 4$

If $r = 1$ then $t = 3$

Therefore,

$$\int_0^1 2r\sqrt{4-r^2}\,dr = \int_4^3 -\sqrt{t}\,dt$$

$$= -\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^3$$ Use the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$.

$$= -\frac{2}{3}\left[t^{\frac{3}{2}}\right]_4^3$$

Now, apply the limits.

$$\int_0^1 2r\sqrt{4-r^2}\,dr = -\frac{2}{3}\left[t^{\frac{3}{2}}\right]_4^3$$

$$= -\frac{2}{3}\left[3^{3/2} - 4^{3/2}\right]$$

$$= \frac{2}{3}\left[4^{3/2} - 3^{3/2}\right]$$

$$= \frac{2}{3}\left[8 - 3^{3/2}\right]$$

Thus, $\int_0^1 2r\sqrt{4-r^2}\,dr = \frac{2}{3}\left[8-3^{3/2}\right]$ ...... **(1)**

Now,

$$V = \int_0^{2\pi} d\theta \int_0^1 2r\sqrt{4-r^2}\,dr$$ Since from (1),

$$= (\theta)_0^{2\pi} \cdot \left(\frac{2}{3}\left[8-3^{3/2}\right]\right)$$

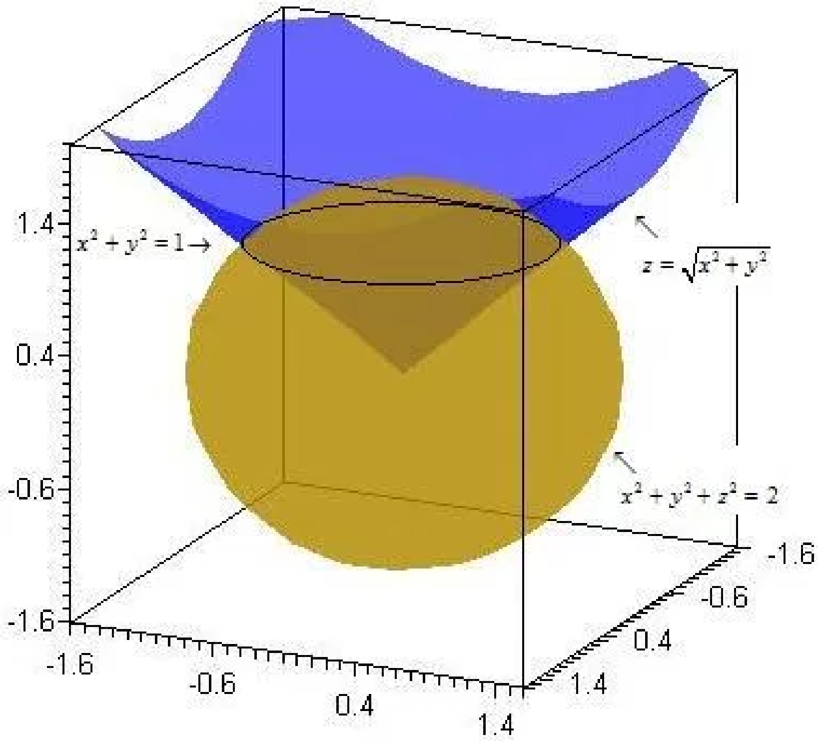
$$= (2\pi)\cdot\left(\frac{2}{3}\left[8-3^{3/2}\right]\right)$$ Simplify,

$$= \frac{4\pi}{3}\left(8-3^{3/2}\right)$$

Hence, the required volume is $\boxed{V = \frac{4\pi}{3}\left(8-3^{3/2}\right)}$.

Chapter 15 Multiple Integrals 15.8 23E

Consider the sphere $x^2 + y^2 + z^2 = 2$ and the cone $z = \sqrt{x^2 + y^2}$.



The region is bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the cone $z = \sqrt{x^2 + y^2}$.

We have $\sqrt{x^2 + y^2} \le z \le \sqrt{2 - x^2 - y^2}$.

And $x^2 + y^2 \le 1$

In polar coordinates, this region $x^2 + y^2 \le 1$ is $R = \{(r, \theta) : 0 \le r \le 1, 0 \le \theta \le 2\pi\}$ and also $\sqrt{2 - x^2 - y^2} = \sqrt{2 - r^2}$ and $\sqrt{x^2 + y^2} = \sqrt{r^2} = r$.

Now, we can compute the volume by finding the volume under the graph of $\sqrt{2-r^2}$ above the disk $R=\{(r,\theta):0\le r\le 1, 0\le\theta\le 2\pi\}$ and subtracting the volume under the graph of r above R.

Therefore, the required volume is

$$V=\int_0^{2\pi}\int_0^1\left(\sqrt{2-r^2}-r\right)r\,dr\,d\theta$$
$$=\int_0^{2\pi}\int_0^1\left(r\sqrt{2-r^2}-r^2\right)dr\,d\theta$$
$$=\int_0^{2\pi}\left[-\frac{1}{3}\left(2-r^2\right)^{\frac{3}{2}}-\frac{1}{3}r^3\right]_0^1 d\theta$$
$$=\int_0^{2\pi}\left[-\frac{1}{3}\left(2-r^2\right)^{\frac{3}{2}}-\frac{1}{3}r^3\right]_0^1 d\theta$$

Continuing the above step,

$$V=\int_0^{2\pi}\left[\left(-\frac{1}{3}-\frac{1}{3}\right)-\left(-\frac{1}{3}2^{\frac{3}{2}}\right)\right]d\theta$$
$$=\int_0^{2\pi}\left[-\frac{2}{3}+\frac{1}{3}2^{\frac{3}{2}}\right]d\theta$$
$$=\frac{2^{\frac{3}{2}}-2}{3}\left[\theta\right]_0^{2\pi}$$
$$=\frac{2^{\frac{3}{2}}-2}{3}\cdot 2\pi$$
$$=\frac{4}{3}\pi\left(\sqrt{2}-1\right)$$

Therefore, the volume of the enclosed region is $\boxed{\frac{4}{3}\pi\left(\sqrt{2}-1\right)}$.

Chapter 15 Multiple Integrals 15.8 24E

Consider the following paraboloid and the sphere:

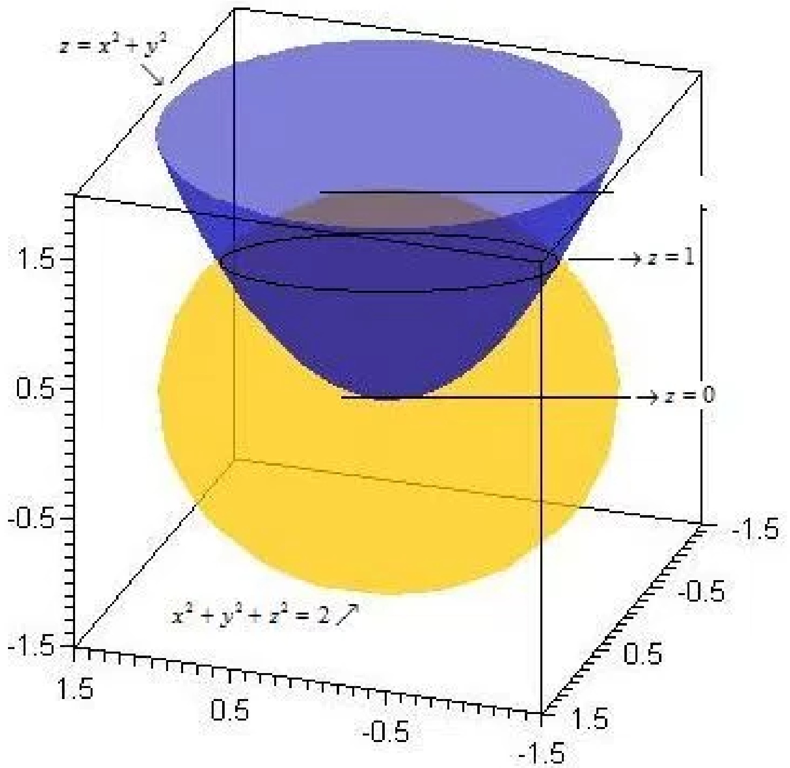
$z = x^2 + y^2$ and $x^2 + y^2 + z^2 = 2$

The objective is to find the volume of the solid.

Find the intersection points of the paraboloid and the sphere as follows:

$$
\begin{aligned}
x^2 + y^2 + z^2 &= 2 \\
z + z^2 &= 2 \qquad \text{Since } z = x^2 + y^2 \\
z^2 + z - 2 &= 0 \\
(z+2)(z-1) &= 0 \\
z &= 1, -2
\end{aligned}
$$

Sketch the graph of the paraboloid and the sphere as follows:



Find the volume of the solid as follows:

The intersection of the paraboloid and the sphere is the circle, $x^2+y^2=1, z=1$

The projection of the region on the $xy-$plane is a circle defined by the equation $x^2+y^2=1$.

Thus, the limits of the integration is,

$$E=\left\{(x,y,z)\,|\,-1\le x\le 1, 0\le y\le \sqrt{1-x^2}, x^2+y^2\le z\le \sqrt{2-x^2-y^2}\right\}$$

The volume of the solid region is expressed by the integral:

$$\begin{aligned}V&=\iiint_E dxdydz\\&=\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dxdydz \qquad ......(1)\end{aligned}$$

Use cylindrical coordinates to simplify the integral as follows:

$$x=r\cos\theta, y=r\sin\theta, z=z$$

Find the region of the integration as follows:

From cylindrical coordinates $x^2+y^2=r^2$.

Substitute $x^2+y^2=r^2$ in the equations $z=x^2+y^2$ and $x^2+y^2+z^2=2$.

$$\begin{aligned}&r^2+z^2=2;\ z=r^2\\&\qquad z=\sqrt{2-r^2};\ z=r^2\end{aligned}$$

In cylindrical coordinates, the region of the integration is,

$$E=\left\{(r,\theta,z)\,|\,0\le r\le 1, 0\le\theta\le 2\pi, r^2\le z\le\sqrt{2-r^2}\right\}$$

Substitute these values in the equation (1).

The volume of the solid is,

$$V = \iiint_E r\,dr\,dz\,d\theta$$
$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$$
$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r\left(\sqrt{2-r^2} - r^2\right) dr\,d\theta$$
$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left(r\sqrt{2-r^2} - r^3\right) dr\,d\theta$$
$$= \int_0^{2\pi} \left(-\frac{1}{3}\left(-r^2+2\right)^{3/2} - \frac{1}{4}r^4\right)_0^1 d\theta$$
$$= \int_0^{2\pi} \left(\frac{-7}{12} + \frac{2}{3}\sqrt{2}\right) d\theta$$
$$= \left(\frac{-7}{12} + \frac{2}{3}\sqrt{2}\right) \int_0^{2\pi} d\theta$$

Continue the above integral.

$$V = \left(\frac{-7}{12} + \frac{2}{3}\sqrt{2}\right) \int_0^{2\pi} d\theta$$
$$= \left(\frac{-7}{12} + \frac{2}{3}\sqrt{2}\right) (\theta)_0^{2\pi}$$
$$= \left(\frac{-7}{12} + \frac{2}{3}\sqrt{2}\right) (2\pi - 0)$$
$$= \frac{4}{3}\sqrt{2}\pi - \frac{7}{6}\pi$$

Therefore, the volume of the solid is,

$$\boxed{V = \frac{4}{3}\sqrt{2}\pi - \frac{7}{6}\pi}.$$

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(a)

The sphere of radius a has equation $r^2+z^2=a^2$.

Solve this for z to get $z=\pm\sqrt{a^2-r^2}$.

Note that the circle $r=a\cos\theta$ is complete for $0\le\theta\le\pi$.

The volume is,

$$V=\int_0^{\pi}\int_0^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$

Complete the inner integration.

$$V=\int_0^{\pi}\int_0^{a\cos\theta} rz\Big|_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} dr\,d\theta$$

$$=\int_0^{\pi}\int_0^{a\cos\theta} 2r\sqrt{a^2-r^2}\,dr\,d\theta$$

Substitute $u=a^2-r^2$ with $xu=a^2-r^2$,

$$V=\int_0^{\pi}\int_{a^2}^{a^2(1-\cos^2\theta)} -u^{\frac{1}{2}}\,du\,d\theta$$

$$=\int_0^{\pi}-\frac{2}{3}u^{\frac{3}{2}}\Big|_{a^2}^{a^2\sin^2\theta}\,d\theta$$

$$=-\frac{2}{3}\int_0^{\pi}\left(a^2\sin^2\theta\right)^{\frac{3}{2}}-\left(a^2\right)^{\frac{3}{2}}\,d\theta$$

$$=-\frac{2a^3}{3}\int_0^{\pi}\sin^3\theta-1\,d\theta$$

Continuous from the previous step,

$$V=-\frac{2a^3}{3}\int_0^{\pi}\sin^2\theta\sin\theta-1\,d\theta$$

$$=-\frac{2a^3}{3}\int_0^{\pi}\left(1-\cos^2\theta\right)\sin\theta-1\,d\theta$$

$$=-\frac{2a^3}{3}\int_0^{\pi}\sin\theta-\cos^2\theta\sin\theta-1\,d\theta$$

$$V=-\frac{2a^3}{3}\left(-\cos\theta+\frac{1}{3}\cos^3\theta-\theta\Big|_0^{\pi}\right.$$

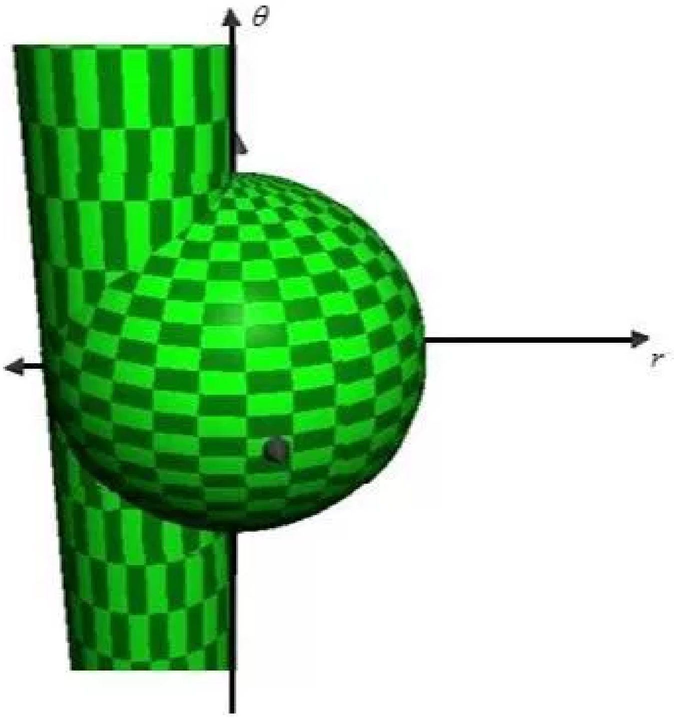
$$=-\frac{2a^3}{3}\left(\left(1-\frac{1}{3}-\pi\right)-\left(-1+\frac{1}{3}-0\right)\right)$$

$$=-\frac{2a^3}{3}\left(\frac{4}{3}-\pi\right)$$

$$=\boxed{\frac{2a^3}{3}\left(\pi-\frac{4}{3}\right)}$$

(b)

The graph of the sphere and cylinder on the same screen is as shown below.



Chapter 15 Multiple Integrals 15.8 27E

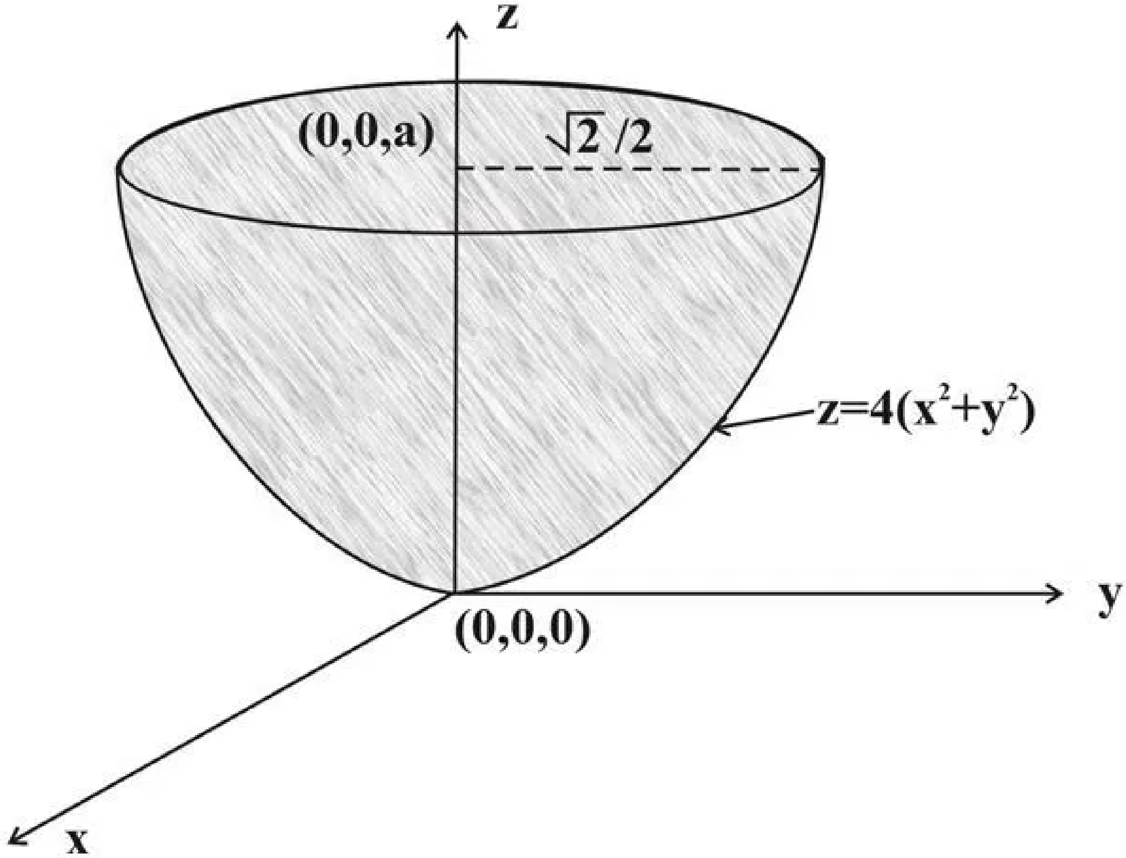
The paraboloid $z = 4x^2 + 4y^2$ meets plane $z = a$ $(a > 0)$ in a circle with equation

$$a = 4x^2 + 4y^2$$

$$a = 4\left(x^2 + y^2\right)$$

$$x^2 + y^2 = \frac{a}{4}$$

The graph of a paraboloid



Cylindrical Coordinates:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

$$\begin{aligned} x^2 + y^2 &= r^2\cos^2(\theta) + r^2\sin^2(\theta) \\ &= r^2\left(\cos^2(\theta) + \sin^2(\theta)\right) \\ &= r^2(1) \\ &= r^2 \end{aligned}$$

$$dV = r dz dr d\theta$$

Here $x^2 + y^2 = \frac{a}{4}$ and $x^2 + y^2 = r^2$

Then

$$r^2 = \frac{a}{4}$$

$r = \frac{\sqrt{a}}{2}$ Since radius is non-negative

Then the solid S in cylindrical co – ordinates

$$E = \left\{(r,\theta,z):\ 0 \le \theta \le 2\pi, 0 \le r \le \frac{\sqrt{a}}{2}, 4r^2 \le z \le a\right\}$$

Now, density $\rho(x,y,z) = k$ (constant)

Then the mass is given by

$$m = \iiint_E \rho(x,y,z)\, dV$$

Convert to cylindrical coordinates, then

$$m = k\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}\int_{4r^2}^{a} r\, dz\, dr\, d\theta$$

$$= k\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}} r(z)_{4r^2}^{a} .dr\, d\theta$$

$$= k\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}} \left(a - 4r^2\right) r dr\, d\theta$$

$$= k\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}} \left(ar - 4r^3\right) dr\, d\theta$$

On calculating the mass

$$m = k\int_0^{2\pi}\left(\frac{ar^2}{2} - r^4\right)_0^{\frac{\sqrt{a}}{2}} d\theta$$

$$= k\int_0^{2\pi}\left(\frac{a^2}{8} - \frac{a^2}{16}\right) d\theta$$

$$= k\frac{a^2}{16}\int_0^{2\pi} d\theta$$

$$= k\frac{a^2}{16}(2\pi)$$

$$= k\frac{a^2\pi}{8}$$

Therefore mass $m = \boxed{\frac{k\pi a^2}{8}}$

Now, moments about the three coordinate planes are

$$M_{yz} = \iiint_E x\,\rho(x,y,z)\,dV$$

$$= k\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}\int_{4r^2}^{a} r\cos\theta\; r\,dz\,dr\,d\theta$$

$$= k\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}} r^2\cos\theta\,(z)_{4r^2}^{a}\,dr\,d\theta$$

$$= k\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}} r^2\cos\theta\left(a - 4r^2\right)dr\,d\theta$$

$$= k\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}\left(a\cos\theta.r^2 - 4\cos\theta.r^4\right)dr\,d\theta$$

Now, on calculating

$$M_{yz} = k\int_0^{2\pi}\left(a\cos\theta.\frac{r^3}{3}-\frac{4}{5}\cos\theta\, r^5\right)_0^{\frac{\sqrt{a}}{2}} d\theta$$

$$= k\int_0^{2\pi}\left(\frac{a^{\frac{5}{2}}}{24}-\frac{4}{160}a^{\frac{5}{2}}\right)\cos\theta.d\theta$$

$$= k\frac{1}{60}a^{\frac{5}{2}}\int_0^{2\pi}\cos\theta\, d\theta$$

$$= k\frac{1}{60}a^{\frac{5}{2}}\left(\sin\theta\right)_0^{2\pi}$$

$$= k\frac{1}{60}a^{\frac{5}{2}}\left(\sin 2\pi-\sin 0\right)$$

$$= 0$$

And

$$M_{xz} = \iiint_E y\,\rho(x,y,z)\,dV$$

$$= k\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}\int_{4r^2}^{a} r\sin\theta.r\, dz\, dr\, d\theta$$

$$= k\int_0^{2\pi}\int_0^{\sqrt{a}/2} r^2\sin\theta.\left(a-4r^2\right)dr\, d\theta$$

$$= k\int_0^{2\pi}\sin\theta\, d\theta\int_0^{\frac{\sqrt{a}}{2}}\left(ar^2-4r^4\right)dr$$

$$= k\left(-\cos\theta\right)_0^{2\pi}\left(\frac{a}{3}r^3-\frac{4}{5}r^5\right)_0^{\frac{\sqrt{a}}{2}}$$

$$= -k\left(\cos 2\pi-\cos 0\right)\left(\frac{1}{60}a^{\frac{5}{2}}\right)$$

$$= -k\left(1-1\right)\frac{1}{60}a^{\frac{5}{2}}$$

$$= 0$$

And

$$M_{xy} = \iiint_E z\,\rho(x,y,z)\,dV$$

$$= k\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}\int_{4r^2}^{a} z.r\,dz\,dr\,d\theta$$

$$= \frac{k}{2}\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}} r\left(z^2\right)_{4r^2}^{a}\,dr\,d\theta$$

$$= \frac{k}{2}\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}} r\left(a^2 - 16r^4\right)dr\,d\theta$$

On calculating the moment

$$M_{xy} = \frac{k}{2}\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}\left(a^2 r - 16r^5\right)dr\,d\theta$$

$$= \frac{k}{2}\int_0^{2\pi}\left(\frac{a^2r^2}{2} - \frac{16r^6}{6}\right)_0^{\frac{\sqrt{a}}{2}} d\theta$$

$$= \frac{k}{2}\left(\frac{a^3}{8} - \frac{8a^3}{3\times 64}\right)(\theta)_0^{2\pi}$$

$$= \frac{k}{2}\frac{2a^3}{(3)(8)}(2\pi)$$

$$= \frac{2}{24}k\pi a^3$$

Now

$$\overline{x} = \frac{M_{yz}}{m} = 0,\quad \overline{y} = \frac{M_{xz}}{m} = 0,\quad \overline{z} = \frac{M_{xy}}{m}$$

$$= \frac{2}{24}k\pi a^3 \times \frac{8}{k\pi a^2} \quad = \frac{2}{3}a$$

Hence the mass is

$$m = \boxed{\frac{k\pi a^2}{8}}$$

And center of mass is $\left(\overline{x},\overline{y},\overline{z}\right) = \boxed{\left(0,0,\frac{2a}{3}\right)}$

Chapter 15 Multiple Integrals 15.8 28E

The density at any point is proportional to its distance from the z – axis i.e.

$$\rho(x,y,z)\,\alpha\sqrt{x^2+y^2}$$

Taking constant of proportionality to be k we have

$$\rho(x,y,z)=k\sqrt{x^2+y^2}$$

Now the region of integration i.e. the ball B can be written in cylindrical co – ordinates as

$$B=\left\{(r,\theta,z):\,0\le\theta\le 2\pi,0\le r\le a,-\sqrt{a^2-r^2}\le z\le\sqrt{a^2-r^2}\right\}$$

Then the mass of the ball is

$$m=\iiint_B \rho(x,y,z)\,dV$$

$$=k\int_0^{2\pi}\int_0^{a}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r.r\,dz\,dr\,d\theta$$

$$=k\int_0^{2\pi}\int_0^{a}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^2\,dz\,dr\,d\theta$$

$$=k\int_0^{2\pi}\int_0^{a} r^2\,(z)_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}}\,dr\,d\theta$$

$$=k_2\int_0^{2\pi}\int_0^{a} r^2\sqrt{a^2-r^2}\,dr\,d\theta$$

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Chapter 15 Multiple Integrals 15.8 29E

Consider the following iterated integral,

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy$$

This is a triple integral over a solid region,

$$E=\left\{(x,y,z)\,|\,-2\le y\le 2,-\sqrt{4-y^2}\le x\le\sqrt{4-y^2},\sqrt{x^2+y^2}\le z\le 2\right\}$$

The lower surface of the plane is the cone $z=\sqrt{x^2+y^2}$ and its upper surface is the plane $z=2$.

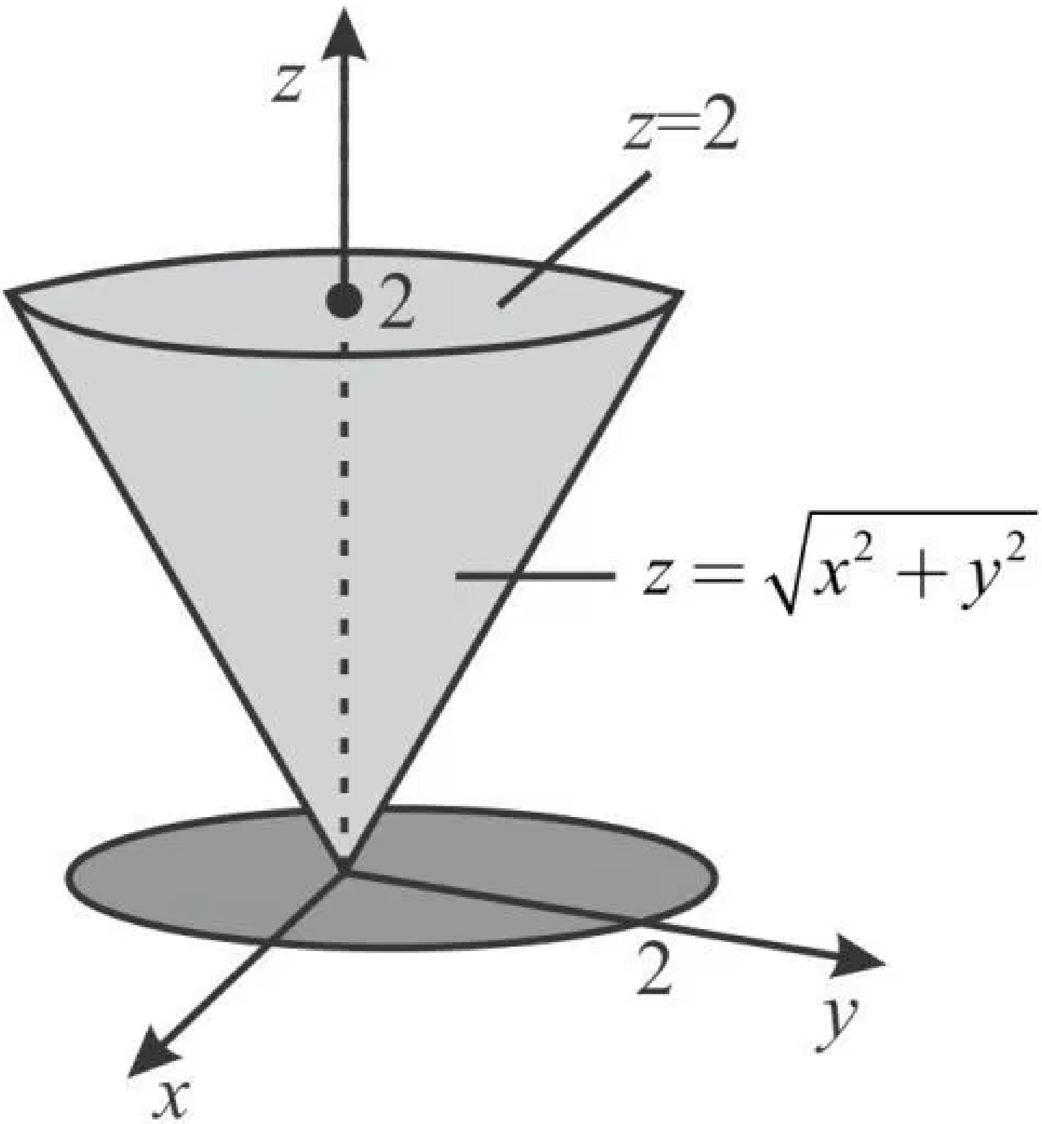
Equate the z terms.

$$\sqrt{x^2+y^2}=2$$

$$x^2+y^2=4$$

Therefore, the projection of E onto the xy – plane is the disk $x^2+y^2\le 4$.

Sketch the region E.



Convert the rectangular coordinates (x, y, z) to cylindrical coordinates (r, θ, z), using the conversions $r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}$, and $z = z$.

Here, $x = r\cos\theta, y = r\sin\theta$, and $z = z$.

Translating the surfaces and the region into cylindrical form,

$$\begin{aligned}\sqrt{x^2+y^2} &= \sqrt{r^2} \\ &= r\end{aligned}$$

So, the limits of integration is $r \le z \le 2, 0 \le r \le 2$, and $0 \le \theta \le 2\pi$.

Therefore, the description of the solid E in cylindrical coordinates is as follows:

$$E = \{(r, \theta, z) | r \le z \le 2, 0 \le r \le 2, 0 \le \theta \le 2\pi\}$$

Rewrite the integral $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xzdzdxdy$ in cylindrical form.

$$\begin{aligned}\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xzdzdxdy &= \int_0^{2\pi}\int_0^2\int_r^2 (r\cos\theta)\, zrdzdrd\theta \\ &= \int_0^{2\pi}\int_0^2\int_r^2 (r^2 z\cos\theta)dzdrd\theta\end{aligned}$$

First evaluate the inner integral with respect to z.

$$\begin{aligned}\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xzdzdxdy &= \int_0^{2\pi}\int_0^2 (r^2\cos\theta)\left(\frac{z^2}{2}\right)_r^2 drd\theta \\ &= \int_0^{2\pi}\int_0^2 (r^2\cos\theta)\left(\frac{4}{2}-\frac{r^2}{2}\right)drd\theta \\ &= \int_0^{2\pi}\int_0^2 \frac{1}{2}\cos\theta\left(4r^2-r^4\right)drd\theta\end{aligned}$$

Apply the limits of z, and evaluate the middle integral with respect to r.

$$\begin{aligned}\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xzdzdxdy &= \int_0^{2\pi}\frac{1}{2}\cos\theta\left(\left(\frac{4r^3}{3}\right)_0^2-\left(\frac{r^5}{5}\right)_0^2\right)d\theta \\ &= \int_0^{2\pi}\frac{1}{2}\cos\theta\left(\frac{4}{3}(8-0)-\frac{1}{5}(32-0)\right)d\theta \\ &= \int_0^{2\pi}\frac{1}{2}\cos\theta\left(\frac{32}{3}-\frac{32}{5}\right)d\theta \\ &= \int_0^{2\pi}\frac{1}{2}\left(\frac{64}{15}\right)\cos\theta d\theta\end{aligned}$$

Finally, evaluate the outer integral with respect to θ.

$$\begin{aligned}\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xzdzdxdy &= \frac{32}{15}\int_0^{2\pi}\cos\theta\, d\theta \\ &= \frac{32}{15}(\sin\theta)_0^{2\pi} \\ &= \frac{32}{15}(\sin 2\pi-\sin 0) \\ &= 0\end{aligned}$$

Therefore, the value of $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xzdzdxdy$ is $\boxed{0}$.

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Consider the following iterated integral:

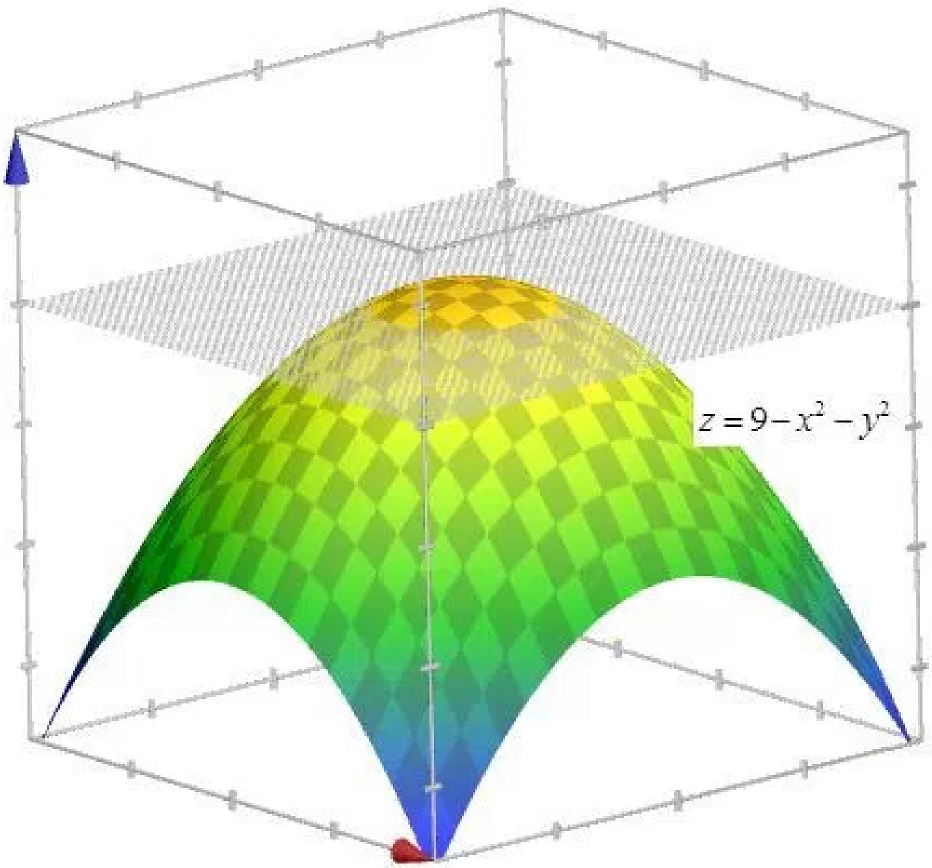
$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx.$$

The objective is to evaluate the iterated integral using cylindrical coordinates.

This is a triple integral over a solid region:

$$E=\left\{(x,y,z)\mid -3\le x\le 3, 0\le y\le \sqrt{9-x^2}, 0\le z\le 9-x^2-y^2\right\}.$$

The lower surface of the solid is a plane $z=0$ and its upper surface is the paraboloid opening downward $z=9-x^2-y^2$ that is shown is as below.



Equate the z terms.

$$9-x^2-y^2=0$$
$$x^2+y^2=9$$

Therefore, the projection of E onto the $xy-$plane is the disk $x^2+y^2\le 9.$

Convert the rectangular coordinates (x, y, z) to cylindrical coordinates (r, θ, z), using the conversions $r^2 = x^2 + y^2, \tan\theta = \frac{y}{x},$ and $z = z.$

Here, $x = r\cos\theta, y = r\sin\theta,$ and $z = z.$

Translating the surfaces and the region into cylindrical form,

$$\begin{aligned}\sqrt{x^2+y^2} &= \sqrt{r^2} \\ &= r\end{aligned}$$

So, the limits of integration is $0 \le z \le 9 - r^2, 0 \le r \le 3,$ and $0 \le \theta \le 2\pi.$

Therefore, the description of the solid E in cylindrical coordinates is as follows:

$$E = \left\{(r,\theta,z) \middle| 0 \le z \le 9 - r^2, 0 \le r \le 3, \text{ and } 0 \le \theta \le 2\pi\right\}.$$

Rewrite the integral $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$ in cylindrical form.

$$\begin{aligned}\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx &= \int_{0}^{2\pi}\int_{0}^{3}\int_{0}^{9-r^2}\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}\,r\,dz\,dr\,d\theta \\ &= \int_{0}^{2\pi}\int_{0}^{3}\int_{0}^{9-r^2}\sqrt{r^2\left(\cos^2\theta+\sin^2\theta\right)}\,r\,dz\,dr\,d\theta \\ &= \int_{0}^{2\pi}\int_{0}^{3}\int_{0}^{9-r^2} r^2\,dz\,dr\,d\theta\end{aligned}$$

First evaluate the inner integral with respect to $z.$

$$\begin{aligned}\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx &= \int_{0}^{2\pi}\int_{0}^{3} r^2\left[z\right]_0^{9-r^2}\,dr\,d\theta \\ &= \int_{0}^{2\pi}\int_{0}^{3} r^2\left[\left(9-r^2\right)-0\right]dr\,d\theta \\ &= \int_{0}^{2\pi}\int_{0}^{3}\left[9r^2-r^4\right]dr\,d\theta\end{aligned}$$

Apply the limits of z, and evaluate the middle integral with respect to $r.$

$$\begin{aligned}\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx &= \int_{0}^{2\pi}\left[9\left(\frac{r^3}{3}\right)_0^3-\left(\frac{r^5}{5}\right)_0^3\right]d\theta \\ &= \int_{0}^{2\pi}\left[3(27-0)-\frac{1}{5}(243-0)\right]d\theta \\ &= \int_{0}^{2\pi}\left[\frac{162}{5}\right]d\theta \\ &= \frac{162}{5}\int_{0}^{2\pi}1\,d\theta\end{aligned}$$

Finally, evaluate the outer integral with respect to $\theta.$

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx=\frac{162}{5}(\theta)_0^{2\pi}$$
$$=\frac{162}{5}(2\pi-0)$$
$$=\frac{324\pi}{5}$$

Therefore, the value of $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$ is $\boxed{\frac{324\pi}{5}}$.

Chapter 15 Multiple Integrals 15.8 31E

Let the conical region occupied by the mountain be C.

Write the formula to find the work done in lifting a small volume of material.

$$\text{Work}=\text{Height}\times\text{Density}\times\text{Volume}$$

Divide the whole mountain into smaller volumes of material.

Use this formula to find the work done in lifting a small volume of mountain.

$$\Delta W=h(P)g(P)\Delta V$$

Now, the total volume of the mountain will be obtained by the following integral.

$$W=\boxed{\iiint_C h(P)g(P)\Delta V}.$$

Radius of the conical mountain is $R=62{,}000$ ft.

Height of the conical mountain is $H=12{,}400$ ft.

Density of the conical mountain is $g(P)=200\text{ lb/ft}^3$.

In the cylindrical coordinates, the region C is as follows:

$$C=\left\{(r,\theta,z)\mid 0\le\theta\le2\pi,0\le z\le12{,}400,0\le r\le62{,}000\left(1-\frac{z}{12{,}400}\right)\right\}$$

Use the cylindrical coordinates to set up the integral.

$$W = \iiint_C h(P)\,g(P)\,\Delta V$$

$$= \int_0^{2\pi}\int_0^{12,400}\int_0^{62,000\left(1-\frac{z}{12,400}\right)} (z\cdot 200r)\,dr\,dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^{12,400}\int_0^{62,000\left(1-\frac{z}{12,400}\right)} (z\cdot 200r)\,dr\,dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^{12,400} 100z\left[r^2\right]_0^{62,000\left(1-\frac{z}{12,400}\right)} dz\,d\theta$$

$$= 100\cdot 62,000^2\int_0^{2\pi}\left[\int_0^{12,400} z\left(1-\frac{z}{12,400}\right)^2 dz\right]d\theta$$

$$= \frac{100\cdot 62,000^2}{12,400^2}\int_0^{2\pi}\left[\int_0^{12,400}\left(12,400^2 z + z^3 + 24,800z^2\right)dz\right]d\theta$$

$$= \frac{100\cdot 62,000^2}{12,400^2}\int_0^{2\pi}\left[12,400^2\frac{z^2}{2} + \frac{z^4}{4} + 24,800\frac{z^3}{3}\right]_0^{12,400} d\theta$$

$$= \frac{100\cdot 62,000^2}{12,400^2}\left(\frac{12,400^4}{2} + \frac{12,400^4}{4} + \frac{24,800\cdot 12,400^3}{3}\right)\int_0^{2\pi} d\theta$$

$$= \frac{100\cdot 62,000^2}{12,400^2}\left(\frac{12,400^4}{2} + \frac{12,400^4}{4} + \frac{24,800\cdot 12,400^3}{3}\right)2\pi$$

$$= \frac{50\pi\cdot 62,000^2\cdot 12,400^2}{3}\ \text{ft/lb}$$

$$\approx 3.1\times 10^{19}\ \text{ft/lb}$$

Hence the amount of work done is $\boxed{3.1\times 10^{19}\ \text{ft/lb}}$.