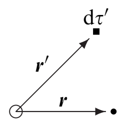
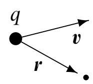
7.3 Electromagnetic fields (general)

Field relationships

Conservation of charge	$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}$	(7.39)	$\boldsymbol{J}$ current density ρ charge density t time
Magnetic vector potential	$\boldsymbol{B} = \nabla \times \boldsymbol{A}$	(7.40)	$\boldsymbol{A}$ vector potential
Electric field from potentials	$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \phi$	(7.41)	ϕ electrical potential
Coulomb gauge condition	$\nabla \cdot \boldsymbol{A} = 0$	(7.42)	
Lorenz gauge condition	$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$	(7.43)	c speed of light
Potential field equations[a]	$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \frac{\rho}{\epsilon_0}$	(7.44)	
	$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}$	(7.45)	
Expression for ϕ in terms of ρ[a]	$\phi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int_{\text{volume}} \frac{\rho(\boldsymbol{r}',t-\lvert\boldsymbol{r}-\boldsymbol{r}'\rvert/c)}{\lvert\boldsymbol{r}-\boldsymbol{r}'\rvert}\,d\tau'$	(7.46)	$d\tau'$ volume element $\boldsymbol{r}'$ position vector of $d\tau'$
Expression for $\boldsymbol{A}$ in terms of $\boldsymbol{J}$[a]	$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int_{\text{volume}} \frac{\boldsymbol{J}(\boldsymbol{r}',t-\lvert\boldsymbol{r}-\boldsymbol{r}'\rvert/c)}{\lvert\boldsymbol{r}-\boldsymbol{r}'\rvert}\,d\tau'$	(7.47)	μ_0 permeability of free space

[a]Assumes the Lorenz gauge.

Liénard–Wiechert potentials[a]

Electrical potential of a moving point charge	$\phi = \frac{q}{4\pi\epsilon_0(\lvert\boldsymbol{r}\rvert - \boldsymbol{v}\cdot\boldsymbol{r}/c)}$	(7.48)	q charge $\boldsymbol{r}$ vector from charge to point of observation $\boldsymbol{v}$ particle velocity
Magnetic vector potential of a moving point charge	$\boldsymbol{A} = \frac{\mu_0 q \boldsymbol{v}}{4\pi(\lvert\boldsymbol{r}\rvert - \boldsymbol{v}\cdot\boldsymbol{r}/c)}$	(7.49)	

[a]In free space. The right-hand sides of these equations are evaluated at retarded times, i.e., at $t' = t - \lvert\boldsymbol{r}'\rvert/c$, where $\boldsymbol{r}'$ is the vector from the charge to the observation point at time t'.

Maxwell's equations

Differential form:		Integral form:	
$\nabla\cdot\boldsymbol{E}=\dfrac{\rho}{\epsilon_0}$	(7.50)	$\displaystyle\oint_{\text{closed surface}}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{s}=\frac{1}{\epsilon_0}\int_{\text{volume}}\rho\,\mathrm{d}\tau$	(7.51)
$\nabla\cdot\boldsymbol{B}=0$	(7.52)	$\displaystyle\oint_{\text{closed surface}}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{s}=0$	(7.53)
$\nabla\times\boldsymbol{E}=-\dfrac{\partial\boldsymbol{B}}{\partial t}$	(7.54)	$\displaystyle\oint_{\text{loop}}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l}=-\frac{\mathrm{d}\Phi}{\mathrm{d}t}$	(7.55)
$\nabla\times\boldsymbol{B}=\mu_0\boldsymbol{J}+\mu_0\epsilon_0\dfrac{\partial\boldsymbol{E}}{\partial t}$	(7.56)	$\displaystyle\oint_{\text{loop}}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{l}=\mu_0 I+\mu_0\epsilon_0\int_{\text{surface}}\frac{\partial\boldsymbol{E}}{\partial t}\cdot\mathrm{d}\boldsymbol{s}$	(7.57)

	Equation (7.51) is "Gauss's law"	$\mathrm{d}\boldsymbol{s}$	surface element
	Equation (7.55) is "Faraday's law"	$\mathrm{d}\tau$	volume element
$\boldsymbol{E}$	electric field	$\mathrm{d}\boldsymbol{l}$	line element
$\boldsymbol{B}$	magnetic flux density	Φ	linked magnetic flux $(=\int\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{s})$
$\boldsymbol{J}$	current density	I	linked current $(=\int\boldsymbol{J}\cdot\mathrm{d}\boldsymbol{s})$
ρ	charge density	t	time

Maxwell's equations (using $\boldsymbol{D}$ and $\boldsymbol{H}$)

Differential form:		Integral form:	
$\nabla\cdot\boldsymbol{D}=\rho_{\text{free}}$	(7.58)	$\displaystyle\oint_{\text{closed surface}}\boldsymbol{D}\cdot\mathrm{d}\boldsymbol{s}=\int_{\text{volume}}\rho_{\text{free}}\,\mathrm{d}\tau$	(7.59)
$\nabla\cdot\boldsymbol{B}=0$	(7.60)	$\displaystyle\oint_{\text{closed surface}}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{s}=0$	(7.61)
$\nabla\times\boldsymbol{E}=-\dfrac{\partial\boldsymbol{B}}{\partial t}$	(7.62)	$\displaystyle\oint_{\text{loop}}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l}=-\frac{\mathrm{d}\Phi}{\mathrm{d}t}$	(7.63)
$\nabla\times\boldsymbol{H}=\boldsymbol{J}_{\text{free}}+\dfrac{\partial\boldsymbol{D}}{\partial t}$	(7.64)	$\displaystyle\oint_{\text{loop}}\boldsymbol{H}\cdot\mathrm{d}\boldsymbol{l}=I_{\text{free}}+\int_{\text{surface}}\frac{\partial\boldsymbol{D}}{\partial t}\cdot\mathrm{d}\boldsymbol{s}$	(7.65)

$\boldsymbol{D}$	displacement field	$\boldsymbol{E}$	electric field
ρ_{free}	free charge density (in the sense of $\rho=\rho_{\text{induced}}+\rho_{\text{free}}$)	$\mathrm{d}\boldsymbol{s}$	surface element
		$\mathrm{d}\tau$	volume element
$\boldsymbol{B}$	magnetic flux density	$\mathrm{d}\boldsymbol{l}$	line element
$\boldsymbol{H}$	magnetic field strength	Φ	linked magnetic flux $(=\int\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{s})$
$\boldsymbol{J}_{\text{free}}$	free current density (in the sense of $\boldsymbol{J}=\boldsymbol{J}_{\text{induced}}+\boldsymbol{J}_{\text{free}}$)	I_{free}	linked free current $(=\int\boldsymbol{J}_{\text{free}}\cdot\mathrm{d}\boldsymbol{s})$
		t	time

Relativistic electrodynamics

Lorentz transformation of electric and magnetic fields	$\boldsymbol{E}'_\parallel = \boldsymbol{E}_\parallel$ (7.66) $\boldsymbol{E}'_\perp = \gamma(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})_\perp$ (7.67) $\boldsymbol{B}'_\parallel = \boldsymbol{B}_\parallel$ (7.68) $\boldsymbol{B}'_\perp = \gamma(\boldsymbol{B} - \boldsymbol{v}\times\boldsymbol{E}/c^2)_\perp$ (7.69)	$\boldsymbol{E}$ electric field $\boldsymbol{B}$ magnetic flux density $'$ measured in frame moving at relative velocity $\boldsymbol{v}$ γ Lorentz factor $= [1-(v/c)^2]^{-1/2}$ $\parallel$ parallel to $\boldsymbol{v}$ $\perp$ perpendicular to $\boldsymbol{v}$
Lorentz transformation of current and charge densities	$\rho' = \gamma(\rho - vJ_\parallel/c^2)$ (7.70) $J'_\perp = J_\perp$ (7.71) $J'_\parallel = \gamma(J_\parallel - v\rho)$ (7.72)	$\boldsymbol{J}$ current density ρ charge density
Lorentz transformation of potential fields	$\phi' = \gamma(\phi - vA_\parallel)$ (7.73) $A'_\perp = A_\perp$ (7.74) $A'_\parallel = \gamma(A_\parallel - v\phi/c^2)$ (7.75)	ϕ electric potential $\boldsymbol{A}$ magnetic vector potential
Four-vector fields[a]	$\underset{\sim}{\boldsymbol{J}} = (\rho c, \boldsymbol{J})$ (7.76) $\underset{\sim}{\boldsymbol{A}} = \left(\frac{\phi}{c}, \boldsymbol{A}\right)$ (7.77) $\Box^2 = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}, -\nabla^2\right)$ (7.78) $\Box^2 \underset{\sim}{\boldsymbol{A}} = \mu_0 \underset{\sim}{\boldsymbol{J}}$ (7.79)	$\underset{\sim}{\boldsymbol{J}}$ current density four-vector $\underset{\sim}{\boldsymbol{A}}$ potential four-vector $\Box^2$ D'Alembertian operator

[a]Other sign conventions are common here. See page 65 for a general definition of four-vectors.