11. Applications of Derivatives

Exercise 11A

1. Question

The side of a square is increasing at the rate of 0.2 cm/s. Find the rate of increase of the perimeter of the square.

Answer

Let the side of the square be a

Rate of change of side= $\frac{da}{dt} = 0.2 \text{ cm/s}$

Perimeter of the square = 4a

Rate of change of perimeter = $4\frac{da}{dt} = 4 \times 0.2$

$$\frac{dP}{dt} = 0.8 \text{ cm/s}$$

2. Question

The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

Answer

Let the radius of the circle be r

$$\frac{dr}{dt} = 0.7\text{cm/s}$$

Circumference of the circle=$2\pi r$

Rate of change of circumference $= 2\pi\frac{dr}{dt}$

$= 2 \times 3.14 \times 0.7$

$$\frac{dC}{dt} = 4.4 \text{ cm/s}$$

3. Question

The radius of a circle is increasing uniformly at the rate of 0.3 centimetre per second. At what rate is the area increasing when the radius is 10 cm?

(Take π = 3.14.)

Answer

Let the radius of the circle be r

$$\frac{dr}{dt} = 0.3\text{cm/s}$$

Area of the circle=πr^2

Rate of change of Area $= 2\pi r\frac{dr}{dt}$

$= 2 \times 3.14 \times 10 \times 0.3$

$$\frac{dA}{dt} = 18.84 \text{ cm}^2\text{/s}$$

4. Question

The side of a square sheet of metal is increasing at 3 centimetres per minute. At what rate is the area increasing when the side is 10 cm long?

Answer

Let the side of the square be a

Rate of change of side= $\frac{da}{dt} = 3 \text{ cm/s}$

Area of the square = a^2

Rate of change of Area = $2a\frac{da}{dt} = 2 \times 10 \times 3$

$$\frac{dA}{dt} = 60 \text{ cm}^2/\text{s}$$

5. Question

The radius of a circular soap bubble is increasing at the rate of 0.2 cm/s. Find the rate of increase of its surface area when the radius is 7 cm.

Answer

Soap bubble will be in the shape of a sphere

Let the radius of the soap bubble be r

$$\frac{dr}{dt} = 0.2\text{cm/s}$$

Surface area of the soap bubble=$4\pi r^2$

Rate of change of Surface area $= 8\pi r\frac{dr}{dt}$

$= 8 \times 3.14 \times 7 \times 0.2$

$$\frac{dS}{dt} = 35.2 \text{ cm}^2/\text{s}$$

6. Question

The radius of an air bubble is increasing at the rate of 0.5 centimetre per second. At what rate is the volume of the bubble increasing when the radius is 1 centimetre?

Answer

Soap bubble will be in the shape of a sphere

Let the radius of the soap bubble be r

$$\frac{dr}{dt} = 0.5\text{cm/s}$$

Volume of the soap bubble=$\frac{4}{3}\pi r^3$

Rate of change of Volume $= 4\pi r^2\frac{dr}{dt}$

$= 4 \times 3.14 \times 1^2 \times 0.5$

$$\frac{dV}{dt} = 6.28 \text{ cm}^3/\text{s}$$

7. Question

The volume of a spherical balloon is increasing at the rate of 25 cubic centimetres per second. Find the rate of change of its surface at the instant when its radius is 5 cm.

Answer

Let the radius of the balloon be r

Let the volume of the spherical balloon be V

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$25\ cm^3/s = 4 \times \pi \times 5^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{4\pi}$

Surface area of the bubble $=4\pi r^2$

Rate of change of Surface area $= 8\pi r \frac{dr}{dt}$

$= 8 \times \pi \times 5 \times \frac{1}{4\pi}$

$\frac{dS}{dt} = 10\ cm^2/s$

8. Question

A balloon which always remains spherical is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 cm.

Answer

When we pump a balloon its volume changes.

Let the radius of the balloon be r

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$900\ cm^3/s = 4 \times \pi \times 15^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{900}{4 \times 3.14 \times 225}$

$\frac{dr}{dt} = 0.32\ cm/s$

9. Question

The bottom of a rectangular swimming tank is 25 m by 40 m. Water is pumped into the tank at the rate of 500 cubic metres per minute. Find the rate at which the level of water in the tank is rising.

Answer

Let the volume of the water tank be V

$V = l \times b \times h$

$V = 25 \times 40 \times h$

$\frac{dV}{dt} = 1000 \times \frac{dh}{dt}$

$$500 = 1000 \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.5 \text{ m/min}$$

10. Question

A stone is dropped into a quiet lake and waves move in circles at a speed of 3.5 cm per second. At the instant when the radius of the circular wave is 7.5 cm. how fast is the enclosed area increasing? (Take π = 22/7.)

Answer

Let the radius of the circle be r

$$\frac{dr}{dt} = 3.5 \text{ cm/s}$$

Area of the circle=πr^2

Rate of change of Area $= 2\pi r \frac{dr}{dt}$

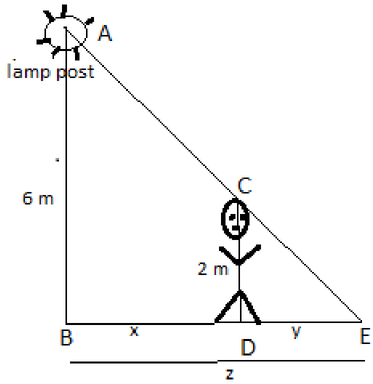
$= 2 \times 3.14 \times 7.5 \times 3.5$

$= 165 \text{ cm}^2/\text{s}$

11. Question

A 2-m tall man walks at a uniform speed of a uniform speed of 5 km per hour away from a 6-metre-high lamp post. Find the rate at which the length of his shadow increases.

Answer



ABE and CDE are similar triangles.

So,

$$\frac{AB}{BE} = \frac{CD}{DE}$$

$$\frac{0.006}{x+y} = \frac{0.002}{y}$$

$$6y = 2(x+y)$$

$$6\frac{dy}{dt} = 2(\frac{dx}{dt} + \frac{dy}{dt})$$

$$6\frac{dy}{dt} = 2(5 + \frac{dy}{dt})$$

$$6\frac{dy}{dt} = 10 + 2\frac{dy}{dt}$$

$$4\frac{dy}{dt} = 10$$

$$\frac{dy}{dt} = 2.5\text{km/h}$$

12. Question

An inverted cone has a depth of 40 cm and a base of radius 5 cm. Water is poured into it at a rate of 1.5 cubic centimetres per minute. Find the rate at which the level of water in the cone is rising when the depth is 4 cm.

Answer

Let the volume of the cone be V

$$\frac{dV}{dt} = 1.5\ \text{cm}^3/\text{s}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi 5^2 h$$

$$V = \frac{25}{3}\pi h$$

$$\frac{dV}{dt} = \frac{25}{3}\pi\frac{dh}{dt}$$

$$\frac{15}{10} = \frac{25}{3}\pi\frac{dh}{dt}$$

13. Question

Sand is pouring from a pipe at the rate of 18cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is one-sixth of the radius of the base. How fast is the height of the sand cone increasing when its height is 3 cm?

Answer

$$h = \frac{1}{6}r$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi(6h)^2 h$$

$$V = 12\pi h^3$$

$$\frac{dV}{dt} = 36\pi h^2\frac{dh}{dt}$$

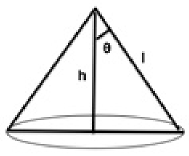
$$18 = 36 \times 9 \times \pi \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{18\pi}\text{cm/s}$$

14. Question

Water is dripping through a tiny hole at the vertex in the bottom of a conical funnel at a uniform rate of 4 cm^3/s. When the slant height of the water is 3 cm, find the rate of decrease of the slant height of the water, given that the vertical angle of the funnel is 120°.

Answer



Let the volume of the cone be V

$$\frac{dV}{dt} = 4cm^3/s$$

$$V = \frac{1}{3}\pi r^2 h$$

$$\cos Q = \frac{h}{l} = \cos 120 = \cos(180 - 60) = -\frac{1}{2}$$

$$\sin Q = \frac{r}{l} = \sin 120 = \sin(180 - 60) = \sin 60 = \frac{\sqrt{3}}{2}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{\sqrt{3}}{2}l\right)^2 (-\frac{1}{2}l)$$

$$V = -\frac{3}{24}\pi l^3$$

$$\frac{dV}{dt} = -\frac{9}{24}\pi l^2 \frac{dl}{dt}$$

$$4 = -\frac{3}{8}\pi 3^2 \frac{dl}{dt}$$

$$-\frac{32}{27\pi}\ cm/s = \frac{dl}{dt}$$

15. Question

Oil is leaking at the rate of 15 mL/s from a vertically kept cylindrical drum containing oil. If the radius of the drum is 7 cm and its height is 60 cm, find the rate at which the level of the oil is changing when the oil level is 18 cm.

Answer

$$\frac{dV}{dt} = 15\ mL/s$$

$$\frac{d}{dt}(\pi r^2 h) = 15$$

$$\frac{d}{dt}(\pi 7^2 h) = 15$$

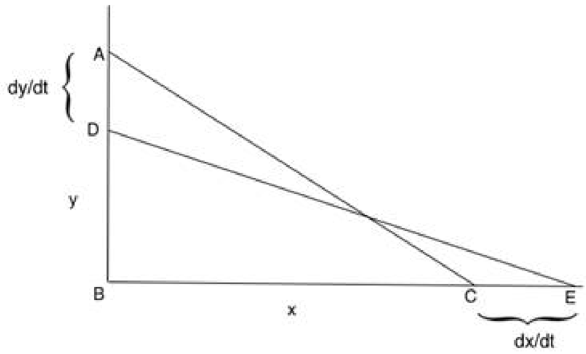
$$49\pi \frac{dh}{dt} = 15$$

$$\frac{dh}{dt} = \frac{15}{49\pi}$$

16. Question

A 13-m long ladder is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the ladder is 5 m away from the wall?

Answer



Let the original ladder be AC and the pulled ladder be DE

Let AB=y and BC=x

Applying Pythagoras Theorem in ABC

$x^2 + y^2 = 13^2$ ...(1)

$5^2 + y^2 = 13^2$

$y = 12\text{cm}$

Differentiating both sides of eqn (1) wrt to t

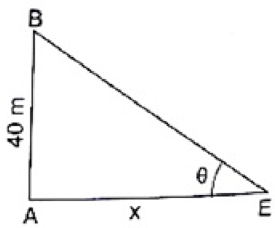
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$5.2 + 12\frac{dy}{dt} = 0$$

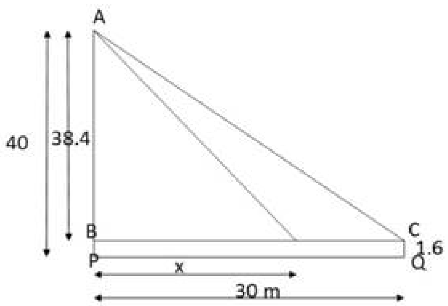
$$\frac{dy}{dt} = -\frac{10}{12} = -\frac{5}{6}\text{cm/s}$$

17. Question

A man is moving away from a 40-m high tower at a speed of 2 m/s. Find the rate is which the angle of elevation of the top of the tower is changing when he is at a distance of 30 metres from the foot of the tower. Assume that the eye level of the man is 1.6 m from the ground.



Answer



$$\frac{dx}{dt} = -2 \text{ cm/s}$$

$$\tan Q = \frac{38.4}{x}$$

$$Q = \tan^{-1}\frac{38.4}{x}$$

$$\frac{dQ}{dt} = \frac{1}{1+\frac{38.4^2}{x^2}}\left(-\frac{1}{x^2}\right).38.4$$

$$\frac{dQ}{dt} = \frac{x^2}{x^2+1474.56}\left(-\frac{1}{x^2}\right).38.4$$

$$\frac{dQ}{dt} = -\frac{1}{30^2+1474.56}.38.4.\frac{dx}{dt}$$

$$\frac{dQ}{dt} = -\frac{1}{30^2+1474.56}.38.4 \times 2$$

$$\frac{dQ}{dt} = -0.032 \text{ radian/second}$$

18. Question

Find an angle x which increases twice as fast as its sine.

Answer

ATQ,

$$\frac{dx}{dt} = 2\frac{d}{dt}(\sin x)$$

$$\frac{dx}{dt} = 2\cos x\frac{dx}{dt}$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}$$

19. Question

The radius of a balloon is increasing at the rate of 10 m/s. At what rate is the surface area of the balloon increasing when the radius is 15 cm?

Answer

$$\frac{dr}{dt} = 10 \text{ m/s}$$

$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi. 15.10$$

$$\frac{dS}{dt} = 1200\pi \text{ cm}^2/\text{s}$$

20. Question

An edge of a variable cube is increasing at the rate of 5 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?

Answer

$$\frac{da}{dt} = 5cm/s$$

$$V = a^3$$

$$\frac{dV}{dt} = 3a^2\frac{da}{dt}$$

$$\frac{dV}{dt} = 3.10^2.5$$

$$\frac{dV}{dt} = 1500\ cm^3/s$$

21. Question

The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. Find the rate at which the area is increasing when the side is 10 cm.

Answer

$$\frac{da}{dt} = 2cm/s$$

$$A = \frac{\sqrt{3}}{4}a^2$$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}2a\frac{da}{dt}$$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2}.10.2$$

$$\frac{dA}{dt} = 10\sqrt{3}\ cm^2/s$$

Exercise 11B

1. Question

Using differentials, find the approximate values of:

find the approximate values of $\sqrt{37}$.

Answer

Let y =√x.

Let x =36 and Δx = 1.

As y = √x.

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{1}{2\sqrt{x}}.\Delta x$

$\Rightarrow \Delta y = \frac{1}{2\sqrt{36}}.1$

$\Rightarrow \Delta y = \frac{1}{12}$

∴ Δy = 0.08

Also,

Δy = f(x+Δx)-f(x)

$\therefore 0.08 = \sqrt{36+1} - \sqrt{36}$

⇒ 0.08 = √37 – 6

⇒ √37 = 6.08

2. Question

Using differentials, find the approximate values of:

Find the approximate values of $\sqrt[3]{29}$.

Answer

Let $y = \sqrt[3]{x}$.

Let x =27 and Δx = 2.

As $y = \sqrt[3]{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{3}x^{\frac{-2}{3}}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{1}{3}x^{\frac{-2}{3}}.\Delta x$

$\Rightarrow \Delta y = \frac{1}{3}27^{\frac{-2}{3}}.2$

$\Rightarrow \Delta y = \frac{2}{27}$

∴ Δy = 0.074

Also,

Δy = f(x+Δx)-f(x)

$\therefore 0.074 = \sqrt[3]{27+2} - \sqrt[3]{27}$

$\Rightarrow 0.074 = \sqrt[3]{29} - 3$

$\Rightarrow \sqrt[3]{29} = 3.074$

3. Question

Using differentials, find the approximate values of:

Find the approximate values of $\sqrt{27}$

Answer

Let y =√x.

Let x =25 and Δx = 2.

As y = √x.

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{1}{2\sqrt{x}}.\Delta x$

$\Rightarrow \Delta y = \frac{1}{2\sqrt{25}}.2$

$\Rightarrow \Delta y = \frac{1}{5}$

∴ Δy = 0.2

Also,

Δy = f(x+Δx)-f(x)

$\therefore 0.2 = \sqrt{25+2} - \sqrt{25}$

⇒ 0.2 = √27 – 5

⇒ √27 = 5.2

4. Question

Using differentials, find the approximate values of:

Find the approximate values of $\sqrt{0.24}$

Answer

Let y =√x.

Let x =0.25 and Δx = -0.01.

As y = √x.

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{1}{2\sqrt{x}}.\Delta x$

$\Rightarrow \Delta y = \frac{1}{2\sqrt{0.25}}.(-0.01)$

$\Rightarrow \Delta y = -0.01$

∴ Δy = -0.01

Also,

Δy = f(x+Δx)-f(x)

$\therefore -0.01 = \sqrt{0.25-0.01} - \sqrt{0.25}$

⇒ -0.01 = √0.24 – 0.5

⇒ √0.24 = 0.49

5. Question

Using differentials, find the approximate values of:

Find the approximate values of $\sqrt{49.5}$

Answer

Let y =√x.

Let x =19 and Δx = 0.5.

As y = √x.

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{1}{2\sqrt{x}}.\Delta x$

$\Rightarrow \Delta y = \frac{1}{2\sqrt{49}}.\,0.5$

$\Rightarrow \Delta y = \frac{0.5}{14}$

∴ Δy = 0.0357

Also,

Δy = f(x+Δx)-f(x)

$\therefore 0.0357 = \sqrt{49 + 0.5} - \sqrt{49}$

⇒ 0.0357 = √49.5 – 7

⇒ √49.5 = 7.0357.

6. Question

Using differentials, find the approximate values of:

Find the approximate values of $\sqrt[4]{15}$

Answer

Let $y = \sqrt[4]{x}$.

Let x =16 and Δx = 1.

As $y = \sqrt[4]{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{4}x^{\frac{-3}{4}}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{1}{4}x^{\frac{-3}{4}}.\Delta x$

$\Rightarrow \Delta y = \frac{1}{4}16^{\frac{-3}{4}}.(-1)$

$\Rightarrow \Delta y = \frac{-1}{32}$

∴ Δy = -0.03125

Also,

Δy = f(x+Δx)-f(x)

$\therefore -0.03125 = \sqrt[4]{16-1} - \sqrt[4]{16}$

$\Rightarrow -0.03125 = \sqrt[4]{15} - 2$

$\Rightarrow \sqrt[4]{15} = 1.96875$

7. Question

find the approximate values of $\frac{1}{(2.002)^2}$

Answer

Let $y = \frac{1}{x^2}$

Let x =2 and Δx = 0.002.

As $y = \frac{1}{x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{-2}{x^3}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{-2}{x^3}.\Delta x$

$\Rightarrow \Delta y = \frac{-2}{8}.(0.002)$

$\Rightarrow \Delta y = \frac{-0.5}{1000}$

∴ Δy = -0.0005

Also,

Δy = f(x+Δx)-f(x)

$\therefore -0.0005 = \frac{1}{(2.002)^2} - \frac{1}{2^2}$

$\Rightarrow -0.0005 = \frac{1}{(2.002)^2} - 0.25$

$\Rightarrow \frac{1}{(2.002)^2} = 0.2495$

8. Question

find the approximate values of $\log_e 10.02$, given that $\log_e 10$ = 2.3026

Answer

Let $y = \log_e x$

Let x =10 and Δx = 0.02.

As $y = \log_e x$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{1}{x}.\Delta x$

$\Rightarrow \Delta y = \frac{1}{10}.(0.02)$

$\Rightarrow \Delta y = \frac{0.02}{10}$

$\therefore \Delta y = 0.002$

Also,

$\Delta y = f(x+\Delta x)-f(x)$

$\therefore 0.002 = \log_e(10+0.02) – \log_e(10)$

$\Rightarrow 0.002 = \log_e(10.02) – 2.3026$

$\Rightarrow \log_e(10.02) = 2.3046.$

9. Question

find the approximate values of $\log_{10}(4.04)$, it being given that $\log_{10}4 = 0.6021$ and $\log_{10}e = 0.4343$

Answer

Let $y = \log_{10} x$

$\therefore y = \frac{\log_e x}{\log_e 10}$

$\therefore y = 0.4343 \log_e x$

Let x =4 and $\Delta x = 0.04$.

As $y = 0.4343 \log_e x$

$\Rightarrow \frac{dy}{dx} = \frac{0.4343}{x}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx} \Delta x$

$\therefore \Delta y = \frac{0.4343}{x}.\Delta x$

$\Rightarrow \Delta y = \frac{0.4343}{4}.(0.04)$

$\Rightarrow \Delta y = \frac{0.017372}{4}$

$\therefore \Delta y = 0.004343$

Also,

$\Delta y = f(x+\Delta x)-f(x)$

$\therefore 0.004343 = \log_e(4+0.04) – \log_e(4)$

$\Rightarrow 0.004343 = \log_e(4.04) – 0.6021$

$\Rightarrow \log_e(4.04) = 0.606443.$

10. Question

find the approximate values of cos 61°, it being given that sin 60° = 0.86603 and 1° = 0.01745 radian

Answer

Let y = cosx

Let x =60° and Δx = 1° .

As y =cos x

$\Rightarrow \frac{dy}{dx} = \sin x$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \sin x \,.\, \Delta x$

$\Rightarrow \Delta y = \ \sin(60)(0.01745)$

$\Rightarrow \Delta y = (0.86603)(0.01745$

∴ Δy = 0.01511

Also,

Δy = f(x+Δx)-f(x)

∴ 0.01511 = cos(60+1) – cos(60)

⇒ 0.01511 = cos61° – 0.5

⇒ cos61° = 0.48489

11. Question

If y = sin x and x changes from $\frac{\pi}{2}$ to $\frac{22}{14}$, what is the approximate change in y?

Answer

Given x is π /2

Value of π is 22/7

22/14 is π /2

Hence there will be no change.

12. Question

A circular metal plate expands under heating so that its radius increases by 2%. Find the approximate increase in the area of the plate, if the radius of the plate before heating is 10 cm.

Answer

Let the radius of the plate 10cm.

Radius increases by 2% by heating

∴ After increment = $\frac{2}{100} \times 10 = 0.2$

Change in radius dr = 0.2

∴ New radius = 10+0.2 = 10.2cm

Area of circular plate = A=πr^2

∴ Change in Area = $\frac{dA}{dr}$

$\Rightarrow \frac{dA}{dr} = 2\pi r dr$

$\Rightarrow \frac{dA}{dr} = 2 \times \pi \times 10 \times 0.2$

$\Rightarrow \frac{dA}{dr} = 4\pi\ cm^2$

13. Question

If the length of a simple pendulum is decreased by 2%, find the percentage decrease in its period T, where $T = 2\pi\sqrt{\frac{l}{g}}$.

Answer

The formula for time period -

$\therefore T = 2\pi\sqrt{\frac{l}{g}}$

Here 2,π,g have no dimensions. So we can eliminate them.

Now $\frac{\Delta T}{T} = \frac{1}{2} \times \frac{\Delta L}{L}$

By representing in percentage error

$\Rightarrow \frac{\Delta T}{T} \times 100\% = \frac{1}{2} \times \frac{\Delta L}{L} \times 100\%$

$\Rightarrow \frac{\Delta T}{T} \times 100\% = \frac{1}{2} \times \frac{\Delta L}{L} \times 100\%$

$\Rightarrow \frac{\Delta T}{T}\% = \frac{1}{2} \times 2\%$

$\Rightarrow \frac{\Delta T}{T}\% = 1\%$

Hence the time period becomes 1 %.

14. Question

The pressure p and the volume V of a gas are connected by the relation, $pV^{1/4} = k$, where k is a constant. Find the percentage increase in the pressure, corresponding to a diminution of 0.5% in the volume.

Answer

Given: $pv^{1/4} = k$

%decrease in the volume = 1/2%s

$\therefore \frac{\Delta V}{V} \times 100 = \frac{-1}{2}$

$pv^{1/4} = k$

taking log on both sides

$\log[pv^{1/4}] = \log a$

log P + 1.4logV = log a

Differentiating both the sides we get

$\Rightarrow \frac{1}{P}dP + \frac{1.4}{V}dV = 0$

$\Rightarrow \frac{dP}{P} + 1.4\frac{dV}{V} = 0$

Multiplying both sides by 100.

$\Rightarrow \frac{dP}{p} \times 100 + 1.4 \times \frac{dV}{V} \times 100 = 0$

$\Rightarrow \frac{dP}{P} \times 100 + 1.4\left(\frac{-1}{2}\right) = 0$

$\Rightarrow \frac{dP}{P} \times 100 = 0.7$

%error in P = 0.7%.

15. Question

The radius of a sphere shrinks from 10 cm to 9.8 cm. Find approximately the decrease in (i) volume, and (ii) surface area.

Answer

Decrease in radius = dr = 10-9.8

∴ dr = 0.2

Volume of the sphere is given by = $V = \frac{4}{3}\pi r^3$

Change in volume = $dV = 4\pi r^2 dr$

$\therefore dV = 4\pi(10)^2 \times 0.2$

$\Rightarrow dV = 80\pi\ cm^3$

Surface area of the sphere is given by = $A = 4\pi r^2$

Change in volume = $dA = 8\pi r dr$

∴ dA = 8π×10× 0.2

∴ dA = 16π.

16. Question

If there is an error of 0.1% in the measurement of the radius of a sphere, find approximately the percentage error in the calculation of the volume of the sphere.

Answer

Volume of the sphere is given by = $V = \frac{4}{3}\pi r^2$

Change in volume = $dV = 4\pi r^2 dr$

Given: Δr = 0.1

$\Rightarrow \Delta r.\frac{dV}{dr} = 4\pi r^2 \Delta r$

$\Rightarrow \Delta V = 4\pi r^2 \Delta r$

Percentage error

$\Rightarrow \frac{\Delta V}{V} = \frac{4\pi r^2}{\frac{4\pi r^3}{3}} \times 0.1$

= 0.3%

17. Question

Show that the relative error in the volume of a sphere, due to an error in measuring the diameter, is three times the relative error in the diameter.

Answer

Let d be the diameter r be the radius and V be the volume of Sphere

Volume of the sphere is given by = $V = \frac{4}{3}\pi r^2$

$\Rightarrow V = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 = \frac{1}{6}\pi D^3$

Let Δd be the error in d and the corresponding error in V be ΔV.

$\therefore \Delta V = \frac{dV}{dd}\Delta d = \frac{1}{2}\pi d^2 \Delta D$

$\therefore \frac{\Delta V}{V} = \frac{\frac{1}{2}\pi d^2 \Delta D}{\frac{1}{6}\pi D^3} = 3\frac{\Delta D}{D}$

Hence Proved

Exercise 11C

1. Question

Verify Rolle's theorem for each of the following functions:

$f(x) = x^2$ on $[-1,1]$

Answer

Condition (1):

Since, $f(x)=x^2$ is a polynomial and we know every polynomial function is continuous for all $x \in R$.

⇒ $f(x)=x^2$ is continuous on [-1,1].

Condition (2):

Here, f'(x)=2x which exist in [-1,1].

So, $f(x)=x^2$ is differentiable on (-1,1).

Condition (3):

Here, $f(-1)=(-1)^2=1$

And $f(1)=1^1=1$

i.e. f(-1)=f(1)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (-1,1)$ such that f'(c)=0

i.e. 2c=0

i.e. c=0

Value of $c=0 \in (-1,1)$

Thus, Rolle's theorem is satisfied.

2. Question

Verify Rolle's theorem for each of the following functions:

$f(x) = x^2 - x - 12$ in $[-3,4]$

Answer

Condition (1):

Since, $f(x)=x^2-x-12$ is a polynomial and we know every polynomial function is continuous for all $x \in R$.

⇒ $f(x)= x^2-x-12$ is continuous on [-3,4].

Condition (2):

Here, f'(x)=2x-1 which exist in [-3,4].

So, $f(x)= x^2-x-12$ is differentiable on (-3,4).

Condition (3):

Here, $f(-3)=(-3)^2-3-12=0$

And f(4)=4^2-4-12=0

i.e. f(-3)=f(4)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one c∊(-3,4) such that f'(c)=0

i.e. 2c-1=0

i.e. $c = \frac{1}{2}$

Value of $c = \frac{1}{2} \epsilon(-3,4)$

Thus, Rolle's theorem is satisfied.

3. Question

Verify Rolle's theorem for each of the following functions:

$f(x) = x^2 - 5x + 6 \text{ in } [2,3]$

Answer

Condition (1):

Since, f(x)=x^2-5x+6 is a polynomial and we know every polynomial function is continuous for all x∊R.

⇒ f(x)= x^2-5x+6 is continuous on [2,3].

Condition (2):

Here, f'(x)=2x-5 which exist in [2,3].

So, f(x)= x^2-5x+6 is differentiable on (2,3).

Condition (3):

Here, f(2)=2^2-5×2+6=0

And f(3)= 3^2-5×3+6=0

i.e. f(2)=f(3)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one c∊(2,3) such that f'(c)=0

i.e. 2c-5=0

i.e. $c = \frac{5}{2}$

Value of $c = \frac{5}{2} \epsilon(2,3)$

Thus, Rolle's theorem is satisfied.

4. Question

Verify Rolle's theorem for each of the following functions:

$f(x) = x^2 - 5x + 6 \text{ in } [-3,6]$

Answer

Condition (1):

Since, f(x)= x^2-5x+6 is a polynomial and we know every polynomial function is continuous for all x∊R.

⇒ f(x)= x^2-5x+6 is continuous on [-3,6].

Condition (2):

Here, f'(x)=2x-5 which exist in [-3,6].

So, f(x)= x^2-5x+6 is differentiable on (-3,6).

Condition (3):

Here, f(-3)=$(-3)^2$-5×(-3)+6=30

And f(6)= 6^2-5×6+6=12

i.e. f(-3)≠f(6)

Conditions (3) of Rolle's theorem is not satisfied.

So, Rolle's theorem is not applicable.

5. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = x^2 - 4x + 3 \text{ in } [1,3]$$

Answer

Condition (1):

Since, f(x)=x^2-4x+3 is a polynomial and we know every polynomial function is continuous for all x∈R.

⇒ f(x)=x^2-4x+3 is continuous on [1,3].

Condition (2):

Here, f'(x)=2x-4 which exist in [1,3].

So, f(x)=x^2-4x+3 is differentiable on (1,3).

Condition (3):

Here, f(1)=$(1)^2$-4(1)+3=0

And f(3)= $(3)^2$-4(3)+3=0

i.e. f(1)=f(3)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one c∈(1,3) such that f'(c)=0

i.e. 2c-4=0

i.e. c=2

Value of c=2 ∈(1,3)

Thus, Rolle's theorem is satisfied.

6. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = x(x-4)^2 \text{ in } [0,4]$$

Answer

Condition (1):

Since, $f(x)=x(x-4)^2$ is a polynomial and we know every polynomial function is continuous for all x∊R.

⇒ $f(x)= x(x-4)^2$ is continuous on [0,4].

Condition (2):

Here, $f'(x)= (x-4)^2+2x(x-4)$ which exist in [0,4].

So, $f(x)= x(x-4)^2$ is differentiable on (0,4).

Condition (3):

Here, $f(0)=0(0-4)^2=0$

And $f(4)= 4(4-4)^2=0$

i.e. f(0)=f(4)

Conditions of Rolle’s theorem are satisfied.

Hence, there exist at least one c∊(0,4) such that f’(c)=0

i.e. $(c-4)^2+2c(c-4)=0$

i.e. (c-4)(3c-4)=0

i.e. c=4 or c=3÷4

Value of $c = \frac{3}{4} \epsilon (0,4)$

Thus, Rolle’s theorem is satisfied.

7. Question

Verify Rolle’s theorem for each of the following functions:

$f(x) = x^3 - 7x^2 + 16x - 12 \text{ in } [2,3]$

Answer

Condition (1):

Since, $f(x)=x^3- 7x^2+16x-12$ is a polynomial and we know every polynomial function is continuous for all x∊R.

⇒ $f(x)= x^3- 7x^2+16x-12$ is continuous on [2,3].

Condition (2):

Here, $f'(x)=3x^2-14x+16$ which exist in [2,3].

So, $f(x)= x^3- 7x^2+16x-12$ is differentiable on (2,3).

Condition (3):

Here, $f(2)= 2^3- 7(2)^2+16(2)-12=0$

And $f(3)= 3^3- 7(3)^2+16(3)-12=0$

i.e. f(2)=f(3)

Conditions of Rolle’s theorem are satisfied.

Hence, there exist at least one c∊(2,3) such that f’(c)=0

i.e. $3c^2-14c+16=0$

i.e. (c-2)(3c-7)=0

i.e. c=2 or c=7÷3

Value of $c = \frac{7}{3} \epsilon (2,3)$

Thus, Rolle's theorem is satisfied.

8. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = x^3 + 3x^2 - 24x - 80 \text{ in } [-4, 5]$$

Answer

Condition (1):

Since, f(x)= $x^3+3x^2-24x-80$ is a polynomial and we know every polynomial function is continuous for all xϵR.

⇒ f(x)= $x^3+3x^2-24x-80$ is continuous on [-4,5].

Condition (2):

Here, f'(x)= $3x^2+6x-24$ which exist in [-4,5].

So, f(x)= $x^3+3x^2-24x-80$ is differentiable on (-4,5).

Condition (3):

Here, f(-4)= $(-4)^3+3(-4)^2-24(-4)-80=0$

And f(5)= $(5)^3+3(5)^2-24(5)-80=0$

i.e. f(-4)=f(5)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one cϵ(-4,5) such that f'(c)=0

i.e. $3c^2+6c-24=0$

i.e. c=-4 or c=2

Value of c=2 ϵ(-4,5)

Thus, Rolle's theorem is satisfied.

9. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = (x-1)(x-2)(x-3) \text{ in } [1, 3]$$

Answer

Condition (1):

Since, f(x)=(x-1)(x-2)(x-3) is a polynomial and we know every polynomial function is continuous for all xϵR.

⇒ f(x)= (x-1)(x-2)(x-3) is continuous on [1,3].

Condition (2):

Here, f'(x)= (x-2)(x-3)+ (x-1)(x-3)+ (x-1)(x-2) which exist in [1,3].

So, f(x)= (x-1)(x-2)(x-3) is differentiable on (1,3).

Condition (3):

Here, f(1)= (1-1)(1-2)(1-3) =0

And f(3)= (3-1)(3-2)(3-3) =0

i.e. f(1)=f(3)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one cϵ(1,3) such that f'(c)=0

i.e. (c-2)(c-3)+ (c-1)(c-3)+ (c-1)(c-2)=0

i.e. (c-3)(2c-3)+(c-1)(c-2)=0

i.e. $(2c^2-9c+9)+(c^2-3c+2)=0$

i.e. $3c^2-12c+11=0$

i.e. $c = \frac{12 \pm \sqrt{12}}{6}$

i.e. c=2.58 or c=1.42

Value of c=1.42ϵ(1,3) and c=2.58ϵ(1,3)

Thus, Rolle's theorem is satisfied.

10. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = (x-1)(x-2)^2 \text{ in } [1,2]$$

Answer

Condition (1):

Since, $f(x)=(x-1)(x-2)^2$ is a polynomial and we know every polynomial function is continuous for all xϵR.

⇒ $f(x)= (x-1)(x-2)^2$ is continuous on [1,2].

Condition (2):

Here, $f'(x)= (x-2)^2+2(x-1)(x-2)$ which exist in [1,2].

So, $f(x)= (x-1)(x-2)^2$ is differentiable on (1,2).

Condition (3):

Here, $f(1)= (1-1)(1-2)^2=0$

And $f(2)= (2-1)(2-2)^2=0$

i.e. f(1)=f(2)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one cϵ(1,2) such that f'(c)=0

i.e. $(c-2)^2+2(c-1)(c-2)=0$

(3c-4)(c-2)=0

i.e. c=2 or c=4÷3

Value of $c = \frac{4}{3} = 1.33\epsilon(1,2)$

Thus, Rolle's theorem is satisfied.

11. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = (x-2)^4 (x-3)^3 \text{ in } [2,3]$$

Answer

Condition (1):

Since, $f(x)=(x-2)^4(x-3)^3$ is a polynomial and we know every polynomial function is continuous for all x∊R.

⇒ $f(x)= (x-2)^4(x-3)^3$ is continuous on [2,3].

Condition (2):

Here, $f'(x)= 4(x-2)^3(x-3)^3+3(x-2)^4(x-3)^2$ which exist in [2,3].

So, $f(x)= (x-2)^4(x-3)^3$ is differentiable on (2,3).

Condition (3):

Here, $f(2)= (2-2)^4(2-3)^3=0$

And $f(3)= (3-2)^4(3-3)^3=0$

i.e. f(2)=f(3)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one c∊(2,3) such that f'(c)=0

i.e. $4(c-2)^3(c-3)^3+3(c-2)^4(c-3)^2=0$

$(c-2)^3(c-3)^2(7c-18)=0$

i.e. c=2 or c=3 or c=18÷7

Value of $c = \frac{18}{7} = 2.57\epsilon(2,3)$

Thus, Rolle's theorem is satisfied.

12. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = \sqrt{1-x^2} \text{ in } [-1,1]$$

Answer

Condition (1):

Since, $f(x) = \sqrt{1-x^2}$ is a polynomial and we know every polynomial function is continuous for all x∊R.

⇒ $f(x) = \sqrt{1-x^2}$ is continuous on [-1,1].

Condition (2):

Here, $f'(x) = -\frac{x}{\sqrt{1-x^2}}$ which exist in [-1,1].

So, $f(x) = \sqrt{1-x^2}$ is differentiable on (-1,1).

Condition (3):

Here, $f(-1) = \sqrt{1-(-1)^2} = 0$

And $f(1) = \sqrt{1-1^2} = 0$

i.e. $f(-1) = f(1)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one c∊(-1,1) such that f'(c)=0

i.e. $-\frac{c}{\sqrt{1-c^2}} = 0$

i.e. c=0

Value of c=0ϵ(-1,1)

Thus, Rolle’s theorem is satisfied.

3. Question

Verify Rolle’s theorem for each of the following functions:

$$f(x) = \cos x \text{ in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Answer

Condition (1):

Since, f(x)=cosx is a trigonometric function and we know every trigonometric function is continuous.

⇒ f(x)=cosx is continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Condition (2):

Here, f’(x)=-sinx which exist in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

So, f(x)=cosx is differentiable on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Condition (3):

Here, $f\left(-\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

And $f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

i.e. $f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)$

Conditions of Rolle’s theorem are satisfied.

Hence, there exist at least one $c\epsilon(-\frac{\pi}{2}, \frac{\pi}{2})$ such that f’(c)=0

i.e. -sinc=0

i.e. c=0

Value of $c = 0\epsilon(-\frac{\pi}{2}, \frac{\pi}{2})$

Thus, Rolle’s theorem is satisfied.

14. Question

Verify Rolle’s theorem for each of the following functions:

$$f(x) = \cos 2x \text{ in } [0, \pi]$$

Answer

Condition (1):

Since, f(x)=cos2x is a trigonometric function and we know every trigonometric function is continuous.

⇒ f(x)= cos2x is continuous on [0,π].

Condition (2):

Here, f’(x)= -2sin2x which exist in [0,π].

So, f(x)=cos2x is differentiable on (0,π).

Condition (3):

Here, f(0)=cos0=1

And f(π)=cos2π=1

i.e. f(0)=f(π)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one c∊(0,π) such that f'(c)=0

i.e. -2sin2c =0

i.e. 2c=π

i.e. $c = \frac{\pi}{2}$

Value of $c = \frac{\pi}{2} \epsilon (0, \pi)$

Thus, Rolle's theorem is satisfied.

15. Question

Verify Rolle's theorem for each of the following functions:

$f(x) = \sin 3x \text{ in } [0, \pi]$

Answer

Condition (1):

Since, f(x)=sin3x is a trigonometric function and we know every trigonometric function is continuous.

⇒ f(x)= sin3x is continuous on [0,π].

Condition (2):

Here, f'(x)= 3cos3x which exist in [0,π].

So, f(x)= sin3x is differentiable on (0,π).

Condition (3):

Here, f(0)=sin0=0

And f(π)=sin3π=0

i.e. f(0)=f(π)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one c∊(0,π) such that f'(c)=0

i.e. 3cos3c =0

i.e. $3c = \frac{\pi}{2}$

i.e. $c = \frac{\pi}{6}$

Value of $c = \frac{\pi}{6} \epsilon (0, \pi)$

Thus, Rolle's theorem is satisfied.

16. Question

Verify Rolle's theorem for each of the following functions:

$f(x) = \sin x + \cos x \text{ in } \left[0, \frac{\pi}{2}\right]$

Answer

Condition (1):

Since, f(x)=sinx+cosx is a trigonometric function and we know every trigonometric function is continuous.

⇒ f(x)= sinx+cosx is continuous on $[0, \frac{\pi}{2}]$.

Condition (2):

Here, f'(x)= cosx-sinx which exist in $[0, \frac{\pi}{2}]$.

So, f(x)= sinx+cosx is differentiable on $(0, \frac{\pi}{2})$

Condition (3):

Here, f(0)=sin0+cos0=1

And $f(\frac{\pi}{2}) = \sin\left(\frac{\pi}{2}\right) + \cos(\frac{\pi}{2}) = 1$

i.e. $f(0) = f(\frac{\pi}{2})$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c\epsilon(0, \frac{\pi}{2})$ such that f'(c)=0

i.e. cosc-sinc =0

i.e. $c = \frac{\pi}{4}$

Value of $c = \frac{\pi}{4}\epsilon(0, \frac{\pi}{2})$

Thus, Rolle's theorem is satisfied.

17. Question

Verify Rolle's theorem for each of the following functions:

$f(x) = e^{-x} \sin x \text{ in } [0, \pi]$

Answer

Condition (1):

Since, f(x)=e^{-x} sinx is a combination of exponential and trigonometric function which is continuous.

⇒ f(x)= e^{-x} sinx is continuous on [0,π].

Condition (2):

Here, f'(x)= e^{-x} (cosx – sinx) which exist in [0,π].

So, f(x)= e^{-x} sinx is differentiable on (0,π)

Condition (3):

Here, f(0)= e^{-0} sin0=0

And f(π)= $e^{-\pi}$sinπ =0

i.e. f(0)=f(π)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one cϵ(0,π) such that f'(c)=0

i.e. e^{-c} (cos c – sin c) =0

i.e. cos c-sin c = 0

i.e. $c = \frac{\pi}{4}$

Value of $c = \frac{\pi}{4} \epsilon (0, \pi)$

Thus, Rolle's theorem is satisfied.

18. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = e^{-x} (\sin x - \cos x) \text{ in } \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

Answer

Condition (1):

Since, f(x)=e^{-x} (sinx-cosx) is a combination of exponential and trigonometric function which is continuous.

⇒ f(x)= e^{-x} (sinx-cosx) is continuous on $[\frac{\pi}{4}, \frac{5\pi}{4}]$.

Condition (2):

Here, f'(x)= e^{-x} (sinx + cosx) - e^{-x} (sinx – cosx)

= e^{-x} cosx which exist in $[\frac{\pi}{4}, \frac{5\pi}{4}]$.

So, f(x)= e^{-x} (sinx-cosx) is differentiable on $(\frac{\pi}{4}, \frac{5\pi}{4})$

Condition (3):

Here, $f(\frac{\pi}{4}) = e^{-\frac{\pi}{4}} (\sin\frac{\pi}{4} - \cos\frac{\pi}{4}) = 0$

And $f(\frac{5\pi}{4}) = e^{-\frac{5\pi}{4}} (\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}) = 0$

i.e. $f(\frac{\pi}{4}) = f(\frac{5\pi}{4})$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \epsilon (\frac{\pi}{4}, \frac{5\pi}{4})$ such that f'(c)=0

i.e. e^{-c} cos c =0

i.e. cos c = 0

i.e. $c = \frac{\pi}{2}$

Value of $c = \frac{\pi}{2} \epsilon (\frac{\pi}{4}, \frac{5\pi}{4})$

Thus, Rolle's theorem is satisfied.

19. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = \sin x - \sin 2x \text{ in } [0, 2\pi]$$

Answer

Condition (1):

Since, f(x) = sinx-sin2x is a trigonometric function and we know every trigonometric function is continuous.

⇒ f(x) = sinx-sin2x is continuous on [0,2π].

Condition (2):

Here, f'(x)= cosx-2cos2x which exist in [0,2π].

So, f(x)= sinx-sin2x is differentiable on (0,2π)

Condition (3):

Here, f(0)= sin0-sin0 = 0

And f(2π)=sin(2π)-sin(4π) =0

i.e. f(0)=f(2π)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one c∈(0,2π) such that f'(c)=0

i.e. cosx-2cos2x =0

i.e. cosx-4cos^{2}x+2=0

i.e. 4cos^{2}x-cosx-2=0

i.e. $\cos x = \frac{1 \pm \sqrt{33}}{8}$

i.e. c=32° 32' or c=126°23'

Value of c=32°32'∈(0,2π)

Thus, Rolle's theorem is satisfied.

20. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = x(x+2)e^x \text{ in } [-2,0]$$

Answer

Condition (1):

Since, f(x)=x(x+2)e^x is a combination of exponential and polynomial function which is continuous for all x∈R.

⇒ f(x)= x(x+2)e^x is continuous on [-2,0].

Condition (2):

Here, f'(x)=(x^2+4x+2)e^x which exist in [-2,0].

So, f(x)=x(x+2)e^x is differentiable on (-2,0).

Condition (3):

Here, f(-2)= (-2)(-2+2)e^{-2} =0

And f(0)= 0(0+2)e^0=0

i.e. f(-2)=f(0)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one c∈(-2,0) such that f'(c)=0

i.e. (c^2+4c+2)e^c =0

i.e. $(c+\sqrt{2})^2=0$

i.e. $c=-\sqrt{2}$

Value of $c=-\sqrt{2} \in (-2,0)$

Thus, Rolle's theorem is satisfied.

21. Question

Verify Rolle's theorem for each of the following functions:

Show that $f(x)=x(x-5)^2$ satisfies Rolle's theorem on [0, 5] and that the value of c is (5/3)

Answer

Condition (1):

Since, $f(x)=x(x-5)^2$ is a polynomial and we know every polynomial function is continuous for all $x\in R$.

⇒ $f(x)= x(x-5)^2$ is continuous on [0,5].

Condition (2):

Here, $f'(x)= (x-5)^2+ 2x(x-5)$ which exist in [0,5].

So, $f(x)= x(x-5)^2$ is differentiable on (0,5).

Condition (3):

Here, $f(0)= 0(0-5)^2=0$

And $f(5)= 5(5-5)^2=0$

i.e. $f(0)=f(5)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c\in(0,5)$ such that $f'(c)=0$

i.e. $(c-5)^2+ 2c(c-5)=0$

i.e. $(c-5)(3c-5)=0$

i.e. $c=\frac{5}{3}$ or $c=5$

Value of $c=\frac{5}{3}\in(0,5)$

Thus, Rolle's theorem is satisfied.

22. Question

Discuss the applicability for Rolle's theorem, when:

$f(x)=(x-1)(2x-3)$, where $1\le x\le 3$

Answer

Condition (1):

Since, $f(x)=(x-1)(2x-3)$ is a polynomial and we know every polynomial function is continuous for all $x\in R$.

⇒ $f(x)= (x-1)(2x-3)$ is continuous on [1,3].

Condition (2):

Here, $f'(x)= (2x-3)+ 2(x-1)$ which exist in [1,3].

So, $f(x)= (x-1)(2x-3)$ is differentiable on (1,3).

Condition (3):

Here, f(1)= (1-1)(2(1)-3)=0

And f(5)= (3-1)(2(3)-3)=6

i.e. f(1)≠f(3)

Condition (3) of Rolle's theorem is not satisfied.

So, Rolle's theorem is not applicable.

23. Question

Discuss the applicability for Rolle's theorem, when:

$$f(x) = x^{1/2} \text{ on } [-1,1]$$

Answer

Condition (1):

Since, f(x)=$x^{1/2}$ is a polynomial and we know every polynomial function is continuous for all x∊R.

⇒ f(x)= $x^{1/2}$ is continuous on [-1,1].

Condition (2):

Here, $f'(x) = \frac{1}{2x^{\frac{1}{2}}}$ which does not exist at x=0 in [-1,1].

f(x)=$x^{1/2}$ is not differentiable on (-1,1).

Condition (2) of Rolle's theorem is not satisfied.

So,Rolle's theorem is not applicable.

24. Question

Discuss the applicability for Rolle's theorem, when:

$$f(x) = 2 + (x-1)^{2/3} \text{ on } [0,2]$$

Answer

Condition (1):

Since, f(x)=2+$(x-1)^{2/3}$ is a polynomial and we know every polynomial function is continuous for all x∊R.

⇒ f(x)= 2+$(x-1)^{2/3}$ is continuous on [0,2].

Condition (2):

Here, $f'(x) = \frac{2}{3(x-1)^{\frac{1}{3}}}$ which does not exist at x=1 in [0,2].

f(x)= 2+$(x-1)^{2/3}$ is not differentiable on (0,2).

Condition (2) of Rolle's theorem is not satisfied.

So,Rolle's theorem is not applicable.

25. Question

Discuss the applicability for Rolle's theorem, when:

$$f(x) = \cos\frac{1}{x} \text{ on } [-1,1]$$

Answer

Condition (1):

Since, $f(x) = \cos\frac{1}{x}$ which is discontinuous at x=0

⇒ $f(x) = \cos\frac{1}{x}$ is not continuous on [-1,1].

Condition (1) of Rolle's theorem is not satisfied.

So,Rolle's theorem is not applicable.

26. Question

Discuss the applicability for Rolle's theorem, when:

$f(x)=[x]$ on $[-1,1]$, where [x] denotes the greatest integer not exceeding x

Answer

Condition (1):

Since, f(x)=[x] which is discontinuous at x=0

⇒ f(x)=[x] is not continuous on [-1,1].

Condition (1) of Rolle's theorem is not satisfied.

So,Rolle's theorem is not applicable.

27. Question

Using Rolle's theorem, find the point on the curve $y = x(x-4), x \in [0,4]$, where the tangent is parallel to the x-axis.

Answer

Condition (1):

Since, y=x(x-4) is a polynomial and we know every polynomial function is continuous for all x∊R.

⇒ y= x(x-4) is continuous on [0,4].

Condition (2):

Here, y'= (x-4)+x which exist in [0,4].

So, y= x(x-4) is differentiable on (0,4).

Condition (3):

Here, y(0)=0(0-4)=0

And y(4)= 4(4-4)=0

i.e. y(0)=y(4)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one c∊(0,4) such that y'(c)=0

i.e. (c-4)+c=0

i.e. 2c-4=0

i.e. c=2

Value of c=2∊(0,4)

So,y(c)=y(2)=2(2-4)=-4

By geometric interpretation, (2,-4) is a point on a curve y=x(x-4),where tangent is parallel to x-axis.

Exercise 11D

1. Question

Verify Lagrange's mean-value theorem for the following function:

$f(x) = x^2 + 2x + 3$ on $[4,6]$

Answer

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [4,6].

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$= \dfrac{(36 + 12 + 3) - (16 + 8 + 3)}{6 - 4}$

$= \dfrac{24}{2}$

=12

⇒ f' (c)=2c+2

⇒ 2c+2=12

⇒ c=5

2. Question

Verify Lagrange's mean-value theorem for the following function:

$f(x) = x^2 + x - 1$ on $[0,4]$

Answer

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [0,4].

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$= \dfrac{(16 + 4 - 1) - (0 + 0 - 1)}{4 - 0}$

=5

⇒ f'(c)=2c+1

⇒ 2c+1=5

⇒ c=2

3. Question

Verify Lagrange's mean-value theorem for the following function:

$f(x) = 2x^2 - 3x + 1$ on $[1,3]$

Answer

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [1,3].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\mathbf{(18 - 9 + 1) - (2 - 3 + 1)}}{\mathbf{3 - 1}}$$

=5

⇒ f'(c)=4c-3

⇒ 4c-3=5

⇒ c=2

4. Question

Verify Lagrange’s mean-value theorem for the following function:

$$f(x) = x^3 + x^2 - 6x \text{ on } [-1, 4]$$

Answer

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [-1,4].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{(64 + 16 - 24) - (-1 + 1 + 6)}{4 + 1}$$

$$= \frac{50}{5}$$

=10

f' (c)=$3c^2$+2c-6

⇒ 3 c^2+2c-6=10

⇒ 3 c^2+2c-16=0

⇒ 3 c^2-6c+8c-16=0

⇒ 3c(c-2)+8(c-2)=0

⇒ (3c+8)(c-2)=0

$$c = 2, \frac{-8}{3}$$

5. Question

Verify Lagrange’s mean-value theorem for the following function:

$$f(x) = (x - 4)(x - 6)(x - 8) \text{ on } [4, 6]$$

Answer

Given:

$\mathbf{f(x) = x^3 - 18x^2 + 104x - 192}$

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [4,6].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \; f'(c) = \frac{(216 - 648 + 624 - 192) - (64 - 288 + 416 - 192)}{6 - 2}$$

=0

⇒ f' (c)=3c^2-36c+104

=3c^2-36c+10

=0

$$\Rightarrow \; c = \frac{36 \pm \sqrt{1296 - 1248}}{6}$$

$$\Rightarrow \; c = \frac{36 \pm \sqrt{48}}{6}$$

$$\Rightarrow \; c = 6 \pm \frac{2}{3}\sqrt{3}$$

6. Question

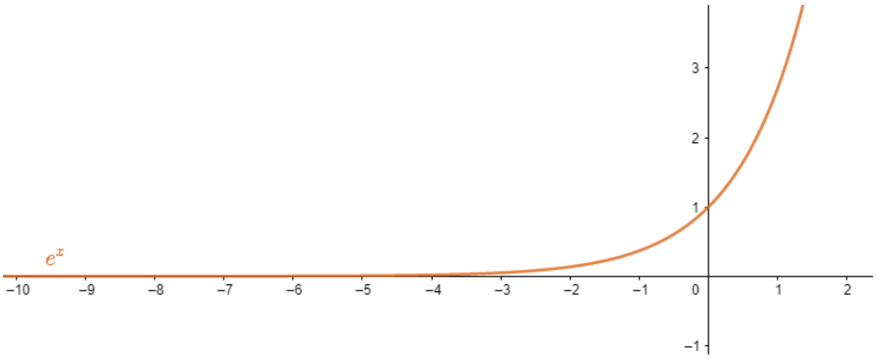
Verify Lagrange’s mean-value theorem for the following function:

$$f(x) = e^x \text{ on } [0,1]$$

Answer

Given:

Since f(c) is continuous as well as differentiable in the interval [0,1].



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{e - 1}{1}$$

⇒ f' (c)=e^c

⇒ e^c =e-1

$$\Rightarrow \; \log_e e^c = \log_e(e - 1)$$

⇒ c=log$_e$(e-1)

7. Question

Verify Lagrange’s mean-value theorem for the following function:

$$f(x) = x^{2/3} \text{ on } [0,1]$$

Answer

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [0,1].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{1 - 0}{1 - 0}$$

=1

$$f'(c) = \frac{2}{3}c^{\frac{1}{3}}$$

$$\Rightarrow \frac{2}{3}c^{\frac{1}{3}} = 1$$

$$\Rightarrow c^{\frac{-1}{3}} = \frac{3}{2}$$

$$\Rightarrow c^{\frac{1}{3}} = \frac{2}{3}$$

$$\Rightarrow c = \frac{8}{27}$$

8. Question

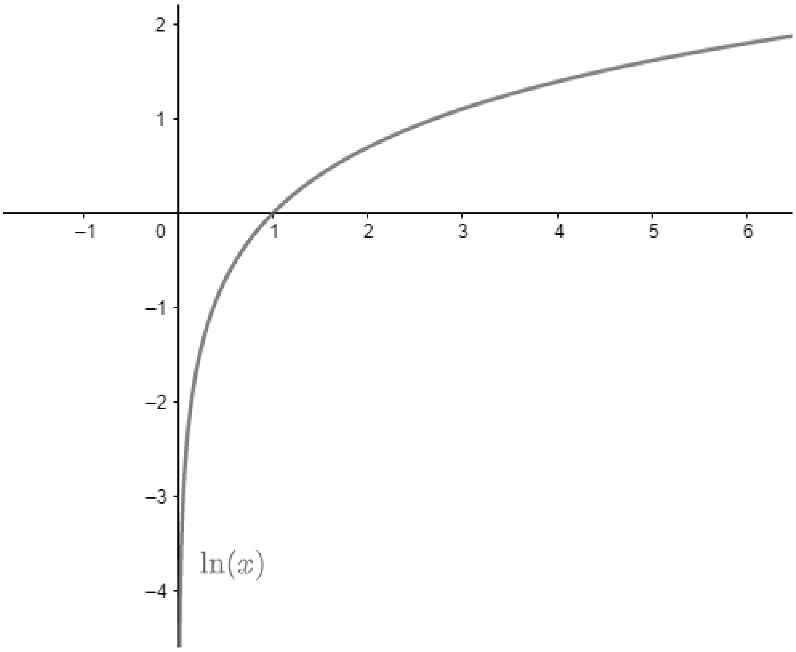
Verify Lagrange’s mean-value theorem for the following function:

$$f(x) = \log x \text{ on } [1,e]$$

Answer

Given:

Since log x is a continuous as well as differentiable function in the interval [1,e].



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\log e - \log 1}{e - 1}$$

$$= \frac{1}{e - 1}$$

$$f'(c) = \frac{1}{c}$$

$$\Rightarrow \frac{1}{e - 1} = \frac{1}{c}$$

c=e-1

9. Question

Verify Lagrange's mean-value theorem for the following function:

$$f(x) = \tan^{-1} x \text{ on } [0,1]$$

Answer

Given:

Since $\tan^{-1} x$ is a continuous as well as differentiable function in the interval [0,1].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\tan^{-1} 1 - \tan^{-1} 0}{1 - 0}$$

$$= \frac{\pi}{4}$$

$$\mathbf{f'(c)} = \frac{1}{1 + c^2}$$

$$\Rightarrow \frac{1}{1 + c^2} = \frac{\pi}{4}$$

$$\Rightarrow 1 + c^2 = \frac{4}{\pi}$$

$\Rightarrow\ c = \sqrt{\frac{4}{\pi} - 1}$

10. Question

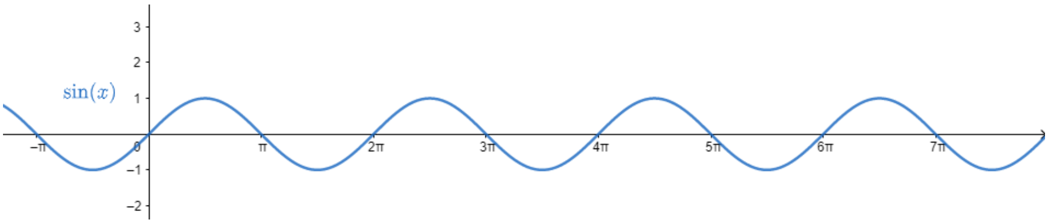
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = \sin x \text{ on } \left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$$

Answer

Given:

Since sin x is a continuous as well as differentiable function in the interval $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$.



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\sin\frac{5\pi}{2} - \sin\frac{\pi}{2}}{\frac{5\pi}{2} - \frac{\pi}{2}}$$

=0

f' (c)=cos x

cos x=0

$$x = \frac{n\pi}{2}, n\epsilon\{1,2,3,4,5\}$$

11. Question

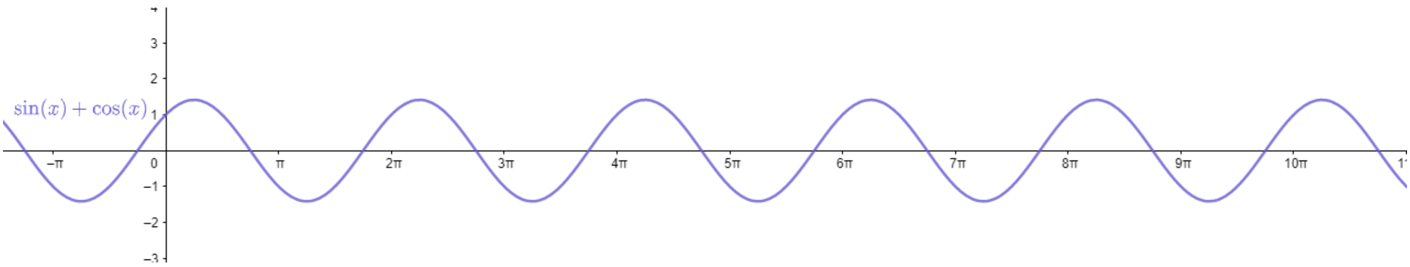
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = (\sin x + \cos x) \text{ on } \left[0, \frac{\pi}{2}\right]$$

Answer

Given:

Since (sin x + cos x) is a continuous as well as differentiable function in the interval $\left[0, \frac{\pi}{2}\right]$.



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\sin\frac{\pi}{2} + \cos\frac{\pi}{2} - \sin 0 - \cos 0}{\frac{\pi}{2} - 0}$$

=0

f' (c)=cos x-sin x

⇒ cos x-sin x=0

$$\Rightarrow \cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4} = 0$$

$$\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = 0$$

$$\Rightarrow \left(x + \frac{\pi}{4}\right) = \cos^{-1} 0$$

$$\Rightarrow \left(x + \frac{\pi}{4}\right) = 1$$

$$\Rightarrow \mathbf{x = 1 - \frac{\pi}{4}}$$

12. Question

Show that Lagrange's mean-value theorem is not applicable to $f(x) = |x|$ on $[-1,1]$.

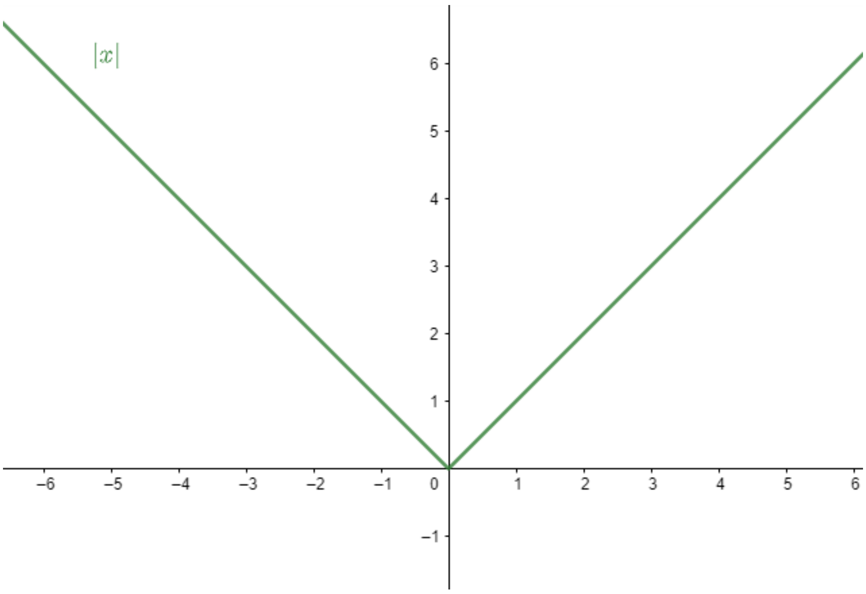
Answer

Given:

Since f(x) is continuous in the interval [-1,1].

But is non differentiable at x=0 due to sharp corner.

So LMVT is not applicable to ƒ(x)=|x|



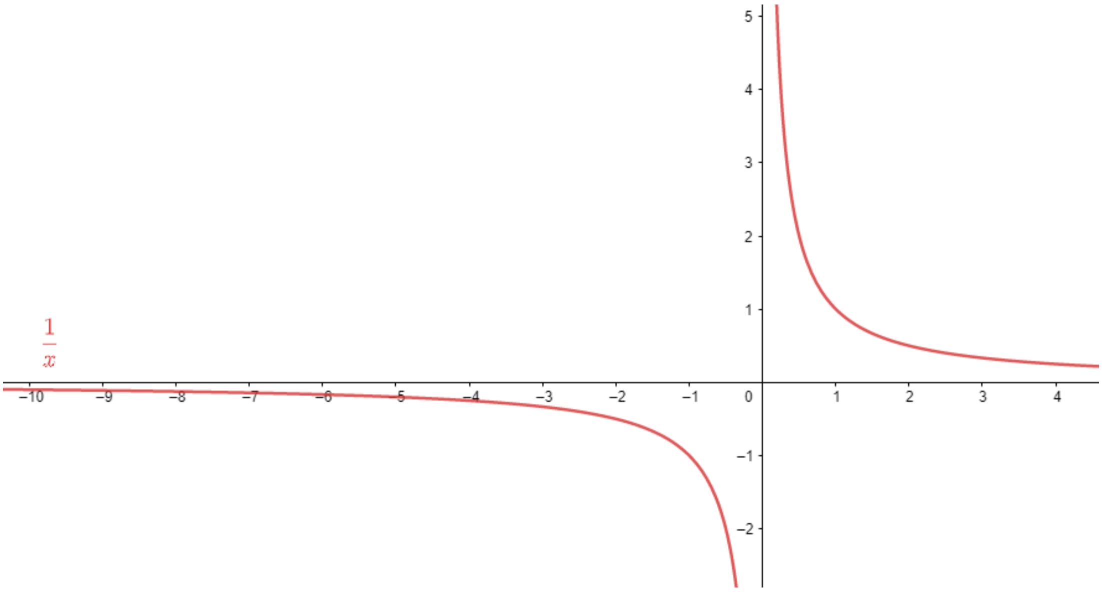
13. Question

Show that Lagrange's mean-value theorem is not applicable to $f(x) = \frac{1}{x}$ on $[-1,1]$

Answer

Given:

Since the graph is discontinuous at x=0 as shown in the graph.



So LMVT is not applicable to the above function.

14 A. Question

Find 'c' of Lagrange's mean-value theorem for

$$f(x) = (x^3 - 3x^2 + 2x) \text{ on } \left[0, \frac{1}{2}\right]$$

Answer

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval $[0,\frac{1}{2}]$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\frac{1}{8} - \frac{3}{4} + 1 - 0}{\frac{1}{2} - 0}$$

$$= \frac{3}{4}$$

$f'(c)=3x^2-6x+2$

$3x^2-6x+2=3/4$

$12x^2-24x+8=3$

$12x^2-24x+5=0$

$$x = \frac{24 \pm \sqrt{576 - 240}}{24}$$

$$x = 1 \pm \sqrt{\frac{336}{576}}$$

$$x = 1 \pm \sqrt{\frac{7}{12}}$$

14 B. Question

Find 'c' of Lagrange's mean-value theorem for

$$f(x)=\sqrt{25-x^2} \text{ on } [1,5]$$

Answer

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [1,5].

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$=\frac{\sqrt{25-25}-\sqrt{25-1}}{5-1}$$

$$=\frac{-\sqrt{24}}{4}$$

$$f'(c)=\frac{1}{2\sqrt{25-c^2}}(-2c)$$

$$\Rightarrow \frac{-c}{\sqrt{25-c^2}}=\frac{-\sqrt{24}}{4}$$

$$\Rightarrow 4c=\sqrt{24(25-c^2)}$$

$\Rightarrow 16c^2=600-24c^2$

$\Rightarrow 40c^2=600$

$\Rightarrow c^2=15$

$$\Rightarrow c=\sqrt{15}$$

14 C. Question

Find 'c' of Lagrange's mean-value theorem for

$$f(x)=\sqrt{x+2} \text{ on } [4,6]$$

Answer

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [4,6].

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$=\frac{\sqrt{8}-\sqrt{6}}{6-4}$$

$$=\frac{\sqrt{8}-\sqrt{6}}{2}$$

$$\mathbf{f'(c)}=\frac{1}{2\sqrt{c+2}}$$

$$\Rightarrow \frac{1}{2\sqrt{c+2}}=\frac{\sqrt{8}-\sqrt{6}}{2}$$

$\Rightarrow \frac{1}{\sqrt{c+2}} = \frac{\sqrt{8}-\sqrt{6}}{1}$

$\Rightarrow \sqrt{c+2} = \frac{1}{\sqrt{8}-\sqrt{6}} \times \frac{\sqrt{8}+\sqrt{6}}{\sqrt{8}+\sqrt{6}}$

$\Rightarrow \sqrt{c+2} = \frac{\sqrt{8}+\sqrt{6}}{2}$

$\Rightarrow \mathbf{c+2} = \frac{1}{4}\left(8+6+2\sqrt{48}\right)$

$\Rightarrow \mathbf{c} = \frac{\mathbf{3}}{\mathbf{2}} + \mathbf{2\sqrt{3}}$

⇒ c=4.964

15. Question

Using Lagrange's mean-value theorem, find a point on the curve $y = x^2$, where the tangent is parallel to the line joining the point (1, 1) and (2, 4)

Answer

Given:

$y=x^2$

Since y is a polynomial function.

It is continuous and differentiable in [1,2]

So, there exists a c such that:

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$= \frac{4-1}{2-1}$

$= 3$

⇒ f' (c)=2c

⇒ 2c=3

$c = \frac{3}{2}$

So, the point is $\left(\frac{3}{2}, \frac{9}{4}\right)$

16. Question

Find a point on the curve $y = x^3$, where the tangent to the curve is parallel to the chord joining the points (1, 1) and (3, 27).

Answer

Given:

$y = x^3$

Since y is a polynomial function.

It is continuous and differentiable in [1,3]

So, there exists a c such that:

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$=\frac{\mathbf{27-1}}{\mathbf{3-1}}$$

$= 13$

⇒ f' (c)=3c^2

⇒ 3c^2=13

$$\Rightarrow\ c=\sqrt{\frac{13}{3}}$$

$$\Rightarrow\ c=\frac{\sqrt{39}}{3}$$

So the point is $\left(\frac{\sqrt{39}}{3},\frac{13\sqrt{39}}{9}\right)$

17. Question

Find the points on the curve $y=x^3-3x$, where the tangent to the curve is parallel to the chord joining (1, −2) and (2, 2).

Answer

Given:

y=x^3-3x

Since y is a polynomial function.

It is continuous and differentiable in [1,2]

So, there exists a c such that:

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$=\frac{\mathbf{(8-6)-(1-3)}}{\mathbf{2-1}}$$

=4

⇒ f' (c)=3c^2-3

⇒ 3 c^2-3=4

⇒ 3c^2=7

$$\Rightarrow\ c^2=\frac{7}{3}$$

$$\Rightarrow\ \mathbf{c=\pm\sqrt{\frac{7}{3}}}$$

So, the points are $\left(\sqrt{\frac{7}{3}},\frac{-2}{3}\sqrt{\frac{7}{3}}\right),\left(\frac{-7}{3}\sqrt{\frac{7}{3}},\frac{2}{3}\sqrt{\frac{7}{3}}\right)$

18. Question

If $f(x)=x(1-\log x)$, where c > 0, show that $(a-b)\log c=b(1-\log b)-a\ (1-\log a)$, where 0 < a < c < b.

Answer

Given:

f(x)=x(1-log x)

Since the function is continuous as well as differentiable

So, there exists c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow\ (1 - \log c) - c \times \frac{1}{c} = \frac{b(1 - \log b) - a(1 - \log a)}{b - a}$$

$$\Rightarrow\ \log c = \frac{b(1 - \log b) - a(1 - \log a)}{b - a}$$

(b-a) log c=b(1-log b)-a(1-log a)

Hence proved.

Exercise 11E

1. Question

Find the maximum or minimum values, if any, without using derivatives, of the function:

$$(5x - 1)^2 + 4.$$

Answer

min. value = 4

Since the square of any no. Is positive, the given function has no maximum value.

The minimum value exists when the quantity $(5x-1)^2=0$

Therefore, minimum value=4

2. Question

Find the maximum or minimum values, if any, without using derivatives, of the function:

$$-(x - 3)^2 + 9$$

Answer

max. value = 9

Since the quantity $(x-3)^2$ has a –ve sign, the max. Value it can have is 9.

Also hence it has no minimum value.

3. Question

Find the maximum or minimum values, if any, without using derivatives, of the function:

$$-|x + 4| + 6$$

Answer

max. value = 6

Since |x+4| is non-negative for all x belonging to R.

Therefore the least value it can have is 0 .

Hence value of function is 6.

It has no minimumvalue as it can have infinitely many.

4. Question

Find the maximum or minimum values, if any, without using derivatives, of the function:

$\sin\ 2x + 5$

Answer

max. value = 4, min. value = 6

f(x)=sin2x+5

We know that,

-1≤sinϴ≤1

-1≤sin2x≤1

Adding 5 on both sides,

-1+5≤sin2x+5≤1+5

4≤sin2x+5≤6

Hence

max value of f(x)=2x+5 will be 6

Min value of f(x) =2x+5 will be 4

5. Question

Find the maximum or minimum values, if any, without using derivatives, of the function:

$|\sin\ 4x + 3|$

Answer

max. value = 4, min. value = 2

We know that

-1≤sinϴ≤1

-1≤sin4x≤1

Adding 3 on both sides,

We get

-1+3≤sin4x+3≤1+3

2≤|sin4x+3|≤4

Hence min.Value is 2 and max value is 4

6. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$f(x) = (x-3)^4$

Answer

local max. value is 0 at x = 3

$F'(x)=4(x-3)^3=0$

⇒X=3

 local max. Vaue is 0.

7. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x) = x^2$$

Answer

local min. value is 0 at x = 0

$F'(x)=2x=0$

$x=0$

 local min.value is 0

8. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x) = 2x^3 - 21x^2 + 36x - 20$$

Answer

local max. value is −3 at x = 1 and local min. value is −128 at x = 6

$F'(x)=6x^2-42x+36=0$

$\Rightarrow 6(x-1)(x-6)=0$

$\Rightarrow x=1,6$

$F''(x)=12x-42$

$F''(1)<0$,1 is the pont of local max.

$F''(6)>0$, 6 is the point of localmin.

$F(1)=2-21+36-20=-3$

$F(6)=-128$

9. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x) = x^3 - 6x^2 + 9x + 15$$

Answer

local max. value is 19 at x = 1 and local min. value is 15 at x = 3

$F'(x)=3x^2-12x+9=0$

$\Rightarrow 3(x-3)(x-1)=0$

$\Rightarrow x=3,1$

$F''(x)=6x-12$

$F''(3)=18-12=6>0$, 3 is the of local min.

$F''(1)<0$, 1 is the point of local max.

$F(3)=15$

F(1)=19

10. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x) = x^4 - 62x^2 + 120x + 9$$

Answer

local max. value is 68 at x = 1 and local min. values are −1647 at x = −6 and −316 at x = 5

F'(x)=4x^3-124x+120=0

⇒ 4(x^3-31x+30)=0

For x=1, the given eq is 0

x-1 is a factor,

4(x-1)(x+6)(x-5)=0

⇒X=1,-6,5

F''(1)<0, 1is the point of max.

F''(-6) and f''(5)>0, -6 and 5 are point of min.

F(1)=68

F(-6)=-1647

F(5)=-316

11. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x) = -x^3 + 12x^2 - 5$$

Answer

local max. value is 251 at x = 8 and local min. value is −5 at x = 0

'}(x)=-3x^2+24x=0

⇒ -3x(x-8)=0

⇒ x=0,8

F''(x)=-6x+24

F''(0)>0, 0 is the point of local min.

F''(8)<0, 8 is the point of local max.

F(8)=251 and f(0)=-5

12. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x) = (x-1)(x+2)^2$$

Answer

local max. value is 0 at x = −2 and local min. value is −4 at x = 0

$f'(x)=(x-1)2(x+2)+(x+2)^2=0$

$x=0,-2$

$f''(0)>0$, 0 is the point of local min.

$f''(-2)<0$, -2 is the point of local max.

$f(0)=-4$

$f(-2)=0$

13. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x)=-(x-1)^3(x+1)^2$$

Answer

local max. value is 0 at each of the points x = 1 and x = −1 and local min. value is $\frac{-3456}{3125}$ at $x=-\frac{1}{5}$

$F'(x)=-(x-1)^3 2(x+1)-3(x-1)^2(x+1)^2=0$

$\Rightarrow\ x=1,-1,-\frac{1}{5}$

Since, $f''(1)$ and $f''(-1)<0$, 1 and -1 are the points of local max.

$F''(-\frac{1}{5})>0$, $-\frac{1}{5}$ is the point of local min.

$F(1)=f(-1)=0$

Also, $f\left(-\frac{1}{5}\right)=-\frac{3456}{3125}$

14. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x)=\frac{x}{2}+\frac{2}{x}, x>0$$

Answer

local min. value is 2 at x = 2

$F'(x)=\frac{1}{2}-\frac{2}{x^2}=0$

$\Rightarrow x^2-4=0$

$\Rightarrow x=\pm 2$

But since $x>0$, $x=2$

$F''(2)=\frac{2}{x^3}$

$=\frac{2}{8}<0$

point of local mini. is 2

$F(2)=\frac{2}{2}+\frac{2}{2}=2$

15. Question

Find the maximum and minimum values of $2x^3 - 24x + 107$ on the interval $[-3,3]$.

Answer

max. value is 139 at x = −2 and min. value is 89 at x = 3

$F'(x)=6x^2-24=0$

$6(x^2-4)=0$

$6(x^2-2^2)=0$

6(x-2)(x+2)=0

X=2,-2

Now, we shall evaluate the value of f at these points and the end points

$F(2)=2(2)^3-24(2)+107=75$

$F(-2)=2(-2)^3-24(-2)+107=139$

$F(-3)=2(-3)^3-24(-3)+107=125$

$F(3)=2(3)^3-24(3)+107=89$

16. Question

Find the maximum and minimum values of $3x^4 - 8x^3 + 12x^2 - 48x + 1$ on the interval $[1,4]$.

Answer

max. value is 257 at x = 4 and min. value is −63 at x = 2

$F'(x)=12x^3-24x^2+24x-48=0$

$12(x^3-2x^2+2x-4)=0$

Since for x=2, $x^3-2x^2+2x-4=0$, x-2 is a factor

On dividing x^3-2x^2+2x-4 by x-2, we get,

$12(x-2)(x^2+2)=0$

X=2,4

Now, we shall evaluate the value of f at these points and the end points

$F(1)=3(1)^4-8(1)^3+12(1)^2-48(1)+1=-40$

$F(2)= 3(2)^4-8(2)^3+12(2)^2-48(2)+1=-63$

$F(4)= 3(4)^4-8(4)^3+12(4)^2-48(4)+1=257$

17. Question

Find the maximum and minimum of

$$f(x)=\left(\sin x+\frac{1}{2}\cos x\right) \text{ in } 0 \le x \le \frac{\pi}{2}$$

Answer

max. value is $\frac{3}{4}$ at $x=\frac{\pi}{6}$ and min. value is $\frac{1}{2}$ at $x=\frac{\pi}{2}$

$F^{I}(x)=\cos x-\frac{1}{2}\sin x=0$

ð 2 cos x=sin x

$\Rightarrow \frac{\pi}{6} = \frac{\pi}{3}$

$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \frac{1}{2}\cos\frac{\pi}{2} = \frac{1}{2}$

$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{6} = \frac{1}{2} + \frac{\sqrt{3}}{4}$

$f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \frac{1}{2}\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{1}{4}$

18. Question

Show that the maximum value of $x^{1/x}$ is $e^{1/e}$

Answer

The given function is

$Y = x^{\frac{1}{x}}$

Now, taking logarithm from both sides, we get..

$\log y = \frac{1}{x}\log x$

Differentiating both sides w.r.t x....

$\frac{1}{y}y' = -\frac{1}{x^2}\ln(x) + \frac{1}{x^2}$

$\Rightarrow y' = \frac{y}{x^2}(1 - \ln(x))$

(1-ln(x))=0

ln(x)=1

x=e

hence the max. point is x=e

max value is $e^{\frac{1}{e}}$.

19. Question

Show that $\left(x + \frac{1}{x}\right)$ has a maximum and minimum, but the maximum value is less than the minimum value.

Answer

$F(x)=x+\frac{1}{x}$

Taking first derivative and equating it to zero to find extreme points.

$F'(x)=1-\frac{1}{x^2} = 0$

$X^2=1$

x=1,x=-1

now to determine which of these is min. And max. We use second derivative.

$f^{||}(x)=\frac{2}{x^3}$

$f^{||}(1)=2$ and $f^{||}(-1)=-2$

since $f^{||}(1)$ is +ve it is minimum point while $f^{||}(-1)$ is –ve it is maximum point

max value-> $f(-1)=-1+\frac{1}{-1}=-2$

min vaue-> $f(1)=1+\frac{1}{1}=2$

hence maximum value is less than minimum value

20. Question

Find the maximum profit that a company can make, if the profit function is given by $p(x)=41+24x-18x^2$
.

Answer

49

$\frac{dp}{dx}=-24-36x$

=0

⇒x=−23

Step 2

$\frac{d^2p}{dx^2}=-36$ is negative

Step 3

maximum profit $=p\left(-\frac{2}{3}\right)$

=49

21. Question

An enemy jet is flying along the curve $y=\left(x^2+2\right)$. A soldier is placed at the point (3, 2). Find the nearest point between the soldier and the jet.

Answer

(1, 3)

Let P(x,y) be the position of the jet and the soldier is placed at A(3,2)

AP=$\sqrt{(x-3)^2+(y-2)^2}$

As $y=x^2+2$ or $y-2=x^2$

 $AP^2=(x-3)^2+x^4=z$ (say)

$\frac{dz}{dx}=2(x-3)+4x^3$

$\frac{dz}{dx}=0$

2 □x-6+4x^3=0

Put x=1

2-6+4=0

□ x-1 is a factor

And $\frac{d^2z}{dx^2} = 12x^2 + 2$

$\frac{dz}{dx} = 0$ or x=1

and $\frac{d^2z}{dx^2}$(at x=1)>0

□ z is minimum when x=1,y=1+2=3

Point is (1,3)

22. Question

Find the maximum and minimum values of

$f(x) = (-x + 2\sin x)$ on $[0, 2\pi]$.

Answer

max. value is $\left(-\frac{\pi}{3} + \sqrt{3}\right)$ at $x = \frac{\pi}{3}$ and min. value is $\left(\frac{5\pi}{3} + \sqrt{3}\right)$ at $x = \frac{5\pi}{3}$

f$'$(x)=-1+2cos x=0

$\Rightarrow \cos x = -\frac{1}{2}$

$x = \frac{2\pi}{3}$

By finding the general solution, we get $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$

Now, by finding the second derivative, we get that $f''(\frac{\pi}{3}) < 0$ and $f''(\frac{5\pi}{3}) > 0$

Therefore, max. value is $\left(-\frac{\pi}{3} + \sqrt{3}\right)$ at $x = \frac{\pi}{3}$ and min. value is $\left(\frac{5\pi}{3} + \sqrt{3}\right)$ at $x = \frac{5\pi}{3}$

Exercise 11F

1. Question

Find two positive number whose product is 49 and the sum is minimum.

Answer

Given,

• The two numbers are positive.

• the product of two numbers is 49.

• the sum of the two numbers is minimum.

Let us consider,

• x and y are the two numbers, such that x > 0 and y > 0

• Product of the numbers : x × y = 49

• Sum of the numbers : S = x + y

Now as,

x × y = 49

$y = \frac{49}{x}$ ------ (1)

Consider,

S = x + y

By substituting (1), we have

$S = x + \frac{49}{x}$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.</u>

Differentiating the equation (2) with x

$$\frac{dS}{dx} = \frac{d}{dx}\left(x + \frac{49}{x}\right)$$

$$\frac{dS}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{49}{x}\right)$$

$\frac{dS}{dx} = 1 + 49\left(\frac{-1}{x^2}\right)$ ----- (3)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dS}{dx} = 1 + 49\left(\frac{-1}{x^2}\right) = 0$$

$$= 1 - \left(\frac{49}{x^2}\right) = 0$$

$$= 1 = \left(\frac{49}{x^2}\right)$$

$= x^2 = 49$

$= x = \pm\sqrt{49}$

As x > 0, then x = 7

Now, for checking if the value of S is maximum or minimum at x=7, <u>we will perform the second differentiation and check the value of $\frac{d^2S}{dx^2}$ at the critical value x = 7.</u>

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\left[1 + 49\left(\frac{-1}{x^2}\right)\right]$$

$$\frac{d^2S}{dx^2} = \frac{d}{dx}[1] + \frac{d}{dx}\left[49\left(\frac{-1}{x^2}\right)\right]$$

$$\frac{d^2S}{dx^2} = 0 + \left[49\left(\frac{-1 \times -2}{x^3}\right)\right]$$

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{d^2S}{dx^2} = 49\left(\frac{2}{x^3}\right) = \frac{98}{x^3}$$

Now when x = 7,

$$\left[\frac{d^2S}{dx^2}\right]_{x=7} = \frac{98}{7^3} = \frac{98}{343} > 0$$

As second differential is positive, hence the critical point x = 7 will be the minimum point of the function S.

Therefore, the function S = sum of the two numbers is minimum at x = 7.

From Equation (1), if x= 7

$$y = \frac{49}{7} = 7$$

Therefore, x = 7 and y = 7 are the two positive numbers whose product is 49 and the sum is minimum.

2. Question

Find two positive numbers whose sum is 16 and the sum of whose squares is minimum.

Answer

Given,

• The two numbers are positive.

• the sum of two numbers is 16.

• the sum of the squares of two numbers is minimum.

Let us consider,

• x and y are the two numbers, such that x > 0 and y > 0

• Sum of the numbers : x + y = 16

• Sum of squares of the numbers : $S = x^2 + y^2$

Now as,

x + y = 16

y = (16-x) ------ (1)

Consider,

$S = x^2 + y^2$

By substituting (1), we have

$S = x^2 + (16-y)^2$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with x

$$\frac{dS}{dx} = \frac{d}{dx}\left[x^2 + (16-x)^2\right]$$

$$\frac{dS}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}\left[(16-x)^2\right]$$

$\frac{dS}{dx} = 2x + 2(16-x)(-1)$ ----- (3)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dS}{dx} = 2x + 2(16 - x)(-1) \ = 0$$

⇒ 2x – 2(16 -x) = 0

⇒ 2x – 32 + 2x = 0

= 4x = 32

$$\Rightarrow x = \frac{32}{4}$$

⇒ x = 8

As x > 0, x = 8

Now, for checking if the value of S is maximum or minimum at x=8, we will perform the second differentiation and check the value of $\frac{d^2S}{dx^2}$ at the critical value x = 8.

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\,[2x + 2(16 - x)(-1)\,]$$

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\,[2x] - 2\,\frac{d}{dx}[16 - x\,]$$

$$\frac{d^2S}{dx^2} = 2 - \ 2[0 - 1\,]$$

[Since $\frac{d}{dx}\,(\text{constant}) = 0$ and $\frac{d}{dx}\,(x^n) = \ nx^{n-1}$]

$$\frac{d^2S}{dx^2} = 2 - 0 + 2 = 4$$

Now when x = 8,

$$\left[\frac{d^2S}{dx^2}\right]_{x=8} = 4 > 0$$

As second differential is positive, hence the critical point x = 8 will be the minimum point of the function S.

Therefore, the function S = sum of the squares of the two numbers is minimum at x = 8.

From Equation (1), if x= 8

y = 16 – 8 = 8

Therefore, x = 8 and y = 8 are the two positive numbers whose su is 16 and the sum of the squares is minimum.

3. Question

Divide 15 into two parts such that the square of one number multiplied with the cube of the other number is maximum.

Answer

Given,

• the number 15 is divided into two numbers.

• the product of the square of one number and cube of another number is maximum.

Let us consider,

• x and y are the two numbers

• Sum of the numbers : x + y = 15

• Product of square of the one number and cube of anther number : P = x^3 y^2

Now as,

x + y = 15

y = (15-x) ------ (1)

Consider,

P = x^3y^2

By substituting (1), we have

P = x^3 × $(15\text{-}x)^2$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.</u>

Differentiating the equation (2) with x

$$\frac{dP}{dx} = \frac{d}{dx}[x^3 \times (15-x)^2]$$

$$\frac{dP}{dx} = (15-x)^2 \frac{d}{dx}(x^3) + x^3 \frac{d}{dx}[(15-x)^2]$$

$$\frac{dP}{dx} = (15-x)^2 (3x^2) + x^3 [2(15-x)(-1)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u \times v) = v \times \frac{d}{dx}(u) + u \times \frac{d}{dx}(v)$]

$$\frac{dP}{dx} = (15-x)^2 (3x^2) + x^3 [-30 + 2x]$$

= 3×[15^2 – 2× (15)×(x) + x^2] x^2 + x^3(2x-30)

= x^2[3× (225 – 30x + x^2)+ x (2x - 30)]

= x^2[675– 90x + $3x^2$+ $2x^2$ – 60x]

= x^2[$5x^2$ – 120x + 675]

= $5x^2$ [x^2 – 24x + 135] ----- (3)

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dP}{dx} = 5x^2[x^2 - 24x + 135] = 0$$

Hence $5x^2$ = 0 (or) x^2 – 24x + 135 = 0

x = 0 (or) $x = \frac{-(-24)\pm\sqrt{(-24)^2-4(1)(135)}}{2 \times 1}$

x = 0 (or) $x = \frac{24\pm\sqrt{576-540)}}{2}$

x = 0 (or) $x = \frac{24\pm\sqrt{36}}{2}$

x = 0 (or) $x = \frac{24\pm6}{2}$

x = 0 (or) $x = \frac{24+6}{2}$ (or) $x = \frac{24-6}{2}$

x = 0 (or) $x = \frac{30}{2}$ (or) $x = \frac{18}{2}$

x = 0 (or) x = 15 (or) x = 9

Now considering the critical values of x = 0,9,15

Now, for checking if the value of P is maximum or minimum at x=0,9,15, we will perform the second differentiation and check the value of $\frac{d^2P}{dx^2}$ at the critical value x = 0,9,15.

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2P}{dx^2} = \frac{d}{dx}[5x^2(x^2 - 24x + 135)]$$

$$\frac{d^2P}{dx^2} = (x^2 - 24x + 135)\frac{d}{dx}[5x^2] + 5x^2\frac{d}{dx}[x^2 - 24x + 135]$$

$= (x^2 - 24x + 135)(5 \times 2x) + 5x^2(2x - 24 + 0)$

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u \times v) = v \times \frac{d}{dx}(u) + u \times \frac{d}{dx}(v)$]

$= (x^2 - 24x + 135)(10x) + 5x^2(2x - 24)$

$= 10x^3 - 240x^2 + 1350x + 10x^3 - 120x^2$

$= 20x^3 - 360x^2 + 1350x$

$= 5x(4x^2 - 72x + 270)$

$$\frac{d^2P}{dx^2} = 5x(4x^2 - 72x + 270)$$

Now when x = 0,

$$\left[\frac{d^2P}{dx^2}\right]_{x=0} = 5 \times 0[4(0)^2 - 72(0) + 270]$$

= 0

So, we reject x = 0

Now when x = 15,

$$\left[\frac{d^2P}{dx^2}\right]_{x=15} = 5 \times 15[4(15)^2 - 72(15) + 270]$$

$= 65[(4 \times 225) - 1080 + 270]$

$= 65[900 - 1080 + 270]$

$= 65[1170 - 1080]$

$= 65 \times (90) > 0$

Hence $\left[\frac{d^2P}{dx^2}\right]_{x=15} > 0$, so at x = 15, the function P is minimum

Now when x = 9,

$$\left[\frac{d^2P}{dx^2}\right]_{x=9} = 5 \times 9[4(9)^2 - 72(9) + 270]$$

$= 45[(4 \times 81) - 648 + 270]$

$= 45[324 - 648 + 270]$

= 45 [594 – 648]

= 45 × (-54)

= -2430 < 0

As second differential is negative, hence at the critical point x = 9 will be the maximum point of the function P.

Therefore, the function P is maximum at x = 9.

From Equation (1), if x= 9

y = 15 – 9 = 6

Therefore, x = 9 and y = 6 are the two positive numbers whose sum is 15 and the product of the square of one number and cube of another number is maximum.

4. Question

Divide 8 into two positive parts such that the sum of the square of one and the cube of the other is minimum.

Answer

Given,

• the number 8 is divided into two numbers.

• the product of the square of one number and cube of another number is minimum.

Let us consider,

• x and y are the two numbers

• Sum of the numbers : x + y = 8

• Product of square of the one number and cube of anther number : $S = x^3 + y^2$

Now as,

x + y = 8

y = (8-x) ------ (1)

Consider,

$S = x^3 + y^2$

By substituting (1), we have

$S = x^3 + (8-x)^2$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f’(c) = 0.</u>

Differentiating the equation (2) with x

$$\frac{dS}{dx} = \frac{d}{dx}[x^3 + (8-x)^2]$$

$$\frac{dS}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}[(8-x)^2]$$

$$\frac{dS}{dx} = (3x^2) + 2(8-x)(-1)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dS}{dx} = 3x^2 - 16 + 2x$$

$= 3x^2 + 2x - 16$ ------ (3)

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dS}{dx} = 3x^2 + 2x - 16 = 0$$

Hence $3x^2 + 2x - 16 = 0$

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(3)(-16)}}{2 \times 3}$$

$$= \frac{-2 \pm \sqrt{4 + 192}}{6}$$

$$= \frac{-2 \pm \sqrt{196}}{6}$$

$$x = \frac{-2 \pm 14}{6}$$

$x = \frac{-2+14}{6}$ (or) $x = \frac{-2-14}{6}$

$x = \frac{12}{6}$ (or) $x = \frac{-16}{6}$

x = 2 (or) x = -2.67

Now considering the critical values of x = 2,-2.67

Now, for checking if the value of P is maximum or minimum at x=2,-2.67, we will perform the second differentiation and check the value of $\frac{d^2S}{dx^2}$ at the critical value x = 2,-2.67.

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\,[3x^2 + 2x - 16\,]$$

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\,[3x^2] + \frac{d}{dx}[2x] - \frac{d}{dx}\,[16]$$

= 3 (2x) + 2 (1) - 0

[Since $\frac{d}{dx}$ (constant) $= 0$ and $\frac{d}{dx}$ $(x^n) = nx^{n-1}$]

= 6x + 2

$$\frac{d^2S}{dx^2} = 6x + 2$$

Now when x = -2.67,

$$\left[\frac{d^2S}{dx^2}\right]_{x=-2.67} = 6\,(-2.67) + 2$$

= - 16.02 + 2 = -14.02

At x = -2.67 $\frac{d^2S}{dx^2} = -14.02 < 0$ hence, the function S will be maximum at this point.

Now consider x = 2,

$$\left[\frac{d^2S}{dx^2}\right]_{x=2} = 6\,(2) + 2$$

= 12 + 2 = 14

Hence $\left[\frac{d^2S}{dx^2}\right]_{x=2} = 14 > 0$, so at x = 2, the function S is minimum

As second differential is positive, hence at the critical point x = 2 will be the maximum point of the function S.

Therefore, the function S is maximum at x = 2.

From Equation (1), if x= 2

y = 8 – 2 = 6

Therefore, x = 2 and y = 6 are the two positive numbers whose sum is 8 and the sum of the square of one number and cube of another number is maximum.

5. Question

Divide a into two parts such that the product of the pth power of one part and the qth power of the second part may be maximum.

Answer

Given,

• the number ‘a’ is divided into two numbers.

• the product of the pth power of one number and qth power of another number is maximum.

Let us consider,

• x and y are the two numbers

• Sum of the numbers : x + y = a

• Product of square of the one number and cube of anther number : $P = x^p y^q$

Now as,

x + y = a

y = (a-x) ------ (1)

Consider,

$P = x^p y^q$

By substituting (1), we have

$P = x^p \times (a-x)^q$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f’(c) = 0.

Differentiating the equation (2) with x

$$\frac{dP}{dx} = \frac{d}{dx}[x^p \times (a-x)^q]$$

$$\frac{dP}{dx} = (a-x)^q \frac{d}{dx}(x^p) + x^p \frac{d}{dx}[(a-x)^q]$$

$$\frac{dP}{dx} = (a-x)^q (px^{p-1}) + x^p [q(a-x)^{q-1}(-1)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u \times v) = v \times \frac{d}{dx}(u) + u \times \frac{d}{dx}(v)$]

$$\frac{dP}{dx} = x^{p-1}(a-x)^{q-1}[(a-x)p - xq]$$

$= x^{p-1}(a-x)^{q-1}[ap-xp-xq]$

$= x^{p-1}(a-x)^{q-1}[ap – x (p+q)]$ ----- (3)

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dP}{dx} = x^{p-1}(a-x)^{q-1}[ap - x(p+q)] = 0$$

Hence $x^{p-1} = 0$ (or) $(a-x)^{q-1}$ (or) ap– x(p+q)= 0

x = 0 (or) x = a (or) $x = \frac{ap}{p+q}$

Now considering the critical values of x = 0,a and $x = \frac{ap}{p+q}$

Now, using the First Derivative test,

For f, a continuous function which has a critical point c, then, function has the local maximum at c, if f'(x) changes the sign from positive to negative as x increases through c, i.e. f'(x)>0 at every point close to the left of c and f'(x)<0 at every point close to the right of c.

Now when x = 0,

$$\left[\frac{dP}{dx}\right]_{x=0} = 0$$

So, we reject x = 0

Now when x = a,

$$\left[\frac{dP}{dx}\right]_{x=a} = 0$$

Hence we reject x = a

Now when $x < \frac{ap}{p+q}$,

$$\left[\frac{dP}{dx}\right]_{x<\frac{ap}{p+q}} = \left(\frac{ap}{p+q}\right)^{p-1}\left(a - \frac{ap}{p+q}\right)^{q-1}\left[ap - \frac{ap}{p+q}(p+q)\right] > 0 \text{ ---- (4)}$$

Now when $x > \frac{ap}{p+q}$,

$$\left[\frac{dP}{dx}\right]_{x>\frac{ap}{p+q}} = \left(\frac{ap}{p+q}\right)^{p-1}\left(a - \frac{ap}{p+q}\right)^{q-1}\left[ap - \frac{ap}{p+q}(p+q)\right] < 0 \text{ ---- (5)}$$

By using first derivative test, from (4) and (5), we can conclude that, the function P has local maximum at $x = \frac{ap}{p+q}$

From Equation (1), if $x = \frac{ap}{p+q}$

$$y = a - \frac{ap}{p+q} = \frac{a(p+q) - ap}{p+q} = \frac{ap}{p+q}$$

Therefore, $x = \frac{ap}{p+q}$ and $y = \frac{aq}{p+q}$ are the two positive numbers whose sum together to give the number 'a' and whose product of the pth power of one number and qth power of the other number is maximum.

6. Question

The rate of working of an engine is given by.

$$R = 15\upsilon + \frac{6000}{\upsilon}, \text{ where } 0 < \upsilon < 30$$

and υ is the speed of the engine. Show that R is the least when υ = 20.

Answer

Given:

Rate of working of an engine R, v is the speed of the engine:

$R = 15v + \frac{6000}{v}$, where 0<v<30

For finding the maximum/ minimum of given function, we can find it by differentiating it with v and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Now, differentiating the function R with respect to v.

$$\frac{dR}{dv} = \frac{d}{dv}\left[15v + \frac{6000}{v}\right]$$

$$\frac{dR}{dv} = \frac{d}{dv}[15v] + \frac{d}{dv}\left[\frac{6000}{v}\right]$$

$\frac{dR}{dv} = 15 + \left[\frac{6000}{v^2}\right](-1) = 15 - \frac{6000}{v^2}$ ----- (1)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Equating equation (1) to zero to find the critical value.

$$\frac{dR}{dv} = 15 - \frac{6000}{v^2} = 0$$

$$15 = \frac{6000}{v^2}$$

$$v^2 = \frac{6000}{15} = 400$$

$v^2 = 400$

$v = \pm\sqrt{400}$

v = 20 (or) v = -20

As given in the question 0<v<30, v = 20

Now, for checking if the value of R is maximum or minimum at v=20, we will perform the second differentiation and check the value of $\frac{d^2R}{dv^2}$ at the critical value v = 20.

Differentiating Equation (1) with respect to v again:

$$\frac{d^2R}{dv^2} = \frac{d}{dx}\left[15 - \frac{6000}{v^2}\right]$$

$$= \frac{d}{dx}[15] - \frac{d}{dx}\left[\frac{6000}{v^2}\right]$$

$$= 0 - (-2)\left[\frac{6000}{v^3}\right]$$

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$= 2\left[\frac{6000}{v^3}\right]$$

$\frac{d^2R}{dv^2} = \left[\frac{12000}{v^3}\right]$ ----- (2)

Now find the value of $\left(\frac{d^2R}{dv^2}\right)_{v=20}$

$$\left(\frac{d^2R}{dv^2}\right)_{v=20} = \left[\frac{12000}{(20)^3}\right] = \frac{12000}{20 \times 20 \times 20} = \frac{3}{2} > 0$$

So, at critical point v = 20. The function R is at its minimum.

Hence, the function R is at its minimum at v = 20.

7. Question

Find the dimensions of the rectangle of area 96 cm^2 whose perimeter is the least. Also, find the perimeter of the rectangle.

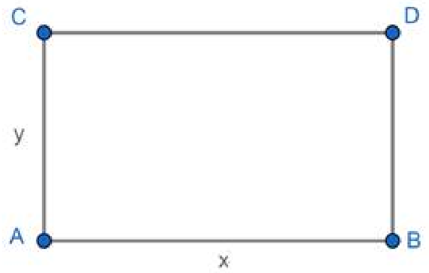
Answer

Given,

• Area of the rectangle is 93 cm^2.

• The perimeter of the rectangle is also fixed.

Let us consider,



• x and y be the lengths of the base and height of the rectangle.

• Area of the rectangle = A = x × y = 96 cm^2

• Perimeter of the rectangle = P = 2 (x + y)

As,

x × y = 96

$y = \frac{96}{x}$ ------ (1)

Consider the perimeter function,

P = 2 (x + y)

Now substituting (1) in P,

$P = 2\left(x + \frac{96}{x}\right)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$$\frac{dP}{dx} = \frac{d}{dx}\left[2\left(x + \frac{96}{x}\right)\right]$$

$$\frac{dP}{dx} = \frac{d}{dx}(2x) + 2\frac{d}{dx}\left(\frac{96}{x}\right)$$

$$\frac{dP}{dx} = 2(1) + 2\left(\frac{96}{x^2}\right)(-1)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$\frac{dP}{dx} = 2 - \left(\frac{192}{x^2}\right)$ ----- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dP}{dx} = 2 - \left(\frac{192}{x^2}\right) = 0$$

$$2 = \left(\frac{192}{x^2}\right)$$

$$x^2 = \left(\frac{192}{2}\right) = 96$$

$$x = \sqrt{96}$$

$$x = \pm 4\sqrt{6}$$

As the length and breadth of a rectangle cannot be negative, hence $x = 4\sqrt{6}$

<u>Now to check if this critical point will determine the least perimeter, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left[2 - \left(\frac{192}{x^2}\right)\right]$$

$$\frac{d^2P}{dx^2} = \frac{d}{dx}(2) - \frac{d}{dx}\left(\frac{192}{x^2}\right)$$

$$\frac{d^2P}{dx^2} = 0 - (-2)\left(\frac{192}{x^3}\right)$$

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$\frac{d^2P}{dx^2} = \left(\frac{2 \times 192}{x^3}\right)$ ------ (4)

Now, consider the value of $\left(\frac{d^2P}{dx^2}\right)_{x=4\sqrt{6}}$

$$\frac{d^2P}{dx^2} = \left(\frac{2 \times 192}{(4\sqrt{6})^3}\right)$$

$$= \left(\frac{2 \times 192}{4\sqrt{6} \times 4\sqrt{6} \times 4\sqrt{6}}\right)$$

$$= \left(\frac{2 \times 192}{4\sqrt{6} \times 4\sqrt{6} \times 4\sqrt{6}}\right) = \frac{1}{\sqrt{6}}$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=4\sqrt{6}} = \frac{1}{\sqrt{6}} > 0$, so the function P is minimum at $x = 4\sqrt{6}$.

Now substituting $x = 4\sqrt{6}$ in equation (1):

$$y = \frac{96}{4\sqrt{6}}$$

$$y = \frac{96\sqrt{6}}{4 \times 6}$$

[By rationalizing he numerator and denominator with $\sqrt{6}$]

$$\therefore\ y = 4\sqrt{6}$$

Hence, area of the rectangle with sides of a rectangle with $x = 4\sqrt{6}$ and $y = 4\sqrt{6}$ is 96cm^2 and has the least perimeter.

Now the perimeter of the rectangle is

$P = 2(4\sqrt{6} + 4\sqrt{6}) = 2\left(8\sqrt{6}\right) = 16\sqrt{6}$ cms

The least perimeter is $16\sqrt{6}$ cms.

8. Question

Prove that the largest rectangle with a given perimeter is a square.

Answer

Given,

• Rectangle with given perimeter.

Let us consider,

• 'p' as the fixed perimeter of the rectangle.

• 'x' and 'y' be the sides of the given rectangle.

• Area of the rectangle, A = x × y.

Now as consider the perimeter of the rectangle,

p = 2(x +y)

p = 2x + 2y

$y = \frac{p-2x}{2}$ ----- (1)

Consider the area of the rectangle,

A = x × y

Substituting (1) in the area of the rectangle,

$A = x \times \left(\frac{p-2x}{2}\right)$

$A = \frac{1}{2} \times (px - 2x^2)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to x:

$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{1}{2}(px - 2x^2)\right]$

$\frac{dA}{dx} = \frac{1}{2}\frac{d}{dx}(px) - \frac{1}{2}\frac{d}{dx}(2x^2)$

$\frac{dA}{dx} = \frac{1}{2}(p) - \frac{2}{2}(2x)$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dA}{dx} = \frac{p}{2} - (2x)$ ----- (3)

To find the critical point, we need to equate equation (3) to zero.

$\frac{dA}{dx} = \frac{p}{2} - (2x) = 0$

$2x = \frac{p}{2}$

$x = \frac{p}{4}$

Now to check if this critical point will determine the largest rectangle, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left[\frac{p}{2} - (2x)\right]$$

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{p}{2}\right) - \frac{d}{dx}(2x)$$

$$\frac{d^2A}{dx^2} = 0 - 2 = -2$$

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{d^2A}{dx^2} = -2$ ------ (4)

Now, consider the value of $\left(\frac{d^2A}{dx^2}\right)_{x=\frac{p}{4}}$

$$\frac{d^2A}{dx^2} = -2 < 0$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=\frac{p}{4}} = -2 < 0$, so the function P is maximum at $x = \frac{p}{4}$.

Now substituting $x = \frac{p}{4}$ in equation (1):

$$y = \frac{p - 2\left(\frac{p}{4}\right)}{2}$$

$$y = \frac{p - \frac{p}{2}}{2} = \frac{p}{4}$$

$$\therefore\ y = \frac{p}{4}$$

As $= y = \frac{p}{4}$ the sides of the taken rectangle are equal, we can clearly say that a largest rectangle which has a given perimeter is a square.

9. Question

Given the perimeter of a rectangle, show that its diagonal is minimum when it is a square.

Answer

Given,

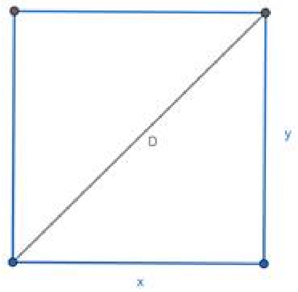
• Rectangle with given perimeter.

Let us consider,

• ‘p’ as the fixed perimeter of the rectangle.

• ‘x’ and ‘y’ be the sides of the given rectangle.

• Diagonal of the rectangle, $D = \sqrt{x^2 + y^2}$. (using the hypotenuse formula)



Now as consider the perimeter of the rectangle,

p = 2(x +y)

p = 2x + 2y

$y = \frac{p-2x}{2}$ ----- (1)

Consider the diagonal of the rectangle,

$D = \sqrt{x^2 + y^2}$

Substituting (1) in the diagonal of the rectangle,

$$D = \sqrt{x^2 + \left(\frac{p-2x}{2}\right)^2}$$

[squaring both sides]

$Z = D^2 = x^2 + \left(\frac{p-2x}{2}\right)^2$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}\left[x^2 + \left(\frac{p-2x}{2}\right)^2\right]$$

$$\frac{dZ}{dx} = \frac{d}{dx}(x^2) + \frac{1}{4}\frac{d}{dx}[(p-2x)^2]$$

$$\frac{dZ}{dx} = 2x + \frac{1}{4}[2(p-2x)(-2)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

= 2x - p + 2x

$\frac{dZ}{dx} = 4x - p$ ------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 4x - p = 0$$

4x –p = 0

4x = p

$$x = \frac{p}{4}$$

Now to check if this critical point will determine the minimum diagonal, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[4x - p]$$

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}(4x) - \frac{d}{dx}(p)$$

= 4 + 0

[Since $\frac{d}{dx}$ (constant) $= 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{d^2Z}{dx^2} = 4$ ------ (4)

Now, consider the value of $\left(\frac{d^2Z}{dx^2}\right)_{x=\frac{p}{4}}$

$$\frac{d^2Z}{dx^2} = 4 > 0$$

As $\left(\frac{d^2Z}{dx^2}\right)_{x=\frac{p}{4}} = 4 > 0$, so the function Z is minimum at $x = \frac{p}{4}$.

Now substituting $x = \frac{p}{4}$ in equation (1):

$$y = \frac{p - 2\left(\frac{p}{4}\right)}{2}$$

$$y = \frac{p - \frac{p}{2}}{2} = \frac{p}{4}$$

$$\therefore\ y = \frac{p}{4}$$

As $x = y = \frac{p}{4}$ the sides of the taken rectangle are equal, we can clearly say that a rectangle with minimum diagonal which has a given perimeter is a square.

10. Question

Show that a rectangle of maximum perimeter which can be inscribed in a circle of radius a is a square of side $\sqrt{2}\,a$.

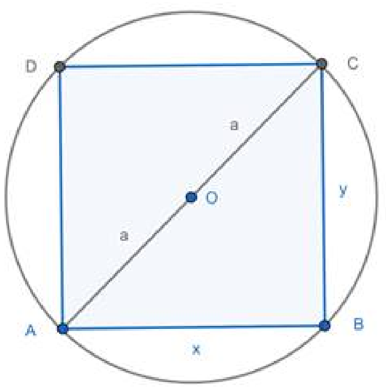
Answer

Given,

• Rectangle is of maximum perimeter.

• The rectangle is inscribed inside a circle.

• The radius of the circle is ‘a’.



Let us consider,

• 'x' and 'y' be the length and breadth of the given rectangle.

• Diagonal $AC^2 = AB^2 + BC^2$ is given by $4a^2 = x^2+y^2$ (as AC = 2a)

• Perimeter of the rectangle, P = 2(x+y)

Consider the diagonal,

$4a^2 = x^2 + y^2$

$y^2 = 4a^2 – x^2$

$y = \sqrt{4a^2 - x^2}$ ---- (1)

Now, perimeter of the rectangle, P

P = 2x + 2y

Substituting (1) in the perimeter of the rectangle.

$P = 2x + 2\sqrt{4a^2 - x^2}$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dP}{dx} = \frac{d}{dx}\left[2x + 2\sqrt{4a^2 - x^2}\right]$$

$$\frac{dP}{dx} = \frac{d}{dx}(2x) + 2\frac{d}{dx}[\sqrt{4a^2 - x^2}]$$

$$\frac{dP}{dx} = 2 + 2[\frac{1}{2}(4a^2 - x^2)^{-\frac{1}{2}}(-2x)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dP}{dx} = 2 - \frac{2x}{\sqrt{4a^2 - x^2}}$$ ------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dP}{dx} = 2 - \frac{2x}{\sqrt{4a^2 - x^2}} = 0$$

$$2 = \frac{2x}{\sqrt{4a^2 - x^2}}$$

$$\sqrt{4a^2 - x^2} = x$$

[squaring on both sides]

$4a^2 – x^2 = x^2$

$2x^2 = 4a^2$

$x^2 = 2a^2$

$$x = \pm a\sqrt{2}$$

$$x = a\sqrt{2}$$

[as x cannot be negative]

Now to check if this critical point will determine the maximum diagonal, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left[2 - \frac{2x}{\sqrt{4a^2 - x^2}}\right]$$

$$\frac{d^2P}{dx^2} = \frac{d}{dx}(2) - \frac{d}{dx}\left(\frac{2x}{\sqrt{4a^2 - x^2}}\right)$$

$$\frac{d^2P}{dx^2} = 0 - \left[\frac{\sqrt{4a^2 - x^2}\,\frac{d}{dx}(2x) - (2x)\frac{d}{dx}(\sqrt{4a^2 - x^2})}{(\sqrt{4a^2 - x^2})^2}\right]$$

[Since $\frac{d}{dx}$ $(\text{constant}) = 0$ and $\frac{d}{dx}$ $(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$]

$$\frac{d^2P}{dx^2} = -\left[\frac{\sqrt{4a^2 - x^2}\,(2) - (2x)\frac{1}{2}(4a^2 - x^2)^{-\frac{1}{2}}(-2x)}{4a^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{\sqrt{4a^2 - x^2}\,(2) + (2x^2)(4a^2 - x^2)^{-\frac{1}{2}}}{4a^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{2\sqrt{4a^2 - x^2} + \frac{2x^2}{\sqrt{4a^2 - x^2}}}{4a^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{2\,(4a^2 - x^2) + 2x^2}{(4a^2 - x^2)^{\frac{3}{2}}}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{8a^2}{(4a^2 - x^2)^{\frac{3}{2}}}\right] \text{ ----- (4)}$$

Now, consider the value of $\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}}$

$$\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}} = -\left[\frac{8a^2}{(4a^2 - (a\sqrt{2})^2)^{\frac{3}{2}}}\right]$$

$$\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}} = -\left[\frac{8a^2}{(4a^2 - 2a^2)^{\frac{3}{2}}}\right] = -\frac{8a^2}{(2a^2)^{\frac{3}{2}}} = -\frac{8a^2}{2\sqrt{2}\,a^3} = -\frac{2\sqrt{2}}{a}$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}} = -\frac{2\sqrt{2}}{a} < 0$, so the function P is maximum at $x = a\sqrt{2}$.

Now substituting $x = a\sqrt{2}$ in equation (1):

$$y = \sqrt{4a^2 - (a\sqrt{2})^2}$$

$$y = \sqrt{4a^2 - 2a^2} = \sqrt{2a^2}$$

$$\therefore\ y = a\sqrt{2}$$

As $x = y = a\sqrt{2}$ the sides of the taken rectangle are equal, we can clearly say that a rectangle with maximum perimeter which is inscribed inside a circle of radius ‘a’ is a square.

11. Question

The sum of the perimeters of a square and a circle is given. Show that the sum of their areas is least when the side of the square is equal to the diameter of the circle.

Answer

Given,

• Sum of perimeter of square and circle.

Let us consider,

• ‘x’ be the side of the square.

• ‘r’ be the radius of the circle.

• Let ‘p’ be the sum of perimeters of square and circle.

p = 4x + 2πr

Consider the sum of the perimeters of square and circle.

p = 4x + 2πr

4x = p – 2πr

$x = \frac{p-2\pi r}{4}$ ---- (1)

Sum of the area of the circle and square is

$A = x^2 + \pi r^2$

Substituting (1) in the sum of the areas,

$$A = \left(\frac{p - 2\pi r}{4}\right)^2 + \pi r^2$$

$A = \frac{1}{16}[p^2 + 4\pi^2 r^2 - 4\pi pr] + \pi r^2$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f’(c) = 0.

Differentiating the equation (2) with respect to r:

$$\frac{dA}{dr} = \frac{d}{dr}\left[\frac{1}{16}[p^2 + 4\pi^2 r^2 - 4\pi pr] + \pi r^2\right]$$

$$\frac{dA}{dr} = \frac{1}{16}\frac{d}{dr}(p^2 + 4\pi^2 r^2 - 4\pi pr) + \frac{d}{dr}[\pi r^2]$$

$$\frac{dA}{dr} = \frac{1}{16}(0 + 8\pi^2 r - 4\pi p) + 2\pi r$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(\text{constant}) = 0$]

$\frac{dA}{dr} = \frac{\pi^2 r}{2} - \frac{\pi p}{4} + 2\pi r$ ------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dA}{dr} = \frac{\pi^2 r}{2} - \frac{\pi p}{4} + 2\pi r = 0$$

$$\left(\frac{\pi^2}{2} + 2\pi\right) r - \frac{\pi p}{4} = 0$$

$$r = \frac{\frac{\pi p}{4}}{\frac{\pi^2}{2} + 2\pi} = \frac{2\pi p}{4(\pi^2 + 4\pi)} = \frac{\pi p}{2(\pi^2 + 4\pi)}$$

$$r = \frac{\pi p}{2\pi(\pi + 4)} = \frac{p}{2(\pi + 4)}$$

$$r = \frac{p}{2(\pi + 4)}$$

<u>Now to check if this critical point will determine the least of the sum of the areas of square and circle, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with r:

$$\frac{d^2A}{dr^2} = \frac{d}{dx}\left[\frac{\pi^2 r}{2} - \frac{\pi p}{4} + 2\pi r\right]$$

$$\frac{d^2A}{dr^2} = \frac{d}{dr}\left(\frac{\pi^2 r}{2}\right) - \frac{d}{dr}\left(\frac{\pi p}{4}\right) + \frac{d}{dr}(2\pi r)$$

$$\frac{d^2A}{dr^2} = \frac{\pi^2}{2} - 0 + 2\pi$$

[Since $\frac{d}{dx}$ (constant) $= 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{d^2A}{dr^2} = \frac{\pi^2}{2} + 2\pi$ ----- (4)

Now, consider the value of $\left(\frac{d^2A}{dr^2}\right)_{r=\frac{p}{2(\pi+4)}}$

$$\left(\frac{d^2A}{dr^2}\right)_{r=\frac{p}{2(\pi+4)}} = \frac{\pi^2}{2} + 2\pi$$

As $\left(\frac{d^2A}{dr^2}\right)_{r=\frac{p}{2(\pi+4)}} = \frac{\pi^2}{2} + 2\pi > 0$, so the function A is minimum at $r = \frac{p}{2(\pi+4)}$.

Now substituting $r = \frac{p}{2(\pi+4)}$ in equation (1):

$$x = \frac{p - 2\pi\left(\frac{p}{2(\pi + 4)}\right)}{4}$$

$$x = \frac{p(\pi + 4) - \pi p}{4 \times (\pi + 4)} = \frac{\pi p + 4p - \pi p}{4\pi + 16} = \frac{4p}{4(\pi + 4)}$$

$$x = \frac{p}{\pi + 4}$$

As the side of the square,

$$x = \frac{p}{\pi + 4}$$

$$x = 2\left[\frac{p}{2(\pi + 4)}\right] = 2r$$

[as $r = \frac{p}{2(\pi+4)}$]

Therefore, side of the square, x = 2r = diameter of the circle.

12. Question

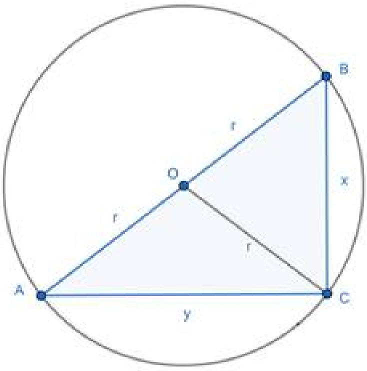
Show that the right triangle of maximum area that can be inscribed in a circle is an isosceles triangle.

Answer

Given,

• A right angle triangle is inscribed inside the circle.

• The radius of the circle is given.



Let us consider,

• ‘r’ is the radius of the circle.

• ‘x’ and ‘y’ be the base and height of the right angle triangle.

• The hypotenuse of the ΔABC = $AB^2 = AC^2 + BC^2$

AB = 2r, AC = y and BC = x

Hence,

$4r^2 = x^2 + y^2$

$y^2 = 4r^2 – x^2$

$y = \sqrt{4r^2 - x^2}$ --- (1)

Now, Area of the ΔABC is

$$A = \frac{1}{2} \times \text{base} \times \text{height}$$

$$A = \frac{1}{2} \times x \times y$$

Now substituting (1) in the area of the triangle,

$$A = \frac{1}{2} x (\sqrt{4r^2 - x^2})$$

[Squaring both sides]

$Z = A^2 = \frac{1}{4} x^2 (4r^2 - x^2)$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f’(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}\left[\frac{1}{4} x^2 (4r^2 - x^2)\right]$$

$$\frac{dZ}{dx} = \frac{1}{4}\left[(4r^2 - x^2) \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(4r^2 - x^2)\right]$$

$$\frac{dZ}{dx} = \frac{1}{4}[(4r^2 - x^2) \times (2x) + x^2 (0 - 2x)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u.v) = v\frac{du}{dx} + u\frac{dv}{dx}$]

$$\frac{dZ}{dx} = \frac{1}{4}[8r^2x - 2x^3 - 2x^3]$$

$$\frac{dZ}{dx} = \frac{1}{4}[8r^2x - 4x^3] = \frac{4x}{4}[2r^2 - x^2]$$

$\frac{dZ}{dx} = 2r^2x - x^3$ ------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 2r^2x - x^3 = 0$$

$$2r^2x = x^3$$

$$x^2 = 2r^2$$

$$x = \pm\sqrt{2r^2}$$

$$x = r\sqrt{2}$$

[as the base of the triangle cannot be negative.]

<u>Now to check if this critical point will determine the maximum area of the triangle, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[2r^2x - x^3]$$

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}(2r^2x) - \frac{d}{dx}(x^3)$$

$\frac{d^2Z}{dx^2} = 2r^2 - 3x^2$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now, consider the value of $\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}}$

$$\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}} = 2r^2 - 3(r\sqrt{2})^2 = 2r^2 - 6r^2 = -4r^2$$

As $\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}} = -4r^2 < 0$, so the function A is maximum at $x = r\sqrt{2}$.

Now substituting $x = r\sqrt{2}$ in equation (1):

$$y = \sqrt{4r^2 - (r\sqrt{2})^2}$$

$$y = \sqrt{4r^2 - 2r^2} = \sqrt{2r^2} = r\sqrt{2}$$

As $x = y = r\sqrt{2}$, the base and height of the triangle are equal, which means that two sides of a right angled triangle are equal,

Hence the given triangle, which is inscribed in a circle, is an isosceles triangle with sides AC and BC equal.

13. Question

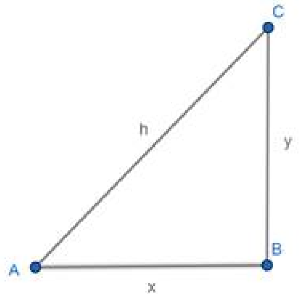
Prove that the perimeter of a right-angled triangle of given hypotenuse is maximum when the triangle is isosceles.

Answer

Given,

• A right angle triangle.

• Hypotenuse of the given triangle is given.



Let us consider,

• ‘h’ is the hypotenuse of the given triangle.

• ‘x’ and ‘y’ be the base and height of the right angle triangle.

• The hypotenuse of the ΔABC = $AC^2 = AB^2 + BC^2$

AC = h, AB = x and BC = y

Hence,

$h^2 = x^2 + y^2$

$y^2 = h^2 - x^2$

$y = \sqrt{h^2 - x^2}$ --- (1)

Now, perimeter of the ΔABC is

P = h + x + y

Now substituting (1) in the area of the triangle,

$P = h + x + \sqrt{h^2 - x^2}$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f’(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dP}{dx} = \frac{d}{dx}\left[h + x + \sqrt{h^2 - x^2}\right]$$

$$\frac{dP}{dx} = \left[\frac{d}{dx}(h) + \frac{d}{dx}(x) + \frac{d}{dx}(\sqrt{h^2 - x^2})\right]$$

$$\frac{dP}{dx} = 0 + 1 + \frac{1}{2}\left(\frac{-2x}{\sqrt{h^2 - x^2}}\right)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dP}{dx} = 1 - \frac{x}{\sqrt{h^2 - x^2}}$ ------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dP}{dx} = 1 - \frac{x}{\sqrt{h^2 - x^2}} = 0$$

$$\frac{x}{\sqrt{h^2 - x^2}} = 1$$

$$x = \sqrt{h^2 - x^2}$$

[squaring on both sides]

$$x^2 = h^2 - x^2$$

$$x^2 = \frac{h^2}{2}$$

$$x = \pm\sqrt{\frac{h^2}{2}}$$

$$x = \frac{h}{\sqrt{2}}$$

[as the base of the triangle cannot be negative.]

Now to check if this critical point will determine the maximum perimeter of the triangle, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left[1 - \frac{x}{\sqrt{h^2 - x^2}}\right]$$

$$\frac{d^2P}{dx^2} = \frac{d}{dx}(1) - \frac{d}{dx}\left(\frac{x}{\sqrt{h^2 - x^2}}\right)$$

$$\frac{d^2P}{dx^2} = 0 - \left[\frac{\sqrt{h^2 - x^2}\,\frac{d}{dx}(x) - x\,\frac{d}{dx}\left(\sqrt{h^2 - x^2}\right)}{\left(\sqrt{h^2 - x^2}\right)^2}\right]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ if u and v are two functions of x, then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$]

$$\frac{d^2P}{dx^2} = -\left[\frac{\sqrt{h^2 - x^2}\,(1) - x\left(\frac{-2x}{2\sqrt{h^2 - x^2}}\right)}{h^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{(\sqrt{h^2 - x^2})^2 + x^2}{h^2 - x^2\,\sqrt{h^2 - x^2}}\right] = -\left[\frac{h^2}{(h^2 - x^2)\,\sqrt{h^2 - x^2}}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{h^2}{(h^2 - x^2)^{\frac{3}{2}}}\right]$$

Now, consider the value of $\left(\frac{d^2P}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}}$

$$\left(\frac{d^2P}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}} = -\left[\frac{h^2}{(h^2 - \left(\frac{h}{\sqrt{2}}\right)^2)^{\frac{3}{2}}}\right] = -\left[\frac{h^2}{(\frac{h^2}{2})^{\frac{3}{2}}}\right] = -\left[\frac{h^2}{(\frac{h^2}{2})^{\frac{3}{2}}}\right] = -\frac{2^{\frac{3}{2}}}{h}$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}} = -\frac{2^{\frac{3}{2}}}{h} < 0$, so the function A is maximum at $x = \frac{h}{\sqrt{2}}$.

Now substituting $x = \frac{h}{\sqrt{2}}$ in equation (1):

$$y = \sqrt{h^2 - \left(\frac{h}{\sqrt{2}}\right)^2}$$

$$y = \sqrt{\frac{h^2}{2}} = \frac{h}{\sqrt{2}}$$

As $x = y = \frac{h}{\sqrt{2}}$, the base and height of the triangle are equal, which means that two sides of a right angled triangle are equal,

Hence the given triangle is an isosceles triangle with sides AB and BC equal.

14. Question

The perimeter of a triangle is 8 cm. If one of the sides of the triangle be 3 cm, what will be the other two sides for maximum area of the triangle?

Answer

Given,

• Perimeter of a triangle is 8 cm.

• One of the sides of the triangle is 3 cm.

• The area of the triangle is maximum.

Let us consider,

• ‘x’ and ‘y’ be the other two sides of the triangle.

Now, perimeter of the ΔABC is

8 = 3 + x + y

y = 8-3-x = 5-x

y = 5-x --- (1)

Consider the Heron’s area of the triangle,

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Where $s = \frac{a+b+c}{2}$

As perimeter = a + b+ c = 8

$$s = \frac{8}{2} = 4$$

Now Area of the triangle is given by

$$A = \sqrt{8\,(8-3)(8-x)(8-y)}$$

Now substituting (1) in the area of the triangle,

$$A = \sqrt{4(4-3)(4-x)(4-(5-x))}$$

$$A = \sqrt{4(4-x)(x-1)}$$

$$A = \sqrt{4(\,4x-4-\,x^2+x)} = \sqrt{4(\,5x-\,x^2-4)}$$

$$A = \sqrt{4(\,5x-\,x^2-4)}$$

[squaring on both sides]

$Z = A^2 = 4(5x - x^2 - 4)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f’(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}\,[4(5x - x^2 - 4)]$$

$$\frac{dZ}{dx} = 4\frac{d}{dx}(5x) - 4\frac{d}{dx}(x^2) - 4\frac{d}{dx}(4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dZ}{dx} = 4(5) - 4(2x) - 0$$

$\frac{dZ}{dx} = 20 - 8x$------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 20 - 8x = 0$$

20 – 8x = 0

8x = 20

$$x = \frac{5}{2}$$

Now to check if this critical point will determine the maximum area of the triangle, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[20 - 8x]$$

$\frac{d^2Z}{dx^2} = -8$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

As $\left(\frac{d^2Z}{dx^2}\right)_{x=\frac{5}{2}} = -8 < 0$, so the function A is maximum at $x = \frac{5}{2}$.

Now substituting $x = \frac{5}{2}$ in equation (1):

y = 5 – 2.5

y = 2.5

As x = y = 2.5, two sides of the triangle are equal,

Hence the given triangle is an isosceles triangle with two sides equal to 2.5 cm and the third side equal to 3cm.

15. Question

A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 metres. Find the dimensions of the windows to admit maximum light through it.

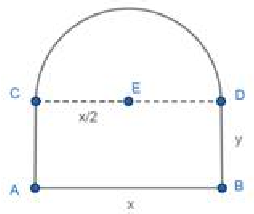
Answer

Given,

• Window is in the form of a rectangle which has a semicircle mounted on it.

• Total Perimeter of the window is 10 metres.

• The total area of the window is maximum.



Let us consider,

• The breadth and height of the rectangle be ‘x’ and ‘y’.

• The radius of the semicircle will be half of the base of the rectangle.

Given Perimeter of the window is 10 meters:

$$10 = (x + 2y) + \frac{1}{2}\left[2\pi\left(\frac{x}{2}\right)\right]$$

[as the perimeter of the window will be equal to one side (x) less to the perimeter of rectangle and the perimeter of the semicircle.]

$$10 = (x + 2y) + \left(\frac{\pi x}{2}\right)$$

From here,

$$2y = 10 - x - \left(\frac{\pi x}{2}\right) = \frac{20 - 2x - \pi x}{2}$$

$y = \frac{20-2x-\pi x}{4}$ ----- (1)

Now consider the area of the window,

Area of the window = area of the semicircle + area of the rectangle

$$A = \frac{1}{2}\left[\pi\left(\frac{x}{2}\right)^2\right] + x\,y$$

Substituting (1) in the area equation:

$$A = \frac{1}{2}\left[\pi\left(\frac{x}{2}\right)^2\right] + x\left(\frac{20 - 2x - \pi x}{4}\right)$$

$$A = \frac{1}{8}\left[\pi x^2\right] + \left(\frac{20x - 2x^2 - \pi x^2}{4}\right)$$

$$A = \frac{\pi x^2 - 2\pi x^2 + 40x - 4x^2}{8}$$

$A = \frac{1}{8}\left[x^2(\pi - 2\pi - 4) + 40x\right]$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f’(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{1}{8}\left[x^2(\pi - 2\pi - 4) + 40x\right]\right]$$

$$\frac{dA}{dx} = \frac{1}{8}\frac{d}{dx}\left(x^2(\pi - 2\pi - 4)\right) + \frac{1}{8}\frac{d}{dx}(40x)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dA}{dx} = \frac{1}{8}\left[2x(-\pi - 4)\right] + \frac{1}{8}(40)$$

$\frac{dA}{dx} = \frac{1}{4}[x(-\pi - 4)] + 5$ ------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dA}{dx} = \frac{1}{4}[x(-\pi - 4)] + 5 = 0$$

$$\frac{1}{4}[x(-\pi - 4)] + 5 = 0$$

$$\frac{1}{4}[x(4 + \pi)] = 5$$

x (4 +π) = 20

$$x = \frac{20}{(4 + \pi)}$$

<u>Now to check if this critical point will determine the maximum area of the window, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left[\frac{1}{4}[x(-\pi - 4)] + 5\right]$$

$$\frac{d^2A}{dx^2} = \frac{d}{dx}[x(-\pi - 4)] + \frac{d}{dx}(5)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{d^2A}{dx^2} = (-\pi - 4)(1) + 0 = -(\pi + 4)$ ------ (4)

As $\left(\frac{d^2A}{dx^2}\right)_{x=\frac{20}{(4+\pi)}} = -(\pi + 4) < 0$, so the function A is maximum at $x = \frac{20}{(4+\pi)}$.

Now substituting $x = \frac{20}{(4+\pi)}$ in equation (1):

$$y = \frac{20 - \left(\frac{20}{(4 + \pi)}\right)(\pi + 2)}{4}$$

$$y = \frac{20(4 + \pi) - (20)(\pi + 2)}{4(4 + \pi)} = \frac{20[4 + \pi - \pi - 2]}{4(4 + \pi)} = \frac{20 \times 2}{4(4 + \pi)}$$

$$y = \frac{5 \times 2}{(4 + \pi)} = \frac{10}{(4 + \pi)}$$

Hence the given window with maximum area has breadth, $x = \frac{20}{(4+\pi)}$ and height, $y = \frac{10}{(4+\pi)}$.

16. Question

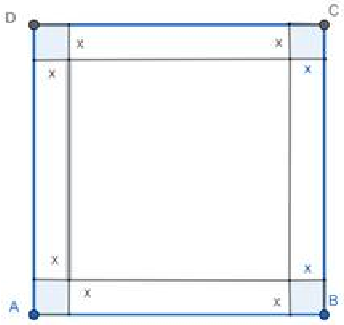
A square piece of tin of side 12 cm is to be made into a box without a lid by cutting a square from each corner and folding up the flaps to form the sides. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find this maximum volume.

Answer

Given,

• Side of the square piece is 12 cms.

• the volume of the formed box is maximum.



Let us consider,

- 'x' be the length and breadth of the piece cut from each vertex of the piece.
- Side of the box now will be (12-2x)
- The height of the new formed box will also be 'x'.

Let the volume of the newly formed box is :

$V = (12\text{-}2x)^2 \times (x)$

$V = (144 + 4x^2 - 48x)\ x$

$V = 4x^3 - 48x^2 + 144x$ ------ (1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (1) with respect to x:

$$\frac{dV}{dx} = \frac{d}{dx}[4x^3 - 48x^2 + 144x]$$

$\frac{dV}{dx} = 12x^2 - 96x + 144$ -------- (2)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dV}{dx} = 12x^2 - 96x + 144 = 0$$

$x^2 - 8x + 12 = 0$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(12)}}{2(1)} = \frac{8 \pm \sqrt{64 - 48}}{2} = \frac{8 \pm \sqrt{16}}{2}$$

$$x = = \frac{8 \pm 4}{2}$$

x = 6 or x =2

x= 2

[as x = 6 is not a possibility, because 12-2x = 12-12= 0]

Now to check if this critical point will determine the maximum area of the box, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2V}{dx^2} = \frac{d}{dx}[12x^2 - 96x + 144]$$

$\frac{d^2V}{dx^2} = 24x - 96$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dx^2}\right)_{x=2} = 24\,(2) - 96 = 48 - 96 = -48$$

As $\left(\frac{d^2V}{dx^2}\right)_{x=2} = -48 < 0$, so the function A is maximum at x = 2

Now substituting x = 2 in 12 – 2x, the side of the considered box:

Side = 12-2x = 12 - 2(2) = 12-4= 8cms

Therefore side of wanted box is 8cms and height of the box is 2cms.

Now, the volume of the box is

$V = (8)^2 \times 2 = 64 \times 2 = 128cm^3$

Hence maximum volume of the box formed by cutting the given 12cms sheet is $128cm^3$ with 8cms side and 2cms height.

17. Question

An open box with a square base is to be made out of a given cardboard of area c^2 (square) units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ (cubic) units.

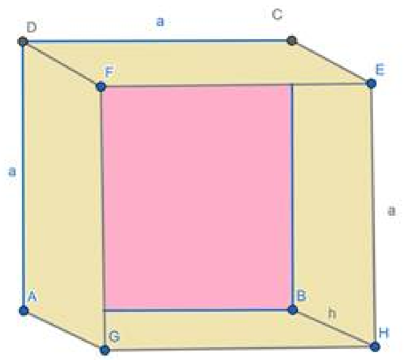
Answer

Given,

• The open box has a square base

• The area of the box is c^2 square units.

• The volume of the box is maximum.



Let us consider,

• The side of the square base of the box be ‘a’ units. (pink coloured in the figure)

• The breadth of the 4 sides of the box will also be ‘a’units (skin coloured part).

• The depth of the box or the length of the sides be ‘h’ units (skin coloured part).

Now, the area of the box =

(area of the base) + 4 (area of each side of the box)

So as area of the box is given c^2,

$c^2 = a^2 + 4ah$

$h = \frac{c^2 - a^2}{4a}$ ---- (1)

Let the volume of the newly formed box is :

V = $(a)^2 \times (h)$

[substituting (1) in the volume formula]

$$V = a^2 \times \left(\frac{c^2 - a^2}{4a}\right)$$

$V = \left(\frac{ac^2 - a^3}{4}\right)$ ------- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with a and then equating it to zero. This is because if the function f(a) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to a:

$$\frac{dV}{da} = \frac{d}{da}\left[\left(\frac{ac^2 - a^3}{4}\right)\right]$$

$\frac{dV}{da} = \frac{c^2}{4} - \frac{3a^2}{4}$ ------- (3)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dV}{da} = \frac{c^2}{4} - \frac{3a^2}{4} = 0$$

$c^2 - 3a^2 = 0$

$$a^2 = \frac{c^2}{3}$$

$$a = \pm\sqrt{\frac{c^2}{3}}$$

$$a = \frac{c}{\sqrt{3}}$$

[as 'a' cannot be negative]

Now to check if this critical point will determine the maximum Volume of the box, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2V}{da^2} = \frac{d}{dx}\left[\frac{c^2}{4} - \frac{3a^2}{4}\right]$$

$\frac{d^2V}{da^2} = 0 - \frac{3 \times 2 \times a}{4} = -\frac{3a}{2}$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{da^2}\right)_{a=\frac{c}{\sqrt{3}}} = -\frac{3\left(\frac{c}{\sqrt{3}}\right)}{2} = -\frac{c\sqrt{3}}{2}$$

As $\left(\frac{d^2V}{da^2}\right)_{a=\frac{c}{\sqrt{3}}} = -48 - \frac{c\sqrt{3}}{2} < 0$, so the function V is maximum at $a = \frac{c}{\sqrt{3}}$

Now substituting a in equation (1)

$$h = \frac{c^2 - \left(\frac{c}{\sqrt{3}}\right)^2}{4\left(\frac{c}{\sqrt{3}}\right)} = \frac{\frac{2c^2}{3}}{\frac{4c}{\sqrt{3}}} = \frac{c\sqrt{3}}{6} = \frac{c}{2\sqrt{3}}$$

$$\therefore\ h = \frac{c}{2\sqrt{3}}$$

Therefore side of wanted box has a base side, $a = \frac{c}{\sqrt{3}}$ is and height of the box, $h = \frac{c}{2\sqrt{3}}$.

Now, the volume of the box is

$$V = \left(\frac{c}{\sqrt{3}}\right)^2 \times \left(\frac{c}{2\sqrt{3}}\right)$$

$$V = \frac{c^2}{3} \times \left(\frac{c}{2\sqrt{3}}\right) = \frac{c^3}{6\sqrt{3}}$$

$$\therefore V = \frac{c^3}{6\sqrt{3}}$$

18. Question

A cylindrical can is to be made to hold 1 litre of oil. Find the dimensions which will minimize the cost of the metal to make the can.

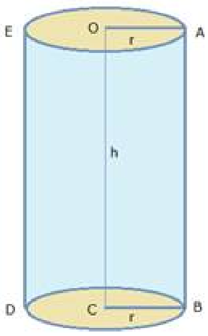
Answer

Given,

• The can is cylindrical with a circular base

• The volume of the cylinder is 1 litre = 1000 cm^2.

• The surface area of the box is minimum as we need to find the minimum dimensions.



Let us consider,

• The radius base and top of the cylinder be ‘r’ units. (skin coloured in the figure)

• The height of the cylinder be ‘h’units.

• As the Volume of cylinder is given, V = $1000cm^3$

The Volume of the cylinder= $\pi r^2 h$

1000 = $\pi r^2 h$

$h = \frac{1000}{\pi r^2}$ ---- (1)

The Surface area cylinder is = area of the circular base + area of the circular top + area of the cylinder

$S = \pi r^2 + \pi r^2 + 2\pi rh$

$S = 2\ \pi r^2 + 2\pi rh$

[substituting (1) in the volume formula]

$$S = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$S = 2\left[\pi r^2 + \left(\frac{1000}{r}\right)\right]$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. This is because if the function f(r) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to r:

$$\frac{dS}{dr} = \frac{d}{dr}\left[2\left[\pi r^2 + \left(\frac{1000}{r}\right)\right]\right]$$

$$\frac{dS}{dr} = 2\,(2\pi r) + \left(\frac{1000}{r^2}\right)(-1)$$

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$ and $\frac{d}{dx}\,(x^{-n}) = -nx^{-n-1}$]

$\frac{dS}{dr} = 2\,(2\pi r) - 2\left(\frac{1000}{r^2}\right)$ ------- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dr} = 2\,(2\pi r) - 2\left(\frac{1000}{r^2}\right) = 0$$

$$2\,(2\pi r) - 2\left(\frac{1000}{r^2}\right) = 0$$

$$2\pi r = \frac{1000}{r^2}$$

$$r^3 = \frac{500}{\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}}$$

Now to check if this critical point will determine the minimum surface area of the box, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with r:

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[2\,(2\pi r) - 2\left(\frac{1000}{r^2}\right)\right]$$

$\frac{d^2S}{dr^2} = 4\pi - \frac{2\times 1000\times(-2)}{r^3} = 4\pi + \frac{4000}{r^3}$ ----- (4)

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$ and $\frac{d}{dx}\,(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{500}{\pi}}} = 4\pi + \frac{4000}{\left(\sqrt[3]{\frac{500}{\pi}}\right)^3} = 4\pi + \frac{4000\ \times\ \pi}{500} = 4\pi + 8\pi = 12\pi$$

As $\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{500}{\pi}}} = 12\pi > 0$, so the function S is minimum at $r = \sqrt[3]{\frac{500}{\pi}}$

Now substituting r in equation (1)

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\pi^{\frac{1}{3}}\,(500)^{\frac{2}{3}}}$$

$$\therefore\ h = \frac{1000}{\pi^{\frac{1}{3}}\,(500)^{\frac{2}{3}}}$$

Therefore the radius of base of the cylinder, $r = \sqrt[3]{\frac{500}{\pi}}$ and height of the cylinder, $h = \frac{1000}{\pi^{\frac{1}{3}}\,(500)^{\frac{2}{3}}}$ where the surface area of the cylinder is minimum.

19. Question

Show that the right circular cone of the least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

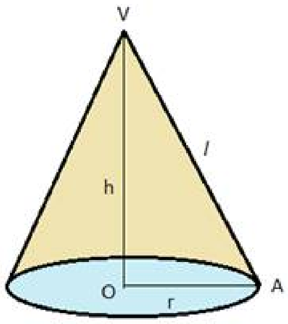
Answer

Given,

• The volume of the cone.

• The cone is right circular cone.

• The cone has least curved surface.



Let us consider,

• The radius of the circular base be ‘r’ cms.

• The height of the cone be ‘h’ cms.

• The slope of the cone be ‘l’ cms.

Given the Volume of the cone = $\pi r^2 l$

$$V = \frac{\pi r^2 h}{3}$$

$h = \frac{3v}{\pi r^2}$ ----- (1)

The Surface area cylinder is = πrl

S = πrl

$$S = \pi r\left(\sqrt{h^2 + r^2}\right)$$

[substituting (1) in the Surface area formula]

$$S = \pi r \left[\sqrt{\left(\frac{3V}{\pi r^2}\right)^2 + r^2}\right]$$

[squaring on both sides]

$$Z = S^2 = \pi^2 r^2 \left(\frac{9V^2}{\pi^2 r^4} + r^2\right)$$

$$Z = \pi^2 \left(\frac{9V^2}{\pi^2 r^2} + r^4\right) \text{----- (2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. This is because if the function Z has a maximum/minimum at a point c then Z'(c) = 0.

Differentiating the equation (2) with respect to r:

$$\frac{dZ}{dr} = \frac{d}{dr}\left[\pi^2 \left(\frac{9V^2}{\pi^2 r^2} + r^4\right)\right]$$

$$\frac{dZ}{dr} = \pi^2 \left(\frac{9V^2}{\pi^2}\right) \frac{d}{dr}\left(\frac{1}{r^2}\right) + \pi^2 \frac{d}{dr}(r^4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{dZ}{dr} = \left(\frac{-18V^2}{r^3}\right) + \pi^2 (4 r^3) \text{------- (3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dr} = \left(\frac{-18V^2}{r^3}\right) + \pi^2 (4 r^3) = 0$$

$$\pi^2 (4 r^3) = \frac{18V^2}{r^3}$$

$$2\pi^2 r^6 = 9V^2 \text{ ---- (4)}$$

Now to check if this critical point will determine the minimum surface area of the cone, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with r:

$$\frac{d^2Z}{dr^2} = \frac{d}{dr}\left[\left(\frac{-18V^2}{r^3}\right) + \pi^2 (4 r^3)\right]$$

$$\frac{d^2Z}{dr^2} = \frac{-18V^2(-3)}{r^4} + \pi^2 (4 \times 3 r^2)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{d^2Z}{dr^2} = \frac{54V^2}{r^4} + \pi^2 (12 r^2)$$

Now let us find the value of

$$\left(\frac{d^2Z}{dr^2}\right) = \frac{54V^2}{r^4} + \pi^2 (12 r^2) > 0$$

As $\left(\frac{d^2Z}{dr^2}\right) > 0$, so the function Z = S^2 is minimum

Now consider, the equation (4),

$$9V^2 = 2\pi^2 r^6$$

Now substitute the volume of the cone formula in the above equation.

$$9\left(\frac{\pi r^2 h}{3}\right)^2 = 2\pi^2 r^6$$

$\pi^2 r^4 h^2 = 2\ \pi^2 r^6$

$2r^2 = h^2$

$$h = r\sqrt{2}$$

Hence, the relation between h and r of the cone is proved when S is the minimum.

20. Question

Find the radius of a closed right circular cylinder of volume 100cm^3 which has the minimum total surface area.

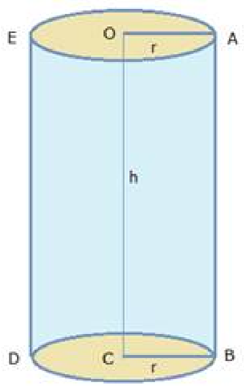
Answer

Given,

• The closed is cylindrical can with a circular base and top.

• The volume of the cylinder is 1 litre = 100 cm^3.

• The surface area of the box is minimum.



Let us consider,

• The radius base and top of the cylinder be ‘r’ units. (skin coloured in the figure)

• The height of the cylinder be ‘h’units.

• As the Volume of cylinder is given, V = $100cm^3$

The Volume of the cylinder= $\pi r^2 h$

$100 = \pi r^2 h$

$h = \frac{100}{\pi r^2}$ ---- (1)

The Surface area cylinder is = area of the circular base + area of the circular top + area of the cylinder

$S = \pi r^2 + \pi r^2 + 2\pi rh$

$S = 2\ \pi r^2 + 2\pi rh$

[substituting (1) in the volume formula]

$$S = 2\pi r^2 + 2\pi r\left(\frac{100}{\pi r^2}\right)$$

$S = 2\left[\pi r^2 + \left(\frac{100}{r}\right)\right]$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. This is because if the function f(r) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to r:

$$\frac{dS}{dr} = \frac{d}{dr}\left[2\left[\pi r^2 + \left(\frac{100}{r}\right)\right]\right]$$

$$\frac{dS}{dr} = 2\,(2\pi r) + \left(\frac{100}{r^2}\right)(-1)$$

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$ and $\frac{d}{dx}\,(x^{-n}) = -nx^{-n-1}$]

$\frac{dS}{dr} = 2\,(2\pi r) - 2\left(\frac{100}{r^2}\right)$ ------- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dr} = 2\,(2\pi r) - 2\left(\frac{100}{r^2}\right) = 0$$

$$2\,(2\pi r) - 2\left(\frac{100}{r^2}\right) = 0$$

$2\pi r = \frac{100}{r^2}$ ---- (4)

Now to check if this critical point will determine the minimum surface area of the box, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with r:

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[2\,(2\pi r) - 2\left(\frac{100}{r^2}\right)\right]$$

$\frac{d^2S}{dr^2} = 4\pi - \frac{2\times 100\times(-2)}{r^3} = 4\pi + \frac{400}{r^3}$ ----- (5)

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$ and $\frac{d}{dx}\,(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{50}{\pi}}} = 4\pi + \frac{400}{\left(\sqrt[3]{\frac{50}{\pi}}\right)^3} = 4\pi + \frac{400\times\pi}{50} = 4\pi + 8\pi = 12\pi$$

As $\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{50}{\pi}}} = 12\pi > 0$, so the function S is minimum at $r = \sqrt[3]{\frac{50}{\pi}}$

As S is minimum from equation (4)

$$2\pi r = \frac{100}{r^2}$$

$$2\pi r = \frac{V}{r^2}$$

$V = 2\pi r^3$

Now in equation (1) we have,

$$h = \frac{V}{\pi r^2} = \frac{2\pi r^3}{\pi r^2}$$

h = 2r = diameter

Therefore when the total surface area of a cone is minimum, then height of the cone is equal to twice the radius or equal to its diameter.

21. Question

Show that the height of a closed cylinder of given volume and the least surface area is equal to its diameter.

Answer

Let r be the radius of the base and h the height of a cylinder.

The surface area is given by,

$S = 2\pi r^2 + 2\pi rh$

$h = \frac{S - 2\pi r^2}{2\pi r}$.........(1)

Let V be the volume of the cylinder.

Therefore, $V = \pi r^2 h$

$V = \pi r^2 \left(\frac{S - 2\pi r^3}{2\pi r}\right)$.......Using equation 1

$V = \frac{Sr - 2\pi r^3}{2}$

Differentiating both sides w.r.t r, we get,

$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2$..........(2)

For maximum or minimum, we have,

$\frac{dV}{dr} = 0$

$\Rightarrow \frac{S}{2} - 3\pi r^2 = 0$

$\Rightarrow S = 6\pi r^2$

$2\pi r^2 + 2\pi rh = 6\pi r^2$

$h = 2r$

Differentiating equation 2, with respect to r to check for maxima and minima, we get,

$\frac{d^2V}{dr^2} = -6\pi r < 0$

Hence, V is maximum when h = 2r or h = diameter

22. Question

Prove that the volume of the largest cone that can be inscribed in a sphere is $\frac{8}{27}$ of the volume of the sphere.

Answer

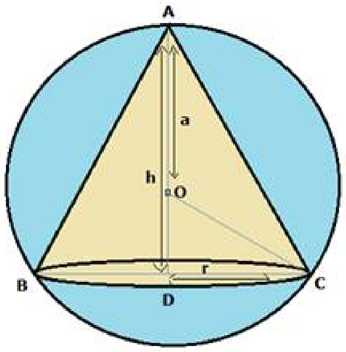
Given,

• Volume of the sphere.

• Volume of the cone.

• Cone is inscribed in the sphere.

• Volume of cone is maximum.



Let us consider,

• The radius of the sphere be ‘a’ units.

• Volume of the inscribed cone be ‘V’.

• Height of the inscribed cone be ‘h’.

• Radius of the base of the cone is ‘r’.

Given volume of the inscribed cone is,

$$V = \frac{\pi r^2 h}{3}$$

Consider OD = (AD-OA) =(h-a)

Now let $OC^2 = OD^2 + DC^2$, here OC = a, OD = (h-a), DC = r,

So $a^2 = (h\text{-}a)^2 + r^2$

$r^2 = a^2 – (h^2 + a^2 – 2ah)$

$r^2 = h\ (2a\ \text{-}h)$ ----- (1)

Let us consider the volume of the cone:

$$V = \frac{1}{3}\ (\pi r^2 h)$$

Now substituting (1) in the volume formula,

$$V = \frac{1}{3}\ (\pi h(2a - h)h)$$

$V = \frac{1}{3}\ (2\pi h^2 a - \ \pi h^3)$ ---- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with h and then equating it to zero. This is because if the function V(r) has a maximum/minimum at a point c then V’(c) = 0.

Differentiating the equation (2) with respect to h:

$$\frac{dV}{dh} = \frac{d}{dh}\left[\frac{1}{3}\ (2\pi h^2 a - \ \pi h^3)\right]$$

$$\frac{dV}{dh} = \frac{1}{3}\ (2\pi a)(2h) - \frac{1}{3}(\pi)(3h^2)$$

[Since $\frac{d}{dx}\ (x^n) = nx^{n-1}$]

$\frac{dV}{dh} = \frac{1}{3}[\ 4\pi ah - \ 3\pi h^2]$ ------- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dV}{dh} = \frac{1}{3}[\ 4\pi ah - \ 3\pi h^2]\ =\ 0$$

$4\pi ah - 3\pi h^2 = 0$

$h(4\pi a - 3\pi h) = 0$

h = 0 (or) $h = \frac{4\pi a}{3\pi} = \frac{4a}{3}$

$$h = \frac{4a}{3}$$

[as h cannot be zero]

Now to check if this critical point will determine the maximum volume of the inscribed cone, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with h:

$$\frac{d^2V}{dh^2} = \frac{d}{dh}\left[\frac{1}{3}[4\pi ah - 3\pi h^2]\right]$$

$\frac{d^2V}{dh^2} = \frac{1}{3}[4\pi a - (3\pi)(2h)] = \frac{\pi}{3}[4a - 6h]$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dh^2}\right)_{h=\frac{4a}{3}} = \frac{\pi}{3}\left[4a - 6\left(\frac{4a}{3}\right)\right] = \frac{4a\pi}{3}[1-2] = -\frac{4a\pi}{3}$$

As $\left(\frac{d^2V}{dh^2}\right)_{h=\frac{4a}{3}} = -\frac{4a\pi}{3} < 0$, so the function V is maximum at $h = \frac{4a}{3}$

Substituting h in equation (1)

$$r^2 = \left(\frac{4a}{3}\right)\left(2a - \frac{4a}{3}\right)$$

$$r^2 = \left(\frac{4a}{3}\right)\left(2a - \frac{4a}{3}\right)$$

$$r^2 = \frac{8a^2}{9}$$

As V is maximum, substituting h and r in the volume formula:

$$V = \frac{1}{3}\pi\left(\frac{8a^2}{9}\right)\left(\frac{4a}{3}\right)$$

$$V = \frac{8}{27}\left(\frac{4}{3}\pi a^3\right)$$

$$V = \frac{8}{27}(\text{volume of the sphere})$$

Therefore when the volume of a inscribed cone is maximum, then it is equal to $\frac{8}{27}$ times of the volume of the sphere in which it is inscribed.

23. Question

Which fraction exceeds its pth power by the greatest number possible?

Answer

Given,

The pth power of a number exceeds by a fraction to be the greatest.

Let us consider,

• ‘x’ be the required fraction.

• The greatest number will be y = x - x^p ------ (1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function y(x)has a maximum/minimum at a point c then y’(c) = 0.

Differentiating the equation (1) with respect to x:

$$\frac{dy}{dx} = \frac{d}{dx}(x - x^p)$$

$\frac{dy}{dx} = 1 - px^{p-1}$ ---- (2)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dy}{dx} = 1 - px^{p-1} = 0$$

1 = px^{p-1}

$$x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$$

Now to check if this critical point will determine the if the number is the greatest, we need to check with second differential which needs to be negative.

Consider differentiating the equation (2) with x:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[1 - px^{p-1}]$$

$\frac{d^2y}{dx^2} = -p(p-1)x^{p-2}$ ----- (3)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2y}{dx^2}\right)_{x=\left(\frac{1}{p}\right)^{\frac{1}{p-1}}} = -p(p-1)\left(\left(\frac{1}{p}\right)^{\frac{1}{p-1}}\right)^{p-2}$$

As $\left(\frac{d^2y}{dx^2}\right)_{x=\left(\frac{1}{p}\right)^{\frac{1}{p-1}}} = -p(p-1)\left(\left(\frac{1}{p}\right)^{\frac{1}{p-1}}\right)^{p-2} < 0$, so the number y is greatest at $x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$

Hence, the y is the greatest number and exceeds by a fraction $x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$

24. Question

Find the point on the curve $y^2 = 4x$ which is nearest to the point (2, −8).

Answer

Given,

• A point is present on a curve y^2 = 4x

• The point is near to the point (2,-8)

Let us consider,

• The co-ordinates of the point be P(x,y)

• As the point P is on the curve, then $y^2 = 4x$

$$x = \frac{y^2}{4}$$

• The distance between the points is given by,

$D^2 = (x-2)^2 + (y+8)^2$

$D^2 = x^2 - 4x + 4 + y^2 + 64 + 16y$

Substituting x in the distance equation

$$D^2 = \left(\frac{y^2}{4}\right)^2 - 4\left(\frac{y^2}{4}\right) + y^2 + 16y + 68$$

$Z = D^2 = \frac{y^4}{16} + 16y + 68$ ---- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with y and then equating it to zero. <u>This is because if the function Z(x) has a maximum/minimum at a point c then Z'(c) = 0.</u>

Differentiating the equation (2) with respect to y:

$$\frac{dZ}{dy} = \frac{d}{dy}\left(\frac{y^4}{16} + 16y + 68\right)$$

$\frac{dZ}{dy} = \frac{4y^3}{16} + 16 = \frac{y^3}{4} + 16$ ---- (2)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dZ}{dy} = \frac{y^3}{4} + 16 = 0$$

$y^3 + 64 = 0$

$(y + 4)(y^2 - 4y + 16) = 0$

$(y+4) = 0$ (or) $y^2 - 4y + 16 = 0$

$y = -4$

(as the roots of the $y^2 - 4y + 16$ are imaginary)

<u>Now to check if this critical point will determine the distance is mimimum, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (2) with y:

$$\frac{d^2Z}{dy^2} = \frac{d}{dy}\left[\frac{y^3}{4} + 16\right]$$

$\frac{d^2Z}{dy^2} = \frac{3y^2}{4}$ ----- (3)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2Z}{dy^2}\right)_{y=-4} = \frac{3}{4}(-4)^2 = 12$$

As $\left(\frac{d^2Z}{dy^2}\right)_{y=-4} = 12 > 0$, so the Distance D^2 is minimum at $y = -4$

Now substituting y in x, we have

$$x = \frac{(-4)^2}{4} = 4$$

So, the point P on the curve $y^2 = 4x$ is (4,-4) which is at nearest from the (2,-8)

25. Question

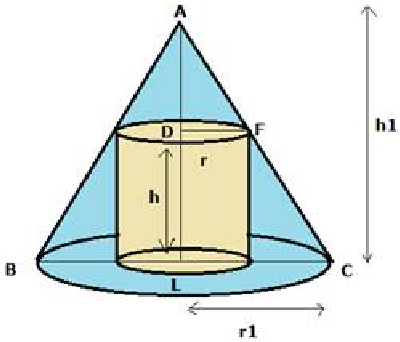
A right circular cylinder is inscribed in a cone. Show that the curved surface area of the cylinder is maximum when the diameter of the cylinder is equal to the radius of the base of the cone.

Answer

Given,

• A right circular cylinder is inscribed inside a cone.

• The curved surface area is maximum.



Let us consider,

• 'r_1' be the radius of the cone.

• 'h_1' be the height of the cone.

• 'r' be the radius of the inscribed cylinder.

• 'h' be the height of the inscribed cylinder.

DF = r, and AD = AL – DL = h_1 – h

Now, here ΔADF and ΔALC are similar,

Then

$$\frac{AD}{AL} = \frac{DF}{LC} \Rightarrow \frac{h_1 - h}{h_1} = \frac{r}{r_1}$$

$$h_1 - h = \frac{rh_1}{r_1}$$

$$h = h_1 - \frac{rh_1}{r_1} = h_1\left(1 - \frac{r}{r_1}\right)$$

$h = h_1\left(1 - \frac{r}{r_1}\right)$ ----- (1)

Now let us consider the curved surface area of the cylinder,

S = 2πrh

Substituting h in the formula,

$$S = 2\pi r\left[h_1\left(1 - \frac{r}{r_1}\right)\right]$$

$S = 2\pi rh_1 - \frac{2\pi h_1 r^2}{r_1}$ ---- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. This is because if the function S(r) has a maximum/minimum at a point c then S'(c) = 0.

Differentiating the equation (2) with respect to r:

$$\frac{dS}{dr} = \frac{d}{dr}\left[2\pi rh_1 - \frac{2\pi h_1 r^2}{r_1}\right]$$

$$\frac{dS}{dr} = 2\pi h_1 - \frac{2\pi h_1 (2r)}{r_1}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dS}{dr} = 2\pi h_1 - \frac{4\pi h_1 r}{r_1}$ ------- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dr} = 2\pi h_1 - \frac{4\pi h_1 r}{r_1} = 0$$

$$\frac{4\pi h_1 r}{r_1} = 2\pi h_1$$

$$r = \frac{2\pi h_1 r_1}{4\pi h_1}$$

$$r = \frac{r_1}{2}$$

Now to check if this critical point will determine the maximum volume of the inscribed cylinder, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with r:

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[2\pi h_1 - \frac{4\pi h_1 r}{r_1}\right]$$

$\frac{d^2S}{dr^2} = 0 - \frac{4\pi h_1}{r_1} = -\frac{4\pi h_1}{r_1}$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\frac{d^2S}{dr^2}_{r=\frac{r_1}{2}} = -\frac{4\pi h_1}{r_1}$$

As $\frac{d^2S}{dr^2}_{r=\frac{r_1}{2}} = -\frac{4\pi h_1}{r_1} < 0$, so the function S is maximum at $r = \frac{r_1}{2}$

Substituting r in equation (1)

$$h = h_1\left(1 - \frac{\frac{r_1}{2}}{r_1}\right)$$

$h = h_1\left(1 - \frac{1}{2}\right) = \frac{h_1}{2}$ --- (5)

As S is maximum, from (5) we can clearly say that $h_1 = 2h$ and

$r_1 = 2r$

this means the radius of the cone is twice the radius of the cylinder or equal to diameter of the cylinder.

26. Question

Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube.

Answer

Given,

• Closed cuboid has square base.

• The volume of the cuboid is given.

• Surface area is minimum.

Let us consider,

• The side of the square base be 'x'.

• The height of the cuboid be 'h'.

• The given volume, $V = x^2h$

$h = \frac{V}{x^2}$ ----- (1)

Consider the surface area of the cuboid,

Surface Area =

2(Area of the square base) + 4(areas of the rectangular sides)

$S = 2x^2 + 4xh$

Now substitute (1) in the Surface Area formula

$$S = 2x^2 + 4x\left(\frac{V}{x^2}\right)$$

$S = 2x^2 + \left(\frac{4V}{x}\right)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function S(x) has a maximum/minimum at a point c then S'(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$$\frac{dS}{dx} = \frac{d}{dx}\left[2x^2 + \left(\frac{4V}{x}\right)\right]$$

$$\frac{dS}{dx} = 2(2x) + 4V\left(\frac{-1}{x^2}\right)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$\frac{dS}{dx} = 4x - \frac{4V}{x^2}$ ------- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dx} = 4x - \frac{4V}{x^2} = 0$$

$$4x = \frac{4V}{x^2}$$

$x^3 = V$

<u>Now to check if this critical point will determine the minimum surface area, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\left[4x - \frac{4V}{x^2}\right]$$

$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3}$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\frac{d^2S}{dx^2}_{x=V^{\frac{1}{3}}} = 4 + \frac{8V}{V} = 12$$

As $\frac{d^2S}{dx^2}_{x=V^{\frac{1}{3}}} = 12 > 0$, so the function S is minimum at $x = \sqrt[3]{V}$

Substituting x in equation (1)

$$h = \frac{V}{x^2} = \frac{x^3}{x^2} = x$$

h = x

As S is minimum and h = x, this means that the cuboid is a cube.

27. Question

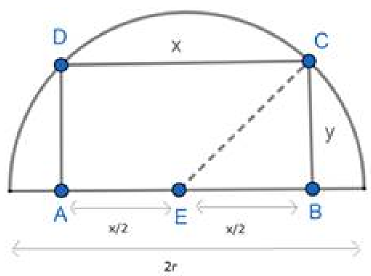
A rectangle is inscribed in a semicircle of radius r with one of its sides on the diameter of the semicircle. Find the dimensions of the rectangle so that its area is maximum. Find also this area.

Answer

Given,

• Radius of the semicircle is ‘r’.

• Area of the rectangle is maximum.



Let us consider,

• The base of the rectangle be ‘x’ and the height be ‘y’.

Consider the ΔCEB,

$CE^2 = EB^2 + BC^2$

As CE = r, $EB = \frac{x}{2}$ and CB = y

$$r^2 = \left(\frac{x}{2}\right)^2 + y^2$$

$y^2 = r^2 - \left(\frac{x}{2}\right)^2$ ---- (1)

Now the area of the rectangle is

A = x × y

Squaring on both sides

$A^2 = x^2 y^2$

Substituting (1) in the above Area equation

$$A^2 = x^2 \left[r^2 - \left(\frac{x}{2}\right)^2\right]$$

$Z = A^2 = x^2 r^2 - x^2 \frac{x^2}{4} = x^2 r^2 - \frac{x^4}{4}$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function Z(x) has a maximum/minimum at a point c then Z'(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}\left[x^2 r^2 - \frac{x^4}{4}\right]$$

$$\frac{dZ}{dx} = r^2\,(2x) - \frac{4x^3}{4}$$

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$]

$\frac{dZ}{dx} = 2xr^2 - x^3$ ------- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 2xr^2 - x^3 = 0$$

$x(2r^2 – x^2) = 0$

$x = 0$ (or) $x^2 = 2r^2$

$x = 0$ (or) $x = r\sqrt{2}$

$x = r\sqrt{2}$

[as x cannot be zero]

Now to check if this critical point will determine the maximum area, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[2xr^2 - x^3]$$

$\frac{d^2Z}{dx^2} = 2r^2 - 3x^2$ ----- (4)

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$]

Now let us find the value of

$$\frac{d^2Z}{dx^2}_{x=r\sqrt{2}} = 2r^2 - 3(r\sqrt{2})^2 = 2r^2 - 6r^2 = -4r^2$$

As $\frac{d^2Z}{dx^2}_{x=r\sqrt{2}} = -4r^2 < 0$, so the function Z is maximum at $x = r\sqrt{2}$

Substituting x in equation (1)

$$y^2 = r^2 - \left(\frac{r\sqrt{2}}{2}\right)^2 = r^2 - \frac{r^2}{2} = \frac{r^2}{2}$$

$$y = \sqrt{\frac{r^2}{2}} = \frac{r}{\sqrt{2}} = \frac{r\sqrt{2}}{2}$$

As the area of the rectangle is maximum, and $x = r\sqrt{2}$ and $y = \frac{r\sqrt{2}}{2}$

So area of the rectangle is

$$A = r\sqrt{2} \times \frac{r\sqrt{2}}{2}$$

$A = r^2$

Hence the maximum area of the rectangle inscribed inside a semicircle is r^2 square units.

28. Question

Two sides of a triangle have lengths a and b and the angle between them is θ. What value of θ will maximize the area of the triangle?

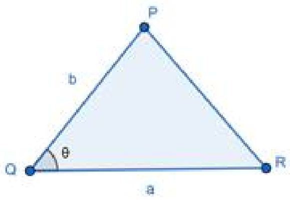
Answer

Given,

• The length two sides of a triangle are 'a' and 'b'

• Angle between the sides 'a' and 'b' is θ.

• The area of the triangle is maximum.



Let us consider,

The area of the ΔPQR is given be

$A = \frac{1}{2}$ ab sinθ ---- (1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with θ and then equating it to zero. <u>This is because if the function A (θ) has a maximum/minimum at a point c then A'(c) = 0.</u>

Differentiating the equation (1) with respect to θ:

$$\frac{dA}{d\theta} = \frac{d}{d\theta}\left[\frac{1}{2} ab \sin\theta\right]$$

$\frac{dA}{d\theta} = \frac{1}{2} ab\cos\theta$ ---- (2)

[Since $\frac{d}{dx}(\sin\theta) = \cos\theta$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dA}{d\theta} = \frac{1}{2} ab\cos\theta = 0$$

Cos θ = 0

$$\theta = \frac{\pi}{2}$$

<u>Now to check if this critical point will determine the maximum area, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (2) with θ :

$$\frac{d^2A}{d\theta^2} = \frac{d}{d\theta}\left[\frac{1}{2}\,ab\cos\theta\right]$$

$\frac{d^2A}{d\theta^2} = -\frac{1}{2}\,ab\sin\theta$ ----- (2)

[Since $\frac{d}{dx}(\cos\theta) = -\sin\theta$]

Now let us find the value of

$$\frac{d^2A}{d\theta^2}_{\theta=\frac{\pi}{2}} = -\frac{1}{2}\,ab\sin\left(\frac{\pi}{2}\right) = -\frac{1}{2}\,ab$$

As $\frac{d^2A}{d\theta^2}_{\theta=\frac{\pi}{2}} = -\frac{1}{2}\,ab < 0$, so the function A is maximum at $\theta = \frac{\pi}{2}$

As the area of the triangle is maximum when $\theta = \frac{\pi}{2}$

29. Question

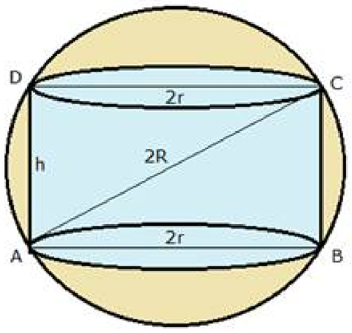
Show that the maximum volume of the cylinder which can be inscribed in a sphere of radius $5\sqrt{3}$ cm is (500π) cm^3.

Answer

Given,

• Radius of the sphere is $5\sqrt{3}$.

• Volume of cylinder is maximum.



Let us consider,

• The radius of the sphere be ‘R’ units.

• Volume of the inscribed cylinder be ‘V’.

• Height of the inscribed cylinder be ‘h’.

• Radius of the cylinder is ‘r’.

Now let $AC^2 = AB^2 + BC^2$, here AC = 2R, AB =2r, BC = h,

So $4R^2 = 4r^2 + h^2$

$r^2 = \frac{1}{4}[4R^2 - h^2]$ ----- (1)

Let us consider, the volume of the cylinder:

$V = \pi r^2 h$

Now substituting (1) in the volume formula,

$V = \pi h\left(\frac{1}{4}[4R^2 - h^2]\right)$

$V = \frac{\pi}{4}(4R^2h - h^3)$ ---- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with h and then equating it to zero. <u>This is because if the function V(h) has a maximum/minimum at a point c then V'(c) = 0.</u>

Differentiating the equation (2) with respect to h:

$\frac{dV}{dh} = \frac{d}{dh}\left[\frac{\pi}{4}(4R^2h - h^3)\right]$

$\frac{dV}{dh} = \frac{4R^2\pi}{4} - \frac{\pi}{4}(3h^2)$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dV}{dh} = R^2\pi - \frac{3h^2\pi}{4}$ ------- (3)

To find the critical point, we need to equate equation (3) to zero.

$\frac{dV}{dh} = R^2\pi - \frac{3h^2\pi}{4} = 0$

$3h^2\pi = 4R^2\pi$

$h^2 = \frac{4}{3}R^2 = \frac{4}{3}(5\sqrt{3})^2 = \frac{4}{3}(25 \times 3) = 100$

h = 10

[as h cannot be negative]

<u>Now to check if this critical point will determine the maximum volume of the inscribed cone, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with h:

$\frac{d^2V}{dh^2} = \frac{d}{dh}\left[R^2\pi - \frac{3h^2\pi}{4}\right]$

$\frac{d^2V}{dh^2} = 0 - \frac{3(2h)\pi}{3} = -2h\pi$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$\left(\frac{d^2V}{dh^2}\right)_{h=10} = -2h\pi = -2(10)\pi = -20\pi$

As $\left(\frac{d^2V}{dh^2}\right)_{h=10} = -20\pi < 0$, so the function V is maximum at h=10

Substituting h in equation (1)

$r^2 = \frac{1}{4}[4(5\sqrt{3})^2 - (10)^2]$

$r^2 = \frac{1}{4}[4(25 \times 3) - 100]$

$r^2 = \frac{300 - 100}{4} = \frac{200}{4} = 50$

As V is maximum, substituting h and r in the volume formula:

V = π (50) (10)

V = 500π cm^3

Therefore when the volume of a inscribed cylinder is maximum and is equal 500π cm^3

30. Question

A square tank of capacity 250cubic meters has to be dug out. The cost of the land is Rs. 50 per square metre. The cost of digging increases with the depth and for the whole tank, it is Rs.$\left(400 \times h^2\right)$, where h metres is the depth of the tank. What should be the dimensions of the tank so that the cost is minimum?

Answer

Given,

• Capacity of the square tank is 250 cubic metres.

• Cost of the land per square meter Rs.50.

• Cost of digging the whole tank is Rs. (400 × h^2).

• Where h is the depth of the tank.

Let us consider,

• Side of the tank is x metres.

• Cost of the digging is; C = $50x^2 + 400h^2$ ---- (1)

• Volume of the tank is; V = x^2h ; 250 =x^2h

$h = \frac{250}{x^2}$ ----- (2)

Substituting (2) in (1),

$$C = 50x^2 + 400\left(\frac{250}{x^2}\right)^2$$

$$C = 50x^2 + \frac{400 \times 62500}{x^4}$$ ----- (3)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function C(x) has a maximum/minimum at a point c then C'(c) = 0.</u>

Differentiating the equation (3) with respect to x:

$$\frac{dC}{dx} = \frac{d}{dx}\left[50x^2 + \frac{400 \times 62500}{x^4}\right]$$

$$\frac{dC}{dx} = 50(2x) + \frac{25000000(-4)}{x^5}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dC}{dx} = 100x - \frac{10^8}{x^5}$ ------- (4)

To find the critical point, we need to equate equation (4) to zero.

$$\frac{dC}{dx} = 100x - \frac{10^8}{x^5} = 0$$

$x^6 = 10^6$

x = 10

<u>Now to check if this critical point will determine the minimum volume of the tank, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (4) with x:

$$\frac{d^2C}{dx^2} = \frac{d}{dx}\left[100x - \frac{10^8}{x^5}\right]$$

$$\frac{d^2C}{dx^2} = 100 - \frac{10^8(-5)}{x^6} = 100 + \frac{10^8(5)}{x^6} \text{ ----- } (5)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2C}{dx^2}\right)_{x=10} = 100 + \frac{10^8(5)}{(10)^6} = 100 + 500 = 600$$

As $\left(\frac{d^2C}{dx^2}\right)_{x=10} = 600 > 0$, so the function C is minimum at x=10

Substituting x in equation (2)

$$h = \frac{250}{(10)^2} = \frac{250}{100} = \frac{5}{2}$$

h = 2.5 m

Therefore when the cost for the digging is minimum, when x = 10m and h = 2.5m

31. Question

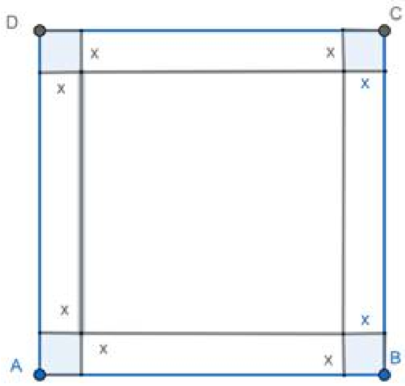
A square piece of tin of side 18 cm is to be made into a box without the top, by cutting a square piece from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find the maximum volume of the box.

Answer

Given,

• Side of the square piece is 18 cms.

• the volume of the formed box is maximum.



Let us consider,

• ‘x’ be the length and breadth of the piece cut from each vertex of the piece.

• Side of the box now will be (18-2x)

• The height of the new formed box will also be ‘x’.

Let the volume of the newly formed box is :

V = $(18-2x)^2 \times (x)$

V = $(324+ 4x^2 - 72x)$ x

V = $4x^3 - 72x^2 + 324x$ ------ (1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function V(x) has a maximum/minimum at a point c then V'(c) = 0.</u>

Differentiating the equation (1) with respect to x:

$$\frac{dV}{dx} = \frac{d}{dx}[4x^3 - 72x^2 + 324x]$$

$$\frac{dV}{dx} = 12x^2 - 144x + 324 \text{ -------- } (2)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dV}{dx} = 12x^2 - 144x + 324 = 0$$

$x^2 - 12x + 27 = 0$

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(1)(27)}}{2(1)} = \frac{12 \pm \sqrt{144 - 108}}{2} = \frac{12 \pm \sqrt{36}}{2}$$

$$x == \frac{12 \pm 6}{2}$$

x = 9 or x =3

x= 2

[as x = 9 is not a possibility, because 18-2x = 18-18= 0]

<u>Now to check if this critical point will determine the maximum area of the box, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2V}{dx^2} = \frac{d}{dx}[12x^2 - 144x + 324]$$

$$\frac{d^2V}{dx^2} = 24x - 144 \text{ ----- } (4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dx^2}\right)_{x=3} = 24(3) - 144 = 72 - 144 = -72$$

As $\left(\frac{d^2V}{dx^2}\right)_{x=3} = -72 < 0$, so the function V is maximum at x = 3cm

Now substituting x = 3 in 18 – 2x, the side of the considered box:

Side = 18-2x = 18 - 2(3) = 18-6= 12cm

Therefore side of wanted box is 12cms and height of the box is 3cms.

Now, the volume of the box is

$V = (12)^2 \times 3 = 144 \times 3 = 432cm^3$

Hence maximum volume of the box formed by cutting the given 18cms sheet is $432cm^3$ with 12cms side and 3cms height.

32. Question

An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when the depth of the tank is half of its width.

Answer

Given,

• The tank is square base open tank.

• The cost of the construction to be least.

Let us consider,

• Side of the tank is x metres.

• Height of the tank be 'h' metres.

• Volume of the tank is; $V = x^2h$

• Surface Area of the tank is $S = x^2 + 4xh$

• Let Rs.P is the price per square.

Volume of the tank,

$h = \frac{V}{x^2}$ ---- (1)

Cost of the construction be:

$C = (x^2 + 4xh)P$ ---- (2)

Substituting (1) in (2),

$$C = \left[x^2 + 4x\frac{V}{x^2}\right]P$$

$C = [x^2 + \frac{4V}{x}]P$ ----- (3)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function C(x) has a maximum/minimum at a point c then C'(c) = 0.

Differentiating the equation (3) with respect to x:

$$\frac{dC}{dx} = \frac{d}{dx}\left[x^2 + \frac{4V}{x}\right]P$$

$$\frac{dC}{dx} = \left[(2x) + \frac{4V(-1)}{x^2}\right]P$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$\frac{dC}{dx} = \left[2x - \frac{4V}{x^2}\right]P$ ------- (4)

To find the critical point, we need to equate equation (4) to zero.

$$\frac{dC}{dx} = \left[2x - \frac{4V}{x^2}\right]P = 0$$

$x^3 = 2V$

Now to check if this critical point will determine the minimum volume of the tank, we need to check with second differential which needs to be positive.

Consider differentiating the equation (4) with x:

$$\frac{d^2C}{dx^2} = P\frac{d}{dx}\left[2x - \frac{4V}{x^2}\right]$$

$\frac{d^2C}{dx^2} = \left[2 - \frac{4V(-2)}{x^3}\right]P = \left[2 + \frac{8V}{x^3}\right]P$ ----- (5)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2C}{dx^2}\right)_{x=(2V)^{\frac{1}{3}}} = \left[2 + \frac{8V}{2V}\right]P = [2+4]P = 6P$$

As $\left(\frac{d^2C}{dx^2}\right)_{x=(2V)^{\frac{1}{3}}} = 6P > 0$, so the function C is minimum at $x = \sqrt[3]{2V}$

Substituting x in equation (2)

$$h = \frac{V}{(2V)^{\frac{2}{3}}} = \frac{V\sqrt[3]{(2V)}}{2V} = \frac{1}{2}\sqrt[3]{2V}$$

$$h = \frac{1}{2}\sqrt[3]{2V}$$

Therefore when the cost for the digging is minimum, when $x = \sqrt[3]{2V}$ and $h = \frac{1}{2}\sqrt[3]{2V}$

33. Question

A wire of length 36 cm is cut into two pieces. One of the pieces is turned in the form of a square and the other in the form of an equilateral triangle. Find the length of each piece so that the sum of the areas of the two be minimum.

Answer

Given,

• Length of the wire is 36 cm.

• The wire is cut into 2 pieces.

• One piece is made to a square.

• Another piece made into a equilateral triangle.

Let us consider,

• The perimeter of the square is x.

• The perimeter of the equilateral triangle is (36-x).

• Side of the square is $\frac{x}{4}$

• Side of the triangle is $\frac{(36-x)}{3}$

Let the Sum of the Area of the square and triangle is

$$A = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{36-x}{3}\right)^2$$

$$A = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(12 - \frac{x}{3}\right)^2 = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right)$$

$$A = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right) \text{--- (1)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function A(x) has a maximum/minimum at a point c then A'(c) = 0.

Differentiating the equation (1) with respect to x:

$$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right)\right]$$

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{\sqrt{3}}{4}\left(0 + \frac{2x}{9} - 8\right)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{\sqrt{3}}{4}\left(\frac{2x}{9} - 8\right) \text{ ----- (2)}$$

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{\sqrt{3}}{4}\left(\frac{2x}{9} - 8\right) = 0$$

$$\frac{2x}{16} = \frac{\sqrt{3}}{4}\left(8 - \frac{2x}{9}\right)$$

$$\frac{2x}{16} = 2\sqrt{3} - \frac{\sqrt{3}x}{18}$$

$$\frac{2x}{16} + \frac{\sqrt{3}x}{18} = 2\sqrt{3}$$

$$x\left(\frac{2\,(9) + \sqrt{3}(8)}{144}\right) = 2\sqrt{3}$$

$$x\left(\frac{18 + 8\sqrt{3}}{144}\right) = 2\sqrt{3}$$

$$x = 2\sqrt{3}\left(\frac{144}{18 + 8\sqrt{3}}\right) = \frac{144\sqrt{3}}{(9 + 4\sqrt{3})}$$

Now to check if this critical point will determine the minimum area, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with x:

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left[\frac{2x}{16} + \frac{\sqrt{3}}{4}\left(\frac{2x}{9} - 8\right)\right]$$

$$\frac{d^2A}{dx^2} = \frac{1}{8} + \frac{\sqrt{3}}{4}\left(\frac{2}{9}\right) = \frac{9+4\sqrt{3}}{72} \text{ ----- (4)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2A}{dx^2}\right)_{x=\frac{144\sqrt{3}}{(9+4\sqrt{3})}} = \frac{9 + 4\sqrt{3}}{72}$$

As $\left(\frac{d^2A}{dx^2}\right)_{x=\frac{144\sqrt{3}}{(9+4\sqrt{3})}} = \frac{9+4\sqrt{3}}{72} > 0$, so the function A is minimum at

$$x = \frac{144\sqrt{3}}{(9 + 4\sqrt{3})}$$

Now, the length of each piece is $x = \frac{144\sqrt{3}}{(9+4\sqrt{3})}$ cm and $36 - x = 36 - \frac{144\sqrt{3}}{(9+4\sqrt{3})} = \frac{324}{(9+4\sqrt{3})}$ cm

34. Question

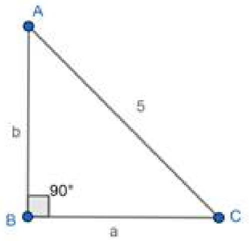
Find the largest possible area of a right-angles triangle whose hypotenuse is 5 cm.

Answer

Given,

• The triangle is right angled triangle.

• Hypotenuse is 5cm.



Let us consider,

• The base of the triangle is ‘a’.

• The adjacent side is ‘b’.

Now $AC^2 = AB^2 + BC^2$

As AC = 5, AB = b and BC = a

$25 = a^2 + b^2$

$b^2 = 25 - a^2$ ---- (1)

Now, the area of the triangle is

$$A = \frac{1}{2}ab$$

Squaring on both sides

$$A^2 = \frac{1}{4}a^2b^2$$

Substituting (1) in the area formula

$Z = A^2 = \frac{1}{4}a^2(25 - a^2)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with a and then equating it to zero. <u>This is because if the function Z (x) has a maximum/minimum at a point c then Z’(c) = 0.</u>

Differentiating the equation (2) with respect to a:

$$\frac{dZ}{da} = \frac{d}{da}\left[\frac{1}{4}a^2(25 - a^2)\right]$$

$$\frac{dZ}{da} = \frac{1}{4}\left[25\,(2a) - 4a^3\right]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dZ}{da} = \frac{25a}{2} - a^3$ ----- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{da} = \frac{25a}{2} - a^3 = 0$$

$$a\left(\frac{25}{2} - a^2\right) = 0$$

a=0 (or) $a = \frac{5}{\sqrt{2}}$

$$a = \frac{5}{\sqrt{2}}$$

[as a cannot be zero]

Now to check if this critical point will determine the maximum area, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with a:

$$\frac{d^2Z}{da^2} = \frac{d}{da}\left[\frac{25a}{2} - a^3\right]$$

$\frac{d^2Z}{da^2} = \frac{25}{2} - 3a^2$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2Z}{da^2}\right)_{a=\frac{5}{\sqrt{2}}} = \frac{25}{2} - 3\left(\frac{5}{\sqrt{2}}\right)^2 = \frac{25}{2} - \frac{(3)25}{2} = -25$$

As $\left(\frac{d^2Z}{da^2}\right)_{a=\frac{5}{\sqrt{2}}} = -25 < 0$, so the function A is maximum at $a = \frac{5}{\sqrt{2}}$

Substituting value of A in (1)

$$b^2 = 25 - \frac{25}{2} = \frac{25}{2}$$

$$b = \frac{5}{\sqrt{2}}$$

Now the maximum area is

$$A = \frac{1}{2}\left(\frac{5}{\sqrt{2}}\right)\left(\frac{5}{\sqrt{2}}\right) = \frac{25}{4}$$

$$\therefore A = \frac{25}{4}\ cm^2$$

Exercise 11G

1. Question

Show that the function $f(x) = 5x - 2$ is a strictly increasing function on R.

Answer

Domain of the function is R

Finding derivative f'(x)=5

Which is greater than 0

Mean strictly increasing in its domain i.e R

2. Question

Show the function $f(x) = -2x + 7$ is a strictly decreasing function on R.

Answer

Domain of the function is R

Finding derivative f'(x)=-2

Which is less than 0

Means strictly decreasing in its domain i.e R

3. Question

Prove that $f(x) = ax + b$, where a and b are constants and a > 0, is a strictly increasing function on R.

Answer

Domain of the function is R

Finding derivative i.e f'(x)=a

As given in question it is given that a>0

Derivative>0

Means strictly increasing in its domain i.e R

4. Question

Prove that the function $f(x) = e^{2x}$ is strictly increasing on R.

Answer

Domain of the function is R

finding derivative i.e f'(x)=2e^{x}

As we know e^{x} is strictly increasing its domain

f'(x)>0

hence f(x) is strictly increasing in its domain

5. Question

Show that the function $f(x) = x^2$ is

a. strictly increasing on [0, ∞[

b. strictly decreasing on [0, ∞[

c. neither strictly increasing nor strictly decreasing on R

Answer

Domain of function is **R.**

f'(x)=2x

for x>0 f'(x)>0 i.e. increasing

for x<0 f'(x)<0 i.e. decreasing

hence it is neither increasing nor decreasing in R

6. Question

Show that the function $f(x) = |x|$ is

a. strictly increasing on]0, ∞[

b. strictly decreasing on] − ∞, 0[

Answer

For x>0

Modulus will open with + sign

f(x)=+x

⇒ f'(x)=+1 which is <0

for x<0

Modulus will open with -ve sign

f(x)=-x => f'(x)=-1 which is >0

hence f(x) is increasing in x>0 and decreasing in x<0

7. Question

Prove that the function $f(x) = \log_e x$ is strictly increasing on]0, ∞[.

Answer

f(x)=ln(x)

$$f'(x) = \frac{1}{x}$$

for x<0

f'(x)=-ve →increasing

for x>0

f'(x)=+ve →decreasing

f(x) in increasing when x>0 i.e x∈(0,∞)

8. Question

Prove that the function $f(x) = \log_a x$ is strictly increasing on]0, ∞[when a > 1 and strictly decreasing on]0, ∞[when 0 < a < 1.

Answer

Consider ƒ(x)=$\log_a$ x

domain of f(x) is x>0

$$f'(x) = \frac{1}{x}\ln(a)$$

⇒ for a>1, ln(a)>0,

hence f'(x) >0 which means strictly increasing.

⇒ for 0<a<1, ln(a)<0,

hence f'(x)<0 which means strictly decreasing.

9. Question

Prove that $f(x) = 3^x$ is strictly increasing on R.

Answer

Consider f(x)=3^x

The domain of f(x) is R.

f'(x)=3^xln(3)

3^x is always greater than 0 and ln(3) is also + ve.

Overall f'(x) is >0 means strictly increasing in its domain i.e. R.

10. Question

Show that $f(x) = x^3 - 15x^2 + 75x - 50$ is increasing on R.

Answer

Consider f(x)=x^3-15x^2+75x-50

Domain of the function is R.

f'(x)=3x^2-30x+75

=3(x^2-10+25)

=3(x-5) (x-5)

=3$(x-5)^2$

f'(x)=0 for x=5

for x<5

f'(x)>5

and

for x>5

f'(x)>5

we can see throughout R the derivative is +ve but at x=5 it is 0 so it is increasing.

11. Question

Show that $f(x) = \left(x - \frac{1}{x}\right)$ is increasing all $x \in R$, where $x \neq 0$.

Answer

$f(x) = \left(x - \frac{1}{x}\right)$

domain of function is R-{0}

$f'(x) = 1 + \frac{1}{x^2}$

f'(x) ∀ x ∈ R is greater than 0.

12. Question

Show that $f(x) = \left(\frac{1}{x} + 5\right)$ is decreasing for all $x \in R$, where $x \neq 0$.

Answer

$f(x) = \frac{1}{x} + 5$

domain of function is R-{0}

$f'(x) = -\frac{1}{x^2}$

for all x, f'(x)<0

Hence function is decreasing.

13. Question

Show that $f(x)=\frac{1}{(1+x^2)}$ is decreasing for all $x \geq 0$

Answer

Consider $f(x)=\frac{1}{(1+x^2)}$,

$f'(x)=-\frac{2x}{(1+x^2)^2}$

for x ≥ 0 ,

f'(x) is -ve.

hence function is decreasing for x ≤ 0

14. Question

Show that $f(x)=\left(x^3+\frac{1}{x^3}\right)$ is decreasing on $]-1,1[$.

Answer

$f(x)=x^3+x^{-3}$

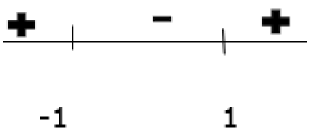
$f'(x)=3x^2-3x^{-4}$

$=3(x^2-1/x^4)$

$=3(\frac{x^3-1}{x^2}.\frac{x^3+1}{x^2})$

$=\frac{3(x-1)(x^2+x+1)(x+1)(x^2-x+1)}{x^4}$

Root of f'(x)=1and-1



Here we can clearly see that f'(x) is decreasing in [-1,1]

So, f(x) is decreasing in interval [-1,1]

15. Question

Show that $f(x)=\frac{x}{\sin x}$ is increasing on $\left]0,\frac{\pi}{2}\right[$.

Answer

Consider $f(x)=\frac{x}{\sin x}$,

$f'(x)=\frac{\sin x - x.\cos x}{\sin^2 x}$

$f'(x)=\frac{\cos x(\tan x - x)}{\sin^2 x}$

in $]0,\frac{\pi}{2}[$ cos>0,

tan x-x>0,

$\sin^2 x > 0$

hence f'(x)>0,

so, function is increasing in the given interval.

16. Question

Prove that the function $f(x) = \log(1+x) - \frac{2x}{(x+2)}$ is increasing for all $x > -1$.

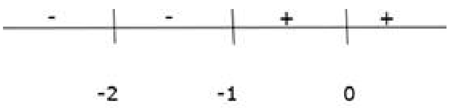
Answer

Consider $f(x) = \log(1+x) - \frac{2x}{(x+2)}$,

$$f'(x) = \frac{1}{1+x} - \frac{4}{(x+2)^2}$$

$$= \frac{(x+2)^2 - 4(x+1)}{(x+1)(x+2)^2}$$

$$= \frac{x^2}{(x+1)(x+2)^2}$$



Clearly we can see that f'(x)>o for x>-1.

Hence function is increasing for all x>-1

17. Question

Let I be an interval disjoint from $]-1,1[$. Prove that the function $f(x) = \left(x + \frac{1}{x}\right)$ is strictly increasing on I.

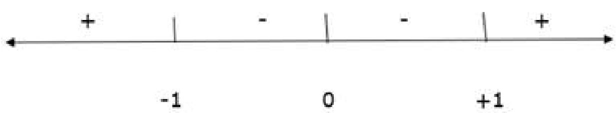
Answer

Consider $f(x) = \left(x + \frac{1}{x}\right)$

$$f'(x) = 1 - \frac{1}{x^2}$$

$$f'(x) = \frac{x^2 - 1}{x^2}$$

$$= \frac{x - 1 . x + 1}{x^2}$$



We can see f'(x) <0 in [-1,1]

i.e. f(x) is decreasing in this interval.

We can see f'(x) >0 in (-∞, -1) ∪(1, ∞)

i.e. f(x) is increasing in this interval.

18. Question

Show that $f\left(x\right)=\frac{\left(x-2\right)}{\left(x+1\right)}$ is increasing for all $x\in R$, except at $x=-1$.

Answer

Consider $f(x)=\frac{(x-2)}{(x+1)}$,

$$f'(x)=\frac{3}{(x+1)^2}$$

f'(x) at x=-1 is not defined

and for all x ∈ R- {-1}

f'(x)>0

hence f(x) is increasing.

19. Question

Find the intervals on which the function $f\left(x\right)=\left(2x^2-3x\right)$ is

(a) strictly increasing

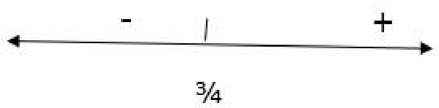
(b) strictly decreasing.

Answer

$f(x)=(2x^2-3x)$

f'(x)=4x-3

f'(x)=0 at x=3/4



Clearly we can see that function is increasing for x∈[3/4, ∞) and is decreasing for x∈(-∞,3/4)

20. Question

Find the intervals on which the function $f\left(x\right)=2x^3-3x^2-36x+7$ is

(a) strictly increasing (b) strictly decreasing.

Answer

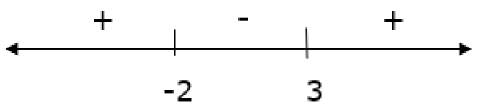
$f(x)=2x^3-3x^2-36x+7$

$f'(x)=6x^2-6x-36$

$f'(x)=6(x^2-x-6)$

f'(x)=6(x-3)(x+2)

f'(x) is 0 at x=3 and x=-2



F'(x)>0 for x ∈ (-∞, -2] ∪ [3, ∞)

hence in this interval function is increasing.

F'(x)<0 for x ∈ (-2 ,3)

hence in this interval function is decreasing.

21. Question

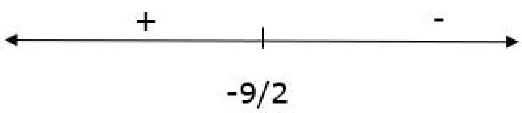
Find the intervals on which the function $f(x) = 6 - 9x - x^2$ is

(a) strictly increasing (b) strictly decreasing.

Answer

f(x)=6-9x-x^2

f'(x)=-(2x+9)



We can see that f(x) is increasing for $x \in \left(-\infty, -\frac{9}{2}\right]$ and decreasing in $x \in \left(-\frac{9}{2}, \infty\right)$

22. Question

Find the intervals on which each of the following functions is (a) increasing (b) decreasing.

$$f(x) = \left(x^4 - \frac{x^3}{3}\right)$$

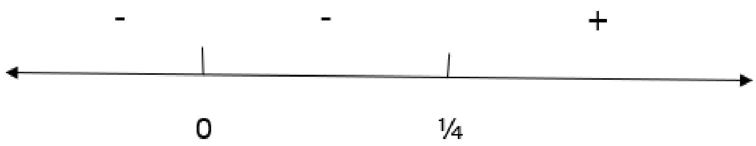
Answer

Consider $f(x) = \left(x^4 - \frac{x^3}{3}\right)$

f'(x)=4x^3-x^2

=x^2(4x-1)

F'(x)=0 for x=0 and x=1/4



Function f(x) is decreasing for x ∈ (-∞,1/4] and increasing in x ∈ (1/4 , ∞)

23. Question

Find the intervals on which each of the following functions is (a) increasing (b) decreasing.

$$f(x) = x^3 - 12x^2 + 36x + 17$$

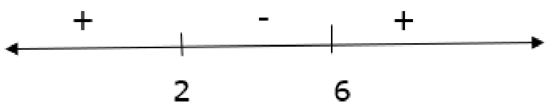
Answer

f(x)=x^3-12x^2+36x+17

f'(x)=3x^2-24x+36

f'(x)=3(x^2-8x+12)

=3(x-6)(x-2)



Function f(x) is decreasing for x ∈ [2,6] and increasing in x ∈ (-∞,2) ∪ (6, ∞)

24. Question

Find the intervals on which each of the following functions is (a) increasing (b) decreasing.

$f(x)=\left(x^3-6x^2+9x+10\right)$

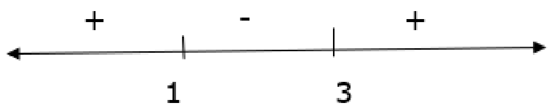
Answer

$f(x)=x^3-6x^2+9x+10$

$f'(x)=3x^2-12x+9$

$f'(x)=3(x^2-4x+3)$

$=3(x-3)(x-1)$



Function f(x) is decreasing for x ∈ [1,3] and increasing in x ∈ (-∞,1) ∪ (3, ∞)

25. Question

Find the intervals on which each of the following functions is (a) increasing (b) decreasing.

$f(x)=\left(6+12x+3x^2-2x^3\right)$

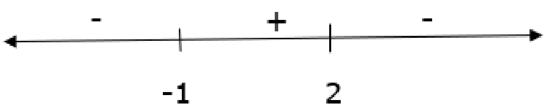
Answer

$f(x)=-2x^3+3x^2+12x+6$

$f'(x)=-6x^2+6x+12$

$f'(x)=-6(x^2-x-2)$

$=-6(x-2)(x+1)$



Function f(x) is increasing for x ∈ [-1,2] and decreasing in x ∈ (-∞,-1) ∪ (2, ∞)

26. Question

Find the intervals on which each of the following functions is (a) increasing (b) decreasing.

$f(x)=2x^3-24x+5$

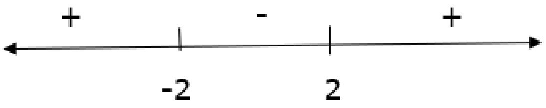
Answer

$f(x)=2x^3-24x+5$

$f'(x)=6x^2-24$

$f'(x)=6(x^2-4)$

$=6(x-2)(x+2)$



Function f(x) is decreasing for x ∈ [-2,2] and increasing in x ∈ (-∞,-2) ∪ (2, ∞)

27. Question

Find the intervals on which each of the following functions is (a) increasing (b) decreasing.

$$f(x)=(x-1)(x-2)^2$$

Answer

$f(x)=(x-1)(x-2)^2$=x2-4x+4 * x-1= x3-4x2+4x-x2+4x-4

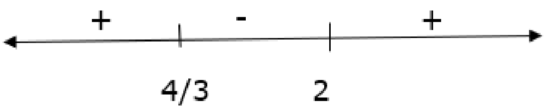
$f(x)=x^3-5x^2+8x-4$

$f'(x)=3x^2-10x+8$

$f'(x)=3x^2-6x-4x+8$

=3x(x-2)-4(x-2)

=(3x-4)(x-2)



Function f(x) is decreasing for x ∈ [4/3,2] and increasing in x ∈ (-∞,4/3) ∪ (2, ∞)

28. Question

Find the intervals on which each of the following functions is (a) increasing (b) decreasing.

$$f(x)=\left(x^4-4x^3+4x^2+15\right)$$

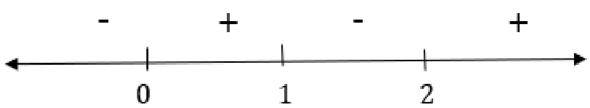
Answer

$f(x)=x^4-4x^3+4x^2+15$

$f'(x)=4x^3-12x^2+8x$

$= 4x(x^2-3x+2)$

=4x(x-1)(x-2)



Function f(x) is decreasing for x ∈ (-∞,0] ∪ [1, 2] and increasing in x ∈ (0,1) ∪ (2, ∞)

29. Question

Find the intervals on which each of the following functions is (a) increasing (b) decreasing.

$$f(x)=2x^3+9x^2+12x+15$$

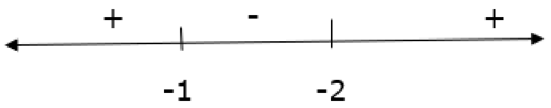
Answer

$f(x)=2x^3+9x^2+12x+15$

$f'(x)=6x^2+18x+12$

f’(x)=6(x^2+3x+2)

=6(x+2)(x+1)



Function f(x) is decreasing for x ∈ [-1,-2] and increasing in x ∈ (-∞,-1) ∪ (-2, ∞)

30. Question

Find the intervals on which each of the following functions is (a) increasing (b) decreasing.

$$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$$

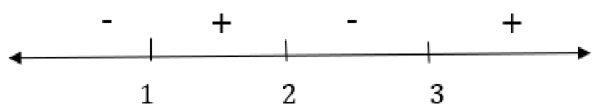
Answer

f(x)=x^4-8x^3+22x^2-24x+21

f’(x)=4x^3-24x^2+44x-24

= 4(x^3 -6x^2+11x-6)

=4(x-3)(x-1)(x-2)



Function f(x) is decreasing for x ∈ (-∞,1] ∪ [2, 3] and increasing in x ∈ (1,2) ∪ (3, ∞)

31. Question

Find the intervals on which each of the following functions is (a) increasing (b) decreasing.

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

Answer

f(x)=3x^4-4x^3-12x^2+5

f’(x)=12x^3-12x^2-24x

=12x(x^2 -x-2)

=12(x)(x+1)(x-2)

Function f(x) is decreasing for x ∈ (-∞,-1] ∪ [0, 2] and increasing in x ∈ (-1,0) ∪ (2, ∞)

32. Question

Find the intervals on which each of the following functions is (a) increasing (b) decreasing.

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$

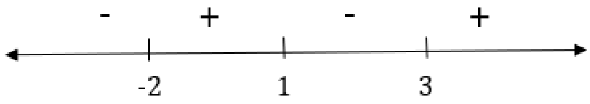
Answer

$$f'(x) = \frac{12x^3}{10} - \frac{12x^2}{5} - 6x + \frac{36}{5}$$

f’(x)=(12x^3-24x^2-60x+72)/10

=1.2(x^3-2x^2-5x+6)

=1.2(x-1)(x-3)(x+2)



Function f(x) is decreasing for $x \in (-\infty,-2] \cup [1, 3]$ and increasing in $x \in (-2,1) \cup (3, \infty)$

Exercise 11H

1. Question

Find the slope of the tangent to the curve

i. $y = \left(x^3 - x\right)$ at $x = 2$

ii. $y = \left(2x^2 + 3\sin x\right)$ at $x = 0$

iii. $y = \left(\sin 2x + \cot x + 2\right)^2$ at $x = \frac{\pi}{2}$

Answer

i. $\frac{dy}{dx} = 3x^2 - 1$

$\frac{dy}{dx}$ at $(x = 2) = 11$

ii. $\frac{dy}{dx} = 4x + 3\cos x$

$\frac{dy}{dx}$ at $(x = 0) = 3$

iii. $\frac{dy}{dx} = 2(\sin 2x + \cot x + 2)(2\cos 2x - \operatorname{cosec}^2 x)$

$\frac{dy}{dx}$ at $\left(x = \frac{\pi}{2}\right) = 2(0 + 0 + 2)(-2 - 1) = -12$

2. Question

Find the equations of the tangent and the normal to the given curve at the indicated point for

$y = x^3 - 2x + 7$ at $(1, 6)$

Answer

$m : \frac{dy}{dx} = 3x^2 - 2$

m at (1, 6) = 1

Tangent : y – b = m(x – a)

y – 6 = 1(x – 1)

x – y + 5 = 0

Normal : $y - b = \frac{-1}{m}(x - a)$

y – 6 = -1(x – 1)

x + y – 7 = 0

3. Question

Find the equations of the tangent and the normal to the given curve at the indicated point for

$y^2 = 4ax \text{ at } \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

Answer

$m: \ 2y\frac{dy}{dx} = 4a$

$m \text{ at } \left(\frac{a}{m^2}, \frac{2a}{m}\right) = m$

Tangent : y – b = m(x – a)

$y - \frac{2a}{m} = m\left(x - \frac{a}{m^2}\right)$

$m^2x - my + a = 0$

Normal : $y - b = \frac{-1}{m}(x - a)$

$y - \frac{2a}{m} = \frac{-1}{m}\left(x - \frac{a}{m^2}\right)$

$m^2x + m^3y - 2am^2 - a = 0$

4. Question

Find the equations of the tangent and the normal to the given curve at the indicated point for

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } (a\cos\theta, b\sin\theta)$

Answer

$m: \ \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$m \text{ at } (a\cos\theta, b\sin\theta) = \frac{-b\cos\theta}{a\sin\theta}$

Tangent : y – b = m(x – a)

$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$bx\cos\theta + ay\sin\theta = ab$

Normal : $y - b = \frac{-1}{m}(x - a)$

$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$

5. Question

Find the equations of the tangent and the normal to the given curve at the indicated point for

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } (a\sec\theta, b\tan\theta)$

Answer

$m: \ \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

$m \text{ at } (a\sec\theta, b\tan\theta) = \frac{b\sec\theta}{a\tan\theta}$

Tangent : y – b = m(x – a)

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$$

$$bx\sec\theta - ay\tan\theta = ab$$

Normal : $y - b = \frac{-1}{m}(x - a)$

$$y - b\sin\theta = \frac{-a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$

$$by\,\mathrm{cosec}\,\theta + ax\sec\theta = (a^2 + b^2)$$

6. Question

Find the equations of the tangent and the normal to the given curve at the indicated point for

$$y = x^3 \text{ at } P(1,1)$$

Answer

$m : \frac{dy}{dx} = 3x^2$

m at (1, 1) = 3

Tangent : y – b = m(x – a)

y – 1 = 3(x – 1)

y = 3x – 2

Normal : $y - b = \frac{-1}{m}(x - a)$

$$y - 1 = \frac{-1}{3}(x - 1)$$

x + 3y = 4

7. Question

Find the equations of the tangent and the normal to the given curve at the indicated point for

$$y^2 = 4ax \text{ at } (at^2, 2at)$$

Answer

$m : 2y\frac{dy}{dx} = 4a$

m at (at^2, 2at) = 1/t

Tangent : y – b = m(x – a)

$$y - 2at = \frac{1}{t}(x - at^2)$$

x – ty + at^2 = 0

Normal : $y - b = \frac{-1}{m}(x - a)$

y – 2at = -t(x – at^2)

tx + y = at^3 + 2at

8. Question

Find the equations of the tangent and the normal to the given curve at the indicated point for

$y = \cot^2 x - 2\cot x + 2$ at $x = \frac{\pi}{4}$

Answer

$m : \frac{dy}{dx} = 2\cot x(-\operatorname{cosec}^2 x) + 2\operatorname{cosec}^2 x$

m at (x = π/4) = 2(-2) + 2(2) = 0

Tangent : y – b = m(x – a)

y – 1 = 0(x – π/4)

y = 1

Normal : $y - b = \frac{-1}{m}(x - a)$

$y - 1 = \frac{-1}{0}\left(x - \frac{\pi}{4}\right)$

x = π/4

9. Question

Find the equations of the tangent and the normal to the given curve at the indicated point for

$16x^2 + 9y^2 = 144$ at $(2, y_1)$, where $y_1 > 0$

Answer

$m : 32x + 18y\frac{dy}{dx} = 0$

$m \text{ at } (2, y_1) = \frac{-32}{9y_1}$

$16(2)^2 + 9(y_1)^2 = 144$

$y_1 = \frac{4\sqrt{5}}{3}$

Tangent : y – b = m(x – a)

$y - \frac{4\sqrt{5}}{3} = \frac{-32}{9\frac{4\sqrt{5}}{3}}(x - 2)$

$8x + 3\sqrt{5}y - 36 = 0$

Normal : $y - b = \frac{-1}{m}(x - a)$

$y - \frac{4\sqrt{5}}{3} = \frac{9\frac{4\sqrt{5}}{3}}{32}(x - 2)$

$9\sqrt{5x} - 24y + 14\sqrt{5} = 0$

10. Question

Find the equations of the tangent and the normal to the given curve at the indicated point for

$y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at the point where x = 1

Answer

$m : \frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$

m at (x = 1) = 2

y at (x = 1) = $(1)^4 - 6(1)^3 + 13(1)^2 - 10(1) + 5 = 3$

Tangent : y – b = m(x – a)

y – 3 = 2(x – 1)

2x – y + 1 = 0

Normal : $y - b = \frac{-1}{m}(x - a)$

$y - 3 = \frac{-1}{2}(x - 1)$

x + 2y – 7 = 0

11. Question

Find the equation of the tangent to the curve $\sqrt{x} + \sqrt{y} = a$ at $\left(\frac{a^2}{4}, \frac{a^2}{4}\right)$

Answer

m : $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$

m at $\left(\frac{a^2}{4}, \frac{a^2}{4}\right) = -1$

y – b = m(x – a)

$y - \frac{a^2}{4} = -1\left(x - \frac{a^2}{4}\right)$

$2(x + y) = a^2$

12. Question

Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Answer

m at $(x_1, y_1) = \frac{b^2x_1}{a^2y_1}$

At (x_1, y_1) : $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \Rightarrow b^2x_1^2 - a^2y_1^2 = a^2b^2$

y – b = m(x – a)

$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$

$a^2y_1y - a^2y_1^2 = b^2x_1x - b^2x_1^2$

$b^2x_1x - a^2y_1y = a^2b^2$

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

13. Question

Find the equation of the tangent to the curve $y = \left(\sec^4 x - \tan^4 x\right)$ at $x = \frac{\pi}{3}$.

Answer

$m: \frac{dy}{dx} = 4\sec^3x(\tan x \sec x) - 4\tan^3x(\sec^2x)$

$m \text{ at } \left(x = \frac{\pi}{3}\right) = 4(2)^3(\sqrt{3} \times 2) - 4(\sqrt{3})^3(2)^2 = 16\sqrt{3}$

At x = π/3 , y = 7

y – b = m(x – a)

$y - 7 = 16\sqrt{3}\left(x - \frac{\pi}{3}\right)$

$3y - 48\sqrt{3}x + 16\sqrt{3}\pi - 21 = 0$

14. Question

Find the equation of the normal to the curve $y = (\sin 2x + \cot x + 2)^2$ at $x = \frac{\pi}{2}$

Answer

$m: \frac{dy}{dx} = 2(\sin 2x + \cot x + 2)(2\cos 2x - \text{cosec}^2x)$

$\frac{dy}{dx} \text{ at } \left(x = \frac{\pi}{2}\right) = 2(0 + 0 + 2)(-2 - 1) = -12$

At x = π/2, y = 4

$y - b = \frac{-1}{m}(x - a)$

$y - 4 = \frac{1}{12}\left(x - \frac{\pi}{2}\right)$

24y – 2x + π – 96 = 0

15. Question

Show that the tangents to the curve $y = 2x^3 - 4$ at the point x = 2 and x = −2 are parallel.

Answer

$m: \frac{dy}{dx} = 6x^2$

m at (x = 2) = 24

m at (x = -2) = 24

We know that if the slope of curve at two different point is

equal then straight lines are parallel at that points.

16. Question

Find the equation of the tangent to the curve $x^2 + 3y = 3$, where is parallel to the line $y - 4x + 5 = 0$.

Answer

We know that if two straight lines are parallel then their slope

are equal. So, slope of required tangent is also equal to 4.

$m: \frac{dy}{dx} = \frac{-2x}{3} = 4$

x = -6 and y = -11

y – b = m(x – a)

y – (-11) = 4(x – (-6))

4x – y + 13 = 0

17. Question

At what point on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, is the tangent parallel to the y-axis?

Answer

If the tangent is parallel to y-axis it means that it's slope is

not defined or 1/0.

$$m: \ 2x + 2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(2x-2)}{(2y-4)} = \frac{1}{0}$$

2y – 4 = 0 ⇒ y = 2

$x^2 + (2)^2 - 2x - 4(2) + 1 = 0$

⇒ $x^2 - 2x - 3 = 0$

⇒ x = 3 and x = -1

So, the requied points are (-1, 2) and (3, 2).

18. Question

Find the point on the curve $x^2 + y^2 - 2x - 3 = 0$ where the tangent is parallel to the x-axis.

Answer

If the tangent is parallel to x-axis it means that it's slope is 0

$$m: \ 2x + 2y\frac{dy}{dx} - 2 = 0$$

2x + 2y(0) – 2 = 0

x = 1

$(1)^2 + y^2 - 2(1) - 3 = 0$

⇒ $y^2 = 4$ ⇒ y = 2 and y = -2

So, the requied points are (1, 2) and (1, -2).

19. Question

Prove the tangent to the curve $y = x^2 - 5x + 6$ at the point (2, 0) and (3, 0) are at right angles.

Answer

We know that if the slope of two tangent of a curve are satisfies a relation $m_1m_2 = -1$, then tangents are at right angles

$$m: \frac{dy}{dx} = 2x - 5$$

m_1 at (2, 0) = -1

m_2 at (3, 0) = 1

$m_1m_2 = (-1)(1) = -1$

So, we can say that tangent at (2, 0) and (3, 0) are at right angles.

20. Question

Find the point on the curve $y = x^2 + 3x + 4$ at which the tangent passes through the origin.

Answer

If tangent is pass through origin it means that equation of tangent is y = mx

Let us suppose that tangent is made at point (x_1, y_1)

$y_1 = x_1^2 + 3x_1 + 4$ ...(1)

$m : \frac{dy}{dx} = 2x + 3$

m at $(x_1, y_1) = 2x_1 + 3$

Equation of tangent : $y_1 = (2x_1 + 3)x_1$ ...(2)

On compairing eq(1) and eq(2)

$x_1^2 + 3x_1 + 4 = (2x_1 + 3)x_1$

$x_1^2 - 4 = 0 \Rightarrow x_1 = 2$ and -2

At $x_1 = 2$, $y_1 = 14$

At $x_1 = -2$, $y_1 = 2$

So, required points are (2, 14) and (-2, 2)

21. Question

Find the point on the curve $y = x^3 - 11x + 5$ at which the equation of tangent is $y = x - 11$.

Answer

Slope of y = x – 11 is equal to 1

$m : \frac{dy}{dx} = 3x^2 - 11$

$3x^2 - 11 = 1 \Rightarrow x = 2$ and -2

At x = 2

From the equation of curve, $y = (2)^3 - 11(2) + 5 = -9$

From the equation of tangent, y = 2 – 11 = -9

At x = -2

From the equation of curve, $y = (-2)^3 - 11(-2) + 5 = 19$

From the equation of tangent, y = -2 – 11 = -13

So, the final answer is (2, -9) because at x = -2, y is come different from the equation of curve and tangent which is not possible.

22. Question

Find the equation of the tangents to the curve $2x^2 + 3y^2 = 14$, parallel to the line $x = 3y = 4$.

Answer

If tangent is parallel to the line x + 3y = 4 then it's slope is -1/3.

$$m:\ 4x + 6y\frac{dy}{dx} = 0$$

$$m = \frac{-2x}{3y} = \frac{-2x}{3\sqrt{\frac{14-2x^2}{3}}} = \frac{-1}{3}$$

$$2x = \sqrt{\frac{14-2x^2}{3}}$$

$$4x^2 = \frac{14-2x^2}{3}$$

x = 1 and -1

At x = 1, y = 2 and y = -2 (not possible)

At x = -1, y = -2 and y = 2 (not possible)

y – b = m(x – a)

At (1, 2)

$$y - 2 = \frac{-1}{3}(x-1)$$

3y + x = 7

At (-1, -2)

$$y - (-2) = \frac{-1}{3}(x-(-1))$$

3y + x = -7

23. Question

Find the equation of the tangent to the curve $x^2 + 2y = 8$, which is perpendicular to the line $x - 2y + 1 = 0$.

Answer

.: If tangent is perpendicular to the line x – 2y + 1 = 0 then it's -1/m is -2.

$$m:\ 2x + 2\frac{dy}{dx} = 0$$

m = -x = 1/2

x = -1/2

At x = -1/2, y = 31/8

$$y - b = \frac{-1}{m}(x-a)$$

At (-1/2, 31/8)

$$y - \frac{31}{8} = \frac{-1}{\frac{1}{2}}\left(x - \left(-\frac{1}{2}\right)\right)$$

16x + 8y – 23 = 0

24. Question

Find the point on the curve $y = 2x^2 - 6x - 4$ at which the tangent is parallel to the x-axis.

Answer

We know that if tangent is parallel to x-axis then it's slope is equal to 0.

$$m : \frac{dy}{dx} = 4x - 6$$

4x – 6 = 0 ⇒ x = 3/2

At x = 3/2 , y = -17/2

So, the required points are $\left(\frac{3}{2}, \frac{-17}{2}\right)$.

25. Question

Find the point on the parabola $y = (x-3)^2$, where the tangent is parallel to the chord joining the point (3, 0) and (4, 1).

Answer

If the tangent is parallel to chord joining the points (3, 0) and (4, 1) then slope of tangent is equal to slope of chord.

$$m = \frac{1-0}{4-3} = 1$$

$$m : \frac{dy}{dx} = 2(x-3)$$

2(x – 3) = 1 ⇒ x = 7/2

At x = 7/2, y = 1/4

So, the required points are $\left(\frac{7}{2}, \frac{1}{4}\right)$.

26. Question

Show that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

Answer

If curves cut at right angle if $8k^2 = 1$ then vice versa also true. So, we have to prove that $8k^2 = 1$ if curve cut at right angles.

If curve cut at right angle then the slope of tangent at their intersecting point satisfies the relation $m_1 m_2 = -1$

We have to find intersecting point of two curves.

$x = y^2$ and xy = k then $y = k^{\frac{1}{3}}$ and $x = k^{\frac{2}{3}}$

$$m_1 : \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$m_1 \text{ at } \left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right) = \frac{1}{2k^{\frac{1}{3}}}$$

$$m_2 : \frac{dy}{dx} = \frac{-k}{x^2}$$

$$m_2 \text{ at } \left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right) = \frac{-k}{k^{\frac{4}{3}}} = -\frac{1}{k^{\frac{1}{3}}}$$

$$m_1 m_2 = -1$$

$$\left(\frac{1}{2k^{\frac{1}{3}}}\right)\left(-\frac{1}{k^{\frac{1}{3}}}\right) = -1$$

$$k^{\frac{2}{3}} = \frac{1}{2} \Rightarrow k^2 = \frac{1}{8} \Rightarrow 8k^2 = 1$$

27. Question

Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.

Answer

If the two curve touch each other then the tangent at their intersecting point formed a angle of 0.

We have to find the intersecting point of these two curves.

$xy = a^2$ and $x^2 + y^2 = 2a^2$

$\Rightarrow x^2 + \left(\frac{a^2}{x}\right)^2 = 2a^2$

$\Rightarrow x^4 - 2a^2x^2 + a^4 = 0$

$\Rightarrow (x^2 - a^2) = 0$

$\Rightarrow x = +a$ and $-a$

At $x = a$, $y = a$

At $x = -a$, $y = -a$

$m_1 : \frac{dy}{dx} = \frac{-a^2}{x^2}$

m_1 at $(a, a) = -1$

m_1 at $(-a, -a) = -1$

$m_2 : 2x + 2y\frac{dy}{dx} = 0$

m_2 at $(a, a) = -1$

m_2 at $(-a, -a) = -1$

At (a, a)

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$

$$\tan\theta = \frac{(-1)-(-1)}{1+(-1)(-1)} = 0 \Rightarrow \theta = 0$$

At (-a, -a)

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$

$$\tan\theta = \frac{(-1)-(-1)}{1+(-1)(-1)} = 0 \Rightarrow \theta = 0$$

So, we can say that two curves touch each other because the angle between two tangent at their intersecting point is equal to 0.

28. Question

Show that the curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ cut orthogonally.

Answer

If the two curve cut orthogonally then angle between their tangent at intersecting point is equal to 90⁰.

We have to find their intersecting point.

$x^3 - 3xy^2 + 2 = 0$ ...(1) and $3x^2y - y^3 - 2 = 0$ ...(2)

On adding eq (1) and eq (2)

$x^3 - 3xy^2 + 2 + 3x^2y - y^3 - 2 = 0$

$x^3 - y^3 - 3xy^2 + 3x^2y = 0$

$(x - y)^3 = 0 \Rightarrow x = y$

Put x = y in eq (1)

$y^3 - 3y^3 + 2 = 0 \Rightarrow y = 1$

At y = 1, x = 1

$$m_1 : 3x^2 - 3\left(x \times 2y\frac{dy}{dx} + y^2\right) = 0$$

m_1 at (1, 1) = 0

$$m_2 : 3\left(x^2\frac{dy}{dx} + 2xy\right) - 3y^2\frac{dy}{dx} = 0$$

m_2 at (1, 1) = -2/0

At (1, 1)

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan\theta = \frac{m_2\left(1 - \frac{m_1}{m_2}\right)}{m_2\left(\frac{1}{m_2} + m_1\right)}$$

$$\tan\theta = \frac{(1-0)}{(0+0)} = \text{not defined} \Rightarrow \theta = \frac{\pi}{2}$$

So, we can say that two curve cut each other orthogonally because angle between two tangent at their intersecting point is equal to 90⁰.

29. Question

Find the equation of tangent to the curve $x = (\theta + \sin\theta), y = (1 + \cos\theta)$ at $\theta = \frac{\pi}{4}$.

Answer

$$m : \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-\sin\theta}{1+\cos\theta}$$

$$m \text{ at } \left(\theta = \frac{\pi}{4}\right) = \frac{-1}{1+\sqrt{2}} = 1 - \sqrt{2}$$

At $\theta = \frac{\pi}{4}, x = \left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)$ and $y = \left(1 + \frac{1}{\sqrt{2}}\right)$

y – b = m(x – a)

$$y - \left(1 + \frac{1}{\sqrt{2}}\right) = (1 - \sqrt{2})\left(x - \left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)\right)$$

$y = (1 - \sqrt{2})x + \frac{(\sqrt{2} - 1)\pi}{4} + 2$

30. Question

Find the equation of the tangent at $t = \frac{\pi}{4}$ for the curve $x = \sin 3t, y = \cos 2t$.

Answer

$m : \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin 2t}{3\cos 3t}$

$m \text{ at } \left(t = \frac{\pi}{4}\right) = \frac{2\sqrt{2}}{3}$

At $t = \frac{\pi}{4}, x = \frac{1}{\sqrt{2}}$ and $y = 0$

y – b = m(x – a)

$y - 0 = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right)$

$4x - 3\sqrt{2}y - 2\sqrt{2} = 0$

Objective Questions

1. Question

Mark (√) against the correct answer in the following:

If $y = 2^x$ then $\frac{dy}{dx} = ?$

A. $x\left(2^{x-1}\right)$

B. $\frac{2^x}{(\log 2)}$

C. $2^x(\log 2)$

D. none of these

Answer

Given that $y=2^x$

Taking log both sides, we get

$\log_e y = x\log_e 2$ (Since $\log_a b^c = c\log_a b$)

Differentiating with respect to x, we get

$\frac{1}{y}\frac{dy}{dx} = \log_e 2 \text{ or } \frac{dy}{dx} = \log_e 2 \times y$

Hence $\frac{dy}{dx} = 2^x \log_e 2$

2. Question

Mark (√) against the correct answer in the following:

If $y = \log_{10} x$ then $\frac{dy}{dx} = ?$

A. $\frac{1}{x}$

B. $\frac{1}{x}(\log 10)$

C. $\frac{1}{x(\log 10)}$

D. none of these

Answer

Given that $y = \log_{10} x$

Using the property that $\log_a b = \frac{\log_e b}{\log_e a}$, we get

$y = \frac{\log_e x}{\log_e 10}$

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{1}{x \log_e 10}$

3. Question

Mark (√) against the correct answer in the following:

If $y = e^{1/x}$ then $\frac{dy}{dx} = ?$

A. $\frac{1}{x} \cdot e^{(1/x-1)}$

B. $\frac{-e^{1/x}}{x^2}$

C. $e^{1/x} \log x$

D. none of these

Answer

Given that $y = e^{\frac{1}{x}}$

Taking log both sides, we get

$\log_e y = \frac{1}{x}$ (Since $\log_a b^c = c \log_a b$)

Differentiating with respect to x, we get

$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x^2}$ or $\frac{dy}{dx} = -\frac{1}{x^2} \times y$

Hence $\frac{dy}{dx} = -\frac{1}{x^2} \times e^{\frac{1}{x}}$

4. Question

Mark (√) against the correct answer in the following:

If $y = x^x$ then $\frac{dy}{dx} = ?$

A. $x^x \log x$

B. $x^x(1 + \log x)$

C. $x(1 + \log x)$

D. none of these

Answer

Let y=f(x)=x^x

Taking log both sides, we get

$\log_e y = x \times \log_e x$-(1) (Since $\log_a b^c = c \log_a b$)

Differentiating (1) with respect to x, we get

$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \log_e x \times 1$

$\Rightarrow \frac{dy}{dx} = y \times (1 + \log_e x)$

$\Rightarrow \frac{dy}{dx} = f'(x) = x^x(1 + \log_e x)$

5. Question

Mark (√) against the correct answer in the following:

If $y = x^{\sin x}$ then $\frac{dy}{dx} = ?$

A. $(\sin\ x) \cdot x^{(\sin x - 1)}$

B. $(\sin\ x \cos x) \cdot x^{(\sin x - 1)}$

C. $x^{\sin x}\left\{\frac{\sin x + x \log x \cdot \cos x}{x}\right\}$

D. none of these

Answer

Let y=f(x)=$x^{\sin x}$

Taking log both sides, we get

$\log_e y = \sin x \times \log_e x$-(1) (Since $\log_a b^c = c \log_a b$)

Differentiating (1) with respect to x, we get

$\frac{1}{y}\frac{dy}{dx} = \sin x \times \frac{1}{x} + \log_e x \times \cos x$

$\Rightarrow \frac{dy}{dx} = y \times \left(\frac{\sin x}{x} + \log_e x \cos x\right)$

$\Rightarrow \frac{dy}{dx} = f'(x) = x^x \left(\frac{\sin x + x \log_e x \cos x}{x}\right)$

6. Question

Mark (√) against the correct answer in the following:

f $y = x^{\sqrt{x}}$ then $\frac{dy}{dx} = ?$

A. $\sqrt{x} \cdot x^{(\sqrt{x}-1)}$

B. $\frac{x^{\sqrt{x}} \log x}{2\sqrt{x}}$

C. $x^{\sqrt{x}} \left\{\frac{2 + \log x}{2\sqrt{x}}\right\}$

D. none of these

Answer

Let $y = f(x) = x^{\sqrt{x}}$

Taking log both sides, we get

$\log_e y = \sqrt{x} \times \log_e x$ -(1)

(Since $\log_a b^c = c \log_a b$)

Differentiating (1) with respect to x, we get

$\frac{1}{y}\frac{dy}{dx} = \sqrt{x} \times \frac{1}{x} + \log_e x \times \frac{1}{2\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = y \times \left(\frac{2 + \log_e x}{2\sqrt{x}}\right)$

$\Rightarrow \frac{dy}{dx} = f'(x)$

$= x^{\sqrt{x}} \left(\frac{2 + \log_e x}{2\sqrt{x}}\right)$

7. Question

Mark (√) against the correct answer in the following:

If $y = e^{\sin \sqrt{x}}$ then $\frac{dy}{dx} = ?$

A. $e^{\sin \sqrt{x}} \cdot \cos \sqrt{x}$

B. $\frac{e^{\sin \sqrt{x}} \cos \sqrt{x}}{2\sqrt{x}}$

C. $\frac{e^{\sin\sqrt{x}}}{2\sqrt{x}}$

D. none of these

Answer

Given that $y = e^{\sin\sqrt{x}}$

Taking log both sides, we get

$\log_e y = \sin\sqrt{x}$

$(\text{Since } \log_a b^c = c\log_a b)$

Differentiating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \cos\sqrt{x} \times \frac{1}{2\sqrt{x}}$$

Or

$$\frac{dy}{dx} = \cos\sqrt{x} \times \frac{1}{2\sqrt{x}} \times y$$

Hence $\frac{dy}{dx} = \frac{e^{\sin\sqrt{x}}\cos\sqrt{x}}{2\sqrt{x}}$

8. Question

Mark (√) against the correct answer in the following:

If $y = (\tan x)^{\cot x}$ then $\frac{dy}{dx} = ?$

A. $\cot x \cdot (\tan x)^{\cot x - 1} \cdot \sec^2 x$

B. $-(\tan x)^{\cot x} \cdot \text{cosec}^2 x$

C. $(\tan x)^{\cot x} \cdot \text{cosec}^2 x (1 - \log \tan x)$

D. none of these

Answer

Given that $y = (\tan x)^{\cot x}$

Taking log both sides, we get

$\log_e y = \cot x \times \log_e \tan x$ (Since $\log_a b^c = c\log_a b$)

Differentiating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \cot x \times \frac{1}{\tan x} \times \sec^2 x - \log_e \tan x \times \text{cosec}^2 x = \text{cosec}^2 x(1 - \log_e \tan x)$$

Hence, $\frac{dy}{dx} = \text{cosec}^2 x(1 - \log_e \tan x \times y = \text{cosec}^2 x(1 - \log_e \tan x)(\tan x)^{\cot x}$

9. Question

Mark (√) against the correct answer in the following:

If $y=(\sin x)^{\log x}$ then $\frac{dy}{dx}=?$

A. $(\log x)\cdot(\sin x)^{(\log x-1)}\cdot\cos x$

B. $(\sin x)^{\log x}\cdot\left\{\frac{x\log x+\log\sin x}{x}\right\}$

C. $(\sin x)^{\log x}\cdot\left\{\frac{(x\log x)\cot x+\log\sin x}{x}\right\}$

D. none of these

Answer

Given that $y=(\sin x)^{\log_e x}$

Taking log both sides, we get

$\log_e y=\log_e x\times\log_e \sin x$ (Since $\log_a b^c=c\log_a b$)

Differentiating with respect to x, we get

$\frac{1}{y}\frac{dy}{dx}=\log_e x\times\frac{1}{\sin x}\times\cos x+\log_e \sin x\times\frac{1}{x}$

$=\frac{x\cot x\log_e x+\log_e \sin x}{x}$

Hence, $\frac{dy}{dx}=\frac{x\cot x\log_e x+\log_e \sin x}{x}\times y$

$=\frac{x\cot x\log_e x+\log_e \sin x}{x}(\sin x)^{\log_e x}$

10. Question

Mark (√) against the correct answer in the following:

If $y=\sin(x^x)$ then $\frac{dy}{dx}=?$

A. $x^x\cos(x^x)$

B. $x^x\cos x^x(1+\log x)$

C. $x^x\cos x^x\log x$

D. none of these

Answer

Given that y=sin(x^x)

Let x^x=u, then y=sin u

Differentiating with respect to x, we get

$\frac{dy}{dx}=\cos u\times\frac{du}{dx}=\cos(x^x)\frac{du}{dx}$ -(1)

Also, u=x^x

Taking log both sides, we get

$\text{Log}_e u = x \times \log_e x$

$(\text{Since } \log_a b^c = c \log_a b)$

Differentiating with respect to x, we get

$\frac{1}{u}\frac{du}{dx} = x \times \frac{1}{x} + \log_e x \times 1$

$\Rightarrow \frac{du}{dx} = u \times (1 + \log_e x)$

$\Rightarrow \frac{du}{dx} = x^x(1 + \log_e x)$ -(2)

From (1) and (2), we get

$\frac{dy}{dx} = \cos(x^x)\, x^x(1 + \log_e x)$

11. Question

Mark (√) against the correct answer in the following:

If $y = \sqrt{x \sin x}$ then $\frac{dy}{dx} = ?$

A. $\frac{(x \cos x + \sin x)}{2\sqrt{x \sin x}}$

B. $\frac{1}{2}(x \cos x + \sin x) \cdot \sqrt{x \sin x}$

C. $\frac{1}{2\sqrt{x \sin x}}$

D. none of these

Answer

Given that $y = \sqrt{x\sin x}$

Squaring both sides, we get

$y^2 = x\sin x$

Differentiating with respect to x, we get

$2y\frac{dy}{dx} = x\cos x + \sin x$ or $\frac{dy}{dx} = \frac{x\cos x + \sin x}{2y}$

Hence, $\frac{dy}{dx} = \frac{x\cos x + \sin x}{2\sqrt{x\sin x}}$

12. Question

Mark (√) against the correct answer in the following:

If $e^{x+y} = xy$ then $\frac{dy}{dx} = ?$

A. $\frac{x(1-y)}{y(x-1)}$

B. $\frac{y(1-x)}{x(y-1)}$

C. $\frac{(x-xy)}{(xy-y)}$

D. none of these

Answer

Given that $xy=e^{x+y}$

Taking log both sides, we get

$\log_e xy = x + y$ (Since $\log_a b^c = c\log_a b$)

Since $\log_a bc = \log_a b + \log_a c$, we get

$\log_e x + \log_e y = x + y$

Differentiating with respect to x, we get

$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 + \frac{dy}{dx}$

Or

$\frac{dy}{dx}\left(\frac{y-1}{y}\right) = \frac{1-x}{x}$

Hence, $\frac{dy}{dx} = \frac{y(1-x)}{x(y-1)}$

13. Question

Mark (√) against the correct answer in the following:

If $(x+y) = \sin(x+y)$ then $\frac{dy}{dx} = ?$

A. -1

B. 1

C. $\frac{1-\cos(x+y)}{\cos^2(x+y)}$

D. none of these

Answer

Given that x+y=sin(x+y)

Differentiating with respect to x, we get

$1 + \frac{dy}{dx} = \cos(x+y)\left(1 + \frac{dy}{dx}\right)$ or $(\cos(x+y) - 1)\left(1 + \frac{dy}{dx}\right) = 0$

Hence, cos(x+y)=1 or $\frac{dy}{dx} = -1$

If cos(x+y)=1 then, x+y=2nπ, n∈ℤ

Hence x+y=sin(2nπ)=0 or y=-x

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -1$$

Hence, $\frac{dy}{dx} = -1$

14. Question

Mark (√) against the correct answer in the following:

If $\sqrt{x} + \sqrt{y} = \sqrt{a}$ then $\frac{dy}{dx} = ?$

A. $\frac{-\sqrt{x}}{\sqrt{y}}$

B. $-\frac{1}{2} \cdot \frac{\sqrt{y}}{\sqrt{x}}$

C. $\frac{-\sqrt{y}}{\sqrt{x}}$

D. None of these

Answer

Given that $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Differentiating with respect to x, we get

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

Or

$$\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

15. Question

Mark (√) against the correct answer in the following:

If $x^y = y^x$ then $\frac{dy}{dx} = ?$

A. $\frac{(y - x \log y)}{(x - y \log x)}$

B. $\frac{y(y - x \log y)}{x(x - y \log x)}$

C. $\frac{y(y + x \log y)}{x(x + y \log x)}$

D. none of these

Answer

Given that $x^y = y^x$

Taking log both sides, we get

$y\log_e x = x\log_e y$

$(\text{Since } \log_a b^c = c\log_a b)$

Differentiating with respect to x, we get

$$\frac{y}{x} + \log_e x\frac{dy}{dx} = \frac{x}{y}\frac{dy}{dx} + \log_e y$$

$$\Rightarrow \frac{x - y\log_e x}{y}\frac{dy}{dx} = \frac{y - x\log_e y}{x}$$

Hence $\frac{dy}{dx} = \frac{y(y - x\log_e y)}{x(x - y\log_e x)}$

16. Question

Mark (√) against the correct answer in the following:

If $x^p y^q = (x+y)^{(p+q)}$ then $\frac{dy}{dx} = ?$

A. $\frac{x}{y}$

B. $\frac{y}{x}$

C. $\frac{x^{p-1}}{y^{q-1}}$

D. none of these

Answer

Given that $x^p y^q = (x+y)^{p+q}$

Taking log both sides, we get

$\log_e x^p y^q = (p+q)\log_e(x+y)$

$(\text{Since } \log_a b^c = c\log_a b)$

Since $\log_a bc = \log_a b + \log_a c$, we get

$\log_e x^p + \log_e y^q = (p+q)\log_e(x+y)$

$p\log_e x + q\log_e y = (p+q)\log_e(x+y)$

Differentiating with respect to x, we get

$$\frac{p}{x} + \frac{q}{y}\frac{dy}{dx} = \frac{p+q}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{xq - yp}{y(x+y)}\right) = \frac{xq - yp}{x(x+y)}$$

Hence, $\frac{dy}{dx} = \frac{y}{x}$

17. Question

Mark (√) against the correct answer in the following:

If $y = x^2 \sin\frac{1}{x}$ then $\frac{dy}{dx} = ?$

A. $x\sin\frac{1}{x} - \cos\frac{1}{x}$

B. $-\cos\frac{1}{x} + 2x\sin\frac{1}{x}$

C. $-x\sin\frac{1}{x} + \cos\frac{1}{x}$

D. None of these

Answer

Given that $y = x^2 \sin\frac{1}{x}$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = x^2\cos\frac{1}{x} \times -\frac{1}{x^2} + 2x\sin\frac{1}{x} = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

18. Question

Mark (√) against the correct answer in the following:

If $y = \cos^2 x^3$ then $\frac{dy}{dx} = ?$

A. $-3x^2 \sin$

B. $-3x^2 \sin^2 x^3$

C. $-3x^2 \cos^2\left(2x^3\right)$

D. none of these

Answer

$y = \cos^2 x^3 = (\cos(x^3))^2$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 2\cos(x^3) \times -\sin(x^3) \times 3x^2$$

Using 2sinAcosA=sin2A

$$\frac{dy}{dx} = -3x^2\sin(2x^3)$$

19. Question

Mark (√) against the correct answer in the following:

If $y = \log\left(x + \sqrt{x^2 + a^2}\right)$ then $\frac{dy}{dx} = ?$

A. $\frac{1}{2\left(x + \sqrt{x^2 + a^2}\right)}$

B. $\dfrac{-1}{\sqrt{x^2+a^2}}$

C. $\dfrac{1}{\sqrt{x^2+a^2}}$

D. none of these

Answer

Given that $y=\log_e\left(x+\sqrt{x^2+a^2}\right)$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\left(1+\frac{1}{2\sqrt{x^2+a^2}}\times 2x\right)$$

Hence, $\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\times\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}=\frac{1}{\sqrt{x^2+a^2}}$

20. Question

Mark (√) against the correct answer in the following:

If $y=\log\left(\dfrac{1+\sqrt{x}}{1-\sqrt{x}}\right)$ then $\dfrac{dy}{dx}=?$

A. $\dfrac{1}{\sqrt{x}(1-x)}$

B. $\dfrac{-1}{x\left(1-\sqrt{x}\right)^2}$

C. $\dfrac{-\sqrt{x}}{2\left(1-\sqrt{x}\right)}$

D. none of these

Answer

Given that $y=\log_e\frac{1+\sqrt{x}}{1-\sqrt{x}}$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=\frac{1-\sqrt{x}}{1+\sqrt{x}}\times\frac{(1-\sqrt{x})\times\frac{1}{2\sqrt{x}}-(1+\sqrt{x})\times-\frac{1}{2\sqrt{x}}}{(1-\sqrt{x})^2}=\frac{1}{(1-x)\sqrt{x}}$$

21. Question

Mark (√) against the correct answer in the following:

If $y=\log\left(\dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right)$ then $\dfrac{dy}{dx}=?$

A. $\frac{2}{\sqrt{1+x^2}}$

B. $\frac{2\sqrt{1+x^2}}{x^2}$

C. $\frac{-2}{\sqrt{1+x^2}}$

D. none of these

Answer

Given that $y = \log_e\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right)$

Differentiating with respect to x, we get

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}+x} \times \frac{\left(\sqrt{1+x^2}-x\right)\times\left(\frac{1}{2\sqrt{1+x^2}}\times 2x+1\right)-\left(\sqrt{1+x^2}+x\right)\times\left(\frac{1}{2\sqrt{1+x^2}}\times 2x-1\right)}{\left(\sqrt{1+x^2}-x\right)^2}$$

Hence, $\frac{dy}{dx} = \frac{2}{\sqrt{1+x^2}}$

22. Question

Mark (√) against the correct answer in the following:

If $y = \sqrt{\frac{1+\sin x}{1-\sin x}}$ then $\frac{dy}{dx} = ?$

A. $\frac{1}{2}\sec^2\left(\frac{\pi}{4}-\frac{\pi}{2}\right)$

B. $\frac{1}{2}\operatorname{cosec}^2\left(\frac{\pi}{4}-\frac{\pi}{2}\right)$

C. $\frac{1}{2}\operatorname{cosec}^2\left(\frac{\pi}{4}-\frac{\pi}{2}\right)\cot\left(\frac{\pi}{4}-\frac{\pi}{2}\right)$

D. none of these

Answer

Given that $y = \sqrt{\frac{1+\sin x}{1-\sin x}}$

Using, $\cos^2\theta + \sin^2\theta = 1$ and $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$y = \sqrt{\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}$$

$$= \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}$$

Dividing by $\sin\frac{x}{2}$ in numerator and denominator, we get

$y = \frac{\cot\frac{x}{2} + 1}{\cot\frac{x}{2} - 1} = \cot\left(\frac{\pi}{4} - \frac{x}{2}\right)$

$\left(\text{Using } \cot\left(\frac{\pi}{4} - A\right) = \frac{\cot A + 1}{\cot A - 1}\right)$

Differentiating with respect to x, we get

$\frac{dy}{dx} = -\text{cosec}^2\left(\frac{\pi}{4} - \frac{x}{2}\right) \times -\frac{1}{2}$

Hence, $\frac{dy}{dx} = \frac{1}{2}\text{cosec}^2\left(\frac{\pi}{4} - \frac{x}{2}\right)$

23. Question

Mark (√) against the correct answer in the following:

If $y = \sqrt{\frac{\sec x - 1}{\sec x + 1}}$ then $\frac{dy}{dx} = ?$

A. $\sec^2 x$

B. $\frac{1}{2}\sec^2\frac{x}{2}$

C. $\frac{-1}{2}\text{cosec}^2\frac{x}{2}$

D. none of these

Answer

Given that $y = \sqrt{\frac{\sec x - 1}{\sec x + 1}}$

Multiplying by cos x in numerator and denominator, we get

$y = \sqrt{\frac{1 - \cos x}{1 + \cos x}}$

Using $1 - \cos x = 2\sin^2\frac{x}{2}$ and $1 + \cos x = 2\cos^2\frac{x}{2}$, we get

$y = \sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}$

$= \tan\left(\frac{x}{2}\right)$

Differentiating with respect to x, we get

$y = \sec^2\frac{x}{2} \times \frac{1}{2}$

$= \frac{1}{2}\sec^2\frac{x}{2}$

24. Question

Mark (√) against the correct answer in the following:

If $y=\sqrt{\frac{1+\tan x}{1-\tan x}}$ then $\frac{dy}{dx}=?$

A. $\frac{1}{2}\sec^2 x\cdot\tan\left(x+\frac{\pi}{4}\right)$

B. $\frac{\sec^2\left(x+\frac{\pi}{4}\right)}{2\sqrt{\tan\left(x+\frac{\pi}{4}\right)}}$

C. $\frac{\sec^2\left(\frac{x}{4}\right)}{\sqrt{\tan\left(x+\frac{\pi}{4}\right)}}$

D. none of these

Answer

Given that $y=\sqrt{\frac{1+\tan x}{1-\tan x}}$

Using $\tan\left(\frac{\pi}{4}+x\right)=\frac{1+\tan x}{1-\tan x}$, we get

$$y=\sqrt{\tan\left(\frac{\pi}{4}+x\right)}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\tan\left(\frac{\pi}{4}+x\right)}}\times\sec^2\left(\frac{\pi}{4}+x\right)\times 1$$

Hence, $\frac{dy}{dx}=\frac{\sec^2\left(\frac{\pi}{4}+x\right)}{2\sqrt{\tan\left(\frac{\pi}{4}+x\right)}}$

25. Question

Mark (√) against the correct answer in the following:

If $y=\tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$ then $\frac{dy}{dx}=?$

A. 1

B. -1

C. $\frac{1}{2}$

D. $\frac{-1}{2}$

Answer

Given that $y=\tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$

Using $1-\cos x = 2\sin^2\frac{x}{2}$ and Using $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$, we get

$y = \tan^{-1}\left(\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$ or $y = \tan^{-1}\tan\frac{x}{2}$

$y = \frac{x}{2}$

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{1}{2}$

26. Question

Mark (√) against the correct answer in the following:

If $y = \tan^{-1}\left\{\frac{\cos x + \sin x}{\cos x - \sin x}\right\}$ then $\frac{dy}{dx} = ?$

A. 1

B. -1

C. $\frac{1}{2}$

D. $\frac{-1}{2}$

Answer

Given that $y = \tan^{-1}\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)$

Dividing numerator and denominator with cosx, we get

$y = \tan^{-1}\left(\frac{1+\tan x}{1-\tan x}\right)$

Using $\tan\left(\frac{\pi}{4}+x\right) = \frac{1+\tan x}{1-\tan x}$, we get

$y = \tan^{-1}\tan\left(\frac{\pi}{4}+x\right) = \frac{\pi}{4}+x$

Differentiating with respect to x, we get

$\frac{dy}{dx} = 1$

27. Question

Mark (√) against the correct answer in the following:

If $y = \tan^{-1}\left\{\frac{\cos x}{1+\sin x}\right\}$ then $\frac{dy}{dx} = ?$

A. $\frac{1}{2}$

B. $\frac{-1}{2}$

C. 1

D. -1

Answer

Given that $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

Using $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$, $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $\cos^2\theta + \sin^2\theta = 1$

Hence, $y = \tan^{-1}\left(\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}\right) = \tan^{-1}\left(\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\right)$

$$\Rightarrow\ y = \tan^{-1}\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}$$

Dividing by $\cos\frac{x}{2}$ in numerator and denominator, we get

$$y = \tan^{-1}\frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}$$

Using $\tan\left(\frac{\pi}{4} - x\right) = \frac{1-\tan x}{1+\tan x}$, we get

$$y = \tan^{-1}\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

$$= \frac{\pi}{4} - \frac{x}{2}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -\frac{1}{2}$$

28. Question

Mark (√) against the correct answer in the following:

If $y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$ then $\frac{dy}{dx} =$

A. $\frac{1}{2}$

B. $\frac{-1}{2}$

C. $\frac{1}{\left(1+x^2\right)}$

D. none of these

Answer

Given that $y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$

Using $1 - \cos x = 2\sin^2\frac{x}{2}$ and $1 + \cos x = 2\cos^2\frac{x}{2}$, we get

$$y = \tan^{-1}\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}} = \tan^{-1}\tan\left(\frac{x}{2}\right) = \frac{x}{2}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{2}$$

29. Question

Mark (√) against the correct answer in the following:

If $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$ then $\frac{dy}{dx} = ?$

A. $\frac{a}{b}$

B. $\frac{-b}{a}$

C. 1

D. -1

Answer

Given that $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$

Dividing by bcosx in numerator and denominator, we get

$$y = \tan^{-1}\left(\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right)$$

Let $\frac{a}{b} = \tan\alpha \Rightarrow \alpha = \tan^{-1}\frac{a}{b}$

Then $y = \tan^{-1}\left(\frac{\tan\alpha - \tan x}{1 + \tan\alpha\tan x}\right)$

Using $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$, we get

$$y = \tan^{-1}\tan(\alpha - x) = \alpha - x = \tan^{-1}\frac{a}{b} - x$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -1$$

30. Question

Mark (√) against the correct answer in the following:

If $y = \sin^{-1}\left(3x - 4x^3\right)$ then $\frac{dy}{dx} = ?$

A. $\frac{3}{\sqrt{1 - x^2}}$

B. $\frac{-4}{\sqrt{1-x^2}}$

C. $\frac{3}{\sqrt{1+x^2}}$

D. none of these

Answer

Given that $y=\sin^{-1}(3x-4x^3)$

Let $x=\sin\theta$

$\Rightarrow \theta=\sin^{-1}x$

Then, $y=\sin^{-1}(3\sin\theta-4\sin^3\theta)$

Using $\sin3\theta=3\sin\theta-4\sin^3\theta$, we get

$y=\sin^{-1}(\sin3\theta)=3\theta=3\sin^{-1}x$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=\frac{3}{\sqrt{1-x^2}}$$

31. Question

Mark (√) against the correct answer in the following:

If $y=\cos^{-1}\left(4x^3-3x\right)$ then $\frac{dy}{dx}=?$

A. $\frac{3}{\sqrt{1-x^2}}$

B. $\frac{-3}{\sqrt{1-x^2}}$

C. $\frac{4}{\sqrt{1-x^2}}$

D. $\frac{4}{\left(3x^2-1\right)}$

Answer

Given that $y=\cos^{-1}(4x^3-3x)$

Let $x=\cos\theta$

$\Rightarrow \theta=\cos^{-1}x$

Then, $y=\cos^{-1}(4\cos^3\theta-3\cos\theta)$

Using $\cos3\theta=4\cos^3\theta-3\cos\theta$, we get

$y=\cos^{-1}(\cos3\theta)=3=3\cos^{-1}x$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}$$

32. Question

Mark (√) against the correct answer in the following:

If $y = \tan^{-1}\left(\frac{\sqrt{a}+\sqrt{x}}{1-\sqrt{ax}}\right)$ then $\frac{dy}{dx} = ?$

A. $\frac{1}{(1+x)}$

B. $\frac{1}{\sqrt{x}(1+x)}$

C. $\frac{2}{\sqrt{x}(1+x)}$

D. $\frac{1}{2\sqrt{x}(1+x)}$

Answer

Given that $y = \tan^{-1}\frac{\sqrt{a}+\sqrt{x}}{1-\sqrt{ax}}$

Let $\sqrt{a} = \tan A$ and $\sqrt{x} = \tan B$, then $A = \tan^{-1}\sqrt{a}$ and $B = \tan^{-1}\sqrt{x}$

Hence, $y = \tan^{-1}\frac{\tan A+\tan B}{1-\tan A\tan B}$

Using $\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$, we get

$y=\tan^{-1}\tan(A+B)=A+B$

$= \tan^{-1}\sqrt{a} + \tan^{-1}\sqrt{x}$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 0 + \frac{1}{1+(\sqrt{x})^2} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(1+x)}$$

33. Question

Mark (√) against the correct answer in the following:

If $y = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$ then $\frac{dy}{dx} = ?$

A. $\frac{2}{(1+x^2)}$

B. $\frac{-2}{(1+x^2)}$

C. $\frac{2x}{\left(1+x^2\right)}$

D. none of these

Answer

Given that $y = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$

$\Rightarrow \cos y = \frac{x^2-1}{x^2+1}$ or $\sec y = \frac{x^2+1}{x^2-1}$

Since $\tan^2 x = \sec^2 x - 1$, therefore

$$\tan^2 y = \left(\frac{x^2+1}{x^2-1}\right)^2 - 1$$

$$= \frac{4x^2}{(x^2-1)^2}$$

Hence, $\tan y = -\frac{2x}{1-x^2}$ or $y = \tan^{-1}\left(-\frac{2x}{1-x^2}\right)$

Let x=tanθ

$\Rightarrow \theta = \tan^{-1} x$

Hence, $y = \tan^{-1}\left(-\frac{2\tan\theta}{1-\tan^2\theta}\right)$

Using $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$, we get

$y = \tan^{-1}(-\tan 2\theta)$

Using -tan x = tan(-x), we get

$y = \tan^{-1}(\tan(-2\theta))$

$= -2\theta$

$= -2\tan^{-1} x$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{-2}{1+x^2}$$

34. Question

Mark (√) against the correct answer in the following:

If $y = \tan^{-1}\left(\frac{1+x^2}{1-x^2}\right)$ then $\frac{dy}{dx} = ?$

A. $\frac{2x}{\left(1+x^4\right)}$

B. $\frac{-2x}{\left(1+x^4\right)}$

C. $\frac{x}{\left(1+x^4\right)}$

D. none of these

Answer

Given that $y=\tan^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

Let $x^2=\tan\theta$

$\Rightarrow \theta=\tan^{-1}x^2$

Hence, $y=\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$

Using $\tan\left(\frac{\pi}{4}+x\right)=\frac{1+\tan x}{1-\tan x}$, we get

$$y=\tan^{-1}\tan\left(\frac{\pi}{4}+\theta\right)=\frac{\pi}{4}+\theta=\frac{\pi}{4}+\tan^{-1}(x^2)$$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=\frac{1}{1+x^4}\times 2x=\frac{2x}{1+x^4}$$

35. Question

Mark (√) against the correct answer in the following:

If $y=\tan^{-1}\left(-\sqrt{x}\right)$

A. $\frac{-1}{(1+x)}$

B. $\frac{2}{\sqrt{(1+x)}}$

C. $\frac{-1}{2\sqrt{x}\,(1+x)}$

D. none of these

Answer

Given that $y=\tan^{-1}(-\sqrt{x})$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=\frac{1}{1+\left(-\sqrt{x}\right)^2}\times\frac{-1}{2\sqrt{x}}=\frac{-1}{2\sqrt{x}(1+x)}$$

36. Question

Mark (√) against the correct answer in the following:

If $y=\cos^{-1}x^3$ then $\frac{dy}{dx}=?$

A. $\frac{-1}{\sqrt{1-x^6}}$

B. $\dfrac{-3x^2}{\sqrt{1-x^6}}$

C. $\dfrac{-3}{x^2\sqrt{1-x^6}}$

D. none of these

Answer

Given that $y=\cos^{-1}x^3$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=\frac{-1}{\sqrt{1-(x^3)^2}}\times 3x^2=\frac{-3x^2}{\sqrt{1-x^6}}$$

37. Question

Mark (√) against the correct answer in the following:

If $y=\tan^{-1}(\sec x+\tan x)$ then $\dfrac{dy}{dx}=?$

A. $\dfrac{1}{2}$

B. $\dfrac{-1}{2}$

C. 1

D. none of these

Answer

Given that $y=\tan^{-1}(\sec x + \tan x)$

Hence, $y=\tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)$

Using $\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}$, $\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$ and $\cos^2\theta+\sin^2\theta=1$

Hence, $y=\tan^{-1}\left(\frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}\right)=\tan^{-1}\left(\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}\right)$

$\Rightarrow y=\tan^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right)$

Dividing by $\cos\frac{x}{2}$ in numerator and denominator, we get

$$y=\tan^{-1}\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$$

Using $\tan\left(\frac{\pi}{4}+x\right)=\frac{1+\tan x}{1-\tan x}$, we get

$$y=\tan^{-1}\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=\frac{\pi}{4}+\frac{x}{2}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{2}$$

38. Question

Mark (√) against the correct answer in the following:

If $y = \cot^{-1}\left(\frac{1-x}{1+x}\right)$ then $\frac{dy}{dx} = ?$

A. $\frac{-1}{(1+x^2)}$

B. $\frac{1}{(1+x^2)}$

C. $\frac{-1}{(1+x^2)^{3/2}}$

D. none of these

Answer

Given that $y = \cot^{-1}\left(\frac{1-x}{1+x}\right)$

Let x=tanθ ⇒ θ=$\tan^{-1}$x and using $\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

Hence, $y = \frac{\pi}{2} - \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$

Using $\tan\left(\frac{\pi}{4} - x\right) = \frac{1-\tan x}{1+\tan x}$, we get

$$y = \frac{\pi}{2} - \tan^{-1}\tan\left(\frac{\pi}{4} - \theta\right) = \frac{\pi}{2} - \left(\frac{\pi}{4} - \theta\right) = \frac{\pi}{4} + \theta = \frac{\pi}{4} + \tan^{-1}x$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

39. Question

Mark (√) against the correct answer in the following:

If $y = \sqrt{\frac{1+x}{1-x}}$ then $\frac{dy}{dx} = ?$

A. $\frac{2}{(1-x)^2}$

B. $\frac{x}{(1-x)^{3/2}}$

C. $\frac{1}{(1-x)^{3/2} \cdot (1+x)^{1/2}}$

D. none of these

Answer

Given that $y = \sqrt{\frac{1+x}{1-x}}$

Let x=-cosθ ⇒ θ=$\cos^{-1}$(-x).

Using $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$ and $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$, we get

$$y = \sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}} = \tan\left(\frac{\theta}{2}\right)$$

Differentiating with respect to x, we get

$\frac{dy}{dx} = \sec^2\left(\frac{\theta}{2}\right) \times \frac{1}{2}\frac{d\theta}{dx}$ -(1)

Since, x=-cosθ ⇒ $2\cos^2\frac{\theta}{2} = 1 + \cos\theta = 1 - x$ or $\sec^2\left(\frac{\theta}{2}\right) = \frac{2}{1-x}$ -(2)

Also, since θ=$\cos^{-1}$(-x), therefore $\frac{d\theta}{dx} = \frac{1}{\sqrt{1-x^2}}$ -(3)e

Substituting (2) and (3) in (1), we get

$$\frac{dy}{dx} = \frac{2}{1-x} \times \frac{1}{2} \times \frac{1}{\sqrt{1-x^2}} = \frac{1}{(1-x)\sqrt{1-x^2}} = \frac{1}{(1-x)^{\frac{3}{2}}(1+x)^{\frac{1}{2}}}$$

40. Question

Mark (√) against the correct answer in the following:

If $y = \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$ then $\frac{dy}{dx} = ?$

A. $\frac{-2}{(1+x^2)}$

B. $\frac{2}{(1+x^2)}$

C. $\frac{-1}{(1+x^2)}$

D. none of these

Answer

Given that $y = \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

$$\Rightarrow \sec y = \frac{x^2+1}{x^2-1}$$

Since $\tan^2x=\sec^2x-1$, therefore

$$\tan^2 y = \left(\frac{x^2+1}{x^2-1}\right)^2 - 1 = \frac{4x^2}{(x^2-1)^2}$$

Hence, $\tan y = -\frac{2x}{1-x^2}$ or $y = \tan^{-1}\left(-\frac{2x}{1-x^2}\right)$

Let x=tanθ ⇒ θ=$\tan^{-1}$x

Hence, $y = \tan^{-1}\left(-\frac{2\tan\theta}{1-\tan^2\theta}\right)$

Using $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$, we get

$y = \tan^{-1}(-\tan 2\theta)$

Using -tan x=tan(-x), we get

$y = \tan^{-1}(\tan(-2\theta)) = -2\theta = -2\tan^{-1}x$

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{-2}{1+x^2}$

41. Question

Mark (√) against the correct answer in the following:

If $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$ then $\frac{dy}{dx} = ?$

A. $\frac{-2}{(1+x^2)}$

B. $\frac{-2}{(1-x^2)}$

C. $\frac{-2}{\sqrt{1+x^2}}$

D. none of these

Answer

$\Rightarrow\ y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$\Rightarrow\ \sec y\ = \frac{1}{2x^2-1}$

⇒ cos y=2x^2 -1

⇒ y=$\cos^{-1}$ (2x^2 -1)

Put x = cos θ

⇒ y = $\cos^{-1}$(2 $\cos^2$ θ – 1)

⇒ y = cos $^{-1}$(cos 2 θ)

⇒ y = 2θ

But θ = $\cos^{-1}$x.

$\Rightarrow \frac{dy}{dx} = \frac{d(\cos^{-1}x)}{dx}$

$\Rightarrow \frac{dy}{dx} = 2.\frac{d(\cos^{-1}x)}{dx}$

$$\Rightarrow \frac{dy}{dx} = 2.\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

42. Question

Mark (√) against the correct answer in the following:

If $y = \tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}$ then $\frac{dy}{dx} = ?$

A. $\frac{1}{\left(1+x^2\right)}$

B. $\frac{2}{\left(1+x^2\right)}$

C. $\frac{1}{2\left(1+x^2\right)}$

D. none of these

Answer

Put x = tan θ

$$\Rightarrow y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\Rightarrow y = \frac{\theta}{2}$$

$\theta = \tan^{-1}x$

$$\Rightarrow y = \frac{\tan^{-1}x}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2(1+x^2)}$$

43. Question

Mark (√) against the correct answer in the following:

If $y = \sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\}$ then $\frac{dy}{dx} = ?$

A. $\frac{-1}{2\sqrt{1-x^2}}$

B. $\frac{1}{2\sqrt{1-x^2}}$

C. $\frac{1}{2\sqrt{1+x^2}}$

D. none of these

Answer

Put x = cos2θ

$\Rightarrow\ y = \sin^{-1}\left(\frac{\sqrt{1+\cos 2\theta}}{2} + \frac{\sqrt{1-\cos 2\theta}}{2}\right)$

$\Rightarrow\ y = \sin^{-1}\left(\frac{\sqrt{2\cos^2 2\theta}}{2} + \frac{\sqrt{2.\sin^2\theta}}{2}\right)$

$\Rightarrow\ y = \sin^{-1}\left(\frac{\cos 2\theta}{\sqrt{2}} + \frac{\sin 2\theta}{\sqrt{2}}\right)$

$\Rightarrow\ y = \sin^{-1}(\sin\left(\frac{\pi}{4} + 2\theta\right)$

$\Rightarrow\ y = \frac{\pi}{4} + 2\theta.$

$\Rightarrow\ \frac{dy}{d\theta} = 2$

Put $\theta = \frac{\cos^{-1}x}{2}$

$\Rightarrow\ \frac{d\theta}{dx} = \frac{-1}{4\sqrt{1-x^2}}$

$\therefore\ \frac{dy}{dx} = \frac{-1}{2\sqrt{1-x^2}}$

44. Question

Mark (√) against the correct answer in the following:

If $x = at^2, y = 2at$ then $\frac{dy}{dx} = ?$

A. $\frac{1}{t}$

B. $\frac{-1}{t^2}$

C. $\frac{-2}{t}$

D. none of these

Answer

x = at^2

$\therefore \frac{dx}{dt} = 2at$

$\therefore \frac{dt}{dx} = \frac{1}{2at}$

Y = 2at

$\therefore \frac{dy}{dt} = 2a$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\Rightarrow \frac{dy}{dx} = 2a \times \frac{1}{2at}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{t}$

45. Question

Mark (√) against the correct answer in the following:

If $x = a \sec \theta, y = b \tan \theta$ then $\frac{dy}{dx} = ?$

A. $\frac{b}{a} \sec \theta$

B. $\frac{b}{a} \operatorname{cosec} \theta$

C. $\frac{b}{a} \cot \theta$

D. none of these

Answer

x = a sec θ

$\therefore \frac{dx}{d\theta} = a\sec\theta . \tan\theta$

$\therefore \frac{d\theta}{dx} = \frac{1}{a\sec\theta . \tan\theta}$

y = b tan θ

$\therefore \frac{dy}{d\theta} = b. \sec^2\theta$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$\Rightarrow \frac{dy}{dx} = b. \sec^2\theta \times \frac{1}{a\sec\theta . \tan\theta}$

$\Rightarrow \frac{dy}{dx} = \frac{b \sec\theta}{a \tan\theta}$

$\Rightarrow \frac{dy}{dx} = \frac{b.\frac{1}{\cos\theta}}{a.\frac{\sin\theta}{\cos\theta}}$

$\Rightarrow \frac{dy}{dx} = \frac{b}{a}\csc\theta$

46. Question

Mark (√) against the correct answer in the following:

If $x = a\cos^2\theta, y = b\sin^2\theta$ then $\frac{dy}{dx} = ?$

A. $\frac{-a}{b}$

B. $\frac{-a}{b}\cot\theta$

C. $\frac{-b}{a}$

D. none of these

Answer

x = a.$\cos^2\theta$

$\therefore \frac{dx}{d\theta} = -2a\cos\theta.\sin\theta$

$\Rightarrow \frac{d\theta}{dx} = \frac{-1}{2a.\cos\theta.\sin\theta}$

y = b.$\sin^2\theta$

$\therefore \frac{dy}{d\theta} = 2b\sin\theta.\cos\theta$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$\Rightarrow \frac{dy}{dx} = 2b\sin\theta.\cos\theta \times \frac{-1}{2a\cos\theta.\sin\theta}$

$\Rightarrow \frac{dy}{dx} = \frac{-b}{a}$

47. Question

Mark (√) against the correct answer in the following:

If $x = \theta(\cos\theta + \sin\theta)$ and $y=a(\sin\theta - \theta\cos\theta)$ then $\frac{dy}{dx} = ?$

A. cot θ

B. tan θ

C. a cot θ

D. a tan θ

Answer

x = a(cos θ + θ sin θ)

$\therefore \frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta)$

$\Rightarrow \frac{d\theta}{dx} = \frac{1}{a\theta\cos\theta}$

y = a(sin θ – θ cos θ)

$\therefore \frac{dy}{d\theta} = a(\cos\theta - (\cos\theta + \theta(-\sin\theta))$

$\Rightarrow \frac{dy}{d\theta} = a\cos\theta - a\cos\theta + \theta a\sin\theta$

$\Rightarrow \frac{dy}{d\theta} = a\theta\sin\theta$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$\Rightarrow \frac{dy}{dx} = a\theta\sin\theta \times \frac{1}{a\theta\cos\theta}$

$\Rightarrow \frac{dy}{dx} = \tan\theta$

48. Question

Mark (√) against the correct answer in the following:

If $y = x^{x^{x\ldots\infty}}$ then $\frac{dy}{dx} = ?$

A. $\frac{y}{x(1-\log x)}$

B. $\frac{y^2}{x(1-\log x)}$

C. $\frac{y}{x(1-y\log x)}$

D. none of these

Answer

Given:

$\Rightarrow y = x^{x^{x^{x\ldots\infty}}}$

We can write it as

⇒ $y=x^y$

Taking log of both sides we get

log y = y log x

Differentiating

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\log x + y.\frac{1}{x}$

$\Rightarrow \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left(\frac{y}{1 - \log x}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1 - \log x)}$

49. Question

Mark (√) against the correct answer in the following:

If $y = \sqrt{x + \sqrt{x + \sqrt{x +}}} \ldots\infty$ then $\frac{dy}{dx} = ?$

A. $\frac{1}{(2y - 1)}$

B. $\frac{1}{(y^2 - 1)}$

C. $\frac{2y}{(y^2 - 1)}$

D. none of these

Answer

Given:

$\Rightarrow y = \sqrt{x + \sqrt{x + \sqrt{x +}}} \ldots\infty$

We can write it as

$\Rightarrow y = \sqrt{x + y}$

Squaring we get

$\Rightarrow y^2 = x + y$

Differentiating

$\Rightarrow 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{(2y - 1)}$

50. Question

Mark (√) against the correct answer in the following:

If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x +}}} \ldots\infty$ then $\frac{dy}{dx} = ?$

A. $\frac{\sin x}{(2y - 1)}$

B. $\frac{\cos x}{(y-1)}$

C. $\frac{\cos x}{(2y-1)}$

D. none of these

Answer

Given:

$$\Rightarrow\ y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+}}}\ldots\infty$$

We can write it as

$$\Rightarrow\ y=\sqrt{\sin x+y}$$

Squaring we get

⇒ y^2=sin x + y

Differentiating

$$\Rightarrow\ 2y\frac{dy}{dx}=\cos x+\frac{dy}{dx}$$

$$\Rightarrow\ \frac{dy}{dx}=\frac{\cos x}{(2y-1)}$$

51. Question

Mark (√) against the correct answer in the following:

If $y=e^{x}+e^{x+\ldots\infty}$ then $\frac{dy}{dx}=?$

A. $\frac{1}{(1-y)}$

B. $\frac{y}{(1-y)}$

C. $\frac{y}{(y-1)}$

D. none of these

Answer

We can write it as

⇒ $y=e^{x+y}$

logy = (x + y) log e

Differentiating

$$\Rightarrow\ \frac{1}{y}\frac{dy}{dx}=1+\frac{dy}{dx}$$

$\Rightarrow \left(\frac{1}{y} - 1\right)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = 1\left(\frac{y}{1-y}\right)$

52. Question

Mark (√) against the correct answer in the following:

The value of k for which $f(x) = \begin{cases} \frac{\sin 5x}{3x}, \text{ if } x \neq 0 \\ k, \text{if } x = 0 \end{cases}$ is continuous at x = 0 is

A. $\frac{1}{3}$

B. 0

C. $\frac{3}{5}$

D. $\frac{5}{3}$

Answer

Since f(x) is continuous on 0.

$\Rightarrow \lim_{x \to 0} \frac{\sin 5x}{3x} = f(0)$

$\Rightarrow \lim_{x \to 0} \frac{\sin 5x}{3x} \times \frac{5x}{5x} = f(0)$

$\Rightarrow \lim_{x \to 0} \frac{\sin 5x}{5x} \times \frac{5x}{3x} = f(0)$

$\Rightarrow f(0) = \frac{5}{3}$

$\Rightarrow k = \frac{5}{3}$

53. Question

Mark (√) against the correct answer in the following:

Let $f(x) = \begin{cases} x \sin\frac{1}{x}, \text{ if } x \neq 0 \\ 0, \text{ when } x = 0. \end{cases}$

Then, which of the following is the true statement?

A. $f(x)$ is not defined at x = 0

B. $\lim_{x \to 0} f(x)$ does not exist

C. $f(x)$ is continuous at x = 0

D. $f(x)$ is discontinuous at x = 0

Answer

Left hand limit =

$\Rightarrow \lim_{x \to 0^-} f(x)$

$\Rightarrow \lim_{h \to 0} f(0-h)$

$\Rightarrow \lim_{h \to 0} h.\sin\left(\frac{-1}{h}\right)$

$\Rightarrow \lim_{h \to 0} -h.\frac{\sin\left(\frac{-1}{h}\right)}{-\frac{1}{h}} \times \frac{-1}{h} = 1$

Right hand limit =

$\Rightarrow \lim_{x \to 0^+} f(x)$

$\Rightarrow \lim_{h \to 0} f(0+h)$

$\Rightarrow \lim_{h \to 0} h.\sin\left(\frac{1}{h}\right)$

$\Rightarrow \lim_{h \to 0} h.\frac{\sin\left(\frac{1}{h}\right)}{\frac{1}{h}} \times \frac{1}{h}$

= 1

As L.H.L = R.H.L

F(x) is continuous.

54. Question

Mark (√) against the correct answer in the following:

The value of k for which $f(x) = \begin{cases} \frac{3x + 4\tan x}{2}, & \text{when } x \neq 0 \\ k, & \text{when } x = 0 \end{cases}$ is continuous at x = 0, is

A. 7

B. 4

C. 3

D. none of these

Answer

$\Rightarrow f(x) = \frac{3x+4\tan x}{x}$ is continuous at x = 0.

$\Rightarrow f(x) = \lim_{x \to 0} \frac{3x + 4\tan x}{x}$

$\Rightarrow f(x) = \lim_{x \to 0} \frac{3x}{x} + \frac{4\tan x}{x}$

$\Rightarrow f(x) = 3 + 4\lim_{x \to 0} \frac{\tan x}{x}$

⇒ f(x) = 3+4

∴ K = 7.

55. Question

Mark (√) against the correct answer in the following:

Let $f(x) = x^{3/2}$. Then, $f'(0) = ?$

A. $\frac{3}{2}$

B. $\frac{1}{2}$

C. does not exist

D. none of these

Answer

f(x) = $x^{3/2}$

$\Rightarrow f'(x) = \frac{3}{2\sqrt{x}}$

As x→0, f'(x)→∞

∴ f'(x) does not exist.

56. Question

Mark (√) against the correct answer in the following:

The function $f(x) = |x| \forall x \in R$ is

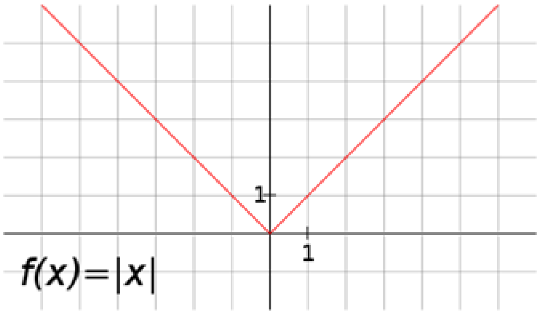
A. continuous but not differentiable at x = 0

B. differentiable but not continuous at x = 0

C. neither continuous nor differentiable at x = 0

D. none of these

Answer



(Sometimes it's easier to get the answer by graphs)

Now in the above graph

We can see f(x) is Continuous on 0.

But it has sharp curve on x = 0 which implies it is not differentiable.

57. Question

Mark (√) against the correct answer in the following:

The function $f\left(x\right)=\begin{cases}1+x, \text{ when } x\le 2\\ 5-x, \text{ when } x>2\end{cases}$ is

A. continuous as well as differentiable at x = 2

B. continuous but not differentiable at x = 2

C. differentiable but not continuous at x = 2

D. none of these

Answer

For continuity left hand limit must be equal to right hand limit and value at the point.

Continuity at x =2.

For continuity at x=2,

L.H.L = $\lim\limits_{x\to 2^-}(1+x)=3$

R.H.L = $\lim\limits_{x\to 2^+}(5-x)=3$

f(2) = 1+2 = 3

∴ f(x) is continuous at x = 2

Now for differentiability.

$\Rightarrow\ f'(2^-)=\lim\limits_{x\to 2^-}\frac{f(x)-f(2)}{x-2}$

$\Rightarrow\ f'(2^-)=\lim\limits_{h\to 0}\frac{f(2-h)-f(2)}{2-h-2}$

$\Rightarrow\ f'(2^-)=\lim\limits_{h\to 0}\frac{1+2-h-3}{2-h-2}=\lim\limits_{h\to 0}\frac{-h}{-h}=1.$

$\Rightarrow\ f'(2^+)=\lim\limits_{x\to 2^-}\frac{f(x)-f(2)}{x-2}$

$\Rightarrow\ f'(2^+)=\lim\limits_{h\to 0}\frac{f(2+h)-f(2)}{2+h-2}$

$\Rightarrow\ f'(2^-)=\lim\limits_{h\to 0}\frac{5-(2-h)-3}{2+h-2}$

$=\lim\limits_{h\to 0}\frac{h}{-h}$

=-1

As, f'(2^-) is not equal to f(2^+)

∴ f(x) is not differentiable.

58. Question

Mark (√) against the correct answer in the following:

If $f\left(x\right)=\begin{cases}kx+5, \text{ when } x\le 2\\ x+1, \text{ when } x>2\end{cases}$ is continuous at x = 2 then k = ?

A. 2

B. -2

C. 3

D. -3

Answer

For continuity left hand limit must be equal to right hand limit and value at the point.

Continuous at x =2.

L.H.L = $\lim_{x \to 2^-}(kx+5)$

⇒ $\lim_{h \to 0}(k(2-h)+5)$

⇒ k(2-0)+5 = 2k+5

R.H.L = $\lim_{x \to 2^+}(x+1)$

⇒ $\lim_{h \to 0}(2+h+1)$

⇒ 2+0+1

=3

As f(x) is continuous

∴ 2k+5 = 3

K = -1.

59. Question

Mark (√) against the correct answer in the following:

If the function $f(x)=\begin{cases}\dfrac{1-\cos 4x}{8x^2}, x \neq 0 \\ k, x=0\end{cases}$ is continuous at x = 0 and then k = ?

A. 1

B. 2

C. $\frac{1}{2}$

D. $\frac{-1}{2}$

Answer

Given:

⇒ $f(x) = \frac{1-\cos 4x}{8x^2}$ is continuous at x = 0.

⇒ 1-cos4x = $2\sin^2 2x$

⇒ $f(x) = \lim_{x \to 0}\frac{2\sin^2 2x}{8x^2}$

⇒ $f(x) = \lim_{x \to 0}\frac{2\sin^2 2x}{2 \times 4x^2}$

⇒ $f(x) = \lim_{x \to 0}\left(\frac{\sin 2x}{2x}\right)^2$

⇒ f(x) = 1

∴ Kk= 1

60. Question

Mark (√) against the correct answer in the following:

If the function $f(x)=\begin{cases}\frac{\sin^2 ax}{x^2}, \text{ when } x \neq 0\\ k, \text{ when } x = 0\end{cases}$ is continuous at x = 0 then k = ?

A. a

B. a^2

C. -2

D. -4

Answer

F(x) is continuous at x = 0.

$\Rightarrow\ f(x) = \lim_{x\to 0}\frac{\sin^2 ax}{x^2}$

$\Rightarrow\ f(x) = \lim_{x\to 0}\frac{\sin^2 ax}{x^2}\times\frac{a^2}{a^2}$

$\Rightarrow\ f(x) = \lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)^2\times a^2$

⇒ f(x) = a^2

∴ k = a^2

61. Question

Mark (√) against the correct answer in the following:

If the function $f(x)=\begin{cases}\frac{k\cos x}{(\pi-2x)}, \text{ when } x \neq \frac{\pi}{2}\\ 3, \text{ when } x = \frac{\pi}{2}\end{cases}$ be continuous at $x=\frac{\pi}{2}$, then the value of k is

A. 3

B. -3

C. -5

D. 6

Answer

Given: f(x) is continuous at x= π/2.

$\therefore\ \text{L.H.L} = \lim_{x\to\frac{\pi}{2}^-} f(x)$

$= \lim_{x\to\frac{\pi}{2}^-}\frac{k\cos x}{\pi-2x}$

Putting$x = \frac{\pi}{2} - h;$

As $x \to \frac{\pi}{2}^{-}$ then $h \to 0$.

$$\therefore \lim_{x \to \frac{\pi}{2}} \frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi - 2\left(\frac{\pi}{2}-h\right)} = k.\lim_{h \to 0}\frac{\sin h}{h}$$

∴ L.H.L = k

As it is continuous which implies right hand limit equals left hand limit equals the value at that point.

∴ k=3.

62. Question

Mark (√) against the correct answer in the following:

At $x = 2,\ f(x) = |x|$ is

A. continuous but not differentiable

B. differentiable but not continuous

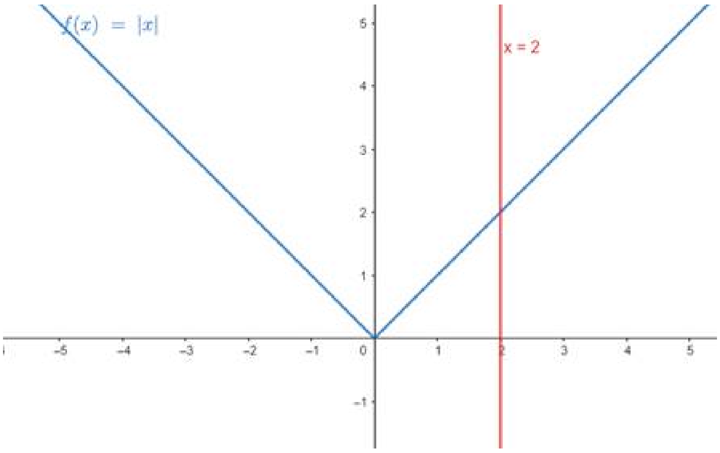
C. continuous as well as differentiable

D. none of these

Answer

Given:

Let us see that graph of the modulus function.



We can see that f(x) = |x| is neither continuous and nor differentiable at x = 2. Hence, D is the correct answer.

63. Question

Mark (√) against the correct answer in the following:

Let $f(x) = \begin{cases} \dfrac{x^2 - 2x - 3}{x + 1^2}, & \text{when } x \neq -1 \\ k, & \text{when } x = -1 \end{cases}$

If $f(x)$ is continuous at x = -1 then k = ?

A. 4

B. -4

C. -3

D. 2

Answer

$\Rightarrow\ f(x) = \frac{x^2-2x-3}{x+1}$ is continuous at x = 0.

$\Rightarrow\ f(x) = \lim_{x\to -1} \frac{(x+1)(x-3)}{x+1}$

$\Rightarrow\ f(x) = \lim_{x\to -1} x-3$

⇒ f(x) = -4

∴ K = 1.

64. Question

Mark (√) against the correct answer in the following:

The function $f(x) = x^3 + 6x^2 + 15x - 12$ is

A. strictly decreasing on R

B. strictly increasing on R

C. increasing in $(-\infty, 2)$ and decreasing in $(2, \infty)$

D. none of these

Answer

Given:

f(x) = $x^3+6x^2+15x-12$.

f'(x) = $3x^2+12x+15$

f'(x) = $3x^2+12x+12+3$

f'(x) = $3(x^2+4x+4)+3$

f'(x) = $3(x+2)^2+3$

As square is a positive number

∴ f'(x) will be always positive for every real number

Hence f'(x) >0 for all x ϵ R

∴ f(x) is strictly increasing.

65. Question

Mark (√) against the correct answer in the following:

The function $f(x) = 4 - 3x + 3x^2 - x^3$ is

A. decreasing on R

B. increasing on R

C. strictly decreasing on R

D. strictly increasing on R

Answer

f(x) = $-x^3+3x^2-3x+4$.

f'(x) = $-3x^2+6x-3$

f’(x) = -3(x^2-2x+1)

f’(x) = -3$(x-1)^2$

As f’(x) has –ve sign before 3

⇒ f’(x) is decreasing over R.

66. Question

Mark (√) against the correct answer in the following:

The function $f(x) = 3x + \cos 3x$ is

A. increasing on R

B. decreasing on R

C. strictly increasing on R

D. strictly decreasing on R

Answer

Given:

f(x) = 3x+cos3x

f’(x) = 3-3sin3x

f’(x) = 3(1-sin3x)

sin3x varies from[-1,1]

when sin3x is 1 f’(x) = 0 and sin3x is -1 f’(x) = 6

As the function is increasing in 0 to 6.

∴ The function is increasing on R.

67. Question

Mark (√) against the correct answer in the following:

The function $f(x) = x^3 + 6x^2 + 9x + 3$ is decreasing for

A. 1 < x < 3

B. x > 1

C. x < 1

D. x < 1 or x > 3

Answer

Given:

f(x) = x^3+6x^2+9x+3.

f’(x) = 3x^2+12x+9 = 0

f’(x) = 3(x^2+4x+3) =0

f’(x) = 3(x+1)(x+3) = 0

x = -1 or x =-3

for x>-1 f(x) is increasing

for x<-3 f(x) is increasing

But for -1<x<-3 it is decreasing.

68. Question

Mark (√) against the correct answer in the following:

The function $f(x) = x^3 - 27x + 8$ is increasing when

A. $|x| < 3$

B. $|x| > 3$

C. -3 < x < 3

D. none of these

Answer

Given:

f(x) = x^3-27x+8.

f'(x) = $3x^2$-27x = 0

f'(x) = 3(x^2-9) =0

f'(x) = 3(x-3)(x+3) = 0

x = 3 or x =-3

for x>3 f(x) is increasing

for x<-3 f(x) is increasing

∴ for |x|>3 f(x) is increasing.

69. Question

Mark (√) against the correct answer in the following:

$f(x) = \sin x$ is increasing in

A. $\left(\frac{\pi}{2}, \pi\right)$

B. $\left(\pi, \frac{3\pi}{2}\right)$

C. $(0, \pi)$

D. $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Answer

Given: f(x) is sin x

∴ f'(x) = cos x

⇒ f'(x) =cos x

= 0

⇒ for $x \epsilon \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

f'(x) is increasing

$\therefore$ f(x) is increasing in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

70. Question

Mark (√) against the correct answer in the following:

$f(x) = \frac{2x}{\log x}$ is increasing in

A. (0, 1)

B. (1, e)

C. (e, ∞)

D. (-∞, e)

Answer

$\Rightarrow f(x) = \frac{2x}{\log x}$

$\Rightarrow f'(x) = \frac{2.\log x - 2}{\log^2 x}$

Put f'(x) = 0

We get

$\Rightarrow \frac{2.\log x - 2}{\log^2 x} = 0$

⇒ 2.logx = 2

log x = 1

⇒ x= e

We only have one critical point

So, we can directly say x>e f(x) would be increasing

∴ f(x) will be increasing in (e, ∞)

71. Question

Mark (√) against the correct answer in the following:

$f(x) = (\sin x - \cos x)$ is decreasing in

A. $\left(0, \frac{3\pi}{4}\right)$

B. $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$

C. $\left(\frac{7\pi}{4}, 2\pi\right)$

D. none of these

Answer

Given:

f(x) = sin x – cos x

f'(x) = cos x + sin x

Multiply and divide by √2.

$$\Rightarrow \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$$

$$\Rightarrow \sqrt{2}\left(\sin\frac{\pi}{4}.\cos x + \cos\frac{\pi}{4}.\sin x\right)$$

$$\Rightarrow \sqrt{2}(\sin\left(\frac{\pi}{4} + x\right))$$

$$\Rightarrow f'(x) = \sqrt{2}\sin\left(\frac{\pi}{4} + x\right)$$

For f(x) to be decreasing .f'(x)<0

$$\Rightarrow f'(x) = \sqrt{2}\sin\left(\frac{\pi}{4} + x\right) < 0$$

$$\Rightarrow \pi < x + \frac{\pi}{4} < 2\pi$$

(∵ sin θ <0 for π <θ <2π)

$$\Rightarrow \pi - \frac{\pi}{4} < x < 2\pi - \frac{\pi}{4}$$

$$\Rightarrow \frac{3\pi}{4} < x < \frac{7\pi}{4}$$

∴ f(x) decreases in the interval.

$$\Rightarrow \left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$$

72. Question

Mark (√) against the correct answer in the following:

$f(x) = \frac{x}{\sin x}$ is

A. increasing in (0, 1)

B. decreasing in (0, 1)

C. increasing in $\left(0, \frac{1}{2}\right)$ and decreasing in $\left(\frac{1}{2}, 1\right)$

D. none of these

Answer

$$\Rightarrow f(x) = \frac{x}{\sin x}$$

$$\Rightarrow f'(x) = \frac{\sin x - x\cos x}{\sin x}$$

Now see

In (0,1) sin x is increasing and cos x is decreasing

sin x – x cos x will be increasing

∴ f(x) is increasing in (0,1)

73. Question

Mark (√) against the correct answer in the following:

$f(x) = x^x$ is decreasing in the interval

A. (0, e)

B. $\left(0, \frac{1}{e}\right)$

C. (0,1)

D. none of these

Answer

Given: $f(x) = x^x$.

⇒ $f'(x)=(\log x+1)\, x^x$

⇒ keeping f'(x) = 0

We get

⇒ $x = 0 \text{ or } x = \frac{1}{e}$

Now

When x>1/e the function is increasing

x<0 function is increasing.

But in the interval (0,1/e) the function is decreasing.

74. Question

Mark (√) against the correct answer in the following:

$f(x) = x^2 e^{-x}$ is increasing in

A. (-2, 0)

B. (0, 2)

C. (2, ∞)

D. (-∞, ∞)

Answer

Given $f(x) = x^2.e^{-x}$

⇒ $f'(x) = 2x.\, e^{-x} – x^2\, e^{-x}$

⇒ Put f'(x) = 0

⇒ $- (x^2 – 2x)e^{-x} = 0$

⇒ x = 0 or x =2.

Now as there is a -ve sign before f'(x)

When x>2 the function is decreasing

x<0 function is decreasing

But in the interval (0,2) the function is increasing.

75. Question

Mark (√) against the correct answer in the following:

$f(x) = \sin x - kx$ is decreasing for all $x \notin R$, when

A. k < 1

B. k ≤ 1

C. k > 1

D. k ≤ 1

Answer

f(x) =sin x - kx

f'(x) = cos x –k

∴ f decreases, if f'(x)≤ 0

⇒ cos x – k ≤ 0

⇒ cos x≤ k

So, for decreasing k≥ 1.

76. Question

Mark (√) against the correct answer in the following:

$f(x) = (x+1)^3 (x-3)^3$ is increasing in

A. (-∞, 1)

B. (-1, 3)

C. (3, ∞)

D. (1, ∞)

Answer

Given:

⇒ $f(x) = (x+1)^3.(x-3)^3$

⇒ $f'(x) = 3(x+1)^2(x-3)^3 + 3(x-3)^3 (x+1)^3$

Put f'(x) = 0

⇒ $3(x+1)^2(x-3)^3 = -3(x-3)^2(x+1)^3$

⇒ x-3 = -(x+1)

⇒ 2x = 2

⇒ x =1

When x>1 the function is increasing.

x<1 function is decreasing.

So, f(x) is increasing in (1, ∞).

77. Question

Mark (√) against the correct answer in the following:

$f(x)=[x(x-3)]^2$ is increasing in

A. (0, ∞)

B. (-∞, 0)

C. (1, 3)

D. $\left(0,\frac{3}{2}\right)\cup(3,\infty)$

Answer

⇒ $f(x)=[x(x-3)]^2$

⇒ f' (x)=2[x(x-3)] =0

⇒ $x = 3 \text{ and } x = \frac{3}{2}$

When x> 3/2 the function is increasing

X<3 function is increasing.

⇒ $\left(0,\frac{3}{2}\right) \cup (3,\infty)$ Function is increasing.

78. Question

Mark (√) against the correct answer in the following:

If $f(x)=kx^3-9x^2+9x+3$ is increasing for every real number x, then

A. k > 3

B. k ≥ 3

C. k < 3

D. k ≤ 3

Answer

Given $f(x) = kx^3 - 9x^2 + 9x+3$

⇒ $f'(x) = 3kx^2-18x+9$

⇒ $f'(x) = 3(kx^2 - 6x + 3)>0$

⇒ $kx^2 - 6x + 3 >0$

For quadratic equation to be greater than 0. a>0 and D<0.

⇒ k>0 and $(-6)^2 - 4(k)(3)<0$

⇒ 36 – 12k<0

⇒ 12k>36

⇒ k>3

∴ k>3.

79. Question

Mark (√) against the correct answer in the following:

$f(x) = \frac{x}{(x^2 - 1)}$ is increasing in

A. (-1, 1)

B. (-1, ∞)

C. $(-\infty, -1) \cup (1, \infty)$

D. none of these

Answer

$\Rightarrow \ f(x) = \frac{x}{x^2 + 1}$

$\Rightarrow \ f'(x) = \frac{x^2 - 2x^2 + 1}{x^2 + 1}$

$\Rightarrow \ f'(x) = -\frac{x^2 - 1}{x^2 + 1}$

⇒ For critical points f'(x) = 0

When f'(x) = 0

We get x = 1 or x = -1

When we plot them on number line as f'(x) is multiplied by –ve sign we get

For x>1 function is decreasing

For x<-1 function is decreasing

But between -1 to 1 function is increasing.

∴ Function is increasing in(-1,1).

80. Question

Mark (√) against the correct answer in the following:

The least value of k for which $f(x) = x^2 + kx + 1$ is increasing on (1, 2), is

A. -2

B. -1

C. 1

D. 2

Answer

f(x) $=x^2$+kx+1

For increasing

f'(x) = 2x+k

k≥ -2x

thus,

k≥ -2.

Least value of -2.

81. Question

Mark (√) against the correct answer in the following:

$f(x) = |x|$ has

A. minimum at x = 0

B. maximum x = 0

C. neither a maximum nor a minimum at x = 0

D. none f these

Answer

f(x) = |x|

Now to check the maxima and minima at x =0.

It can be easily seen through the option.

See |x| is x for x>0 and –x for x<0

That is no matter if you put a number greater than zero or number less than zero you will get positive answer.

∴ for x = 0 we will get minima.

82. Question

Mark (√) against the correct answer in the following:

When x is positive, the minimum value of x^x is

A. e^e

B. $e^{1/e}$

C. $e^{-1/e}$

D. $(1/e)$

Answer

Given: f(x) = x^x.

⇒ f' (x)=(log x+1) x^x

⇒ keeping f'(x) = 0

We get

$$\Rightarrow\ x = 0 \text{ or } x = \frac{1}{e}$$

$$\Rightarrow\ f''(x) = x^x(1 + \log x)\left[1 + \log x + \frac{1}{x(1 + \log x)}\right]$$

When x is greater than zero,

We get a maximum value as the function will be negative.

Therefore,

F(x) = x^x

F(e) = $\left(\frac{1}{e}\right)^{1/e} = e^{-\frac{1}{e}}$

Hence, C is the correct answer.

83. Question

Mark (√) against the correct answer in the following:

The maximum value of $\left(\frac{\log x}{x}\right)$ is

A. $\left(\frac{1}{e}\right)$

B. $\frac{2}{e}$

C. e

D. 1

Answer

$\Rightarrow f(x) = \frac{\log x}{x}$

$\therefore f'(x) = \frac{\log x - x.\frac{1}{x}}{x^2}$

⇒ f'(x) = log x – 1

⇒ Put f'(x) = 0

We get x =e

F''(x) = 1/x

Put x =e in f''(X)

1/e is point of maxima

∴ The max value is 1/e.

84. Question

Mark (√) against the correct answer in the following:

$f(x) = \operatorname{cosec} x$ in (-π, 0) has a maxima at

A. x = 0

B. $x = \frac{-\pi}{4}$

C. $x = \frac{-\pi}{3}$

D. $x = \frac{-\pi}{2}$

Answer

We can go through options for this question

Option a is wrong because 0 is not included in (-π,0)

At x = -π/4 value of f(x) is -√2 = -1.41

At x = -π/3 value of f(x) is -2.

At x = -π/2 value of f(x) is -1.

∴ f(x) has max value at x =-π/2.

Which is -1.

85. Question

Mark (√) against the correct answer in the following:

If x > 0 and xy = 1, the minimum value of (x + y) is

A. -2

B. 1

C. 2

D. none of these

Answer

Given: x>0 and xy = 1

We need to find the minimum value of (x + y).

$\Rightarrow\ y = \frac{1}{x}$

$\Rightarrow\ f(x) = x + \frac{1}{x}$

$\Rightarrow\ f(x) = \frac{x^2 + 1}{x}$

$\Rightarrow\ f'(x) = \frac{x.2x - (x^2 + 1).1}{x^2}$

$\Rightarrow\ f'(x) = \frac{x^2 - 1}{x^2}$

$\Rightarrow\ f''(x) = \frac{x^2(2x) - (x^2 - 1).2x}{x^4}$

$\Rightarrow\ f''(x) = \frac{2x}{x^4}$

$\Rightarrow\ f''(x) = \frac{2}{x^3}$

For maximum or minimum value f’(x) = 0.

$\therefore\ \frac{x^2 - 1}{x^2} = 0$

∴ x = 1 or x =-1

f’’(x) at x = 1.

∴ f’’(x) = 2.

F’’(x)>0 it is decreasing and has minimum value at x = 1

At x = -1

f’’(x) = -2

f’’(x)<0 it is increasing and has maximum value at x = -1.

∴ Substituting x = 1 in f(x) we get

f(x) = 2.

∴ The minimum value of given function is 2.

86. Question

Mark (√) against the correct answer in the following:

The minimum value of $\left(x^2 + \frac{250}{x}\right)$ is

A. 0

B. 25

C. 50

D. 75

Answer

$\Rightarrow f(x) = x^2 + \frac{250}{x}$

$\Rightarrow f'(x) = 2x - \frac{250}{x^2} = 0$

⇒ $2x^3 = 250$

⇒ $x^3 = 125$

⇒ x = 5

Substituting x = 5 in f(x) we get

f(x) = 25+50

f(x) = 75.

87. Question

Mark (√) against the correct answer in the following:

The minimum value of $f(x) = 3x^4 - 8x^3 - 48x + 25$ on [0, 3] is

A. 16

B. 25

C. -39

D. none of these

Answer

Given:

$f(x) = 3x^4 - 8x^3 - 48x + 25$.

$F'(x) = 12x^3 - 24x^2 - 48 = 0$

$F'(x) = 12(x^3 - 2x^2 - 4) = 0$

Differentiating again, we get,

$F''(x) = 3x^2 – 4x = 0$

x(3x – 4) = 0

x = 0 or x = 4/3

Putting the value in equation, we get,

f(x) = -39

Hence, C is the correct answer.

88. Question

Mark (√) against the correct answer in the following:

The maximum value of $f(x) = (x-2)(x-3)^2$ is

A. $\frac{7}{3}$

B. 3

C. $\frac{4}{27}$

D. 0

Answer

$f(x) = (x-2)(x-3)^2$

$f(x) = (x-2)(x^2-6x+9)$

$f(x) = x^3-8x^2+21x-18.$

$f'(x) = 3x^2-16x+21$

$f''(x) = 6x-16$

For maximum or minimum value f'(x) = 0.

$\therefore 3x^2-9x-7x+21 = 0$

⇒ 3x(x-3)-7(x-3)=0

⇒ x = 3 or x =7/3.

f''(x) at x = 3.

∴ f''(x) = 2

f''(x)>0 it is decreasing and has minimum value at x = 3

At x = 7/3

F''(x) = -2

F''(x)<0 it is increasing and has maximum value at x = 7/3.

Substituting x = 7/3 in f(x) we get

$\Rightarrow \left(\frac{7}{3}-2\right)\left(\frac{7}{3}-3\right)^2$

$\Rightarrow \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)^2$

$\Rightarrow \frac{4}{27}$

89. Question

Mark (√) against the correct answer in the following:

The least value of $f(x) = \left(e^x + e^{-x}\right)$ is

A. -2

B. 0

C. 2

D. none of these

Answer

f(x) = $e^X + e^{-X}$

$\Rightarrow\ f(x) = e^x + \frac{1}{e^x}$

$\Rightarrow\ f(x) = \frac{e^{2x}+1}{e^x}$

f(x) is always increasing at x = 0it has the least value

$\Rightarrow\ f(x) = \frac{1+1}{1} = 2$

∴ The least value is 2.