CLASS: XI : CHAPTER : PRINCIPLE OF MATHEMATICAL INDUCTION

EXERCISE 4.1

Prove the following by using the Principle of Mathematical induction for all $n \in N$

QNo. 1 $$1+3+3^2+\cdots 3^{n-1} = \frac{3^n-1}{2}$$

Sol. We are to prove that

$$1+3+3^2+\cdots 3^{n-1} = \frac{3^n-1}{2} \quad \text{-----}(1)$$

for $n=1$ LHS $=1$

RHS $= \frac{3^1-1}{2} = \frac{3-1}{2} = \frac{2}{2} = 1$

$\therefore$ LHS = RHS.

$\therefore$ Result (1) is true for $n=1$

Assume that result (1) is true for $n=k$.

$$\therefore \quad 1+3+3^2+\cdots 3^{k-1} = \frac{3^k-1}{2}$$

Adding $(k+1)^{th}$ term $= 3^{k+1-1} = 3^k$ on both sides, we get

$$1+3+3^2+\cdots 3^{k-1}+3^k = \frac{3^k-1}{2}+3^k = \frac{3^k-1+2.3^k}{2}$$

$$= \frac{3.3^k-1}{2} = \frac{3^{k+1}-1}{2}$$

$\Rightarrow$ The result is true for $n=k+1$

$\therefore$ By Method of Induction (1) is true for all $n \in N$

as if result is true for $n=k$, then it is also true for $n=k+1$

Q No 2. $1^3+2^3+3^3+ \cdots\cdots +n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Sol. We have to prove that

$$1^3+2^3+3^3+ \cdots\cdots n^3 = \left[\frac{n(n+1)}{2}\right]^2 \quad \cdots\cdots(1)$$

for $n=1$

$$LHS = 1^3 = 1$$

$$RHS = \frac{1^2(1+1)^2}{4} = \frac{1\times4}{4} = 1.$$

$\therefore$ LHS = RHS

$\therefore$ Result is true for $n=1$

Assume the result (1) is true for $n=k$.

$$\therefore 1^3+2^3+3^3+\cdots\cdots k^3 = \left[\frac{k(k+1)}{2}\right]^2$$

Adding $(k+1)^{th}$ term $=(k+1)^3$ on both sides.

$$1^3+2^3+3^3+\cdots\cdots k^3+(k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2+(k+1)^3$$

$$=(k+1)^2\left[\frac{k^2}{4}+(k+1)\right] = (k+1)^2\left[\frac{k^2+4k+4}{4}\right]$$

$$=(k+1)^2\left[\frac{k^2+4k+4}{4}\right] = \frac{(k+1)^2(k+2)^2}{4}$$

$$=\frac{(k+1)^2[(k+1)+1]^2}{2^2} = \left[\frac{(k+1)[(k+1)+1]}{2}\right]^2$$

$\Rightarrow$ (1) is true for $n=k+1$ also.

$\therefore$ By Method of Induction the result (1) is true $\forall n \in N$.

QNo 3. $1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots\frac{1}{1+2+3+\cdots n}=\frac{2n}{(n+1)}$

Sol. We have to prove

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots\frac{1}{1+2+3+\cdots n}=\frac{2n}{(n+1)} \quad \text{-----(1)}$$

For $n=1$ LHS $=1$

RHS $=\frac{2\times1}{1+1}=\frac{2}{2}=1$

$\therefore$ LHS $=$ RHS

$\therefore$ Result (1) is true for $n=1$

Assume the result (1) is true for $n=k$.

$$\therefore \quad 1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots\frac{1}{1+2+3+\cdots k}=\frac{2k}{k+1}.$$

Adding next term $=\frac{1}{1+2+3+\cdots k+(k+1)}$ to both sides, we get

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots\frac{1}{1+2+3\cdots k}+\frac{1}{1+2+3+\cdots k+(k+1)}=\frac{2k}{k+1}+\frac{1}{1+2+3+\cdots(k+1)}$$

$$=\frac{2k}{k+1}+\frac{1}{\frac{(k+1)(k+2)}{2}} \qquad \left[\because 1+2+3+\cdots n=\frac{n(n+1)}{2}\right]$$

$$=\frac{2k}{k+1}+\frac{2}{(k+1)(k+2)}=\frac{2}{k+1}\left[k+\frac{2}{k+2}\right]$$

$$=\frac{2}{k+1}\left[\frac{k(k+2)+2}{k+2}\right]=\frac{2}{(k+1)}\left[\frac{k^2+2k+2}{k+2}\right]$$

$$=\frac{2}{(k+1)}\left(\frac{(k+1)^2}{k+2}\right)=\frac{2(k+1)}{k+2}=\frac{2(k+1)}{(k+1)+1}.$$

$\therefore$ (1) is true for $n=k+1$ also

Hence by principle of Mathematical Induction (1) is true $\forall\, n\in N$.

Q No. 4. $1.2.3 + 2.3.4 + \ldots n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

Sol. We have to prove $1.2.3+2.3.4+\ldots n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ --- (1)

For $n=1$ LHS $= 1.2.3 = 1\times2\times3 = 6$

RHS $= \frac{1(1+1)(1+2)(1+3)}{4} = \frac{1\times2\times3\times4}{4} = 6$

$\therefore$ LHS = RHS.

$\therefore$ (1) is true for $n=1$

Assume (1) is true for $n=k$.

$\therefore 1.2.3+2.3.4+\ldots k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$.

Adding next term $(k+1)(k+2)(k+3)$ to both sides we get.

$1.2.3+2.3.4+\ldots k(k+1)(k+2)+(k+1)(k+2)(k+3)$

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$

$$= (k+1)(k+2)(k+3)\left[\frac{k}{4}+1\right]$$

$$= (k+1)(k+2)(k+3)\left(\frac{k+4}{4}\right)$$

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

$$= \frac{(k+1)[(k+1)+1][(k+1)+2][(k+1)+3]}{4}$$

$\therefore$ (1) is true for $n=k+1$ also

$\therefore$ If the result (1) is true for $n=k$, the it is also true for $n=k+1$

ie If the result (1) is true for any integer, then it is also true for next higher integer.

$\therefore$ By Method of Induction (1) is true $\forall n \in N$.

QNo 5 $1.3^1 + 2.3^2 + 3.3^3 + 4.3^4 + \cdots\cdots n.3^n = \frac{(2n-1)3^{n+1}+3}{4}$

Sol. We have to prove that

$$1.3 + 2.3^2 + 3.3^3 + \cdots\ n\,3^n = \frac{(2n-1)3^{n+1}+3}{4} \quad \text{---} (1)$$

For $n=1$

LHS $= 1.3 = 3$

RHS $= \frac{(2\times1-1)(3)^{1+1}+3}{4} = \frac{9+3}{4} = \frac{12}{4} = 3$

$\therefore$ LHS $=$ RHS

$\therefore$ (1) is true for $n=1$.

Assume (1) is true for $n=k$.

$$\therefore\ 1.3 + 2.3^2 + 3.3^3 + \cdots\ k\,3^k = \frac{(2k-1)3^{k+1}+3}{4}$$

Adding next term $(k+1)3^{k+1}$ to both sides, we get

$$1.3 + 2.3^2 + 3.3^3 + \cdots\ 3 + (k+1)3^{(k+1)} = \frac{(2k-1)3^{k+1}+3}{4} + (k+1)3^{k+1}$$

$$= \frac{(2k-1)3^{k+1}+3+4(k+1)3^{k+1}}{4}$$

$$= \frac{3^{k+1}\left[(2k-1)+4(k+1)\right]+3}{4} = \frac{3^{k+1}\left[2k-1+4k+4\right]+3}{4}$$

$$= \frac{3^{k+1}\left[6k+3\right]+3}{4} = \frac{3^{k+1}.3\left[2k+1\right]+3}{4} = \frac{3^{(k+1)+1}\left[2k+1-1+1\right]+3}{4}$$

$$= \frac{3^{(k+1)+1}\left[2(k+1)-1\right]+3}{4}$$

$\therefore$ (1) is true for $n=k+1$ also

$\therefore$ If the result is true for $n=k$, then it is true for $n=k+1$ also

Hence by Principle of Mathematical Induction (PMI)

(1) is true $\forall n \in N$.

QNo 6. $1\cdot2 + 2\cdot3 + 3\cdot4 \ldots\ldots n(n+1) = \left[\frac{n(n+1)(n+2)}{3}\right]$

Sol. We have to prove

$$1\cdot2 + 2\cdot3 + 3\cdot4 \ldots\ldots n(n+1) = \frac{n(n+1)(n+2)}{3} \quad \text{---(1)}$$

For $n=1$ LHS $= 1\cdot2 = 1\times2 = 2$

RHS $= \frac{1(1+1)(1+2)}{2} = 2$

$\therefore$ (1) is true for $n=1$

Assume (1) is true for $n=k$.

$$\therefore 1\cdot2 + 2\cdot3 + 3\cdot4 \ldots\ldots k(k+1) = \frac{k(k+1)(k+2)}{3}$$

Adding next term $(k+1)(k+2)$ to both sides, we get

$$1\cdot2 + 2\cdot3 + 3\cdot4 \ldots k(k+1) + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= (k+1)(k+2)\left[\frac{k}{3} + 1\right]$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$$= \frac{(k+1)\left[(k+1)+1\right]\left[(k+1)+2\right]}{3}$$

Which by comparison with (1) shows that result(1) is true for $n = k+1$ also.

$\therefore$ By method of induction (1) is true $\forall n \in N$.

Q No 7. $1\cdot 3 + 3\cdot 5 + 5\cdot 7 + \cdots (2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3}$

Sol: We have to prove that

$$1\cdot 3 + 3\cdot 5 + 5\cdot 7 \cdots (2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3} \quad \text{---} (1)$$

For $n=1$ LHS $= 1\cdot 3 = 3$

RHS $= (2\times 1-1)(2\times 1+1) = 1\times 3 = 3 =$ LHS.

$\therefore$ (1) is true for $n=1$

Assume (1) is true for $n=k$.

$$1\cdot 3 + 3\cdot 5 + 5\cdot 7 \cdots (2k-1)(2k+1) = \frac{k(4k^2+6k-1)}{3}.$$

Adding next term $[2(k+1)-1][2(k+1)+1]$ or $[(2k+1)(2k+3)]$ on both sides, we get

$$1\cdot 3 + 3\cdot 5 + 5\cdot 7 \cdots (2k-1)(2k+1) + (2k+1)(2k+3) = \frac{k(4k^2+6k-1)}{3} + (2k+1)(2k+3)$$

$$= \frac{1}{3}\left[(4k^3+6k^2-k) + 3(2k+1)(2k+3)\right]$$

$$= \frac{1}{3}\left[4k^3+6k^2-k+12k^2+24k+9\right]$$

$$= \frac{1}{3}\left[4k^3+18k^2+23k+9\right]$$

$$= \frac{1}{3}(k+1)(4k^2+14k+9) = \frac{1}{3}(k+1)\left[4(k+1)^2+6(k+1)-1\right]$$

Comparing with result (1) we see that result (1) is true for $n=k+1$ also.

$\therefore$ Result is true for $n=k+1$ whenever it is true for $n=k$

$\therefore$ By method of Induction, result is true $\forall\, n \in N$.

Q No 8. $1.2 + 2.2^2 + 3.2^3 + \dots \; n2^n = (n-1)2^{n+1} + 2$

We have to prove $1.2 + 2.2^2 + 3.2^3 + \dots \; n2^n = (n-1)2^{n+1} + 2$ — (1)

For $n = 1$ $\quad$ LHS $= 1.2 = 1 \times 2 = 2$

RHS $= (1-1)2^{1+1} + 2 = 0 + 2 = 2 =$ LHS.

$\therefore$ Result (1) is true $n = 1$

Assume (1) is true for $n = k$.

$$\therefore \; 1.2 + 2.2^2 + 3.2^3 + \dots \; k2^k = (k-1)2^{k+1} + 2$$

Adding next term $(k+1)2^{k+1}$ to both sides, we get

$$1.2 + 2.2^2 + 3.2^3 + \dots \; k2^k + (k+1)2^{k+1} = (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$$

$$= 2^{k+1}\left[(k-1) + (k+1)\right] + 2$$

$$= 2^{k+1}\left[2k\right] + 2$$

$$= k\, 2^{k+1} . 2^1 + 2$$

$$= k . 2^{(k+1)+1} + 2$$

$$= (k+1-1)\, 2^{(k+1)+1} + 2$$

$$= \left[(k+1) - 1\right] 2^{(k+1)+1} + 2$$

Comparing with result (1) we see (1) is true for $n = k+1$ also

$\therefore$ (1) is true for $n = k+1$, whenever (1) is true for $n = k$

$\therefore$ By method of Induction (1) is true for all $n \in N$.

Q No 9. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\frac{1}{2^n}=1-\frac{1}{2^n}$.

Sol. We are to prove that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\cdots\cdots\frac{1}{2^n}=1-\frac{1}{2^n} \quad \cdots\cdots (1)$$

For $n=1$ LHS $=\frac{1}{2}$

RHS $=1-\frac{1}{2^1}=1-\frac{1}{2}=\frac{1}{2}=$ LHS

$\therefore$ (1) is true for $n=1$

Assume result (1) is true for $n=k$.

$$\therefore \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\frac{1}{2^k}=1-\frac{1}{2^k}.$$

Adding next term $=\frac{1}{2^{k+1}}$ to both sides, we get

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^k}+\frac{1}{2^{k+1}}$$

$$=1-\frac{1}{2^k}+\frac{1}{2^k.2}=1-\frac{1}{2^k}+\frac{1}{2^k}\cdot\frac{1}{2}$$

$$=1-\frac{1}{2^k}\left(1-\frac{1}{2}\right)=1-\frac{1}{2^k}\cdot\frac{1}{2}$$

$$=1-\frac{1}{2^{k+1}}$$

Comparing this with result (1), we see that result (1) is true for $n=k+1$ also

$\therefore$ Result (1) is true for $n=k+1$, whenever it is true for $n=k$.

Hence by PMI, (1) is true for all $n\in N$.

Q No 10 $\frac{1}{2.5}+\frac{1}{5.8}+\cdots\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$

Sol. We have to prove that

$$\frac{1}{2.5}+\frac{1}{5.8}+\cdots\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4} \quad \text{----} (1)$$

For $n=1$ LHS $=\frac{1}{2.5}=\frac{1}{10}$

RHS $=\frac{1}{6\times1+4}=\frac{1}{10}=$ LHS

$\therefore$ (1) is true for $n=1$

Assume (1) is true for $n=k$

$$\therefore \frac{1}{2.5}+\frac{1}{5.8}+\cdots\frac{1}{(3k-1)(3k+2)}=\frac{k}{6k+4}$$

Adding next term $=\frac{1}{(3(k+1)-1)(3(k+1)+2)}=\frac{1}{(3k+2)(3k+5)}$

to both sides we get

$$\frac{1}{2.5}+\frac{1}{5.8}+\cdots\frac{1}{(3k-1)(3k+2)}+\frac{1}{(3k+2)(3k+5)}=\frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{1}{(3k+2)}\left[\frac{k}{2}+\frac{1}{3k+5}\right]=\frac{1}{(3k+2)}\left[\frac{3k^2+5k+2}{2(3k+5)}\right]$$

$$=\frac{3k^2+3k+2k+2}{2(3k+2)(3k+5)}=\frac{3k[k+1]+2[k+1]}{2(3k+2)(3k+5)}$$

$$=\frac{(k+1)(3k+2)}{2(3k+2)(3k+5)}=\frac{k+1}{6k+10}=\frac{k+1}{6k+6+4}$$

$$=\frac{(k+1)}{6(k+1)+4}$$

Comparing with (1) we find that (1) is true for $n=k+1$ also

Hence by PMI, (1) is true for $n=k+1$ also.

Q No 11 $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\cdots\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$

Sol We have to prove that $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\cdots\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$ ---(1)

For $n=1$ LHS $=\frac{1}{1\cdot2\cdot3}=\frac{1}{6}$

RHS $=\frac{1(1+3)}{4(1+1)(1+2)}=\frac{4}{4(2)(3)}=\frac{1}{6}=$ LHS

$\therefore$ (1) is true for $n=1$

Assume (1) is true for $n=k$, then

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\cdots\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)}$$

Adding next term $=\frac{1}{(k+1)(k+2)(k+3)}$ to both sides we get

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\cdots\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3}=$$

$$\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{1}{(k+1)(k+2)}\left[\frac{k(k+3)}{4}+\frac{1}{k+3}\right]=\frac{1}{(k+1)(k+2)}\left[\frac{k(k+3)^2+4}{4(k+3)}\right]$$

$$=\frac{k(k^2+9+6k)+4}{4(k+2)(k+3)(k+1)}=\frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$$

$$=\frac{(k+4)(k^2+1+2k)}{4(k+1)(k+2)(k+3)}=\frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}=\frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$=\frac{(k+1)[(k+1)+3]}{4((k+1)+1)((k+1)+2)}$$

Comparing with (1) we see that (1) is true for $n=k+1$ also

Hence by PMI (1) is true $\forall\, n\in N$.

Q No 12. $a + ar + ar^2 + \cdots ar^{n-1} = \dfrac{a(r^n - 1)}{r-1}$

Sol. We have to prove that

$$a + ar + ar^2 + \cdots ar^{n-1} = \frac{a(r^n-1)}{r-1} \quad \text{------} (1)$$

For $n=1$ LHS $= a$.

$$\text{RHS} = \frac{a(r^1-1)}{r-1} = \frac{a(r-1)}{(r-1)} = a = \text{LHS}.$$

$\therefore$ (1) is true for $n=1$

Assume (1) is true for $n=k$. then.

$$a + ar + \cdots ar^{k-1} = \frac{a(r^k-1)}{r-1}.$$

Adding next term ar^k to both sides, we get

$$a + ar + \cdots ar^{k-1} + ar^k = \frac{a(r^k-1)}{r-1} + ar^k$$

$$= a\left[\frac{r^k-1}{r-1} + r^k\right] = a\left[\frac{r^k - 1 + r^k(r-1)}{r-1}\right]$$

$$= a\left[\frac{r^k - 1 + r^{k+1} - r^k}{r-1}\right]$$

$$= \frac{a\left[r^{k+1} - 1\right]}{r-1}$$

comparing with result (1), we see that (1) is true for $n = k+1$ Also.

Hence by PMI, (1) is true for all $n \in N$.

Q No 13. $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\cdots\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2$.

Sol. We have to prove $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\cdots\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2$ ------(1)

For $n=1$ LHS $= 1+\frac{3}{1} = 1+3 = 4$

RHS $= (1+1)^2 = 2^2 = 4 =$ LHS.

$\therefore$ (1) is true for $n=1$

Assume (1) is true for $n=k$, we get

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\cdots\left(1+\frac{2k+1}{k^2}\right)=(k+1)^2.$$

Multiplying by next term $= \left(1+\frac{2k+3}{(k+1)^2}\right)$ we get

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\cdots\left(1+\frac{2k+1}{k^2}\right)\left(1+\frac{2k+3}{(k+1)^2}\right)$$

$$=(k+1)^2\left(1+\frac{2k+3}{(k+1)^2}\right)$$

$$=(k+1)^2\left[\frac{(k+1)^2+2k+3}{(k+1)^2}\right]$$

$$=k^2+1+2k+2k+4$$

$$=k^2+4k+4$$

$$=(k+2)^2$$

$$=[(k+1)+1]^2$$

Comparison with result (1) shows that (1) is true for $n=k+1$ also.

Hence by PMI, result (1) is true for all $n\in N$.

Q No 14. $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{n}\right)=(n+1)$

Sol. We have to prove that

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right)=n+1. \quad \text{----} \quad (1)$$

For $n=1$ LHS $= 1+\frac{1}{1} = 1+1 = 2.$

RHS $= 1+1 = 2 = $ LHS.

$\therefore$ (1) is true for $n=1$

Assume (1) is true for $n=k$, we get

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\cdots\left(1+\frac{1}{k}\right)=k+1$$

Multiplying both sides by next term $\left(1+\frac{1}{k+1}\right)$

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)+\cdots\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right)=(k+1)\left(1+\frac{1}{k+1}\right)$$

$$=(k+1)\left(\frac{k+1+1}{k+1}\right)$$

$$=\frac{(k+1)(k+2)}{(k+1)}$$

$$=k+2$$

$$=[(k+1)+1]$$

Comparison with (1) shows that (1) is true for $n=k+1$ also

Hence by PMI, result (1) is true $\forall n \in N$, Q No 15 $1^2+3^2+5^2+\dots(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Sol. We have to prove that

$$1^2+3^2+5^2+\dots(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3} \quad \text{---(1)}$$

For $n=1$ LHS $= 1^2 = 1$

RHS $= \frac{1(2\times1-1)(2\times1+1)}{3} = \frac{1\times1\times3}{3} = 1 =$ LHS

$\therefore$ Result (1) is true for $n=1$.

Assume result (1) is true for $n=k$, we get

$$1^2+3^2+5^2\dots(2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$$

Adding next term $[2(k+1)-1]^2 = (2k+1)^2$ to both sides, we get

$$1^2+3^2+5^2+\dots(2k-1)^2+(2k+1)^2 = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= (2k+1)\left[\frac{k(2k-1)}{3} + (2k+1)\right]$$

$$= (2k+1)\left[\frac{k(2k-1)+3(2k+1)}{3}\right] = \frac{(2k+1)[2k^2-k+6k+3]}{3}$$

$$= (2k+1)\left(\frac{2k^2+5k+3}{3}\right) = (2k+1)\left[\frac{2k^2+2k+3k+3}{3}\right]$$

$$= (2k+1)\left[\frac{2k[k+1]+3[k+1]}{3}\right] = \frac{(2k+1)[(k+1)(2k+3)]}{3}$$

$$= \frac{(k+1)[2(k+1)-1][2(k+1)+1]}{3}$$

Which by comparison with result (1) shows that (1) is true for $n=k+1$ also

Hence by PMI, (1) is true for all $n \in N$.

Q No 16 $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$

Sol. We have to prove that

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1} \quad \text{-----(1)}$$

For $n=1$

$$\text{LHS}=\frac{1}{1.4}=\frac{1}{4}$$

$$\text{RHS}=\frac{1}{(3\times1-2)(3\times1+1)}=\frac{1}{1\times4}=\frac{1}{4}=\text{LHS}.$$

$\therefore$ (1) is true for $n=1$

Assume result (1) is true for $n=k$, we get

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}.$$

Adding next term $=\frac{1}{(3k+1)(3k+4)}$ to both sides, we get

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3k+1)(3k+4)}=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k(3k+4)+1}{(3k+1)(3k+4)}=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$=\frac{(k+1)(3k+1)}{(3k+1)(3k+4)}=\frac{k+1}{3k+3+1}=\frac{k+1}{3(k+1)+1}$$

Which by comparison with result (1) shows that result (1) is true for $n=k+1$ also.

Hence by PMI, result (1) is true for all $n\in N$.

Q No. 17. $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\cdots\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$

Sol. We have to prove that

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\cdots\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)} \quad ----(1)$$

For $n=1$ LHS $=\frac{1}{3.5}=\frac{1}{15}$

RHS $=\frac{1}{3(2\times1+3)}=\frac{1}{3\times5}=\frac{1}{15}=$ LHS.

$\therefore$ Result (1) is true for $n=1$

Assume result (1) is true for $n=k$

$$\therefore \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\cdots\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)}$$

Adding next term $=\frac{1}{(2k+3)(2k+5)}$ to both sides, we get

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\cdots\frac{1}{(2k+1)(2k+3)}+\frac{1}{(2k+3)(2k+5)}=\frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)}$$

$$=\frac{k(2k+5)+3(1)}{3(2k+3)(2k+5)}=\frac{2k^2+5k+3}{3(2k+3)(2k+5)}$$

$$=\frac{2k^2+2k+3k+3}{3(2k+3)(2k+5)}=\frac{2k[k+1]+3[k+1]}{3(2k+3)(2k+5)}$$

$$=\frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}=\frac{k+1}{3(2k+5)}$$

$$=\frac{k+1}{3(2k+2+3)}=\frac{k+1}{3[2(k+1)+3]}$$

Which by comparison with result (1) shows that (1) is for $n=k+1$

$\therefore$ By PMI, result (1) is true for all $n\in N$.

Q No 18 $1+2+3+\cdots n < \frac{1}{8}(2n+1)^2$

Sol: We have to prove that $1+2+3+\cdots n < \frac{1}{8}(2n+1)^2$ --(1)

For $n=1$ (1) becomes.

$1 < \frac{1}{8}(2+1)^2$

ie $1 < \frac{9}{8}$ which is true.

$\therefore$ Result (1) is true for $n=1$.

Assume that result (1) is true for $n=k$.

$\therefore 1+2+3+\cdots k < \frac{1}{8}(2k+1)^2$

Adding next term $(k+1)$ on both sides.

$$1+2+3+\cdots k+(k+1) < \frac{1}{8}(2k+1)^2+(k+1)$$

$$= \frac{1}{8}\left[(2k+1)^2+8(k+1)\right]$$

$$= \frac{1}{8}\left[4k^2+1+4k+8k+8\right]$$

$$= \frac{1}{8}\left[4k^2+12k+9\right] = \frac{1}{8}\left[(2k)^2+(3)^2+2\times2k\times3\right]$$

$$= \frac{1}{8}\left[(2k+3)^2\right]$$

$$= \frac{1}{8}\left[(2k+2+1)^2\right] = \frac{1}{8}\left[2(k+1)+1\right]^2$$

$\therefore 1+2+3+\cdots k+(k+1) < \frac{1}{8}\left[2(k+1)+1\right]^2$

$\therefore$ Result is true for $n=k+1$ also

$\therefore$ Hence by PMI, result is true for all $n \in N$.

Q.No. 19 $n(n+1)(n+5)$ is a multiple of 3.

Sol Let $P(n) = n(n+1)(n+5)$

$\therefore$ $P(1) = 1(1+1)(1+5) = 1 \times 2 \times 6 = 12 = 3 \times 4$

Which is multiple of 3.

Assume the result is true for $n=k$.

$\therefore$ $P(k) = k(k+1)(k+5)$ is multiple of 3.

Let $k(k+1)(k+5) = 3m$. Where m is integer. --(1)

Now $P(k+1) = (k+1)(k+1+1)(k+1+5)$

$= (k+1)(k+2)[(k+5)+1]$

$= \left(k(k+1)[(k+5)+1]\right) + \left(2(k+1)[(k+5)+1]\right)$

$= \{k(k+1)(k+5)\} + [k(k+1) + 2(k+1)(k+6)]$

$= k(k+1)(k+5) + (k+1)[k+2(k+6)]$

$= k(k+1)(k+5) + (k+1)(k+2k+12)$

$= k(k+1)(k+5) + 3(k+1)(k+4)$

$= 3m + 3(k+1)(k+4)$ [$\because$ from (1)]

$= 3[m+(k+1)(k+4)]$

Which is multiple of 3.

$\therefore$ Result is true for $n=k+1$

$\therefore$ Result is true for $n=k+1$ whenever it is true for $n=k$.

$\therefore$ By PMI, result is true for all $n \in N$.

Q.No. 20 $\quad 10^{2n-1} + 1$ is dinsible by 11

Sol: Let $P(n) = 10^{2n-1} + 1$

$\therefore P(1) = 10^{2-1} + 1 = 10 + 1 = 11$ which is divisible by 11.

$\therefore$ Result is teue for $n = 1$.

Assume that result is teue for $n = k$.

$\therefore P(k) = 10^{2k-1} + 1$ is dinsible by 11

Let $10^{2k-1} + 1 = 11m$ where m is integer.

$\therefore 10^{2k-1} = 11m - 1. \quad \text{-----}(1)$

$$P(k+1) = 10^{2(k+1)-1} + 1$$

$$= 10^{2k+2-1} + 1$$

$$= 10^{2k-1}\, 10^{2} + 1.$$

$$= (11m-1)(100) + 1 \quad [\text{from } (1)]$$

$$= 11m \times 100 - 100 + 1$$

$$= 11m \times 100 - 99$$

$$= 11(100m - 9)$$

Which is dinsible by 11

$\therefore$ Result is teue $n = k+1$.

$\therefore$ By PMI result is teue for all $n \in N$.

Q.No 21 $x^{2n} - y^{2n}$ is divisible by $x+y$.

Sol. Let $P(n) = x^{2n} - y^{2n}$.

$P(1) = x^2 - y^2 = (x+y)(x-y)$ which is divisible by $x+y$.

$\therefore$ Result is true for $n = 1$.

Assume that the result is true for $n = k$

$\therefore$ $P(k) = x^{2k} - y^{2k}$ is divisible by $x+y$.

Let $x^{2k} - y^{2k} = m(x+y)$ where m is an integer ---(1)

Now $P(k+1) = x^{2(k+1)} - y^{2(k+1)} = x^{2k+2} - y^{2k+2}$

$= x^2 x^{2k} - y^2 y^{2k}$

$= x^2[y^{2k} + m(x+y)] - y^2 y^{2k}$ [From (1) $x^{2k} = y^{2k} + m(x+y)$]

$= x^2 y^{2k} + m x^2 (x+y) - y^2 y^{2k}$

$= (x^2 - y^2) y^{2k} + m x^2 (x+y)$

$= (x+y)[(x-y) y^{2k} + m x^2]$

which is divisible by $(x+y)$

$\therefore$ Result is true for $n = k+1$

$\therefore$ Result is true for $n = k+1$ whenever result is true for $n = k$.

Hence by Principle of Mathematical induction result is true for all $n \in N$

Q No 22. $3^{2n+2} - 8n - 9$ is divisible by 8.

Sol. Let $P(n) = 3^{2n+2} - 8n - 9$

$\therefore P(1) = 3^{2\times1+2} - 8\times1 - 9 = 3^4 - 9 - 8 = 81 - 17 = 64 = 8\times8$

Which is divisible by 8

$\therefore$ Result is true for $n=k$.

Assume that the result is true for $n=k$.

$\therefore P(k) = 3^{2k+2} - 8k - 9$ is divisible by 8

Let $3^{2k+2} - 8k - 9 = 8m$, $m \in Z$ ---(1)

$\therefore P(k+1) = 3^{2(k+1)+2} - 8(k+1) - 9$

$= 3^{2k+2} \cdot 3^2 - 8k - 8 - 9$

$= 9(8m + 8k + 9) - 8k - 8 - 9$ [using (1)]

$= 72m + 72k + 81 - 8k - 17$

$= 72m + 64k + 64$

$= 8(9m + 8k + 8)$

Which is divisible by 8

$\therefore$ Result is true for $n=k+1$ also.

$\therefore$ Result is true for $n=k+1$, whenever result is true for $n=1$

Hence by PMI result is true for all $n \in N$.

QNo 23 $41^n - 14^n$ is multiple of 27.

Sol. Let $P(n) = 41^n - 14^n$

$\therefore P(1) = 41 - 14 = 27$

Which is multiple of 27.

$\therefore$ Result is true for $n=1$.

Assume result is true for $n=k$.

$\therefore P(k) = 41^k - 14^k$ is multiple of 27

Let $41^k - 14^k = 27m \; ; m \in Z$

$\therefore 41^k = 27m + 14^k$.

Now $P(k+1) = 41^{k+1} - 14^{k+1}$

$= 41^k . 41 - 14^{k+1}$

$= 41(27m + 14^k) - 14^k . 14$

$= 41 \times 27m + 41 \cdot 14^k - 14^k . 14$

$= 27(41m) + 14^k(41-14)$

$= 27(41m) + 27 \cdot 14^k$

$= 27(41m + 14^k)$

Which is multiple of 27.

$\therefore$ Result is true for $n = k+1$ also.

Hence by PMI, Result is true $\forall n \in N$.

QNo 24. $(2n+7) < (n+3)^2$

Sol.. We want to prove that $2n+7 < (n+3)^2$ ----(1)

For $n=1$ $\quad 2\times1+7 < (1+3)^2$

ie $\quad 9 < 16$ Which is true

$\therefore$ Result (1) is true for $n=1$.

Assume the result is true for $n=k$.

$$\therefore \quad 2k+7 < (k+3)^2$$

$$\Rightarrow \quad 2(k+1)+7 < (k+3)^2+2$$

$$\Rightarrow \quad 2(k+1)+7 < k^2+6k+11$$

$$\Rightarrow \quad 2(k+1)+7 < k^2+8k+16-2k-5$$

$$\Rightarrow \quad 2(k+1)+7 < (k^2+8k+16)-(2k+5) < k^2+8k+16$$

$$\Rightarrow \quad 2(k+1)+7 < (k+4)^2$$

ie $\quad 2(k+1)+7 < [(k+1)+3]^2$.

$\therefore$ Result is true for $n=k+1$ also.

$\therefore$ By Principle of M.I result is true $\forall\, n \in N$.