Long Answer Questions-I-B (PYQ)

[4 Mark]

Q.1. If $y^x = e^{y-x}$, then prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$.

Ans.

Given, $y^x = e^{y-x}$

Taking logarithm both sides, we get

$$\log y^x = \log e^{y-x}$$

$\Rightarrow \quad x.\log y = (y-x) \,.\, \log e \quad \Rightarrow \quad x \,.\, \log y = (y-x) \qquad [\because \log (m)^n = n \log m]$

$\Rightarrow \quad x\,(1+\log y) = y \qquad \Rightarrow \quad x = \frac{y}{1+\log y}$

Differentiating both sides with respect to y, we get

$$\frac{dx}{dy} = \frac{(1+\log y).1 - y.\left(0+\frac{1}{y}\right)}{(1+\log y)^2}$$

$$= \frac{1+\log y - 1}{(1+\log y)^2} = \frac{\log y}{(1+\log y)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$$

$$\begin{bmatrix} \text{Note : } (i) \ \log_e mn = \log_e m + \log_e n \\ (ii) \ \log_e \frac{m}{n} = \log_e m - \log_e n \\ (iii) \ \log_e m^n = n \log_e m \\ (iv) \ \log e = 1 \end{bmatrix}$$

Q.2. If $x^y = e^{x-y}$, then show that $\frac{dy}{dx} = \frac{\log x}{\{\log (xe)\}^2}$.

Ans.

Given, $x^y = e^{x-y}$

Taking log both sides, we get

$\Rightarrow \quad \log x^y = \log e^{x-y} \quad \Rightarrow \quad y.\log x = (x-y)\,.\log e \quad [\because \log e = 1]$

$\Rightarrow \quad y.\log x = (x-y) \quad \Rightarrow \quad y\log x + y = x$

$\Rightarrow \quad y = \frac{x}{1+\log x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{(1+\log x).1 - x.\left(0+\frac{1}{x}\right)}{(1+\log x)^2}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1+\log x - 1}{(1+\log x)^2} = \frac{\log x}{(\log e+\log x)^2} \quad [\because 1 = \log e]$

$\Rightarrow \quad \frac{dy}{dx} = \frac{\log x}{(\log ex)^2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\log x}{\{\log (ex)\}^2}$

Q.3. **Prove that :** $\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right] = \sqrt{a^2 - x^2}$

Ans.

$$\text{LHS} = \frac{d}{dx}\left(\frac{x}{2}\sqrt{a^2 - x^2}\right) + \frac{d}{dx}\left(\frac{a^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right) = \frac{d}{dx}\left(\frac{x}{2}\sqrt{a^2 - x^2}\right) + \frac{d}{dx}\left(\frac{a^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right)$$

$$= \frac{1}{2}\left\{x.\frac{1}{2\sqrt{a^2 - x^2}} \times -2x + \sqrt{a^2 - x^2}\right\} + \frac{a^2}{2}.\frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \times \frac{1}{a} \quad \left[\begin{array}{l}\text{Apply product rule}\\ \text{and inverse formula}\end{array}\right]$$

$$= \frac{-x^2}{2\sqrt{a^2 - x^2}} + \frac{\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2\sqrt{a^2 - x^2}}$$

$$= \frac{-x^2+a^2 - x^2+a^2}{2\sqrt{a^2 - x^2}} = \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} = \sqrt{a^2 - x^2} = \text{RHS}$$

Q.4. **If $(\cos x)^y = (\cos y)^x$, then find $\frac{dy}{dx}$.**

Ans.

Given, $(\cos x)^y = (\cos y)^x$

Taking logrithm both sides, we get

$$\log(\cos x)^y = \log(\cos y)^x$$

$\Rightarrow \quad y \cdot \log(\cos x) = x \cdot \log(\cos y) \qquad [\because \log m^n = n \log m]$

Differentiating both sides, we get

$$y \cdot \frac{1}{\cos x}(-\sin x) + \log(\cos x) \cdot \frac{dy}{dx} = x \cdot \frac{1}{\cos y} \cdot (-\sin y) \cdot \frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow \quad -\frac{y \sin x}{\cos x} + \log(\cos x) \cdot \frac{dy}{dx} = -\frac{x \sin y}{\cos y} \cdot \frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow \quad \log(\cos x) \cdot \frac{dy}{dx} + \frac{x \sin y}{\cos y} \cdot \frac{dy}{dx} = \log(\cos y) + \frac{y \sin x}{\cos x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\log(\cos x) + \frac{x \sin y}{\cos y}\right] = \log(\cos y) + \frac{y \sin x}{\cos x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log(\cos y) + \frac{y \sin x}{\cos x}}{\log(\cos x) + \frac{x \sin y}{\cos y}} = \frac{\log(\cos y) + y \tan x}{\log(\cos x) + x \tan y}$$

Q.5.

Find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$, if $x = ae^{\theta}(\sin\theta - \cos\theta)$ and $y = ae^{\theta}(\sin\theta + \cos\theta)$.

Ans.

Given, $x = ae^{\theta}(\sin\theta - \cos\theta)$ and $y = ae^{\theta}(\sin\theta + \cos\theta)$

$x = ae^{\theta}(\sin\theta - \cos\theta)$

Differentiating with respect to q, we get

$$\frac{dx}{d\theta} = ae^{\theta}\,(\cos\theta + \sin\theta) + a(\sin\theta - \cos\theta).e^{\theta} = ae^{\theta}\,(\cos\theta + \sin\theta + \sin\theta - \cos\theta$$

$$= 2\,ae^{\theta}\sin\theta \qquad \ldots (i)$$

Again, $\because y = ae^{\theta}(\sin\theta + \cos\theta)$

Differentiating with respect to θ, we get

$$\frac{dy}{d\theta} = ae^{\theta}\,(\cos\theta - \sin\theta) + a(\sin\theta + \cos\theta).e^{\theta} = ae^{\theta}\,(\cos\theta - \sin\theta + \sin\theta + \cos\theta$$

$$= 2\,ae^{\theta}.\cos\theta \qquad \ldots (ii)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\,ae^{\theta}.\cos\theta}{2\,ae^{\theta}.\sin\theta} \qquad [\text{From } (i) \text{ and } (ii)]$$

$$\Rightarrow \quad \frac{dy}{dx} = \cot\theta \quad \Rightarrow \quad \left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

Q.6. **If $\sin y = x\sin(a+y)$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.**

Ans.

Here, $\sin y = x\sin(a+y) \quad \Rightarrow \quad \frac{\sin y}{\sin(a+y)} = x$

$$\Rightarrow \quad \frac{\sin(a+y).\cos y.\frac{dy}{dx} - \sin y.\cos(a+y).\frac{dy}{dx}}{\sin^2(a+y)} = 1$$

$$\Rightarrow \quad \frac{dy}{dx}\{\sin(a+y).\cos y - \sin y.\cos(a+y)\} = \sin^2(a+y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin(a+y-y)} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Q.7. Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ with respect to x.

Ans.

Let $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Put $x = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1} x$

Now, $y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\left[\begin{array}{ll} \because & -\infty\, x\, \infty \\ \Rightarrow & \tan\left(-\frac{\pi}{2}\right)\tan\theta\tan\left(\frac{\pi}{2}\right) \\ \Rightarrow & -\frac{\pi}{2}\theta\frac{\pi}{2} \\ \Rightarrow & -\frac{\pi}{4}\,\frac{\theta}{2}\,\frac{\pi}{4} \\ \Rightarrow & \frac{\theta}{2} \in \left(-\frac{\pi}{4},\frac{\pi}{4}\right) \subset \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \end{array}\right]$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\,.\,\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$\Rightarrow \quad y = \frac{1}{2}\tan^{-1}x \qquad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2(1+x^2)}$$

Q.8. Differentiate the following with respect to *x*:

$(\sin x)^x + (\cos x)^{\sin x}$

Ans.

Let $u = (\sin x)^x$ and $v = (\cos x)^{\sin x}$

$\therefore$ Given differential equation becomes $y = u + v$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(i)$$

Now, $u = (\sin x)^x$

Taking log on both sides, we get

$$\log u = x \log \sin x$$

Differentiating with respect to x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = x \frac{1}{\sin x} . \cos x + \log \sin x \quad \Rightarrow \quad \frac{du}{dx} = u(x \cot x + \log \sin x)$$

$$\Rightarrow \quad \frac{du}{dx} = (\sin x)^x \{x \cot x + \log \sin x\} \qquad ...(ii)$$

Again $v = (\cos x)^{\sin x}$

Taking log on both sides, we get

$$\log v = \sin x . \log \cos x$$

Differentiating both sides with respect to x, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = \sin x . \frac{1}{\cos x} (-\sin x) + \log (\cos x) . \cos x$$

$$\Rightarrow \quad \frac{dv}{dx} = v \left\{ -\frac{\sin^2 x}{\cos x} + \cos x . \log \cos x \right\} = (\cos x)^{\sin x} \left\{ \cos x . \log (\cos x) - \frac{\sin^2 x}{\cos x} \right\}$$

$$\Rightarrow \quad \frac{dv}{dx} = (\cos x)^{1+\sin x} \{\log (\cos x) - \tan^2 x\} \qquad ...(iii)$$

From (i), (ii) and (iii), we get

$$\frac{dy}{dx} = (\sin x)^x \{x \cot x + \log \sin x\} + (\cos x)^{1+\sin x} \{\log (\cos x) - \tan^2 x\}$$

Q.9. If cos *y* = *x* cos (*a* + *y*), with cos *a* ≠ ±1, then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Hence show that $\sin a \frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0$

Ans.

To prove $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$ (Refer Q. 30 Page-208)

Differentiating both sides w.r.t. *x*, we get

$$\frac{d^2y}{dx^2} = \frac{1}{\sin a}\left\{-2\cos(a+y).\sin(a+y).\frac{dy}{dx}\right\}$$

$$\Rightarrow \quad \sin a\frac{d^2y}{dx^2} = -\sin 2(a+y).\frac{dy}{dx}$$

$$\Rightarrow \quad \sin a\frac{d^2y}{dx^2} + \sin 2(a+y).\frac{dy}{dx} = 0$$

Q.10. Differentiate the following with respect to: $\tan^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

Ans.

Let $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=y$

$$y=\tan^{-1}\left(\frac{1-\frac{\sqrt{1-x}}{\sqrt{1+x}}}{1+\frac{\sqrt{1-x}}{\sqrt{1+x}}}\right)$$

$$\Rightarrow \quad y=\tan^{-1}1-\tan^{-1}\left(\frac{\sqrt{1-x}}{\sqrt{1+x}}\right)$$

$$\frac{dy}{dx}=0-\frac{1}{1+\left(\frac{\sqrt{1-x}}{\sqrt{1+x}}\right)^2}\cdot\frac{d}{dx}\left(\frac{\sqrt{1-x}}{\sqrt{1+x}}\right)$$

$$=-\frac{1+x}{2}\left\{\frac{\frac{-1}{2\sqrt{1-x}}\sqrt{1+x}-\frac{1}{2\sqrt{1+x}}\sqrt{1-x}}{1+x}\right\}$$

$$=\frac{1+x}{4}\left\{\frac{\frac{\sqrt{1+x}\times\sqrt{1+x}}{\sqrt{1-x}\times\sqrt{1+x}}+\frac{\sqrt{1-x}\times\sqrt{1-x}}{\sqrt{1+x}\times\sqrt{1-x}}}{1+x}\right\}$$

$$=\frac{1}{4}\cdot\frac{2}{\sqrt{1-x^2}}=\frac{1}{2\sqrt{1-x^2}}$$

Q.11. If $y=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$, then show that $\frac{dy}{dx}-\sec x=0$.

Ans.

Given, $y=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)}\cdot\frac{d}{dx}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]=\cot\left(\frac{\pi}{4}+\frac{x}{2}\right).\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right).\frac{1}{2}$$

$$=\frac{\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)}\cdot\frac{1}{2}\frac{1}{\cos^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}=\frac{1}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right).\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}$$

$$=\frac{1}{\sin 2\left(\frac{\pi}{4}+\frac{x}{2}\right)}=\frac{1}{\sin\left(\frac{\pi}{2}+x\right)}=\frac{1}{\cos x}=\sec x$$

$$\Rightarrow \quad \frac{dy}{dx}-\sec x=0$$

Hence proved.

Q.13. Differentiate the following function with respect to x: $(\log x)^x + x^{\log x}$.

Ans.

Let $y = (\log x)^x + x^{\log x}$

$\Rightarrow \quad y = u + v, \quad \text{where } u = (\log x)^x, \quad v = x^{\log x}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(i)$

Now, $\quad u = (\log x)^x$

Taking logarithm on both sides, we get

$\log u = x \,.\, \log (\log x)$

Differentiating both sides with respect to x, we get

$\frac{1}{u} . \frac{du}{dx} = x . \frac{1}{\log x} . \frac{1}{x} + \log(\log x) \qquad \Rightarrow \qquad \frac{du}{dx} = u\left\{\frac{1}{\log x} + \log(\log x)\right\}$

$\Rightarrow \quad \frac{du}{dx} = (\log x)^x \left\{\frac{1}{\log x} + \log(\log x)\right\} \qquad \ldots(ii)$

Again $\quad v = x^{\log x}$

Taking logarithm of both sides, we get

$\log v = \log x^{\log x}$

$\Rightarrow \quad \log v = \log x \,.\, \log x \qquad \Rightarrow \qquad \log v = (\log x)^2$

Differentiating both sides with respect to x, we get

$\frac{1}{v} \frac{dv}{dx} = 2 \log x . \frac{1}{x}$

$\Rightarrow \quad \frac{dv}{dx} = 2x^{\log x} . \frac{\log x}{x} \qquad \ldots(iii)$

From (i), (ii) and (iii), we get

$\frac{dy}{dx} = (\log x)^x \left\{\frac{1}{\log x} + \log(\log x)\right\} + 2\frac{\log x . x^{\log x}}{x}$

Q.14.

If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, then show that $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$.

Ans.

Given, $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$

Putting $x=\sin\alpha \Rightarrow \alpha=\sin^{-1}x$ and $y=\sin\beta \Rightarrow \beta=\sin^{-1}y$, we get

$$\sqrt{1-\sin^2\alpha}+\sqrt{1-\sin^2\beta}=a(\sin\alpha-\sin\beta)$$

$\Rightarrow \quad \cos\alpha+\cos\beta=a(\sin\alpha-\sin\beta)$

$\Rightarrow \quad 2\cos\frac{(\alpha+\beta)}{2}\cos\left(\frac{\alpha-\beta}{2}\right)=a.2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$\Rightarrow \quad \cot\left(\frac{\alpha-\beta}{2}\right)=a \quad \Rightarrow \quad \frac{\alpha-\beta}{2}=\cot^{-1}a \quad \Rightarrow \quad \alpha-\beta=2\cot^{-1}a$

$\Rightarrow \quad \sin^{-1}x-\sin^{-1}y=2\cot^{-1}a$

Differentiating both sides with respect to x, we get

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$$

Q.15. **Differentiate** $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$ **with respect to** $\sin^{-1}(2x\sqrt{1-x^2})$.

Ans.

Let $u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ and $v = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

We have to determine $\frac{du}{dv}$

Put $x = \sin\theta \quad \Rightarrow \quad \theta = \sin^{-1} x$

Now, $u = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right) \quad \Rightarrow \quad u = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) \quad \Rightarrow \quad u = \theta$

$\Rightarrow \quad u = \sin^{-1} x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

Again, $v = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$\Rightarrow \quad v = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta} = \sin^{-1}\left(2\sin\theta\cos\theta\right)\right.$

$\Rightarrow \quad v = \sin^{-1}(\sin 2\theta) \quad \Rightarrow \quad v = 2\theta$

$\Rightarrow \quad v = 2\sin^{-1}x \quad \Rightarrow \quad \frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$

$$\left[\begin{array}{ll} \because & -\frac{1}{\sqrt{2}} x \frac{1}{\sqrt{2}} \\ \Rightarrow & \sin\left(-\frac{\pi}{4}\right)\sin\theta\sin\left(\frac{\pi}{4}\right) \\ \Rightarrow & -\frac{\pi}{4}\theta\frac{\pi}{4} \\ \Rightarrow & -\frac{\pi}{2}2\theta\frac{\pi}{2} \\ \Rightarrow & 2\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{array}\right]$$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}} = \frac{1}{2}$$

[**Note:** Here the range of x is taken as $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$]

Q.16.

If $x = \cos t(3 - 2\cos^2 t)$ and $y = \sin t\,(3 - 2\sin^2 t)$, then find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$.

Ans.

Given, $x = \cos t(3 - 2\cos^2 t)$

Differentiating both sides with respect to t, we get

$$\frac{dx}{dt} = \cos t\{0 + 4\cos t.\sin t\} + (3 - 2\cos^2 t).(-\sin t)$$

$$= 4\sin t.\cos^2 t - 3\sin t + 2\cos^2 t.\sin t$$

$$= 6\sin t\cos^2 t - 3\sin t = 3\sin t\,(2\cos^2 t - 1) = 3\sin t.\cos 2t$$

Again, $\because\ y = \sin t\,(3 - 2\sin^2 t)$

Differentiating both sides with respect to t, we get

$$\frac{dy}{dt} = \sin t.\{0 - 4\sin t\cos t\} + (3 - 2\sin^2 t\}.\cos t$$

$$= -4\sin^2 t.\cos t + 3\cos t - 2\sin^2 t.\cos t = 3\cos t - 6\sin^2 t.\cos t$$

$$= 3\cos t\,(1 - 2\sin^2 t) = 3\cos t.\cos 2t$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t.\cos 2t}{3\sin t.\cos 2t}$

$$\frac{dy}{dx} = \cot t$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

Q.17.

Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}(2x\sqrt{1-x^2})$, when $x \neq 0$.

Ans.

Let $u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ and $v = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$

We have to determine $\frac{du}{dv}$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

Now, $u = \tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right) \Rightarrow u = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$

$\Rightarrow u = \tan^{-1}(\cot\theta) \Rightarrow u = \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\theta\right)\right]$

$\Rightarrow u = \frac{\pi}{2} - \theta \Rightarrow u = \frac{\pi}{2} - \sin^{-1} x$

$\Rightarrow \frac{du}{dx} = 0 - \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$

Again, $v = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$

$\because \quad x = \sin\theta$

$\therefore \quad v = \cos^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$\Rightarrow v = \cos^{-1}(2\sin\theta \,.\, \cos\theta)$

$\Rightarrow v = \cos^{-1}(\sin 2\theta)$

$\Rightarrow v = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right)$

$\Rightarrow v = \frac{\pi}{2} - 2\theta \Rightarrow v = \frac{\pi}{2} - 2\sin^{-1} x$

$$\Rightarrow \quad \frac{dv}{dx} = 0 - \frac{2}{\sqrt{1-x^2}} \qquad \Rightarrow \qquad \frac{dv}{dx} = -\frac{2}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{2}{\sqrt{1-x^2}}} = \frac{1}{2}$$

$$\left[\begin{array}{ll} \because & -\frac{1}{\sqrt{2}} x \frac{1}{\sqrt{2}} \\ \Rightarrow & \sin\left(-\frac{\pi}{4}\right) \sin\theta \sin\left(\frac{\pi}{4}\right) \\ \Rightarrow & -\frac{\pi}{4} \theta \frac{\pi}{4} \\ \Rightarrow & -\frac{\pi}{2} 2\theta \frac{\pi}{2} \\ \Rightarrow & \frac{\pi}{2} - 2\theta - \frac{\pi}{2} \\ \Rightarrow & \pi\left(\frac{\pi}{2} - 2\theta\right) 0 \\ \Rightarrow & \left(\frac{\pi}{2} - 2\theta\right) \in (0,\pi) \subset [0,\pi] \end{array}\right]$$

[**Note:** Here the range of x is taken as $-\frac{1}{\sqrt{2}} < x > \frac{1}{\sqrt{2}}$]

Q.18. Find $\frac{dy}{dx}$, if $(x^2 + y^2)^2 = xy$.

Ans.

Given, equation is $(x^2 + y^2)^2 = xy$.

Differentiating with respect to x, we get

$$2(x^2 + y^2).\left(2x + 2y\frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad 4x(x^2 + y^2) + 4y.(x^2 + y^2).\frac{dy}{dx} = x.\frac{dy}{dx} + y$$

$$\Rightarrow \quad \{4y(x^2 + y^2) - x\}\frac{dy}{dx} = y - 4x(x^2 + y^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - 4x(x^2+y^2)}{4y(x^2+y^2) - x}$$

Q.19. Differentiate the following function with respect to *x*:

$$y = (\sin x)^x + \sin^{-1}\sqrt{x}$$

Ans.

Given, $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$$y = u + v, \quad \text{where } u = (\sin x)^x, \; v = \sin^{-1}\sqrt{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(i)$$

Now, $u = (\sin x)^x$

Taking log both sides, we get

$$\log u = \log (\sin x)^x \quad \Rightarrow \quad \log u = x \,.\, \log (\sin x)$$

Differentiating both sides with respect to x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = x \cdot \frac{1}{\sin x}\cos x + \log\ \sin x$$

$$\Rightarrow \quad \frac{du}{dx} = u\{x \cot x + \log\ \sin x\}$$

$$\Rightarrow \quad \frac{du}{dx} = (\sin x)^x \{x \cot x + \log\ \sin x\} \qquad \ldots(ii)$$

Also, $v = \sin^{-1}\sqrt{x}$

$$\frac{dv}{dx} = \frac{1}{\sqrt{1 - (\sqrt{x})^2}} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(1-x)}} \qquad \ldots(iii)$$

From (i), (ii) and (iii), we get

$$\therefore \quad \frac{dy}{dx} = (\sin x)^x \{x \cot x + \log\ \sin x\} + \frac{1}{2\sqrt{x(1-x)}}$$

Q.20. If $y = \cos^{-1}\left\{\frac{3x + 4\sqrt{1-x^2}}{5}\right\}$, then find $\frac{dy}{dx}$.

Ans.

Here, $y = \cos^{-1}\left\{\frac{3x+4\sqrt{1-x^2}}{5}\right\}$

Let $x = \cos\alpha \Rightarrow \alpha = \cos^{-1} x$

$\therefore \quad y = \cos^{-1}\left\{\frac{3\cos\alpha}{5} + \frac{4}{5}\sqrt{1-\cos^2\alpha}\right\}$

$y = \cos^{-1}\left\{\frac{3}{5}\cos\alpha + \frac{4}{5}\sin\alpha\right\}$

Let $\frac{3}{5} = \cos\theta \qquad \therefore \quad \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$

Now, $y = \cos^{-1}\{\cos\theta.\cos\alpha + \sin\theta.\sin\alpha\}$

$\therefore \quad y = \cos^{-1}(\cos(\theta-\alpha)) = \theta - \alpha$

$\Rightarrow \quad y = \cos^{-1}\frac{3}{5} - \cos^{-1}x \qquad \left[\because \quad \cos\theta = \frac{3}{5} \Rightarrow \theta = \cos^{-1}\frac{3}{5}\right]$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{-1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$

Q.21. If $y = \cos^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$, then find $\frac{dy}{dx}$.

Ans.

Given, $y = \cos^{-1}\left(\frac{2^x.2}{1+(2^x)^2}\right)$

Let $2^x = \tan\alpha \Rightarrow \alpha = \tan^{-1}(2^x)$

$\therefore \quad y = \cos^{-1}\left(\frac{2\tan\alpha}{1+\tan^2\alpha}\right)$

$= \cos^{-1}(\sin 2\alpha) = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\alpha\right)\right) = \frac{\pi}{2} - 2\alpha$

$\Rightarrow \quad y = \frac{\pi}{2} - 2\tan^{-1}(2^x)$

$\Rightarrow \quad \frac{dy}{dx} = 0 - 2\frac{1}{1+(2^x)^2}.\log_e 2.2^x = -\frac{2.2^x.\log_e 2}{1+4^x} = -\frac{2^{x+1}.\log_e 2}{1+4^x}$

Q.22. Find $\frac{dy}{dx}$, if $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$.

Ans.

Given, $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$

$$= \sin^{-1}[x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}]$$

$\Rightarrow \quad y = \sin^{-1}x - \sin^{-1}\sqrt{x} \qquad [\text{using } \sin^{-1}x - \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x(1-x)}}$$

Q.23. Find $\frac{dy}{dx}$, if $y = (\cos x)^x + (\sin x)^{1/x}$.

Ans.

Given, $y = (\cos x)^{x} + (\sin x)^{1/x}$

$$y = u + v; \qquad \text{where } u = (\cos x)^{x}, \quad v = (\sin x)^{1/x}$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(i)$$

Now $u = (\cos x)^{x}$

Taking log both sides, we get

$$\log u = \log (\cos x)^{x}$$

$$\Rightarrow \quad \log u = x \log (\cos x)$$

Differentiating with respect to x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = - x . \frac{1}{\cos x} \sin x + \log (\cos x)$$

$$\Rightarrow \quad \frac{du}{dx} = u\{ \log (\cos x) - x \tan x\}$$

$$= (\cos x)^{x} \{\log (\cos x) - x \tan x\}$$

Again, $v = (\sin x)^{1/x}$

Taking log both sides, we get

$$\log v = \frac{1}{x}\log(\sin x)$$

Differentiating with respect to x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{1}{x}\cdot\frac{1}{\sin x}.\cos x + \log(\sin x)\left(-\frac{1}{x^2}\right)$$

$$\frac{dv}{dx} = v\left\{\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right\}$$

$$= (\sin x)^{1/x}\left\{\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right\}$$

Putting the value of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (i), we get

$$\frac{dy}{dx} = (\cos x)^x\{\log(\cos x) - x\tan x\} + (\sin x)^{1/x}\left\{\frac{\cot x}{x} - \frac{\log \sin x}{x^2}\right\}$$

Q.24. Differentiate the following function with respect to
$x : f(x) = \tan^{-1}\left(\frac{1-x}{1+x}\right) - \tan^{-1}\left(\frac{x+2}{1-2x}\right).$

Ans.

$$f(x) = \tan^{-1}\left(\frac{1-x}{1+x}\right) - \tan^{-1}\left(\frac{x+2}{1-2x}\right) = \tan^{-1}\left(\frac{1-x}{1+x.1}\right) - \tan^{-1}\left(\frac{x+2}{1-2.x}\right)$$

$$= (\tan^{-1} 1 - \tan^{-1} x) - (\tan^{-1} x + \tan^{-1} 2) \qquad \left(\because \tan^{-1}\frac{a-b}{1+ab} = \tan^{-1} a - \tan^{-1} b\right)$$

$$= \tan^{-1} 1 - \tan^{-1} 2 - 2\tan^{-1} x$$

Differentiating with respect to x, we get

$$f'(x) = -\frac{2}{1+x^2}$$

Q.25. If $x^{13}y^7 = (x+y)^{20}$, then prove that $\frac{dy}{dx} = \frac{y}{x}$.

OR

If $x^m y^n = (x+y)^{m+n}$, then prove that $\frac{dy}{dx} = \frac{y}{x}$

Q.26. Differentiate with respect to *x* :

$$\sin^{-1}\left(\frac{2^{x+1}.3x}{1+(36)^x}\right)$$

Ans.

$$\text{Let } y = \sin^{-1}\left(\frac{2^{x+1}.3x}{1+(36)^x}\right) = \sin^{-1}\left(\frac{2.2^x.3^x}{1+(6^2)^x}\right) = \sin^{-1}\left(\frac{2.6^x}{1+(6^x)^2}\right)$$

$$\text{Let } 6^x = \tan\theta \Rightarrow \theta = \tan^{-1}(6^x)$$

$$\therefore y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \Rightarrow y = \sin^{-1}(\sin 2\theta)$$

$$\Rightarrow y = 2\theta \Rightarrow y = 2.\tan^{-1}(6^x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+(6^x)^2}.\log_e 6.6^x \Rightarrow \frac{dy}{dx} = \frac{2.6^x.\log_e 6}{1+36^x}$$

Q.27. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ **with respect to $\tan^{-1} x$, when $x \neq 0$.**

Ans.

$$\text{Let } u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) \text{ and } v = \tan^{-1} x$$

We have to find $\frac{du}{dv}$

$$\text{Now, } u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

$$\text{Let } x = \tan\theta \Rightarrow \theta = \tan^{-1} x$$

$$\therefore u = \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right] = \tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right] = \tan^{-1}\left[\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right]$$

$$= \tan^{-1}\left[\frac{1-\cos\theta}{\cos\theta}.\frac{\cos\theta}{\sin\theta}\right] = \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right] = \tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right] = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$\therefore \quad u = \frac{1}{2}\tan^{-1} x$

Differentiating both sides with respect to x, we get

$$\frac{du}{dx} = \frac{1}{2(1+x^2)} \qquad ...(i)$$

Also, $v = \tan^{-1} x$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{1+x^2} \qquad ...(ii)$$

$$\therefore \quad \frac{du}{dv} = \frac{1}{2(1+x^2)} \times \frac{1+x^2}{1} = \frac{1}{2}$$

Q.28. **If $x \sin(a+y) + \sin a \cos(a+y) = 0$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.**

Ans.

Given $x \sin(a+y) + \sin a \cos(a+y) = 0$

$$\Rightarrow \quad x = -\frac{\sin a . \cos(a+y)}{\sin(a+y)} \quad \Rightarrow \quad x = -\sin a . \cot(a+y)$$

Differentiating with respect to y, we get

$$\frac{dx}{dy} = +\sin a . \operatorname{cosec}^2(a+y) = \frac{\sin a}{\sin^2(a+y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Q.29. **If $e^x + e^y = e^{x+y}$, then prove that $\frac{dy}{dx} + e^{y-x} = 0$.**

Ans.

Given, $e^x + e^y = e^{x+y}$

Differentiating both sides with respect to x, we get

$$e^x + e^y.\frac{dy}{dx} = e^{x+y}\left\{1 + \frac{dy}{dx}\right\}$$

$$\Rightarrow \quad e^x + e^y.\frac{dy}{dx} = e^{x+y} + e^{x+y}.\frac{dy}{dx} \qquad \Rightarrow \qquad (e^{x+y} - e^y)\frac{dy}{dx} = e^x - e^{x+y}$$

$$\Rightarrow \quad (e^x + e^y - e^y)\frac{dy}{dx} = e^x - e^x - e^y \qquad [\because\ e^x + e^y = e^{x+y}(\text{ given })]$$

$$\Rightarrow \quad e^x.\frac{dy}{dx} = -e^y \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{e^y}{e^x}$$

$$\Rightarrow \quad \frac{dy}{dx} = -e^{y-x} \qquad \Rightarrow \qquad \frac{dy}{dx} + e^{y-x} = 0$$

Q.30. **If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$.**

Ans.

We have

$$x = e^{\cos 2t}$$

Differentiating w.r.t. t, we get

$$\frac{dx}{dt} = e^{\cos 2t}(-\ 2\sin 2t) = 2x\sin 2t$$

Again $y = e^{\sin 2t}$

Differentiating w.r.t. t, we get

$$\frac{dy}{dt} = e^{\sin 2t}.2\cos 2t = 2y\cos 2t$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2y\cos 2t}{-2x\sin 2t}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-y\ \cos\ 2t}{x\ \sin\ 2t} \qquad [\because x = e^{\cos 2t} \Rightarrow \log x = \cos 2t;\ y = e^{\sin 2t} \Rightarrow \log y = \sin 2t]$

$\Rightarrow \quad \frac{dy}{dx} = -\ \frac{y\ \log\ x}{x\ \log y}$

Hence proved.

Long Answer Questions-I-B (OIQ)

[4 Mark]

Q.1. If $y = \sqrt{\frac{(x\ -\ 3)(x^2+4)}{3x^2+4x+5}}$, then find $\frac{dy}{dx}$.

Ans.

Given, $y=\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$

Taking log on both sides, we get

$$\log y=\frac{1}{2}[\log(x-3)+\log(x^2+4)-\log(3x^2+4x+5)]$$

Differentiating both sides with respect to *x*, we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x-3}+\frac{1}{x^2+4}\times 2x-\frac{1}{3x^2+4x+5}\times(6x+4)\right]$$

$$\therefore \quad \frac{dy}{dx}=\frac{y}{2}\left[\frac{1}{x-3}+\frac{2x}{x^2+4}-\frac{6x+4}{3x^2+4x+5}\right]$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\right)\left[\frac{1}{x-3}+\frac{2x}{x^2+4}-\frac{6x+4}{3x^2+4x+5}\right]$$

Q.2. Find the derivative of *y* with respect to *x*, where $y=(x)^{\sin x}+(\sin x)^x$.

Ans.

Given, $y=(x)^{\sin x}+(\sin x)^x \quad \ldots(i)$

Let $u=x^{\sin x}$, and $v=(\sin x)^x$, then (i) becomes $y=u+v$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad \ldots(ii)$$

First consider, $u=x^{\sin x}$

Taking log on both sides, we get $\log u=\sin x \cdot \log x$

Differentiating with respect to *x*, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\cos x.\log x+\sin x.\frac{1}{x} \quad \Rightarrow \quad \frac{du}{dx}=u\left(\cos x\log x+\frac{\sin x}{x}\right)$$

$$\Rightarrow \quad \frac{du}{dx}=(x)^{\sin x}\left[\cos x(\log x)+\frac{\sin x}{x}\right] \quad \ldots(iii)$$

Again consider, $v = (\sin x)^x$

Taking log on both sides, we get $\log v = x \log \sin x$

Differentiating with respect to x, we get

$$\frac{1}{v}\frac{dv}{dx} = 1.\log\ \sin x + x.\frac{1}{\sin x}.\cos x \quad \Rightarrow \quad \frac{dv}{dx} = v(\log\ \sin x + x\cot x)$$

$$\frac{dv}{dx} = (\sin x)^x[\log\ \sin x + x\cot x] \qquad \ldots(iv)$$

From (ii), (iii) and (iv), we get

$$\frac{dy}{dx} = (x)^{\sin x}\left[\cos x(\log x) + \frac{\sin x}{x}\right] + (\sin x)^x[\log\ \sin x + x\cot x]$$

Q.3. **If $y = [x + \sqrt{x^2 + a^2}]^n$, then prove that $\frac{dy}{dx} = \frac{ny}{\sqrt{x^2+a^2}}$.**

We have, $y = [x + \sqrt{x^2 + a^2}]^n \quad \Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}[\{x + \sqrt{x^2 + a^2}\}^n]$

$$= n\{x + \sqrt{x^2 + a^2}\}^{n-1}\frac{d}{dx}\{x + \sqrt{x^2 + a^2}\} \qquad \text{[By chain rule]}$$

$$= n\{x + \sqrt{x^2 + a^2}\}^{n-1}.\left\{\frac{d}{dx}(x) + \frac{d}{dx}\sqrt{x^2 + a^2}\right\}$$

$$= n\{x + \sqrt{x^2 + a^2}\}^{n-1}.\left\{1 + \frac{1}{2}(x^2 + a^2)^{\frac{-1}{2}}.\frac{d}{dx}(x^2 + a^2)\right\}$$

$$= n\{x + \sqrt{x^2 + a^2}\}^{n-1}.\left\{1 + \frac{1}{2\sqrt{x^2+a^2}}.2x\right\} = n\{x + \sqrt{x^2 + a^2}\}^{n-1}.\left\{1 + \frac{x}{\sqrt{x^2+a^2}}\right\}$$

$$= n\{x + \sqrt{x^2 + a^2}\}^{n-1}\left\{\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right\} = \frac{n\{x+\sqrt{x^2+a^2}\}^n}{\sqrt{x^2+a^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{ny}{\sqrt{x^2+a^2}}$$

Q.4.

If $y\sqrt{x^2+1} = \log(\sqrt{x^2+1} - x)$, then show that $(x^2+1)\frac{dy}{dx} + xy + 1 = 0$.

Ans.

Given $y\sqrt{x^2+1} = \log(\sqrt{x^2+1} - x)$

Differentiating with respect to x on both sides, we get

$$y \times \frac{1}{2\sqrt{x^2+1}} \times 2x + \sqrt{x^2+1}\frac{dy}{dx} = \frac{1}{(\sqrt{x^2+1} - x)} \times \left(\frac{1}{2\sqrt{x^2+1}} \times 2x - 1\right)$$

$$\Rightarrow \quad \frac{xy}{\sqrt{x^2+1}} + \sqrt{x^2+1}\frac{dy}{dx} = \frac{\left(\frac{x}{\sqrt{x^2+1}} - 1\right)}{(\sqrt{x^2+1} - x)}$$

$$\Rightarrow \quad \frac{xy}{\sqrt{x^2+1}} + \sqrt{x^2+1}.\frac{dy}{dx} = \frac{(x - \sqrt{x^2+1})}{(\sqrt{x^2+1})(\sqrt{x^2+1} - x)}$$

$$\Rightarrow \quad \frac{xy}{\sqrt{x^2+1}} + \sqrt{x^2+1}.\frac{dy}{dx} = \frac{-1}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad \sqrt{x^2+1}\frac{dy}{dx} = \frac{-1}{\sqrt{x^2+1}} - \frac{xy}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad \sqrt{x^2+1}\frac{dy}{dx} = \frac{-(1+xy)}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad (x^2+1)\frac{dy}{dx} = -(1+xy)$$

Hence, $(x^2+1)\frac{dy}{dx} + xy + 1 = 0$

Q.5. Find $\frac{dy}{dx}$: $y = (\sin x)^x + (\cos x)^{\tan x}$.

Ans.

Given, $y = (\sin x)^x + (\cos x)^{\tan x}$

Let $u = (\sin x)^x$ and $v = (\cos x)^{\tan x}$

We have, $y = u + v$ then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(i)$$

Now, $u = (\sin x)^x$

Taking log on both sides, we get $\log u = x \log \sin x$

Differentiating both sides, with respect to x, we have

$$\frac{1}{u}\frac{du}{dx} = x \times \frac{1}{\sin x} \times \cos x + \log \sin x$$

$$\therefore \quad \frac{du}{dx} = u(x \cot x + \log \sin x)$$

$$\frac{du}{dx} = (\sin x)^x (x \cot x + \log \sin x) \qquad ...(ii)$$

Again, $v = (\cos x)^{\tan x}$

Taking log on both sides, we get

$$\log v = \tan x [\log \cos x]$$

Differentiating both sides with respect to x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx} = \tan x \times \frac{1}{\cos x} \times (-\sin x) + \log \cos x \times \sec^2 x$$

$$= -\tan^2 x + \sec^2 x \log \cos x$$

$$\frac{dv}{dx} = v(\sec^2 x \log \cos x - \tan^2 x)$$

$$= (\cos x)^{\tan x}(\sec^2 x \log \cos x - \tan^2 x) \qquad ...(iii)$$

From (i), (ii) and (iii), we have

$$\frac{dy}{dx} = (\sin x)^x (x \cot x + \log \sin x) + (\cos x)^{\tan x}(\sec^2 x \log \cos x - \tan^2 x)$$

Q.6.

If $x \in R - [-1, 1]$ then prove that the derivative of $\sec^{-1} x$ with respect to x is $\frac{1}{|x|\sqrt{x^2-1}}$.

Ans.

Let $y = \sec^{-1} x$

Then, $\sec y = \sec(\sec^{-1} x) = x$

Differentiating both sides with respect to x, we have

$$\Rightarrow \quad \frac{d}{dx}\sec y = \frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{d}{dy}(\sec y)\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \sec y \tan y \frac{dy}{dx} = 1 \qquad \text{[Using chain rule]}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{|\sec y|.|\tan y|}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{|\sec y|\sqrt{\tan^2 y}} = \frac{1}{|\sec y|\sqrt{\sec^2 y - 1}} = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\begin{bmatrix} \text{If } x1, \text{ then } y \in \left(0, \frac{\pi}{2}\right) \\ \therefore \quad \sec y > 0, \ \tan y > 0 \\ \Rightarrow \quad |\sec y|.|\tan y| = \sec y \tan y \\ \text{If } x - 1, \text{ then} \\ y \in \left(\frac{\pi}{2}, \pi\right) \therefore \quad \sec y < 0, \ \tan \ y < 0 \\ \Rightarrow \quad |\sec y| \tan y| \\ \Rightarrow \quad (-\sec y)(-\tan y) = \sec y \tan y \end{bmatrix}$$