Exercise 6.8

Answer 1E.

Consider the following:

$$\lim_{x\to a} f(x)=0$$
$$\lim_{x\to a} g(x)=0$$
$$\lim_{x\to a} h(x)=1$$
$$\lim_{x\to a} p(x)=\infty$$
$$\lim_{x\to a} q(x)=\infty$$

(a) Consider the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$.

Need to determine whether the given limit is an indeterminate form or not.

Since $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$.

Then

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)}&=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\\&=\frac{0}{0}\qquad\text{Indeterminate form}\end{aligned}$$

Therefore the given limit is an indeterminate form of type $\frac{0}{0}$.

(b) Consider the limit $\lim_{x\to a}\frac{f(x)}{p(x)}$.

Need to determine whether the given limit is an indeterminate form or not.

Since $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} p(x)=\infty$.

Then

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{p(x)}&=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} p(x)}\\&=\frac{0}{\infty}\qquad\text{Not indeterminate form}\\&=0\end{aligned}$$

The given limit is not indeterminate.

Therefore the value of the given limit is 0.

(c) Consider the limit $\lim_{x\to a}\frac{h(x)}{p(x)}$.

Need to determine whether the given limit is an indeterminate form or not.

Since $\lim_{x\to a}h(x)=1$ and $\lim_{x\to a}p(x)=\infty$.

Then

$$\lim_{x\to a}\frac{h(x)}{p(x)}=\frac{\lim_{x\to a}h(x)}{\lim_{x\to a}p(x)}$$
$$=\frac{1}{\infty} \qquad \text{Not indeterminate form}$$
$$=0$$

The given limit is not indeterminate.

Therefore the value of the given limit is 0.

(d) Consider the limit $\lim_{x\to a}\frac{p(x)}{f(x)}$.

Need to determine whether the given limit is an indeterminate form or not.

Since $\lim_{x\to a}p(x)=\infty$ and $\lim_{x\to a}f(x)=0$.

Then

$$\lim_{x\to a}\frac{p(x)}{f(x)}=\frac{\lim_{x\to a}p(x)}{\lim_{x\to a}f(x)}$$
$$=\frac{\infty}{0} \qquad \text{Not indeterminate form}$$

The given limit is not indeterminate.

If the denominator approaches 0 from left, then $\lim_{x\to a}\frac{p(x)}{f(x)}=-\infty$.

And if the denominator approaches 0 from right then $\lim_{x\to a}\frac{p(x)}{f(x)}=\infty$.

Then $\lim_{x\to a}\frac{p(x)}{f(x)}=\infty \text{ or } -\infty$.

Therefore the value of the given limit is $\infty, -\infty$ or does not exist.

(e) Consider the limit $\lim_{x\to a}\frac{p(x)}{q(x)}$.

Need to determine whether the given limit is an indeterminate form or not.

Since $\lim_{x\to a} q(x)=\infty$ and $\lim_{x\to a} p(x)=\infty$,

Then

$$\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{\lim_{x\to a}p(x)}{\lim_{x\to a}q(x)}$$

$$=\frac{\infty}{\infty} \qquad \text{Indeterminate form}$$

Therefore the given limit is an indeterminate form of type $\frac{\infty}{\infty}$.

Answer 2E.

(A) We have been given that $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} p(x)=\infty$

Then $\lim_{x\to a}\left[f(x)p(x)\right]=0\cdot\infty$ indeterminate form

(B) We have been given that $\lim_{x\to a} h(x)=1$ and $\lim_{x\to a} p(x)=\infty$

Then $\lim_{x\to a}\left[h(x)p(x)\right]=1\cdot\infty=\infty$

Answer 3E.

(C) We have been given that $\lim_{x\to a} p(x)=\infty$ and $\lim_{x\to a} q(x)=\infty$

Then $\lim_{x\to a}\left[p(x)+q(x)\right]=\infty$

Answer 4E.

(C) We have been given that $\lim_{x\to a} h(x)=1$ and $\lim_{x\to a} p(x)=\infty$

Then $\lim_{x\to a}\left[h(x)\right]^{p(x)}=1^{\infty}\rightarrow$indeterminate

(D) We have been given that $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} p(x)=\infty$

Then $\lim_{x\to a}\left[p(x)\right]^{f(x)}=\infty^{0}$ $\rightarrow$Inderminate form

(E) We have been given that $\lim_{x\to a} p(x)=\infty$ and $\lim_{x\to a} q(x)=\infty$

Then $\lim_{x\to a}\left[p(x)\right]^{q(x)}=\infty$

(F) We have been given that $\lim_{x\to a} p(x)=\infty$ and $\lim_{x\to a} q(x)=\infty$

Then $\lim_{x\to a}\left[\sqrt[q(x)]{p(x)}\right]=\lim_{x\to a}\left[p(x)\right]^{1/q(x)}=\infty^{0}$ $\rightarrow$Indeterminate form

Answer 5E.

From the graph both $f(x)$ and $g(x)$ tend to zero when x tends to 2.

Then by l' Hospital's rule

$$\underset{x\to 2}{Lt}\frac{f(x)}{g(x)}=\underset{x\to 2}{Lt}\frac{f'(x)}{g'(x)}$$

$$\underset{x\to 2}{Lt}\frac{f(x)}{g(x)}=\underset{x\to 2}{Lt}\frac{1.8(x-2)}{\frac{4}{5}(x-2)}$$

$$\underset{x\to 2}{Lt}\frac{f(x)}{g(x)} = \underset{x\to 2}{Lt}\frac{1.8}{\left(\frac{4}{5}\right)} \text{(By applying L' Hospital's rule)}$$

$$= \underset{x\to 2}{Lt}(1.8)\times\frac{5}{4}$$

$$= \frac{18}{10}\times\frac{5}{4}$$

$$= \frac{9}{5}\times\frac{5}{4}$$

$$= \frac{9}{4}$$

$$= 2.25$$

$$\boxed{\therefore \underset{x\to 2}{Lt}\frac{f(x)}{g(x)} = 2.25}$$

Answer 6E.

From the graph both $f(x)$ and $g(x)$ tend to zero when x tends to 2.

Then by l' Hospital's rule,

$$\underset{x\to 2}{Lt}\frac{f(x)}{g(x)} = \underset{x\to 2}{Lt}\frac{f'(x)}{g'(x)}$$

$$= \underset{x\to 2}{Lt}\frac{1.5(x-2)}{2-x}$$

$$= \underset{x\to 2}{Lt}\frac{(1.5)(1)}{(-1)} \quad \text{(By applying l' Hospital's Rule)}$$

$$= -1.5$$

Therefore $\boxed{\underset{x\to 2}{Lt}\frac{f(x)}{g(x)} = -1.5}$

Answer 7E.

We have to find the limit

$$\lim_{x\to 1}\frac{x^2-1}{x^2-x} \qquad \text{This is the form of } 0/0$$

First we factorize the denominator and numerator

$$\lim_{x\to 1}\frac{x^2-1}{x^2-x} = \lim_{x\to 1}\frac{(x-1)(x+1)}{x(x-1)}$$

$$= \lim_{x\to 1}\frac{(x+1)}{x}$$

$$= \frac{(1+1)}{1} = \boxed{2}$$

Answer 8E.

Consider the limit

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2}$$

Need to find the limit of the function $\lim_{x\to 2}\frac{x^2+x-6}{x-2}$ using L'Hospital's rule or any other elementary method which know. (might not even need to use L'Hospital's Rule!) Always try to factor first, if possible.

Let $f(x)=\frac{x^2+x-6}{x-2}$

Factor the numerator of the above function

$$f(x)=\frac{(x+3)(x-2)}{x-2}.$$

This reduces to $f(x)=x+3$ (where $x\neq 2$).

This means there is a hole at $x=2$.

So, as $x\to 2$, $x+3\to 5$. That means $\lim_{x\to 2}(x+3)=5$.

Answer 9E.

Both functions x^3-2x^2+1 and x^3-1 tend to zero as x tends to 1. Then using l' Hospital's Rule

$$\underset{x\to 1}{Lt}\frac{x^3-2x^2+1}{x^3-1}=\underset{x\to 1}{Lt}\frac{3x^2-4x}{3x^2}$$

$$=\frac{3-4}{3}$$

$$=-\frac{1}{3}$$

Therefore $\boxed{\underset{x\to 1}{Lt}\frac{x^3-2x^2+1}{x^3-1}=-\frac{1}{3}}$

Answer 10E.

Both functions $6x^2+5x-4$ and $4x^2+16x-9$ tend to 0 as $x\to\frac{1}{2}$. Using l' Hospital's rule

$$\underset{x\to\frac{1}{2}}{Lt}\frac{6x^2+5x-4}{4x^2+16x-9}=\underset{x\to\frac{1}{2}}{Lt}\frac{12x+5}{8x+16}$$

$$=\frac{6+5}{4+16}$$

$$=\frac{11}{20}$$

Therefore $\boxed{\underset{x\to\frac{1}{2}}{Lt}\frac{6x^2+5x-4}{4x^2+16x-9}=\frac{11}{20}}$

Answer 12E.

Evaluate the following limit

$$\lim_{x\to 0}\frac{\sin 4x}{\tan 5x}$$

Let's look at the limit of the numerator and denominator separately.

Notice that when $x\to 0$, $\sin 4x\to 0$ and $\tan 5x\to 0$. This is the indeterminate form $\frac{0}{0}$

Because of this; you are allowed to use L'Hospital's Rule to evaluate the limit.

Now simplified the quotient, let's evaluate the limit.

Note that as $x \to 0$, $\cos x \to 1$

$$\lim_{x\to 0} \frac{4\cos 4x}{5\sec^2 5x} = \lim_{x\to 0} \frac{4}{5}\cos 4x \cdot \cos^2 5x$$

As $x \to 0$, $\cos 4x \to 1$ and $\cos^2 5x \to 1$.

Therefore, $\lim_{x\to 0} \frac{4}{5}\cos 4x \cdot \cos^2 5x = \frac{4}{5}$.

Because of L-Hospital's Rule, the limit of $\frac{\sin 4x}{\tan 5x}$ as $x \to 0$ is $\boxed{\frac{4}{5}}$.

Answer 13E.

Both functions $e^{2t} - 1 \to 0$ and $\sin t \to 0$ as $t \to 0$

By using l' Hospital's rule,

$$\underset{x\to 0}{Lt} \frac{e^{2t}-1}{\sin t} = \underset{x\to 0}{Lt} \frac{2e^{2t}}{\cos t}$$

$$= \frac{2.1}{1}$$

$$= 2$$

Therefore $\boxed{\underset{t\to 0}{Lt} \frac{e^{2t}-1}{\sin t} = 2}$

Answer 14E.

Both functions $x^2 \to 0$ and $1-\cos x \to 0$ as $x \to 0$

By using l' Hospital's rule

$$\underset{x\to 0}{Lt} \frac{x^2}{1-\cos x} = \underset{x\to 0}{Lt} \frac{2x}{\sin x}$$

Again $2x \to 0$ and $\sin x \to 0$ as $x \to 0$ then by l' Hospital's rule,

$$\underset{x\to 0}{Lt} \frac{2x}{\sin x} = \underset{x\to 0}{Lt} \frac{2}{\cos x}$$

$$= \frac{2}{1}$$

$$= 2$$

Therefore $\boxed{\underset{x\to 0}{Lt} \frac{x^2}{1-\cos x} = 2}$

Answer 15E.

Both functions $1-\sin\theta \to 0$ and $1+\cos 2\theta \to 0$ as $\theta \to \pi/2$

Then using l' Hospital's rule

$$\underset{\theta\to \pi/2}{Lt} \frac{1-\sin\theta}{1+\cos 2\theta} = \underset{\theta\to \pi/2}{Lt} \frac{-\cos\theta}{-2\sin 2\theta}$$

$$= \underset{\theta\to \pi/2}{Lt} \frac{\cos\theta}{2\sin 2\theta}$$

Again $\cos\theta \to 0$ and $\sin 2\theta \to 0$ as $\theta \to \pi/2$ then by l' Hospital's rule

Answer 17E.

Consider the following limit:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$$

The objective is to evaluate the given limit.

Answer 18E.

Evaluate the following limit

$$\lim_{x\to\infty}\frac{x+x^2}{1-2x^2}$$

Let's look at the limit of the numerator and denominator separately.

Notice that when $x\to\infty$, $x+x^2\to\infty$ and $1-2x^2\to\infty$. This is the indeterminate form $\frac{\infty}{\infty}$

Because of this; you are allowed to use L'Hospital's Rule to evaluate the limit.

L-Hospital's Rule says:

$$\lim_{x\to b}\frac{f(x)}{g(x)}=\lim_{x\to b}\frac{f'(x)}{g'(x)}$$

Take the derivative of the numerator, and then the derivative of the denominator. (Note: do NOT use the quotient rule. Take the derivative of the numerator and denominator independently.)

Let $f(x)=x+x^2$ and $g(x)=1-2x^2$.

Their respective derivatives are

$f'(x)=1+2x$ and $g'(x)=-4x$.

L-Hospital's Rule says:

$$\lim_{x\to b}\frac{f(x)}{g(x)}=\lim_{x\to b}\frac{f'(x)}{g'(x)}$$

$$\begin{aligned}\lim_{x\to\infty}\frac{x+x^2}{1-2x^2}&=\lim_{x\to\infty}\frac{1+2x}{-4x}\\&=-\lim_{x\to\infty}\frac{1}{4x}-\lim_{x\to\infty}\frac{2x}{4x}\\&=-\frac{1}{4(\infty)}-\frac{2}{4}\\&=0-\frac{2}{4}\end{aligned}$$

$$=-\frac{2}{4}$$

$$=-\frac{1}{2}$$

Thus, $\lim_{x\to\infty}\frac{x+x^2}{1-2x^2}=\boxed{-\frac{1}{2}}$

Answer 20E.

Both functions $\ln\sqrt{x}\to 0$ and $x^2\to\infty$ as $x\to\infty$

Then using l' Hospital's Rule,

$$\begin{aligned}\underset{x\to\infty}{Lt}\frac{\ln\sqrt{x}}{x^2}&=\underset{x\to\infty}{Lt}\frac{\frac{1}{2}\cdot\ln x}{x^2}\\&=\frac{1}{2}\underset{x\to\infty}{Lt}\frac{1/x}{2x}\\&=\frac{1}{4}\underset{x\to\infty}{Lt}\frac{1}{x^2}\\&=\frac{1}{4}(0)\\&=0\end{aligned}$$

Therefore $\boxed{\underset{x\to\infty}{Lt}\frac{\ln\sqrt{x}}{x^2}=0}$

Answer 22E.

Both functions $8^t - 5^t \to 0$ and $t \to 0$ as $t \to 0$

Using l' Hospital's Rule

$$\underset{t\to 0}{Lt}\frac{8^t - 5^t}{t} = \underset{t\to 0}{Lt}\frac{8^t \ln 8 - 5^t \ln 5}{1}$$

$$= \ln 8 - \ln 5$$

$$= \ln(8/5)$$

Therefore $\boxed{\underset{t\to 0}{Lt}\frac{8^t - 5^t}{t} = \ln(8/5)}$

Answer 24E.

Consider the following limit;

$$\lim_{u\to\infty}\frac{e^{u/10}}{u^3}.$$

The objective is to evaluate the given limit.

As $u \to \infty, \quad e^{\frac{u}{10}} \to \infty$

As $u \to \infty \ , \ u^3 \to \infty$

So,

$$\lim_{u\to\infty}\frac{e^{u/10}}{u^3} = \frac{\infty}{\infty}$$

The limit $\lim_{u\to\infty}\frac{e^{u/10}}{u^3}$ is an indeterminate form of type $\frac{\infty}{\infty}$, so use L'Hospital's Rule to evaluate

$$\lim_{u\to\infty}\frac{e^{u/10}}{u^3}.$$

Apply L'Hospital's Rule to $\lim_{u\to\infty}\frac{e^{u/10}}{u^3}$.

$$\lim_{u\to\infty}\frac{e^{u/10}}{u^3} = \lim_{u\to\infty}\frac{\frac{d}{du}\left[e^{\frac{1}{10}u}\right]}{\frac{d}{du}\left[u^3\right]}$$

$$= \lim_{u\to\infty}\frac{e^{\frac{1}{10}u}\cdot\frac{d}{du}\left[\frac{1}{10}u\right]}{3u^2}$$

$$= \lim_{u\to\infty}\frac{e^{\frac{1}{10}u}\cdot\frac{1}{10}\frac{d}{du}(u)}{3u^2}$$

$$= \lim_{u\to\infty}\frac{e^{\frac{1}{10}u}\cdot\frac{1}{10}(1)}{3u^2}$$

$$= \frac{1}{30}\lim_{u\to\infty}\frac{e^{\frac{1}{10}u}}{u^2}$$

$$= \frac{\infty}{\infty}$$

The limit $\lim_{u\to\infty}\frac{e^{\frac{1}{10}u}}{u^2}$ is an indeterminate form of type $\frac{\infty}{\infty}$, again use L'Hospital's Rule to evaluate

$$\lim_{u\to\infty}\frac{e^{\frac{1}{10}u}}{u^2}.$$

Apply L'Hospital's Rule to $\lim_{u\to\infty} \frac{e^{\frac{1}{10}u}}{u}$.

$$\lim_{u\to\infty} \frac{e^{u/10}}{u^3} = \frac{1}{600}\lim_{u\to\infty} \frac{\frac{d}{du}\left[e^{\frac{1}{10}u}\right]}{\frac{d}{du}[u]}$$

$$= \frac{1}{600}\lim_{u\to\infty} \frac{\frac{1}{10}e^{\frac{1}{10}u}}{1}$$

$$= \frac{1}{6000}\lim_{u\to\infty} e^{\frac{1}{10}u}$$

$$= \frac{1}{6000}e^{\frac{1}{10}\infty}$$

$$= \frac{1}{6000}e^{\infty}$$

$$= \frac{1}{6000}\cdot\infty$$

$$= \infty$$

Therefore, $\lim_{u\to\infty} \frac{e^{u/10}}{u^3} = \boxed{\infty}$.

Answer 25E.

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} \qquad \left(\frac{0}{0}\right)$$

Using L' Hospital's rule, we get:

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{\frac{d}{dx}\left(e^x - 1 - x\right)}{\frac{d}{dx}\left(x^2\right)}$$

$$= \lim_{x\to 0} \frac{e^x - 1}{2x} \qquad \left(\frac{0}{0}\right)$$

Again using L' Hospital's rule, we get

$$\lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{\frac{d}{dx}\left(e^x - 1\right)}{\frac{d}{dx}(2x)}$$

$$= \lim_{x\to 0} \frac{e^x}{2}$$

$$= \frac{e^0}{2}$$

$$= \frac{1}{2}$$

Thus $\boxed{\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}}$

Answer 26E.

Consider the limit

$$\lim_{x\to 0}\frac{\sinh x - x}{x^3}$$

Evaluate the limit by using L hospital's rule.

$$\lim_{x\to 0}\frac{\sinh x - x}{x^3} = \lim_{x\to 0}\frac{\frac{d}{dx}(\sinh x - x)}{\frac{d}{dx}(x^3)} \quad \left\{\begin{array}{l}\text{Apply L hospital's rule} \\ \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right]\end{array}\right.$$

$$= \lim_{x\to 0}\frac{\cosh x - 1}{3x^2} \quad \left\{\begin{array}{l}\text{If we apply the limit then we get } \frac{0}{0} \text{ form.} \\ \text{so again use the rule} \\ \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right]\end{array}\right.$$

Continuation to the above steps:

Now

$$\lim_{x\to 0}\frac{\cosh x - 1}{3x^2} = \lim_{x\to 0}\frac{\frac{d}{dx}(\cosh x - 1)}{\frac{d}{dx}(3x^2)} \quad \text{Since } \frac{d}{dx}(\cosh x) = \sinh x$$

$$= \lim_{x\to 0}\frac{\sinh x}{6x} \quad \left\{\begin{array}{l}\text{If we apply the limit then we get } \frac{0}{0} \text{ form.} \\ \text{so again use the rule} \\ \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right]\end{array}\right.$$

Answer 27E.

L-Hospital's Rule says:

$$\lim_{x\to b}\frac{f(x)}{g(x)} = \lim_{x\to b}\frac{f'(x)}{g'(x)}$$

Take the derivative of the numerator, and then the derivative of the denominator. (Note: do NOT use the quotient rule. Take the derivative of the numerator and denominator independently.)

Let $f(x) = \tanh x$ and $g(x) = \tan x$.

Their respective derivatives are $f'(x) = \sec h^2 x$ and $g'(x) = \sec^2 x$.

L-Hospital's Rule says: $\lim_{x\to b}\frac{f(x)}{g(x)} = \lim_{x\to b}\frac{f'(x)}{g'(x)}$

$$\lim_{x\to 0}\frac{\tanh x}{\tan x} = \lim_{x\to 0}\frac{\sec h^2 x}{\sec^2 x}$$

Answer 28E.

Consider $\lim_{x\to 0}\frac{x - \sin x}{x - \tan x}$

When $x = 0$,

$$\begin{aligned} x - \sin x &= 0 - \sin 0 \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} x - \tan x &= 0 - \tan 0 \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

Hence $\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}=\lim_{x\to 0}\frac{1-\cos x}{1-\sec^2 x}$ (applying L'Hospital's rule)

(Since f and g are differentiable and $g'(x)\neq 0$ on an open interval I that contains a. Suppose that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$)

$$=\lim_{x\to 0}\frac{\sin x}{-2\sec x\sec x\tan x}$$

$$=\lim_{x\to 0}\frac{\sin x}{-\left(2\sec^3 x\right)\sin x}$$

$$=-\lim_{x\to 0}\frac{1}{2\sec^3 x}$$

$$=-\frac{1}{2}$$

$$\boxed{\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}=-\frac{1}{2}}$$

Answer 30E.

The numerator and denominator are

$\ln x\to\infty$ as $x\to\infty$

$x\to\infty$ as $x\to\infty$

That is the limit gets again the indeterminate form of $\frac{\infty}{\infty}$

From the L' Hospital's rule,

$$2\lim_{x\to\infty}\frac{\ln x}{x}=2\lim_{x\to\infty}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x)}$$

$$=2\lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)}{(1)}$$

$$=2\lim_{x\to\infty}\frac{1}{x}$$

$\frac{1}{x}\to 0$ as $x\to\infty$

$$\boxed{\lim_{x\to\infty}\frac{(\ln x)^2}{x}=0}$$

Answer 31E.

Both functions $x3^x\to 0$ and $3^x-1\to 0$ as $x\to 0$

Using L' Hospitals Rules

$$\underset{x\to 0}{Lt}\frac{x3^x}{3^x-1}=\underset{x\to 0}{Lt}\frac{x3^x\cdot\ln 3+3^x}{3^x\ln 3}$$

$$=\frac{0+1}{\ln 3}$$

$$=\frac{1}{\ln 3}$$

Therefore $\boxed{\underset{x\to 0}{Lt}\frac{x3^x}{3^x-1}=\frac{1}{\ln 3}}$

Answer 32E.

Again using L-Hospital rule, we get

$$\lim_{x\to 0}\frac{-m\sin mx+n\sin nx}{2x}=\lim_{x\to 0}\frac{\frac{d}{dx}[-m\sin mx+n\sin nx]}{\frac{d}{dx}(2x)}$$

$$=\lim_{x\to 0}\frac{-m^2\cos mx+n^2\cos nx}{2}$$

$$=\frac{-m^2\cos 0+n^2\cos 0}{2}$$

$$=\frac{n^2-m^2}{2}$$

Thus $\boxed{\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}=\frac{n^2-m^2}{2}}$

Answer 33E.

Evaluate the limit as shown below:

$$\lim_{x\to 0}\frac{x+\sin x}{x+\cos x}=\frac{0+\sin(0)}{0+\cos(0)}$$

$$=\frac{0+0}{0+1}$$

$$=\frac{0}{1}$$

$$=0$$

Thus, $\lim_{x\to 0}\frac{x+\sin x}{x+\cos x}=\boxed{0}$.

Answer 34E.

$$\lim_{x\to 0}\frac{x}{\tan^{-1}(4x)}\qquad\left(\frac{0}{0}\right)$$

Using L' Hospital's rule, we get

$$\lim_{x\to 0}\frac{x}{\tan^{-1}(4x)}=\lim_{x\to 0}\frac{\frac{d}{dx}(x)}{\frac{d}{dx}\left[\tan^{-1}(4x)\right]}$$

$$=\lim_{x\to 0}\frac{1}{\frac{1}{(1+16x^2)}\times 4}$$

$$=\lim_{x\to 0}\frac{(1+16x^2)}{4}$$

$$=\frac{(1+16\times 0)}{4}$$

Thus $\boxed{\lim_{x\to 0}\frac{x}{\tan^{-1}(4x)}=\frac{1}{4}}$

Answer 35E.

$$\lim_{x\to 1}\frac{1-x+\ln x}{1+\cos\pi x}\qquad\left(\frac{0}{0}\right)$$

Using L' Hospital's rule, we get

$$\lim_{x\to 1}\frac{1-x+\ln x}{1+\cos\pi x}=\lim_{x\to 1}\frac{\frac{d}{dx}(1-x+\ln x)}{\frac{d}{dx}(1+\cos\pi x)}$$

$$=\lim_{x\to 1}\frac{-1+\frac{1}{x}}{-\pi\sin\pi x}\qquad\left(\frac{0}{0}\right)$$

Answer 36E.

Consider the following limit:

$$\lim_{x\to 0^+} \frac{x^x - 1}{\ln x + x - 1}$$

The objective is to evaluate the limit.

Let

$$x^x = f$$
$$\ln x^x = \ln f$$
$$x \ln x = \ln f$$
$$\lim_{x\to 0^+} (\ln f) = \lim_{x\to 0^+} x \ln x$$
$$\ln\left(\lim_{x\to 0^+} f\right) = \lim_{x\to 0^+} x \ln x$$
$$\lim_{x\to 0^+} f = e^{\lim_{x\to 0^+} x \ln x}$$

Consider,

$$\lim_{x\to 0^+} x \ln x$$

So,

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

Thus the above equation have an indeterminate form of type $\frac{\infty}{\infty}$

Answer 38E.

Consider the limit

$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$$

Need to find the given limit using L'Hospital's Rule or any elementary method.

Before evaluate our limit, note that as $x \to 0$, $e^x \to 1$ and $e^{-x} \to 1$.

Let's look at the limit of the numerator and denominator separately.

As $x \to 0$, $e^x - e^{-x} - 2x \to (1 - 1 - 0) = 0$ and $x - \sin x \to 0$. This is the indeterminate form $\frac{0}{0}$. Because of this, you are allowed to use L'Hospital's Rule to evaluate the limit.

L'Hospital's Rule says

$$\lim_{x\to b} \frac{f(x)}{g(x)} = \lim_{x\to b} \frac{f'(x)}{g'(x)}.$$

Now take the derivative of the numerator and the derivative of the denominator. (Note: do not use the quotient rule. Take the derivative of the numerator and denominator independently.)

Let $f(x) = e^x - e^{-x} - 2x$ and $g(x) = x - \sin x$.

Their respective derivatives are

$f'(x) = e^x - (-e^{-x}) - 2$ and $g'(x) = 1 - \cos x$.

Apply L'Hospital's Rule are you get the following:

$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$$

In the second application of L'Hospital's Rule, to get

$f(x)=e^x+e^{-x}-2$ and $g(x)=1-\cos x$.

Their respective derivatives are $f'(x)=e^x-e^{-x}$ and $g'(x)=\sin x$.

Applying L'Hospital's Rule again gives us:

$$\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x}=\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$$

In the third application of L'Hospital's Rule, to get

$f(x)=e^x-e^{-x}$ and $g(x)=\sin x$.

Their respective derivatives are

$f'(x)=e^x+e^{-x}$ and $g'(x)=\cos x$.

Applying L'Hospital's Rule again gives us:

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x}.$$

Answer 39E.

Consider the limit

$$\lim_{x\to 0}\frac{\cos x-1+\frac{1}{2}x^2}{x^4}$$

Need to find the given limit using L'Hospital's Rule or any other elementary method.

Before evaluate our limit note that, as $x\to 0$, $\cos x\to 1$.

First, let's look at the limit of the numerator and denominator (separately).

As $x\to 0$, $\cos x-1+\frac{1}{2}x^2\to 1-1+0$ and $x^4\to 0$. This is the indeterminate form $\frac{0}{0}$.

Because of this, use L'Hospital's Rule to evaluate the limit.

L'Hospital's Rule says

$$\lim_{x\to b}\frac{f(x)}{g(x)}=\lim_{x\to b}\frac{f'(x)}{g'(x)}.$$

Now take the derivative of the numerator and the derivative of the denominator. (Note: do not use the quotient rule. Take the derivative of the numerator and denominator independently.)

Let $f(x)=\cos x-1+\frac{1}{2}x^2$ and $g(x)=x^4$.

So, their respective derivatives are

$f'(x)=-\sin x+x$ and $g'(x)=4x^3$.

Apply L'Hospital's Rule and you will get:

$$\lim_{x\to 0}\frac{\cos x-1+\frac{1}{2}x^2}{x^4}=\lim_{x\to 0}\frac{-\sin x+x}{4x^3}$$

Evaluate this new limit. Look at the limit of the numerator and denominator individually.

As $x\to 0$, $-\sin x+x\to 0$ and $4x^3\to 0$. This is the indeterminate form $\frac{0}{0}$ so use L'Hospital's Rule again. You can use L'Hospital's Rule as many times as you'd like, just as long as you apply it to an indeterminate form (like $\frac{0}{0}$ or $\frac{\infty}{\infty}$).

In the second application of L'Hospital's Rule, let

$f(x) = -\sin x + x$ and $g(x) = 4x^3$.

Their respective derivatives are

$f'(x) = -\cos x + 1$ and $g'(x) = 12x^2$.

Applying L'Hospital's Rule again will give you:

$$\lim_{x\to 0}\frac{-\sin x + x}{4x^3} = \lim_{x\to 0}\frac{-\cos x + 1}{12x^2}$$

Let's evaluate this new limit. As $x \to 0$, it follows that $-\cos x + 1 \to 0$ and $12x^2 \to 0$. This gives us the indeterminate form $\frac{0}{0}$ again, so use L'Hospital's Rule until arrive at a limit.

In the third application of L'Hospital's Rule,

Let $f(x) = -\cos x + 1$ and $g(x) = 12x^2$.

Their respective derivatives are

$f'(x) = -(-\sin x)$ and $g'(x) = 24x$.

Applying L'Hospital's Rule again will give you:

$$\lim_{x\to 0}\frac{-\cos x + 1}{12x^2} = \lim_{x\to 0}\frac{\sin x}{24x}$$

Let's evaluate this new limit. As $x \to 0$, it follows that $\sin x \to 0$ and $24x \to 0$. This gives us the indeterminate form $\frac{0}{0}$ again, so use L'Hospital's Rule until arrive at a limit.

In the fourth application of L'Hospital's Rule,

Let $f(x) = \sin x$ and $g(x) = 24x$.

Their respective derivatives are $f'(x) = \cos x$ and $g'(x) = 24$.

Applying L'Hospital's Rule again will give you:

$$\lim_{x\to 0}\frac{\sin x}{24x} = \lim_{x\to 0}\frac{\cos x}{24}$$

Let's evaluate this new limit. As $x \to 0$, it follows that $\cos x \to 1$ and $24 \to 24$. That means

$$\lim_{x\to 0}\frac{\cos x}{24} = \frac{1}{24}.$$

From multiple applications of L'Hospital's Rule, finally arrived at the limit:

$$\lim_{x\to 0}\frac{\cos x - 1 + \frac{1}{2}x^2}{x^4} = \boxed{\frac{1}{24}}.$$

Answer 40E.

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Answer 41E.

Consider the limit $\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right)$.

As $x \to \infty$, $\frac{\pi}{x} \to 0$, so $\sin\left(\frac{\pi}{x}\right) \to 0$

The limit is indeterminate form of type $\infty \cdot 0$

l'Hospital's rule cannot apply here.

Rewrite the expression as a quotient.

$$\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right) = \lim_{x\to\infty}\frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}}$$

As $x\to\infty$, $\frac{\pi}{x}\to 0$, so $\sin\left(\frac{\pi}{x}\right)\to 0$ and $\frac{1}{x}\to 0$.

The limit is indeterminate form of type $\frac{0}{0}$.

l'Hospital's rule can apply here.

Apply l'Hospital's rule:

$$\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right) = \lim_{x\to\infty}\frac{\cos\left(\frac{\pi}{x}\right)\frac{d}{dx}\left(\frac{\pi}{x}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{-\frac{\pi}{x^2}\cos\left(\frac{\pi}{x}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{\pi}{\cancel{x^2}}\cos\left(\frac{\pi}{x}\right)\cdot\frac{\cancel{x^2}}{1}$$

$$= \lim_{x\to\infty}\pi\cos\left(\frac{\pi}{x}\right)$$

As $x\to\infty$, $\frac{\pi}{x}\to 0$, so $\cos\left(\frac{\pi}{x}\right)\to 1$

So, $\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right) = \pi(1)$

$= \pi$

Therefore $\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right) = \boxed{\pi}$

Answer 42E.

Consider the limit

$$\underset{x\to\infty}{Lt}\sqrt{x}e^{-x/2}$$

$\sqrt{x}\to\infty$ as $x\to\infty$

$e^{-x/2}\to e^{-\infty}\to 0$ as $x\to\infty$

$\underset{x\to\infty}{Lt}\sqrt{x}e^{-x/2}\to 0\cdot\infty$ (indeterminate form)

We can deal with it by writing the product $\sqrt{x}e^{-x/2}$ as a quotient

$$\underset{x\to\infty}{Lt}\sqrt{x}e^{-x/2} = \underset{x\to\infty}{Lt}\frac{\sqrt{x}}{e^{x/2}}$$

$\sqrt{x}\to\infty$ as $x\to\infty$

$e^{x/2}\to e^{\infty}\to\infty$ as $x\to\infty$

$\underset{x\to\infty}{Lt}\frac{\sqrt{x}}{e^{x/2}}\to\frac{\infty}{\infty}$ (indeterminate form)

From l' Hospital's rules gives

$$\underset{x\to\infty}{Lt}\sqrt{x}e^{-x/2} = \underset{x\to\infty}{Lt}\frac{\sqrt{x}}{e^{x/2}}$$

$$= \underset{x\to\infty}{Lt}\frac{1/2\sqrt{x}}{\frac{1}{2}e^{x/2}}$$

$$= \underset{x\to\infty}{Lt}\frac{1/\sqrt{x}}{e^{x/2}}$$

$$= \underset{x\to\infty}{Lt}\frac{1}{\sqrt{x}\cdot e^{x/2}}$$

$\sqrt{x}\to\infty$ as $x\to\infty$

$e^{x/2}\to e^{\infty}\to\infty$ as $x\to\infty$

$\sqrt{x}\cdot e^{x/2}\to\infty$

$$\underset{x\to\infty}{Lt}\frac{1}{\sqrt{x}\cdot e^{x/2}}\to\frac{1}{\infty}\to 0$$

Therefore $\boxed{\underset{x\to\infty}{Lt}\sqrt{x}e^{-x/2}=0}$

Answer 43E.

$$\lim_{x\to 0}\cot 2x\sin 6x$$

$$= \lim_{x\to 0}\frac{\sin 6x}{\tan 2x} \qquad \left(\frac{0}{0}\right)$$

Using L' Hospital's rule, we get:-

$$\lim_{x\to 0}\frac{\sin 6x}{\tan 2x} = \lim_{x\to 0}\frac{\frac{d}{dx}(\sin 6x)}{\frac{d}{dx}(\tan 2x)}$$

$$= \lim_{x\to 0}\frac{6\cos 6x}{2\sec^2 2x}$$

$$= \frac{6\cos 0}{2\sec^2 0}$$

$$\boxed{=3}$$

Answer 44E.

We have to evaluate $\lim_{x\to 0^+}\sin x\ \ln x$

This is the form of $(0.(-\infty))$

For making it as of indeterminate form (∞/∞), we rewrite the given limit as

$$\lim_{x\to 0^+}\sin x\ \ln x = \lim_{x\to 0^+}\frac{\ln x}{(1/\sin x)}$$

$$= \lim_{x\to 0^+}\frac{\ln x}{\csc x}$$

Now this is the form of $(-\infty/\infty)$

Applying L-Hospital rule

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(\csc x)}$$

$$= \lim_{x\to 0^+} \frac{(1/x)}{(-\csc x \cot x)}$$

$$= \lim_{x\to 0^+} \frac{(1/x)}{\left(-\frac{1}{\sin x}\cdot\frac{1}{\tan x}\right)}$$

$$= \lim_{x\to 0^+} \frac{\sin x \tan x}{-x}$$

$$= \left(-\lim_{x\to 0^+} \frac{\sin x}{x}\right)\left(\lim_{x\to 0^+} \tan x\right)$$

$$= (-1)\left(\lim_{x\to 0^+} \tan x\right) \qquad \text{Since } \lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

$$= (-1)(\tan 0) = 0 \qquad \text{[Direct substitution]}$$

Thus $\boxed{\lim_{x\to 0^+} \sin x \ln x = 0}$

Answer 45E.

Consider the expression

$$\lim_{x\to\infty} x^3 e^{-x^2}$$

Then we can rewrite as

$$\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty} \frac{x^3}{e^{x^2}}$$

$$= \frac{\infty^3}{e^\infty}$$

$$= \frac{\infty}{\infty}$$

L' Hospital's Rule:

Suppose f and g are differentiable and $g'(x) \neq 0$ near a (expect possibly at a). Suppose that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ or that $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ if the limit on the right side exists (or is ∞ or $-\infty$)

Using L' Hospital's Rule, we get

$$\lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \lim_{x\to\infty} \frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(e^{x^2})} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to\infty} \frac{3x^2}{e^{x^2}\cdot\frac{d}{dx}(x^2)} \quad \text{Use the Chain Rule}$$

$$= \lim_{x\to\infty} \frac{3x^2}{e^{x^2}\cdot(2x)}$$

$$= \lim_{x\to\infty} \frac{3x}{2e^{x^2}} \quad \text{Simplify.}$$

Since $\lim_{x\to\infty}\frac{3x}{2e^{x^2}}=\frac{\infty}{\infty}$, again using L' Hospital's Rule, we get

$$\lim_{x\to\infty}\frac{3x}{2e^{x^2}}=\lim_{x\to\infty}\frac{\frac{d}{dx}(3x)}{\frac{d}{dx}\left(2e^{x^2}\right)}\quad\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$=\lim_{x\to\infty}\frac{3}{2e^{x^2}\cdot\frac{d}{dx}\left(x^2\right)}\quad\text{Use the Chain Rule}$$

$$=\lim_{x\to\infty}\frac{3}{2e^{x^2}\cdot(2x)}$$

$$=\lim_{x\to\infty}\frac{3}{4xe^{x^2}}\quad\text{Simplify.}$$

Continuing the previous steps:

$$\begin{aligned}\lim_{x\to\infty}x^3e^{-x^2}&=\frac{3}{4}\cdot\lim_{x\to\infty}\frac{1}{xe^{x^2}}\\&=\frac{3}{4}\cdot\left(\lim_{x\to\infty}\frac{1}{x}\right)\left(\lim_{x\to\infty}\frac{1}{e^{x^2}}\right)\\&=\frac{3}{4}\cdot(0)\left(\lim_{x\to\infty}\frac{1}{e^{x^2}}\right)\qquad\text{Since }\lim_{x\to\infty}\frac{1}{x}=0.\\&=0\end{aligned}$$

Therefore, $\lim_{x\to\infty}x^3e^{-x^2}=\boxed{0}$

Answer 46E.

L'Hospital's rule:

Suppose f and g are differentiable and $g'(x)\neq 0$ on an open interval I that contain a.

Suppose that $\lim_{x\to a}f(x)=0$ and $\lim_{x\to a}g(x)=0$

Or that

$\lim_{x\to a}f(x)=\pm\infty$ and $\lim_{x\to a}g(x)=\pm\infty$

Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

If the limit on the right side exists.

Consider the limit, $\lim_{x\to\infty}x\tan\left(\frac{1}{x}\right)$

Rewrite as the above limit is,

$$\begin{aligned}\lim_{x\to\infty}x\tan\left(\frac{1}{x}\right)&=\lim_{x\to\infty}\frac{\tan\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}\\&=\frac{\tan\left(\frac{1}{\infty}\right)}{\left(\frac{1}{\infty}\right)}\\&=\frac{\tan(0)}{(0)}\\&=\frac{0}{0}\end{aligned}$$

Applying L'Hospital's rule:

$$\lim_{x\to\infty}\frac{\tan\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}=\lim_{x\to\infty}\frac{\frac{d}{dx}\tan\left(\frac{1}{x}\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$

$$=\lim_{x\to\infty}\frac{\left(-\frac{1}{x^2}\right)\sec^2\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)}$$

$$=\lim_{x\to\infty}\sec^2\left(\frac{1}{x}\right)$$

$$=\sec^2\left(\frac{1}{\infty}\right)$$

$$=\sec^2(0)$$

$$=1$$

Therefore, $\boxed{\lim_{x\to\infty}x\tan\left(\frac{1}{x}\right)=1}$

Answer 47E.

We have to evaluate $\lim_{x\to1^+}\ln x\tan\left(\frac{\pi x}{2}\right)$

This is the form of $(0\cdot\infty)$, so we rewrite this limit as

$$\lim_{x\to1^+}\ln x\tan\left(\frac{\pi x}{2}\right)=\lim_{x\to1^+}\frac{\ln x}{\cot(\pi x/2)}$$

Now this the indeterminate form of $(0/0)$

Applying L-Hospital rule

$$\lim_{x\to1^+}\frac{\ln x}{\cot(\pi x/2)}\overset{H}{=}\lim_{x\to1^+}\frac{1/x}{-(\pi/2)\csc^2(\pi x/2)}$$

$$=\frac{1/1}{-(\pi/2)\csc^2(\pi/2)}\qquad\text{[Direct substitution]}$$

$$=\boxed{-\frac{2}{\pi}}$$

Answer 48E.

Consider the limit $\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\cos x\cdot\sec 5x$

Consider the function

$$\cos x\cdot\sec 5x=\cos x\cdot\frac{1}{\cos 5x}\text{ or }\frac{1}{\sec x}\cdot\sec 5x$$

The given function can be expressed as function of cosine or secant function

Use cosine form of function

$$\cos x\cdot\sec 5x=\cos x\cdot\frac{1}{\cos 5x}$$

As $x\to\frac{\pi}{2}^-$, $\cos x\to\cos\frac{\pi}{2}\to 0$ and $\cos 5x\to\cos\frac{5\pi}{2}\to 0$

$$\cos x\cdot\frac{1}{\cos 5x}\to\frac{0}{0}(\text{interminate form})\text{ as }x\to\frac{\pi}{2}^-$$

By using L' Hospital rule

$$\lim_{x\to\frac{\pi}{2}^-} \cos x\cdot\sec 5x = \lim_{x\to\frac{\pi}{2}^-}\frac{\cos x}{\cos 5x}$$

$$= \lim_{x\to\frac{\pi}{2}^-}\frac{-\sin x}{-5\sin 5x}$$

$$= \frac{-1}{-5(1)}$$

$$= \boxed{\frac{1}{5}}$$

Use secant form of a function

$$\cos x\cdot\sec 5x = \frac{\sec 5x}{\sec x}$$

As $x\to\frac{\pi}{2}^-$, $\sec x\to\sec\frac{\pi}{2}\to\infty$ and $\sec 5x\to\sec\frac{5\pi}{2}\to\infty$

$$\frac{\sec 5x}{\sec x}\to\frac{\infty}{\infty}(\text{interminate form }) \text{ as } x\to\frac{\pi}{2}^-$$

So L' Hospital rule gives

$$\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\cos x\sec 5x = \underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{\sec 5x}{\sec x}$$

$$= \underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{5\sec 5x\cdot\tan 5x}{\sec x\tan x}$$

In this case we get again indeterminate form second application of L' Hospital rule gives

$$\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{5\sec 5x\cdot\tan 5x}{\sec x\tan x} = 5\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{\sin 5x\cdot\cos^2 x}{\cos^2 5x\cdot\sin x}$$

$$= 5\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{\cos x(\sin 5x\cdot\cos x)}{\cos 5x(\cos 5x\cdot\sin x)}$$

$$= 5\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{\cos x(2\sin 5x\cdot\cos x)}{\cos 5x(2\cos 5x\cdot\sin x)}$$

$$= 5\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{\cos x(\sin 6x+\sin 4x)}{\cos 5x(\sin 6x-\sin 4x)}$$

$$= 5\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{2\cos x\sin 6x+2\sin 4x\cos x}{2\sin 6x\cos 5x-2\cos 5x\sin 4x}$$

$$= 5\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{\sin 7x+\sin 5x+\sin 5x+\sin 3x}{\sin 11x+\sin x-\sin 9x+\sin x}$$

In this case we get again indeterminate form $\left(\frac{0}{0}\right)$ second application of L' Hospital rule gives

$$5\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{\sin 7x+\sin 5x+\sin 5x+\sin 3x}{\sin 11x+\sin x-\sin 9x+\sin x} = 5\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{7\cos 7x+5\cos 5x+5\cos 5x+3\cos 3x}{11\cos 11x+\cos x-9\cos 9x+\cos x}$$

$$= 5\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{-7^2\sin 7x-5^2\sin 5x-5^2\sin 5x-3^2\sin 3x}{-11^2\sin 11x-\sin x+9^2\sin 9x-\sin x}$$

$$= 5\left(\frac{-7^2(-1)-5^2(1)-5^2(1)-3^2(-1)}{-11^2(-1)-1+9^2(1)-1}\right)$$

$$= 5\left(\frac{8}{200}\right) = \boxed{\frac{1}{5}}$$

Answer 49E.

Consider the limit $\lim_{x\to1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$.

As $x\to1$, $\frac{x}{(x-1)}\to\infty$ and $\frac{1}{\ln x}$ 8

The limit is indeterminate of type $\infty-\infty$

l'Hospital's rule cannot apply here.

Rewrite the expression as a quotient.

$$\lim_{x\to1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to1}\left[\frac{x\ln x-x+1}{(x-1)\ln x}\right]$$

As $x\to1$, $x\ln x-x+1\to0$ and $(x-1)\ln x\to0$.

The limit is indeterminate of type $\frac{0}{0}$

l'Hospital's rule can apply here.

Apply l'Hospital's rule:

$$\begin{aligned}\lim_{x\to1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)&=\lim_{x\to1}\frac{\frac{d}{dx}(x\ln x-x+1)}{\frac{d}{dx}[(x-1)\ln x]}\\&=\lim_{x\to1}\frac{\frac{x}{x}+\ln x-1}{\frac{x-1}{x}+\ln x}\\&=\lim_{x\to1}\frac{\ln x}{\frac{x-1}{x}+\ln x}\end{aligned}$$

As $x\to1$, $\ln x\to0$ and $\frac{x-1}{x}+\ln x\to0$.

The limit is indeterminate of type $\frac{0}{0}$

Again apply l'Hospital's rule.

Apply l'Hospital's rule:

$$\begin{aligned}\lim_{x\to1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)&=\lim_{x\to1}\frac{\frac{1}{x}}{\frac{x-(x-1)}{x^2}+\frac{1}{x}}\\&=\lim_{x\to1}\frac{\frac{1}{x}}{\frac{x+1}{x^2}}\\&=\lim_{x\to1}\frac{x}{x+1}\\&=\boxed{\frac{1}{2}}\end{aligned}$$

Answer 50E.

$$\begin{aligned}&\lim_{x\to0}(\csc x-\cot x) &&(\infty-\infty)\\&=\lim_{x\to0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)\\&=\lim_{x\to0}\left(\frac{1-\cos x}{\sin x}\right) &&\left(\frac{0}{0}\right)\end{aligned}$$

Using L' Hospital's rule, we get:-

$$\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right)=\lim_{x\to 0}\frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(\sin x)}$$
$$=\lim_{x\to 0}\frac{\sin x}{\cos x}$$
$$=\lim_{x\to 0}\tan x$$
$$=0$$

Thus $\boxed{\lim_{x\to 0}(\csc x-\cot x)=0}$

Answer 51E.

Consider $\underset{x\to 0^+}{Lt}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\underset{x\to 0^+}{Lt}\left(\frac{e^x-1-x}{x\left(e^x-1\right)}\right)$

$$=\underset{x\to 0^+}{Lt}\left(\frac{e^x-1}{xe^x+e^x-1}\right)\quad \text{(because using l' Hospital's rule)}$$
$$=\underset{x\to 0^+}{Lt}\left(\frac{e^x}{xe^x+e^x+e^x}\right)\quad \text{(again using l' Hospitals rule)}$$
$$=\frac{1}{0+1+1}$$
$$=\frac{1}{2}$$

Therefore $\boxed{\underset{x\to 0^+}{Lt}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]=\frac{1}{2}}$

Answer 52E.

Consider the limit $\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)$.

As $x\to 0$, $\cot x\to\infty$ and $\frac{1}{x}\to\infty$.

The limit is indeterminate form of type $\infty-\infty$

l'Hospital's rule cannot apply here.

Rewrite the expression as a quotient.

$$\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)=\lim_{x\to 0}\left(\frac{\cos x}{\sin(x)}-\frac{1}{x}\right)$$
$$=\lim_{x\to 0}\left(\frac{x\cos(x)-\sin(x)}{x\sin(x)}\right)$$

As $x\to 0$, $x\cos(x)-\sin(x)\to 0$ and $x\sin(x)\to 0$

The limit is indeterminate form of type $\frac{0}{0}$.

l'Hospital's rule can apply here.

Apply l'Hospital's rule:

$$\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)=\lim_{x\to 0}\left(\frac{-x\sin(x)+\cos(x)-\cos(x)}{x\cos(x)+\sin(x)}\right)$$
$$=\lim_{x\to 0}\left(\frac{-x\sin(x)}{x\cos(x)+\sin(x)}\right)$$

As $x\to 0$, $-x\sin(x)\to 0$ and $x\cos(x)+\sin(x)\to 0$

The limit is indeterminate of type $\frac{0}{0}$

So, again l'Hospital apply here.

Apply l'Hospital's rule:

$$\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)=\lim_{x\to 0}\left(\frac{-[x\cos(x)+\sin(x)]}{-x\sin x+\cos x+\cos(x)}\right)$$
$$=\left(\frac{-[0\cos(0)+\sin(0)]}{-(0)\sin(0)+2\cos(0)}\right)$$
$$=\frac{-(0-0)}{-0+2}$$
$$=0$$

Therefore, $\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)=\boxed{0}$.

Answer 53E.

We have to evaluate $\underset{x\to\infty}{Lim}(x-\ln x)$.

We observe that *as* x $\to\infty$,x $\to\infty$ and lnx $\to\infty$. Therefore this is in the indeterminate form $(\infty-\infty)$. Now we have to reduce the given limit into the indeterminate form $\left(\frac{\infty}{\infty}\right)$ and then we apply l'Hospital' Rule.

Now,

$$\underset{x\to\infty}{Lim}(x-\ln x)=\underset{x\to\infty}{Lim}\,x\left(1-\frac{\ln x}{x}\right)$$
$$=\underset{x\to\infty}{Lim}\,x.\,\underset{x\to\infty}{Lim}\left(1-\frac{\ln x}{x}\right)$$
$$=\underset{x\to\infty}{Lim}\,x.\left[1-\underset{x\to\infty}{Lim}\frac{\ln x}{x}\right]\qquad\ldots(1)$$

First we evaluate $\underset{x\to\infty}{Lim}\frac{\ln x}{x}$. Clearly *as* x $\to\infty$,x $\to\infty$ and lnx $\to\infty$ which is of the $\frac{\infty}{\infty}$. So we use l'Hospital's Rule.

$$\text{Therefore}\ \underset{x\to\infty}{Lim}\frac{\ln x}{x}=\underset{x\to\infty}{Lim}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x)}$$
$$=\underset{x\to\infty}{Lim}\frac{1/x}{1}$$
$$=\underset{x\to\infty}{Lim}\frac{1}{x}$$
$$=0.$$

Therefore from (1)

$$\underset{x\to\infty}{Lim}(x-\ln x)=\underset{x\to\infty}{Lim}\,x\left[1-\underset{x\to\infty}{Lim}\frac{\ln x}{x}\right]$$

$$=\underset{x\to\infty}{Lim}\,x\,[1-0]$$

$$=\underset{x\to\infty}{Lim}\,x$$

$$=\infty.$$

This is not in the indeterminate form.

Hence

$$\boxed{\underset{x\to\infty}{Lim}\,(x-\ln x)=\infty}.$$

Answer 54E.

Consider $\underset{x\to 1^+}{Lt}\left[\ln\left(x^7-1\right)-\ln\left(x^5-1\right)\right]=\underset{x\to 1^+}{Lt}\ln\left(\frac{x^7-1}{x^5-1}\right)$

$$=\underset{x\to 1^+}{Lt}\ln\left[\frac{(x-1)\left(x^6+x^5+x^4+x^3+x^2+x+1\right)}{(x-1)\left(x^4+x^3+x^2+x+1\right)}\right]$$

$$=\underset{x\to 1^+}{Lt}\ln\left[\frac{x^6+x^5+x^4+x^3+x^2+x+1}{x^4+x^3+x^2+x+1}\right]$$

$$=\ln\left[\frac{7}{5}\right]$$

$$=\ln 7-\ln 5$$

$$=\ln\left(\frac{7}{5}\right)$$

Therefore $\boxed{\underset{x\to 1^+}{Lt}\left[\ln\left(x^7-1\right)-\ln\left(x^5-1\right)\right]=\ln(7/5)}$

Answer 55E.

We have $x^{\sqrt{x}}=e^{\sqrt{x}\ln x}$

Using 1'Hospitals rule

$$\underset{x\to 0^+}{Lt}\sqrt{x}\ln x=\underset{x\to 0^+}{Lt}\frac{\ln x}{1/\sqrt{x}}$$

$$=\underset{x\to 0^+}{Lt}\frac{1/x}{\frac{1}{2}x^{-3/2}}$$

$$=\underset{x\to 0^+}{Lt}\frac{-2x^{3/2}}{x}$$

$$=\underset{x\to 0^+}{Lt}-2\sqrt{x}$$

$$=0$$

Therefore $\underset{x\to 0^+}{Lt}x^{\sqrt{x}}=\underset{x\to 0^+}{Lt}e^{\sqrt{x}\ln x}$

$$=e^0$$

$$=1$$

Therefore $\boxed{\underset{x\to 0}{Lt}x^{\sqrt{x}}=1}$

Answer 56E.

Consider the limit $\lim_{x\to0^+}(\tan 2x)^x$.

Recollect the L'Hospital's Rule: $\lim_{x\to b}\frac{f(x)}{g(x)}=\lim_{x\to b}\frac{f'(x)}{g'(x)}$

To evaluate the limit, take

$y=(\tan 2x)^x$

Then

$\ln y=\ln(\tan 2x)^x$

$\ln y = x\ln(\tan 2x)$ Use $\ln(x^r)=r\cdot\ln x$

$=\lim_{x\to0^+}\frac{\ln(\tan 2x)}{x^{-1}}$

$=\lim_{x\to0^+}\frac{\ln(\tan 2x)}{\frac{1}{x}}$ As $x\to0^+$, $\frac{1}{x}\to\infty$ and $\ln(\tan 2x)\to-\infty$

$=\frac{\infty}{\infty}$

Now use L'Hospital's Rule to evaluate the limit.

$$\lim_{x\to0^+}\frac{\ln(\tan 2x)}{\frac{1}{x}}=\lim_{x\to0^+}\frac{(\ln(\tan 2x))'}{\left(\frac{1}{x}\right)'}$$

Let $f(x)=\ln(\tan 2x)$ and $g(x)=x^{-1}$.

$$f'(x)=\frac{1}{\tan 2x}(\sec^2 2x)\cdot 2$$

$$=2\cdot\frac{\cos 2x}{\sin 2x}\left(\frac{1}{\cos^2 2x}\right)$$ Use chain rule $\frac{d}{dx}(\ln u)=\frac{1}{u}\frac{du}{dx}$

$$=\frac{2}{\sin 2x\cos 2x}$$

And $g'(x)=-x^{-2}$ Use $\frac{d}{dx}(x^n)=nx^{n-1}$

Now,

$$\lim_{x\to0^+}\frac{\ln(\tan 2x)}{x^{-1}}=\lim_{x\to0^+}\frac{\frac{2}{\sin 2x\cos 2x}}{-1x^{-2}}$$

$$=\lim_{x\to0^+}\frac{2}{\sin 2x\cos 2x}\cdot\frac{1}{-x^{-2}}$$

$$=\lim_{x\to0^+}\frac{-2x^2}{\sin 2x\cos 2x}$$

$$=\lim_{x\to0^+}\frac{-2x^2}{\sin 2x\cos 2x}\cdot\frac{2}{2}$$

$$=\lim_{x\to0^+}\left(\frac{-4x^2}{\sin 4x}\right)$$ As $x\to0^+$, $-4x^2\to0$ and $\sin 4x\to0$

$$=\frac{0}{0}$$

Again use L'Hospital's Rule to evaluate the limit.

$$\lim_{x\to 0^+}\frac{-4x^2}{\sin 4x}=\lim_{x\to 0^+}\frac{\left(-4x^2\right)'}{\left(\sin 4x\right)'}$$

Let $f(x)=-4x^2$ and $g(x)=\sin 4x$.

$f'(x)=-8x$ Use $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$

And

$g'(x)=4\cos 4x$ Since $\frac{d}{dx}(\sin x)=\cos x$

Now,

$$\lim_{x\to 0^+}\frac{\left(-4x^2\right)'}{\left(\sin 4x\right)'}=\frac{-8x}{4\cos 4x}$$

$$=\frac{-2x}{\cos 4x}$$

$=\frac{0}{1}$ As $x\to 0^+$, $-2x\to 0$ and $\cos 4x\to 1$

$$=0$$

Thus $\lim_{x\to 0^+}\ln y=0.$

Since $\lim_{x\to 0^+}\ln y=0$, then

$$e^{\lim_{x\to 0^+}(\ln y)}=e^0$$

$$\lim_{x\to 0^+}e^{(\ln y)}=1$$

$$\lim_{x\to 0^+}y=1$$

$\lim_{x\to 0^+}\left(\tan 2x\right)^x=\boxed{1}$ Back substitute $y=\left(\tan 2x\right)^x$

Answer 57E.

We have to evaluate $\lim_{x\to 0}(1-2x)^{1/x}$

Let $y=(1-2x)^{1/x}$

Then $\ln y=\frac{1}{x}\ln(1-2x)$

Taking limit

$$\lim_{x\to 0}(\ln y)=\lim_{x\to 0}\frac{\ln(1-2x)}{x} \qquad \left[\text{Form: } \frac{0}{0}\right]$$

Applying L-Hospital rule

$$\lim_{x\to 0}\frac{\ln(1-2x)}{x}\overset{H}{=}\lim_{x\to 0}\frac{\frac{-2}{1-2x}}{1}$$

$$=\lim_{x\to 0}\left(\frac{-2}{1-2x}\right)=-2 \qquad \text{[Direct substitution]}$$

Thus $\lim_{x\to 0}(\ln y)=-2$

Then $e^{\lim_{x\to 0}(\ln y)}=e^{-2}$

Or $\lim_{x\to 0}e^{(\ln y)}=e^{-2}$

Or $\lim_{x\to 0}y=e^{-2}$

Therefore, $\boxed{\lim_{x\to 0}(1-2x)^{1/x}=e^{-2}}$

Answer 58E.

Consider the following limit:

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$$

Write,

$$y=\left(1+\frac{a}{x}\right)^{bx}$$

To evaluate the limit $\lim_{x\to\infty} y$, take natural log both sides:

$$\ln y = bx\ln\left(1+\frac{a}{x}\right) \quad \ldots\ldots (1)$$

Now, apply limits to (1):

$$\lim_{x\to\infty}(\ln y)=\lim_{x\to\infty}\frac{b\ln(1+a/x)}{1/x}$$

Apply the limit and it gives a $\frac{0}{0}$ form, so, apply L-Hospital rule:

$$\lim_{x\to\infty}\frac{b\ln(1+a/x)}{1/x}=\lim_{x\to\infty}\frac{\frac{b}{(1+a/x)}(-a/x^2)}{(-1/x^2)}$$

$$=\lim_{x\to\infty}\frac{ab}{(1+a/x)}$$

Now, apply the limit:

$$\lim_{x\to\infty}(\ln y)=\frac{ab}{(1+0)}$$

Thus,

$$\lim_{x\to\infty}(\ln y)=ab$$

Use exponential function:

$$e^{\lim_{x\to\infty}(\ln y)}=e^{ab}$$

$$\lim_{x\to\infty} e^{(\ln y)}=e^{ab}$$

$$\lim_{x\to\infty} y=e^{ab}$$

Therefore the answer is:

$$\boxed{\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=e^{ab}}$$

Answer 59E.

We have $e^{\frac{1}{1-x}}=e^{\frac{1}{1-x}\ln x}$

By L' Hospitals rule,

$$\underset{x\to 1+}{Lt}\frac{1}{1-x}\ln x=\underset{x\to 1+}{Lt}\frac{1/x}{-1}$$

$$=-1$$

$$\therefore \underset{x\to 1+}{Lt} x^{1/1-x}=\underset{x\to 1+}{Lt} e^{\frac{1}{1-x}\ln x}$$

$$=e^{-1}$$

$$=\frac{1}{e}$$

$$\therefore \boxed{\underset{x\to 1+}{Lt} x^{1/(1-x)}=1/e}$$

Answer 60E.

We have to evaluate $\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)}$

Let $y = x^{(\ln 2)/(1+\ln x)}$

Taking ln on both sides

$$\ln y = \frac{\ln 2}{(1+\ln x)}\ln(x)$$

Taking limits:

$$\lim_{x\to\infty}(\ln y) = \lim_{x\to\infty}\frac{(\ln 2)(\ln x)}{(1+\ln x)} \quad [\text{Form of } \infty/\infty]$$

Applying L-Hospital's rule:

$$\lim_{x\to\infty}\frac{(\ln 2)(\ln x)}{(1+\ln x)} \overset{H}{=} \lim_{x\to\infty}\frac{(\ln 2)(1/x)}{(1/x)}$$

$$= \lim_{x\to\infty}(\ln 2)$$

$$= (\ln 2)$$

Thus $\lim_{x\to\infty}(\ln y) = \ln 2$

Then $\lim_{x\to\infty} e^{(\ln y)} = e^{\ln 2}$

Or $\lim_{x\to\infty} y = 2$

Therefore,

$$\boxed{\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)} = 2}$$

Answer 61E.

Evaluate the limit using L Hospital's rule: $\lim_{x\to\infty} x^{1/x}$

If we apply the limit directly, we get $\lim_{x\to\infty} x^{1/x} = \infty^{\frac{1}{\infty}}$ which is an undefined form. But the L Hospital's rule works only for the undetermined forms of the type $\frac{0}{0}$ and $\frac{\infty}{\infty}$. So before applying the L Hospital's rule we need to rewrite the function in such way that it should be of an undefined form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Rewrite the given function as

$$x^{\frac{1}{x}} = e^{\ln x^{\frac{1}{x}}} \quad \text{Use the formula: } a = e^{\ln a}$$

$$= e^{\frac{\ln x}{x}} \quad \text{Use the formula: } \ln a^n = n\ln a$$

Now write the given limit as

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\frac{\ln x}{x}}$$

$$= e^{\lim_{x\to\infty}\frac{\ln x}{x}} \quad \text{Use the rule: } \lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)}$$

If we apply the limit to the exponent in the above function $e^{\lim_{x\to\infty}\frac{\ln x}{x}}$ we get $\frac{\infty}{\infty}$.

Hence the L Hospitals rule can be applied to the exponent.

According to this rule:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$ If the limit on the right hand side exists.

If it does not exist, then we write that $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f''(x)}{g''(x)}$ provided that the right hand side exist, suppose if it doesn't exist then we look forward to third derivative of both numerator and denominator in the function, this process will be continued until the limit of the right hand side exist.

Now apply the L Hospital's rule to the exponent.

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{(\ln x)'}{(x)'}$$ Apply the rule: $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$

$$=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}$$ Use the formulae: $\frac{d}{dx}(\ln x)=\frac{1}{x}$ and $\frac{d}{dx}(x)=1$.

$$=\frac{\frac{1}{\infty}}{1}$$ Apply the limit, that is substitute ∞ for x.

$$=\frac{0}{1}$$ Since $\frac{1}{\infty}=0$

$$=0$$

Continue the second step,

$$\lim_{x\to\infty}x^{1/x}=\lim_{x\to\infty}e^{\frac{\ln x}{x}}$$

$$=e^{\lim_{x\to\infty}\frac{\ln x}{x}}$$ Use the rule: $\lim_{x\to a}e^{f(x)}=e^{\lim_{x\to a}f(x)}$

$$=e^0$$ Since $\lim_{x\to\infty}\frac{\ln x}{x}=0$

$$=1$$

Therefore $\lim_{x\to\infty}x^{1/x}=\boxed{1}$.

Answer 62E.

We have to evaluate $\lim_{x\to\infty}\left(e^x+x\right)^{1/x}$

Let $y=\left(e^x+x\right)^{1/x}$

Then $\ln y=\frac{1}{x}\ln\left(e^x+x\right)$

Taking limit

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\left[\frac{\ln\left(e^x+x\right)}{x}\right]$$ [Form of ∞/∞]

Applying L-Hospital's rule.

$$\lim_{x\to\infty}\left[\frac{\ln\left(e^x+x\right)}{x}\right]\overset{H}{=}\lim_{x\to\infty}\left[\frac{\frac{1}{\left(e^x+x\right)}\times\left(e^x+1\right)}{1}\right]$$

$$=\lim_{x\to\infty}\frac{e^x+1}{e^x+x}$$ $\left[\text{Form of }\frac{\infty}{\infty}\right]$

Again applying L-Hospital's rule

$$\lim_{x\to\infty}\frac{e^x+1}{e^x+x}\overset{H}{=}\lim_{x\to\infty}\frac{e^x}{e^x+1} \qquad \left[\text{Form of } \frac{\infty}{\infty}\right]$$

Again applying L-Hospital rule:

$$\lim_{x\to\infty}\frac{e^x}{e^x+1}\overset{H}{=}\lim_{x\to\infty}\frac{e^x}{e^x}=1$$

Thus $\lim_{x\to\infty}\ln y=1$

Then $\lim_{x\to\infty}e^{\ln y}=e^1$

Or $\lim_{x\to\infty}y=e$

So, $\boxed{\lim_{x\to\infty}\left(e^x+x\right)^{1/x}=e}$

Answer 63E.

Consider the limit $\lim_{x\to 0^+}(4x+1)^{\cot x}$.

Consider $y=(4x+1)^{\cot x}$.

Apply logarithm on both sides.

$$\ln y=\ln(4x+1)^{\cot x}$$

$$\ln y=\cot x\ln(4x+1) \qquad (\text{Since } \ln a^m=m\ln a)$$

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\cot x\ln(4x+1)$$

As $x\to 0^+$, $\cot x\to\infty$, and $\ln(4x+1)\to 0$

The limit is indeterminate form of type $\infty\cdot 0$

l'Hospital's rule cannot apply here.

Rewrite the expression as a quotient.

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln(4x+1)}{\tan x}$$

As $x\to 0^+$, $\tan x\to 0$, and $\ln(4x+1)\to 0$

The limit is indeterminate form of type $\frac{0}{0}$.

l'Hospital's rule can apply here.

Apply l'Hospital's rule:

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\frac{1}{(4x+1)}\frac{d}{dx}(4x+1)}{\sec^2 x}$$

$$=\lim_{x\to 0^+}\frac{\frac{4}{(4x+1)}}{\sec^2 x}$$

$$=\frac{\frac{4}{(4(0)+1)}}{\sec^2(0)}$$

$$=\frac{\frac{4}{1}}{1}$$

$$=4$$

Therefore, $\lim_{x\to 0^+}\ln y=4$.

To find $\lim_{x\to 0^+} \ln y = 4$,Use the fact $y = e^{\ln y}$

$$\begin{aligned}\lim_{x\to 0^+}(4x+1)^{\cot x} &= \lim_{x\to 0^+} y\\ &= \lim_{x\to 0^+} e^{\ln y}\\ &= e^4\end{aligned}$$

Therefore $\lim_{x\to 0^+}(4x+1)^{\cot x} = \boxed{e^4}$

Answer 64E.

Consider the limit $\lim_{x\to 1}(2-x)^{\tan\left(\frac{\pi x}{2}\right)}$.

Consider $y = (2-x)^{\tan\left(\frac{\pi x}{2}\right)}$.

Apply logarithm on both sides.

$$\ln y = \ln(2-x)^{\tan\left(\frac{\pi x}{2}\right)}$$

$$\ln y = \tan\left(\frac{\pi x}{2}\right)\ln(2-x) \qquad \left(\text{Since } \ln a^m = m\ln a\right)$$

$$\lim_{x\to 1}\ln y = \lim_{x\to 1}\left[\tan\left(\frac{\pi x}{2}\right)\ln(2-x)\right]$$

As $x \to 1$, $\tan\left(\frac{\pi x}{2}\right) \to \infty$, and $\ln(2-x) \to 0$.

The limit is indeterminate form of type $\infty \cdot 0$

l'Hospital's rule cannot apply here.

Rewrite the expression as a quotient.

$$\lim_{x\to 1}\ln y = \lim_{x\to 1}\left[\frac{\ln(2-x)}{\cot\left(\frac{\pi x}{2}\right)}\right]$$

As $x \to 1$, $\cot\left(\frac{\pi x}{2}\right) \to 0$, and $\ln(2-x) \to 0$.

The limit is indeterminate form of type $\frac{0}{0}$.

l'Hospital's rule can apply here.

Apply l'Hospital's rule:

$$\begin{aligned}\lim_{x\to 1}\ln y &= \lim_{x\to 1}\frac{\frac{1}{(2-x)}\frac{d}{dx}(2-x)}{-\csc^2\left(\frac{\pi x}{2}\right)\frac{d}{dx}\left(\frac{\pi x}{2}\right)}\\ &= \lim_{x\to 1}\frac{\frac{-1}{(2-x)}}{-\csc^2\left(\frac{\pi x}{2}\right)}\frac{2}{\pi}\\ &= \frac{\frac{1}{(2-1)}}{\csc^2\left(\frac{\pi(1)}{2}\right)}\frac{2}{\pi}\\ &= \frac{2}{\pi}\end{aligned}$$

Therefore $\lim_{x\to 1}\ln y = \frac{2}{\pi}$

To find $\lim_{x\to 0^+} \ln y = 4$,Use the fact $y = e^{\ln y}$

$$\lim_{x\to 1}(2-x)^{\tan\left(\frac{\pi x}{2}\right)} = \lim_{x\to 0^+} y$$
$$= \lim_{x\to 0^+} e^{\ln y}$$
$$= e^{\frac{2}{\pi}}$$

Therefore $\lim_{x\to 1}(2-x)^{\tan\left(\frac{\pi x}{2}\right)} = \boxed{e^{\frac{2}{\pi}}}$

Answer 65E.

We have to evaluate $\lim_{x\to 0^+}(\cos x)^{1/x^2}$

Let $y = (\cos x)^{1/x^2}$

Then $\ln y = \frac{1}{x^2}\ln(\cos x)$

Taking limit

$$\lim_{x\to 0^+}(\ln y) = \lim_{x\to 0^+}\frac{\ln(\cos x)}{x^2} \quad \left[\text{Form } \frac{0}{0}\right]$$

Applying L-Hospital's rule:

$$\lim_{x\to 0^+}\frac{\ln(\cos x)}{x^2} \overset{H}{=} \lim_{x\to 0^+}\left[\frac{(1/\cos x)\times(-\sin x)}{2x}\right]$$
$$= \lim_{x\to 0^+}\frac{(-\tan x)}{2x} \quad \left[\text{Form of } \frac{0}{0}\right]$$

Again using L-Hospital's rule:

$$\lim_{x\to 0^+}\frac{(-\tan x)}{2x} \overset{H}{=} \lim_{x\to 0^+}\frac{-\sec^2 x}{2}$$
$$= \frac{-\sec^2 0}{2} = -1/2$$

Thus we have

$$\lim_{x\to 0^+}(\ln y) = -1/2$$

Then $\lim_{x\to 0^+} e^{(\ln y)} = e^{-1/2}$

Or $\lim_{x\to 0^+} y = e^{-1/2}$

Therefore,

$$\boxed{\lim_{x\to 0^+}(\cos x)^{1/x^2} = e^{-1/2} = 1/\sqrt{e}}$$

Answer 66E.

We have to evaluate $\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}$

Let $y = \left(\frac{2x-3}{2x+5}\right)^{2x+1}$

Then $\ln y = (2x+1)\ln\left(\frac{2x-3}{2x+5}\right)$

Taking limit

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(\frac{2x-3}{2x+5}\right)}{1/(2x+1)} = \lim_{x\to\infty} \frac{\ln(2x-3)-\ln(2x+5)}{1/(2x+1)} \qquad \left[\text{form of } \frac{0}{0}\right]$$

Applying L- Hospital's rule:

$$\lim_{x\to\infty} \frac{\ln(2x-3)-\ln(2x+5)}{1/(2x+1)} \overset{H}{=} \lim_{x\to\infty} \frac{2/(2x-3)-2/(2x+5)}{-2/(2x+1)^2}$$

$$= \lim_{x\to\infty} \frac{[2(2x+5)-2(2x-3)](2x+1)^2}{-2(2x-3)(2x+5)}$$

$$= \lim_{x\to\infty} \frac{[4x+10-4x+6)](2x+1)^2}{-2(2x-3)(2x+5)}$$

$$= \lim_{x\to\infty} \frac{16(2x+1)^2}{-2(2x-3)(2x+5)}$$

$$= \lim_{x\to\infty} \frac{16x^2(2+1/x)^2}{-2x^2(2-3/x)(2+5/x)}$$

$$= \lim_{x\to\infty} \frac{16(2+1/x)^2}{-2(2-3/x)(2+5/x)}$$

$$= \frac{16(2+0)^2}{-2(2-0)(2+0)} = \frac{64}{-8} = -8$$

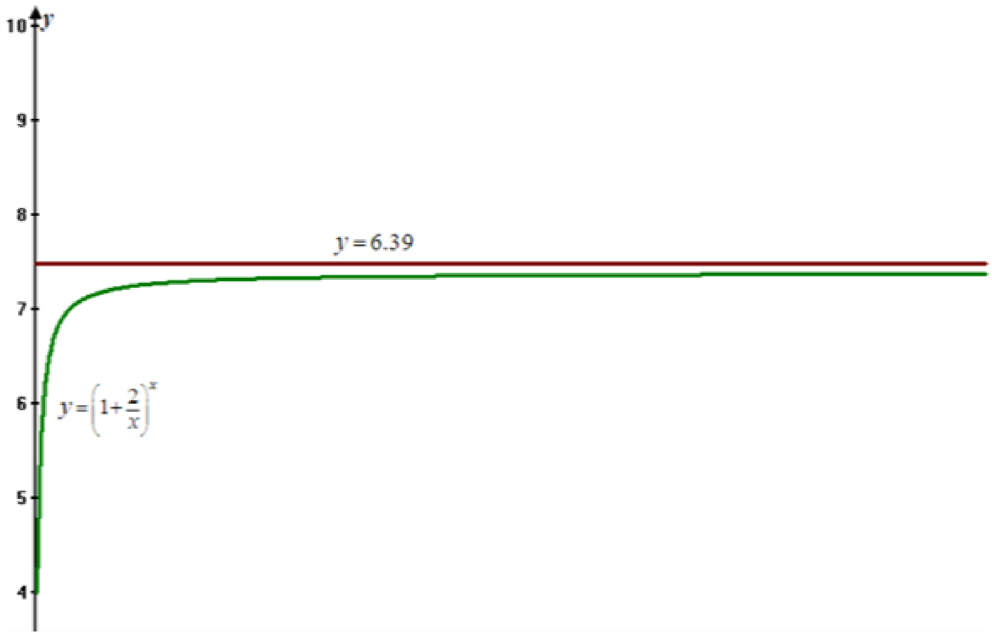
Thus we have $\lim_{x\to\infty} \ln y = -8$

Then $\lim_{x\to\infty} y = e^{-8}$

Or $\boxed{\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1} = e^{-8}}$

Answer 67E.

Use CAS, sketch the graph of the function and then evaluate $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$.



From the above graph if x tends to infinity, then the graph reaches the value approximately 6.39.

Therefore $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x \approx \boxed{6.39}$

Consider the limit of the function $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

Use Hospital's Rule to evaluate $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$.

As $x\to\infty, 1+\frac{2}{x}\to 1+0=1$

This limit is of the Indeterminate Form 1^∞.

So L'Hospital's rule cannot use here.

Take the natural log of both sides and then evaluate the limit.

$$\ln y=\ln\left(1+\frac{2}{x}\right)^x$$

$$\ln y = x\ln\left(1+\frac{2}{x}\right) \qquad \left(\text{Since } \log_b x^r = r\log_b x\right)$$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}x\ln\left(1+\frac{2}{x}\right) \qquad \text{Apply limit on both sides}$$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}$$

$$=\frac{0}{0}$$

Now, L'Hospital's rule can use here.

Apply the L'Hospital's rule.

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\frac{d}{dx}\left[\ln\left(1+\frac{2}{x}\right)\right]}{\frac{d}{dx}\left[\frac{1}{x}\right]}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{2}{x}}\frac{d}{dx}\left(1+\frac{2}{x}\right)}{-\frac{1}{x^2}}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{2}{x}}\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}}$$

$$=\lim_{x\to\infty}\frac{2}{1+\frac{2}{x}}$$

$$=\frac{2}{1+0}$$

$$=2$$

$$\lim_{x\to\infty}\ln y=2$$

To find the limit of y, use the fact that $y = e^{\ln y}$

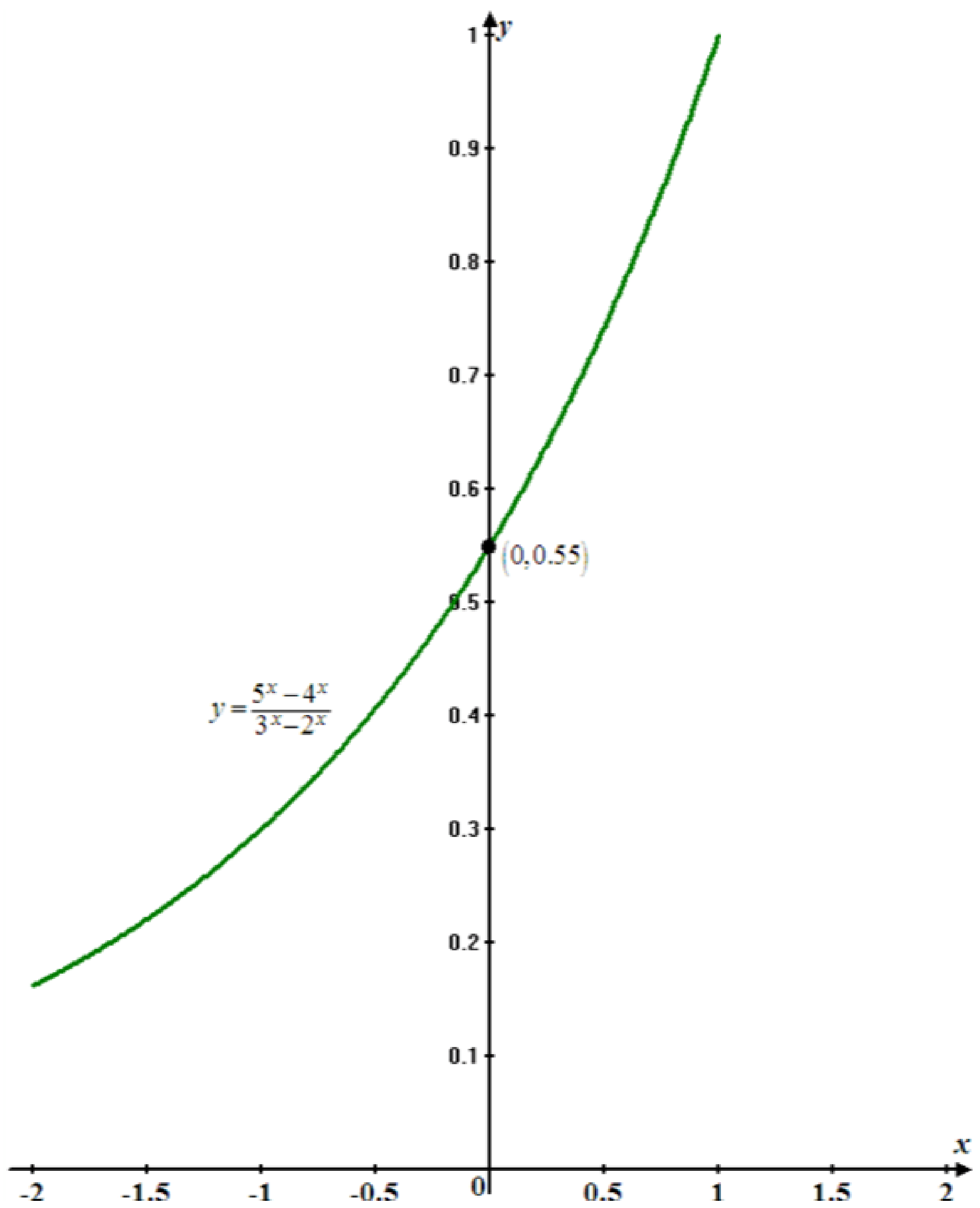
$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x &= \lim_{x\to\infty} y \\ &= \lim_{x\to\infty} e^{\ln y} \\ &= \lim_{x\to\infty} e^2 \\ &= e^2\end{aligned}$$

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x &= e^2 \\ &\approx 7.39\end{aligned}$$

Therefore $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x \approx \boxed{7.39}$

Answer 68E.

Use CAS, sketch the graph of the function and then estimate the value of $\lim_{x\to 0}\frac{5^x-4^x}{3^x-2^x}$.



From the above graph values of x tends to 0, then the function value reaches to 0.55

Consider the limit of the function $\lim_{x\to 0}\frac{5^x-4^x}{3^x-2^x}$

Use L'Hospital's Rule to evaluate $\lim_{x\to 0}\frac{5^x-4^x}{3^x-2^x}$.

As $x\to 0,\ 5^x-4^x\to 1-1=0$

And as $x\to 0, 3^x-2^x\to 1-1=0$

This limit is of the Indeterminate Form $\frac{0}{0}$.

So, L'Hospital's Rule can use here.

Apply the L'Hospital's rule.

$$\begin{aligned}\lim_{x\to 0}\frac{5^x-4^x}{3^x-2^x} &= \lim_{x\to 0}\frac{\frac{d}{dx}\left[5^x-4^x\right]}{\frac{d}{dx}\left[3^x-2^x\right]}\\ &= \lim_{x\to 0}\frac{5^x\ln 5-4^x\ln 4}{3^x\ln 3-2^x\ln 2}\\ &= \frac{5^0\ln 5-4^0\ln 4}{3^0\ln 3-2^0\ln 2}\\ &= \frac{(1\cdot\ln 5)-(1\cdot\ln 4)}{(1\cdot\ln 3)-(1\cdot\ln 2)}\end{aligned}$$

$$=\frac{\ln 5-\ln 4}{\ln 3-\ln 2}$$

$$=\frac{\ln\left(\frac{5}{4}\right)}{\ln\left(\frac{3}{2}\right)} \qquad \left(\text{Since } \ln\left(\frac{a}{b}\right)=\ln(a)-\ln(b)\right)$$

$$=\frac{\ln(1.25)}{\ln(1.5)}$$

$$\approx 0.55$$

Therefore $\lim_{x\to 0}\frac{5^x-4^x}{3^x-2^x}\approx \boxed{0.55}$

Answer 69E.

We have $\quad f(x)=e^x-1, \qquad g(x)=x^3+4x$

Then $\quad f'(x)=e^x, \qquad g'(x)=3x^2+4$

Now we sketch the curves $y=\frac{f(x)}{g(x)}$ and $y=\frac{f'(x)}{g'(x)}$

On the same set of axis near $x=0$

We see that these ratios have same limit as $x\to 0$ so this verifies L' Hospitals rule (which is ≈ 0.25).

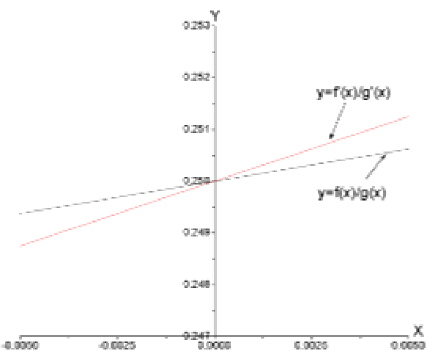


Fig.1

Now we calculate the $\lim_{x \to 0} \frac{f(x)}{g(x)}$

By L- hospitals rule

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x \to 0} \frac{e^x}{3x^2 + 4}$$

$$= \frac{e^0}{0+4}$$

$$= \frac{1}{4}$$

So $\boxed{\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{1}{4}}$

Answer 70E.

We have $f(x) = 2x \sin x$ and $g(x) = \sec x - 1$

Then $f'(x) = 2(x \cos x + \sin x)$ and $g'(x) = \sec x \tan x$

Now we sketch the curve $y = \frac{f(x)}{g(x)}$ and $y = \frac{f'(x)}{g'(x)}$

On the same set of axis near $x = 0$

We see that these ratios have same limit as $x \to 0$

So this verifies L-hospital rule (which is ≈ 4)

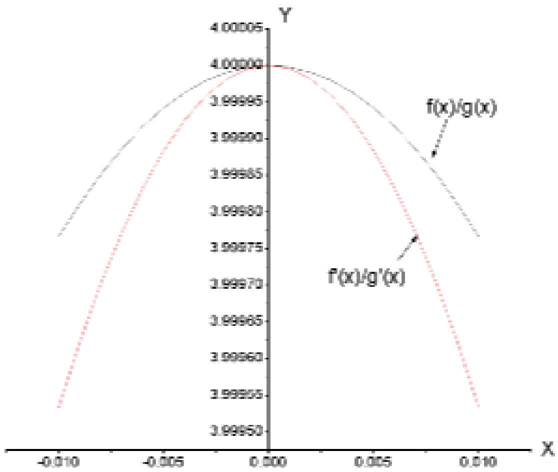


Fig.1

Now we calculate the limit

$$\lim_{x \to 0} \frac{f(x)}{g(x)}$$

By L-Hospital rule

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x \to 0} \frac{2(x \cos x + \sin x)}{(\sec x \tan x)}$$

But this is again in the form of $\frac{0}{0}$ so again we use L-Hospital rule

$$\lim_{x\to0}\frac{2(x\cos x+\sin x)}{(\sec x\tan x)}=\lim_{x\to0}\frac{2\frac{d}{dx}(x\cos x+\sin x)}{\frac{d}{dx}(\sec x\tan x)}$$

$$=\lim_{x\to0}\frac{2(-x\sin x+\cos x+\cos x)}{(\sec x.\sec^2 x+\sec x\tan x\tan x)}$$

$$=\lim_{x\to0}\frac{2(2\cos x-x\sin x)}{(\sec^3 x+\sec x\tan^2 x)}$$

$$=\frac{2(2\times1-0)}{(1+0)}$$

$$=2(2)\ \boxed{=4}$$

Answer 71E.

We have to prove that $\lim_{x\to\infty}\frac{e^x}{x^n}=\infty$

Left hand side $=\lim_{x\to\infty}\frac{e^x}{x^n}\quad\left(\frac{\infty}{\infty}\right)$

Using L – hospitals rule, we get:-

$$\lim_{x\to\infty}\frac{e^x}{x^n}\overset{H}{=}\lim_{x\to\infty}\frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^n)}$$

$$=\lim_{x\to\infty}\frac{e^x}{nx^{n-1}}\qquad\left(\frac{\infty}{\infty}\right)$$

$$\overset{H}{=}\lim_{x\to\infty}\frac{e^x}{n(n-1)x^{n-2}}\qquad\left(\frac{\infty}{\infty}\right)$$

"

"

$\lim_{x\to\infty}\frac{e^x}{n!}=\infty$ Hence proved

Answer 72E.

We have to find $\lim_{x\to\infty}\frac{\ln x}{x^p}$ for p > 0

This is the form of $\frac{\infty}{\infty}$ so we can use L – hospital rule

Answer 73E.

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}=\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}=--$$

using this in the case of the given function, we get $\lim_{x\to\infty}\frac{1}{\frac{\sqrt{x^2+1}}{x}}=\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x}$

in such case , we use $\lim_{t\to0^+}\frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)}=\lim_{t\to0^+}\frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)}$

$$i.e.\lim_{t\to0^+}\frac{\left(\frac{1}{t}\right)}{\sqrt{\left(\frac{1}{t}\right)^2+1}}=\lim_{t\to0^+}\frac{1}{\sqrt{1+t^2}}=1$$

Given $\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{\sec x}{\tan x}$

Using L' Hospitals rule,

$$\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{\sec x}{\tan x}=\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{\sec x\tan x}{\sec^2 x}$$

$$=\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{\tan x}{\sec x}$$, Still in the indeterminate.

Even on applying L' Hospitals rule, for every time we get an indeterminate form.

Now consider,

$$\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{\sec x}{\tan x}=\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{1/\cos x}{\sin x/\cos x}$$

$$=\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}1/\sin x$$

$$=\frac{1}{\sin \pi/2}$$

$$=\frac{1}{1}$$

$$=1$$

$$\therefore \boxed{\underset{x\to\left(\frac{\pi}{2}\right)^-}{Lt}\frac{\sec x}{\tan x}=1}$$

Answer 75E.

We have $y=f(x)=xe^{-x}$

(1) Domain = R

(2) y – intercept is 0

$[\Rightarrow y=0]$

And x – intercept is 0 $\left[xe^{-x}=0\Rightarrow x=0\right]$

(3) $f(-x)=-xe^{-(-x)}=-xe^{x}\neq -f(x)\neq f(x)$

So there is no symmetry

(4) $\lim_{x\to\infty} xe^{-x}=\lim_{x\to\infty}\frac{x}{e^x}$

Using L-hospital rule

$\lim_{x\to\infty}\frac{x}{e^x}\overset{H}{=}\lim_{x\to\infty}\frac{1}{e^x}=0$

So horizontal asymptote is y = 0

There is no vertical asymptote

(5) $f(x)=xe^{-x}$ so $f'(x)=\frac{d}{dx}\left(xe^{-x}\right)$

Then $f'(x)=e^{-x}-xe^{-x}$ [By product and chain rule]

Or $f'(x)=(1-x)e^{-x}$

$f'(x)=0$ when $1-x=0$ or $x=1$

Since $f'(x)>0$ when $x<1$

So $f(x)$ is increasing on $(-\infty,1)$

And $f'(x)<0$ when $x>1$

So $f(x)$ is decreasing on $(1,\infty)$

(6)

$f(x)$ has local maximum $f(1)=\frac{1}{e}$

(7) $f'(x)=(1-x)e^{-x}$

Then $f''(x)=(1-x)(-1)e^{-x}+e^{-x}(-1)$ [By product rule]

$$=(x-1-1)e^{-x}$$

$$=(x-2)e^{-x}$$

$f''(x)=0$ when $x-2=0$ or $x=2$.

Since $f''(x)>0$ when $x>2$

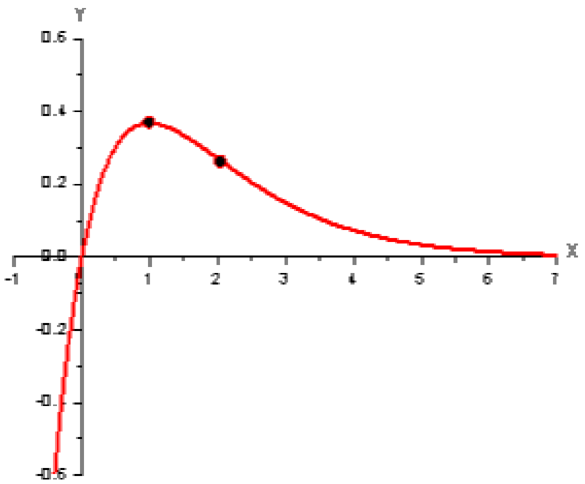
And $f''(x)<0$ when $x<2$

So $f(x)$ is concave upward on $(2,\infty)$

And downward on $(-\infty,2)$

Inflection point is $(2,\ 2e^{-2})$

(8)



Answer 76E.

Consider the curve,

$$y = \frac{\ln x}{x^2}.$$

Let $f(x) = \frac{\ln x}{x^2}$

(a)

Domain:

The domain of the function is the set of values of *x* for which $f(x)$ defined.

The domain is $\{x \mid x^2 \neq 0\} = \{x \mid x \neq 0\}$

$$= \boxed{R - \{0\}}.$$

(b)

Intercepts:

The x-intercepts are values obtained by substituting $y = 0$.

If $f(x) = 0$ that is,

$$\frac{\ln x}{x^2} = 0$$
$$\ln x = 0$$
$$x = e^0$$
$$x = 1$$

So the $x-$intercept is $x = 1$.

The y-intercepts are values obtained by substituting $x = 0$.

$$f(0) = \frac{\ln(0)}{0}$$
$$= \text{does not exist}$$

So the *y*-intercept does not exist.

(c)

Symmetry:

Consider,

$$f(-x) = \frac{\ln(-x)}{(-x)^2}$$
$$= \frac{\ln(-x)}{x^2}$$
$$\neq f(x)$$

And

$$f(-x) = \frac{\ln(-x)}{(-x)^2}$$
$$= \frac{\ln(-x)}{x^2}$$
$$= -\frac{\ln x}{x^2}$$
$$= -f(x)$$

Since $f(-x) = -f(x)$ so the function is an odd function.

(d)

Asymptotes:

Vertical Asymptote is the *x*-value at which the function is undefined.

Since the function $f(x)=\frac{\ln x}{x^2}$ is undefined at $x=0$.

Therefore the vertical asymptote is $\boxed{x=0}$.

Horizontal Asymptote is the value the function cannot take.

Since $f(x)\to\infty$ as $x\to\infty$, so there is no horizontal asymptote.

(e)

Intervals of increase or decrease:

The intervals of increase and decrease are determined by the critical points of the function. The critical points of the function occur when the derivative of the function equals zero.

Find the derivative of the function $f(x)=\frac{\ln x}{x^2}$.

$$f(x)=\frac{\ln x}{x^2}$$

$$f'(x)=\frac{1}{x^3}-\frac{2\ln x}{x^3} \quad \text{Use } \int u\,dv = uv-\int v\,du$$

Equate $f'(x)$ to zero:

$$\begin{aligned}
f'(x)&=0\\
\frac{1}{x^3}-\frac{2\ln x}{x^3}&=0\\
1-2\ln x&=0\\
2\ln x&=1\\
\ln x&=\frac{1}{2}\\
x&=e^{\frac{1}{2}}
\end{aligned}$$

The critical point divides the domain of the function into two intervals $\left(0,e^{\frac{1}{2}}\right)$ and $\left(e^{\frac{1}{2}},\infty\right)$.

Observe the behavior of $f'(x)$ in both the intervals.

For $x=1$,

$$\begin{aligned}
f'(1)&=\frac{1}{(1)^3}-\frac{2\ln(1)}{(1)^3}\\
&=1\\
&>0
\end{aligned}$$

Thus the function is increasing in the interval $\boxed{\left(0,e^{\frac{1}{2}}\right)}$.

For $x=2$,

$$\begin{aligned}
f'(2)&=\frac{1}{(2)^3}-\frac{2\ln(2)}{(2)^3}\\
&=-0.04829\\
&<0
\end{aligned}$$

Thus the function is increasing in the interval $\boxed{\left(e^{\frac{1}{2}},\infty\right)}$.

(f)

The value of f' changes from positive to negative at the critical point $x=2.$

So the function have local maximum at 2.

The local maximum is

$$f(2)=\frac{\ln(2)}{(2)^2}$$

$$=0.17329$$

(g)

The intervals of concavity and the inflection points are determined by the second derivative of the function. The inflection points occur where the second derivative of the function equals zero.

Find the second derivative of the function $f(x)=\frac{\ln x}{x^2}$

Differentiate $f'(x)=\frac{1}{x^3}-\frac{2\ln x}{x^3}$ with respect to x on both sides.

$$f''(x)=-\frac{5}{x^4}+\frac{6\ln(x)}{x^4}$$

Equate $f''(x)$ to zero:

$$f''(x)=0$$

$$-\frac{5}{x^4}+\frac{6\ln(x)}{x^4}=0$$

$$-5+6\ln(x)=0$$

$$\ln x=\frac{5}{6}$$

$$x=e^{\frac{5}{6}}$$

The second derivative test,

The curve changes is concave upward if $f''(x)>0$ and concave downward if $f''(x)<0$

And the inflection points are the points at which the curve changes its direction of concavity.

Concavity: To check concavity, apply the second derivative test.

For $x<e^{\frac{5}{6}}$

$$\begin{aligned} f''(2) &= -\frac{5}{(2)^4}+\frac{6\ln(2)}{(2)^4} \\ &= -0.05257 \\ &< 0 \end{aligned}$$

For $x>e^{\frac{5}{6}}$

Let $x=10$

$$\begin{aligned} f''(10) &= -\frac{5}{(10)^4}+\frac{6\ln(10)}{(10)^4} \\ &= 0.00088 \\ &> 0 \end{aligned}$$

The curve is concave upward for $x<e^{\frac{5}{6}}$ and concave down ward for $x>e^{\frac{5}{6}}$.

The curve changes its position at $x=e^{\frac{5}{6}}$.

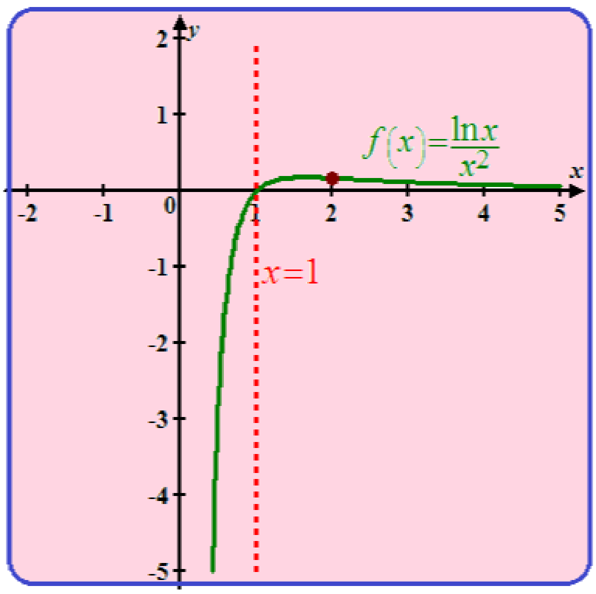
$$\begin{aligned} f\left(e^{\frac{5}{6}}\right) &= \frac{\ln\left(\frac{5}{6}\right)}{\left(\frac{5}{6}\right)^2} \\ &= -0.31507 \end{aligned}$$

Therefore, the inflection point is $\boxed{\left(e^{\frac{5}{6}},-0.31507\right)=(2.3010,-0.31507)}$.

(h)

Graphing:

Use the information from A to G to sketch curve of the function.



Answer 77E.

We have $y=f(x)=xe^{-x^2}$

(1) Domain = R

(2) y – Intercept is 0

$[\Rightarrow y = 0]$

And x – intercept is 0 $\left[xe^{-x^2} = 0 \Rightarrow x = 0\right]$

(3) $f(-x) = -xe^{-x^2} = -f(x)$

So this is an odd function and symmetric about the origin

(4) $\lim_{x\to\infty} xe^{-x^2} = \lim_{x\to\infty} \frac{x}{e^{x^2}}$

Using L-hospital rule

$\lim_{x\to\infty} \frac{x}{e^{x^2}} \overset{H}{=} \lim_{x\to\infty} \frac{1}{e^{x^2}.2x} = 0$

So horizontal asymptote is y = 0

There is no vertical asymptote

(5) $f(x) = xe^{-x^2}$ so $f'(x) = \frac{d}{dx}\left(xe^{-x^2}\right)$

Then $f'(x) = e^{-x^2} - 2x^2e^{-x^2}$ [By product and chain rule]

Or $f'(x) = \left(1-2x^2\right)e^{-x^2}$

$f'(x) = 0$ when $1-2x^2 = 0$ or $x = \pm\frac{1}{\sqrt{2}}$

Since $f'(x) > 0$ when $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

So $f(x)$ is increasing on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

And decreasing on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and on $\left(\frac{1}{\sqrt{2}}, \infty\right)$

(6) $f(x)$ has local minimum $f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2e}}$

$f(x)$ has local maximum $f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2e}}$

(7) $f'(x) = \left(1-2x^2\right)e^{-x^2}$

Then $f''(x) = \left(1-2x^2\right)(-2x)e^{-x^2} + e^{-x^2}(-4x)$ [By product rule]

$= \left(4x^3 - 2x - 4x\right)e^{-x^2}$

$= \left(4x^3 - 6x\right)e^{-x^2}$

$f''(x) = 0$ when $4x^3 - 6x = 0$ or $4x^2 - 6 = 0$ or x = 0.

$\Rightarrow x^2 = \frac{6}{4} \Rightarrow x = \pm\sqrt{\frac{3}{2}}$ or x = 0

Since $f''(x) > 0$ when $-\sqrt{\frac{3}{2}} < x < 0$ and $\sqrt{\frac{3}{2}} < x < \infty$

And $f''(x) < 0$ when $-\infty < x < -\sqrt{\frac{3}{2}}$ and $0 < x < \sqrt{\frac{3}{2}}$

So $f(x)$ is concave upward on $\left(-\sqrt{\frac{3}{2}}, 0\right)$ and $\left(\sqrt{\frac{3}{2}}, \infty\right)$

And downward on $\left(-\infty, -\sqrt{\frac{3}{2}}\right)$ and $\left(0, \sqrt{\frac{3}{2}}\right)$

Inflection points are (0, 0), $\left(\pm\sqrt{\frac{3}{2}}, \pm\sqrt{\frac{3}{2}}e^{-3/2}\right)$

(8)

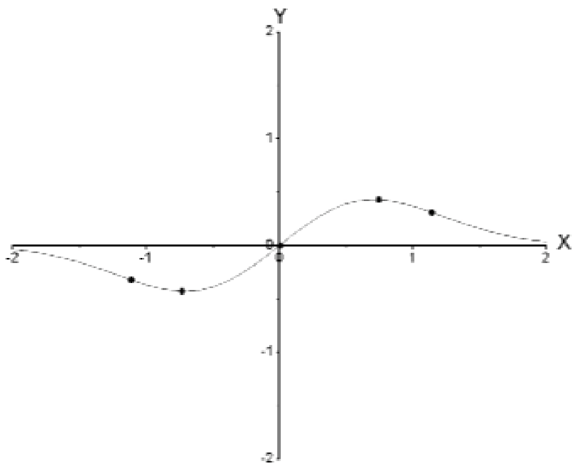


Fig.1

Answer 78E.

We have $y = f(x) = \dfrac{e^x}{x}$

(1) Domain $= \{x / x \neq 0\}$

(2) For x – intercept $\dfrac{e^x}{x} = 0 \Rightarrow e^x = 0$ which is not true

So x – intercept $\Rightarrow$ none

y – intercept $\Rightarrow$ none

(3) Symmetry – none

(4) $\lim_{x\to\infty} \dfrac{e^x}{x} = \lim_{x\to\infty} \dfrac{\frac{d}{dx}e^x}{\frac{d}{dx}x}$ [By L-hospital rule]

$= \lim_{x\to\infty} e^x = \infty$

And $\lim_{x\to-\infty} \dfrac{e^x}{x} = \lim_{x\to-\infty} \dfrac{\frac{d}{dx}e^x}{\frac{d}{dx}x} = \lim_{x\to-\infty} e^x = 0$

So horizontal asymptote is y = 0

Now $\lim_{x\to0^-} \dfrac{e^x}{x} = -\infty$

And $\lim_{x\to0^+} \dfrac{e^x}{x} = \infty$ so x = 0 is a vertical asymptote

(5) $f(x) = \dfrac{e^x}{x}$

Then $f'(x) = \dfrac{xe^x - e^x}{x^2}$ by Quotient rule

$= e^x \dfrac{(x-1)}{x^2}$

$f'(x) = 0$ when $x - 1 = 0$ or $x = 1$

$f'(x) < 0$ when $-\infty < x < 0$ and $0 < x < 1$

and $f'(x) > 0$ when $1 < x < \infty$

So $f(x)$ is increasing on $(1, \infty)$

And decreasing on $(-\infty, 0)$ and $(0, 1)$

(6) $f(x)$ has local minimum $f(1) = e$

(7) $f'(x) = \frac{e^x}{x} - \frac{e^x}{x^2}$

Then $f''(x) = \frac{e^x(x-1)}{x^2} - \frac{x^2e^x - 2xe^x}{x^4}$

$$f''(x) = \frac{\left[x^2(x-1) - (x^2 - 2x)\right]e^x}{x^4}$$

$$= \frac{\left[x^3 - x^2 - x^2 + 2x\right]e^x}{x^4}$$

Or $f''(x) = \frac{(x^3 - 2x^2 + 2x)e^x}{x^4}$

$$= \frac{(x^2 - 2x + 2)e^x}{x^3}$$

Since $x^2 - 2x + 2 > 0$ for all x

So $f''(x) < 0$ fore $x < 0$

And $f''(x) > 0$ for $x > 0$

So $f(x)$ is concave downward on $(-\infty, 0)$

And $f(x)$ is concave upward on $(0, \infty)$

There is no inflection point

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(8)

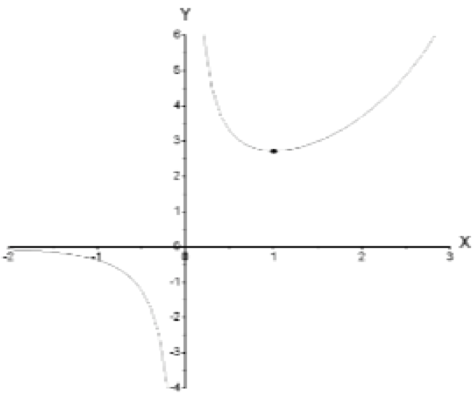


Fig.1

Answer 79E.

We have $y = f(x) = x - \ln(1 + x)$

(1) $f(x)$ is defined when $1 + x > 0$ or $x > -1$

So domain is $= (-1, \infty)$

(2) For y – intercept, putting $x = 0$

$$y = 0 - \ln(1+0) = 0 \Rightarrow y = 0$$

So y – intercept is 0

For x – intercept, putting $y = 0$

$$0 = x - \ln(1+x)$$

$$\Rightarrow \ln(1+x) = x$$

$$\Rightarrow 1 + x = e^x$$

$$\Rightarrow x = e^x - 1$$

$$\Rightarrow e^x - x - 1 = 0 \Rightarrow x = 0$$

So x-intercept is 0

(3) This is not symmetric

(4) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(x - \ln(1+x)\right)$

$$= \lim_{x\to\infty} x\left(1 - \frac{\ln(1+x)}{x}\right) = \infty$$

Since by L-hospital rule,

$$\lim_{x\to\infty}\frac{\ln(1+x)}{x} \overset{H}{=} \lim_{x\to\infty}\frac{1}{(1+x)} = 0$$

And we have $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}\left(x - \ln(1+x)\right)$

Since $\lim_{x\to -1^+}\ln(1+x) = -\infty$ so $\lim_{x\to -1^+}\left(x - \ln(1+x)\right) = \infty$

So x = -1 is a vertical asymptote.

(5) $f(x) = x - \ln(1+x)$

Then $f'(x) = 1 - \dfrac{1}{1+x}$

$f'(x) = 0$ When $1 - \dfrac{1}{1+x} = 0 \Rightarrow \dfrac{1}{1+x} = 1$

$$\Rightarrow 1 + x = 1$$

$$\Rightarrow x = 0$$

So $f'(x) > 0$ when $0 < x < \infty$ and $f'(x) < 0$ when $-1 < x < 0$

Thus $f(x)$ is decreasing on $(-1, 0)$

And $f(x)$ is increasing on $(0, \infty)$

(6) $f(x)$ has local minimum $f(0) = 0$

(7) $f'(x) = 1 - \dfrac{1}{(1+x)}$

Then $f''(x) = -1.\dfrac{(-1)}{(1+x)^2} = \dfrac{1}{(1+x)^2}$

Since $f''(x) > 0$ for all $x > -1$

So $f(x)$ is concave upward on $(-1, \infty)$

(8)

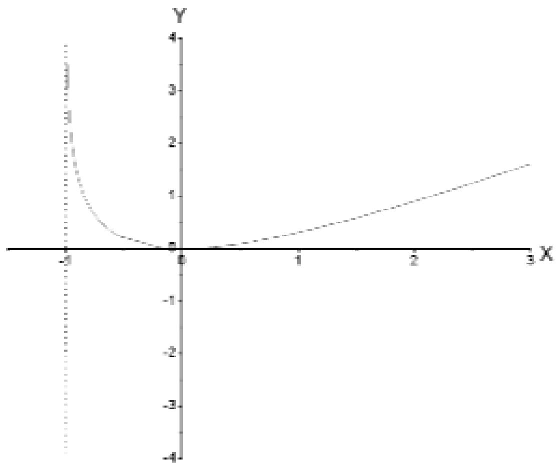


Fig.1

Answer 80E.

Consider the curve,

$y=(x^2-3)e^{-x}$ ...... (1)

Let $f(x)=(x^2-3)e^{-x}$.

A.

Domain:

Domain is set of all values that x take so that the function $f(x)$ is defined.

The function is defined for all real numbers.

Domain of the function is the set of all real numbers.

B.

Intercepts:

The x-intercepts are values obtained by substituting $y=0$.

Substitute $y=0$ in (1)

$$\begin{aligned}0&=(x^2-3)e^{-x}\\x^2-3&=0\\x^2&=3\\x&=\pm\sqrt{3}\end{aligned}$$

The y-intercepts are values obtained by substituting $x=0$.

Substitute $x=0$ in (1)

$$\begin{aligned}y&=((0)^2-3)e^{-0}\\&=-3(1)\\&=-3\end{aligned}$$

Therefore, the x-intercepts are $\boxed{x=\pm\sqrt{3}}$ and y-intercept is -3.

C.

Symmetry:

Consider,

$$f(-x) = \left((-x)^2 - 3\right)e^{-(-x)}$$
$$= \left(x^2 - 3\right)e^{x}$$
$$\neq f(x)$$

And

$$f(-x) = \left((-x)^2 - 3\right)e^{-(-x)}$$
$$= \left(x^2 - 3\right)e^{x}$$
$$\neq -f(x)$$

Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$.

Therefore the function has no symmetry.

D.

Asymptotes:

Vertical Asymptote is the *x*-value at which the function is undefined.

The function is defined for all real values of x.

Vertical Asymptotes are none.

Horizontal Asymptote is the value the function cannot take. And also the function has no horizontal asymptotes since the function can take all the values for all x values.

E.

Intervals of increase or decrease:

The intervals of increase and decrease are determined by the critical points of the function. The critical points of the function occur when the derivative of the function equals zero.

Find the derivative of the function $y = \left(x^2 - 3\right)e^{-x}$

$f(x) = \left(x^2 - 3\right)e^{-x}$ $y = f(x)$

$f'(x) = 2xe^{-x} - \left(x^2 - 3\right)e^{-x}$ ...... (2)

Equate $f'(x)$ to zero:

$$f'(x)=0$$
$$2xe^{-x}-(x^2-3)e^{-x}=0$$
$$(2x-(x^2-3))e^{-x}=0$$
$$2x-x^2+3=0$$
$$-x^2+2x+3=0$$
$$x^2-2x-3=0$$
$$x^2-3x+x-3=0$$
$$x(x-3)+1(x-3)=0$$
$$(x-3)(x+1)=0$$
$$x=3,-1$$

For $x<3$

Plug $x=0$ in $f'(x)=2xe^{-x}-(x^2-3)e^{-x}$

$$\begin{aligned}f'(0)&=2(0)e^{-0}-((0)^2-3)e^{-0}\\&=3\\&>0\end{aligned}$$

For $-1<x<3$

Plug $x=2$ in $f'(x)=2xe^{-x}-(x^2-3)e^{-x}$

$$\begin{aligned}f'(2)&=2(2)e^{-2}-((2)^2-3)e^{-2}\\&=4e^{-2}-e^{-2}\\&=0.4062\\&>0\end{aligned}$$

For $x>3$

Plug $x=5$ in $f'(x)=2xe^{-x}-(x^2-3)e^{-x}$

$$\begin{aligned}f'(5)&=2(5)e^{-5}-((5)^2-3)e^{-5}\\&=10e^{-5}-22e^{-5}\\&=-11e^{-5}\\&<0\end{aligned}$$

Therefore, the function is increasing on the intervals $\boxed{(-\infty,3)(-1,3)}$.

And the function is decrease in the interval $\boxed{(3,\infty)}$.

F.

The value of f' changes from positive to negative at the critical point $x=5$

So the function have local maximum at $\boxed{5}$.

The local maximum is

$$\begin{aligned}f(x)&=((5)^2-3)e^{-5}\\&=(25-3)e^{-5}\\&=22e^{-5}\\&=0.14823\end{aligned}$$

G.

The intervals of concavity and the inflection points are determined by the second derivative of the function. The inflection points occur where the second derivative of the function equals zero.

Find the second derivative of the function (1)

Differentiate (2) with respect to *x* on both sides.

$$f'(x) = 2xe^{-x} - (x^2 - 3)e^{-x}$$

$$f''(x) = -e^{-x}(-x^3 + 2x + 3)(-3x^2 + 2)$$

Equate $f''(x)$ to zero:

$$-e^{-x}(-x^3 + 2x + 3)(-3x^2 + 2) = 0$$
$$(-x^3 + 2x + 3)(-3x^2 + 2) = 0$$
$$(-3x^2 + 2) = 0$$

$$3x^2 = 2$$
$$x^2 = \frac{2}{3}$$
$$x = \pm\sqrt{\frac{2}{3}}$$

The second derivative test,

The curve changes is concave upward if $f''(x) > 0$ and concave downward if $f''(x) < 0$

And the inflection points are the points at which the curve changes its direction of concavity.

Concavity: To check concavity, apply the second derivative test.

For $x < \sqrt{\frac{2}{3}}$

$$f''(-1) = -e^{-(-1)}\left(-(-1)^3 + 2(-1) + 3\right)\left(-3(-1)^2 + 2\right)$$
$$= -e(1 - 2 + 3)(-3 + 2)$$
$$= 2e$$
$$> 0$$

For $x > \sqrt{\frac{2}{3}}$

$$f''(1) = -e^{-(1)}\left(-(1)^3 + 2(1) + 3\right)\left(-3(1)^2 + 2\right)$$
$$= -\frac{1}{e}(-1 + 2 + 3)(-3 + 2)$$
$$= \frac{4}{e}$$
$$> 0$$

The curve is concave upward for $x > \sqrt{\frac{2}{3}}$ and $x < \sqrt{\frac{2}{3}}$.

The curve changes its position at $x = \sqrt{\frac{2}{3}}$.

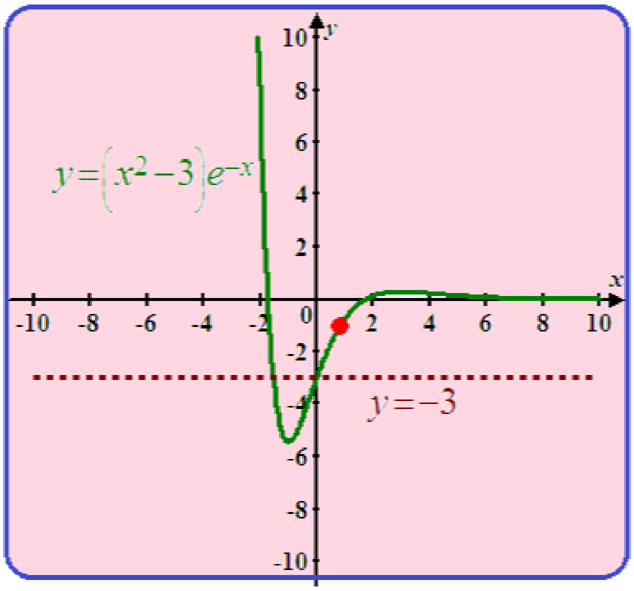
$$y = \left(\left(\sqrt{\frac{2}{3}}\right)^2 - 3\right)e^{-\left(\sqrt{\frac{2}{3}}\right)}$$
$$= \left(\frac{2}{3} - 3\right)e^{-\left(\sqrt{\frac{2}{3}}\right)}$$
$$= -\frac{7}{3}e^{-\left(\sqrt{\frac{2}{3}}\right)}$$
$$= -1.0313$$

Therefore, the inflection point is $\boxed{0.81649, -1.0313}$

(h)

Graphing:

Use the information from A to G to sketch curve of the function.



Answer 81E.

(A) We sketch the graph of $f(x)=x^{-x}$ in figure 1

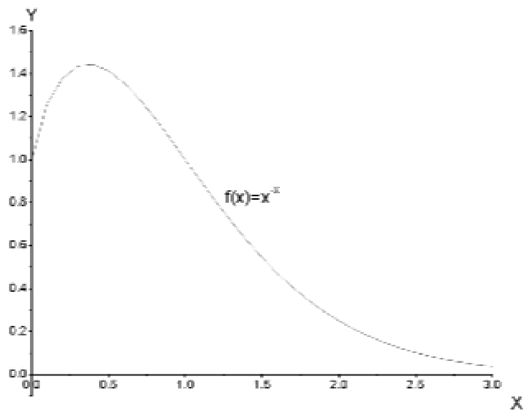


Fig.1

(B) $\lim_{x\to0^+} x^{-x} = \lim_{x\to0^+}\frac{1}{x^x} = 1$ Because $x^x \to 1$ as $x\to0^+$

Using L – hospital rule

Let $y = x^{-x}$

$\Rightarrow \ln y = -x\ln x$

$= -\frac{\ln x}{1/x}$

$\lim_{x\to0^+}\ln y = \lim_{x\to0^+}\left[-\frac{\ln x}{1/x}\right]$

$\overset{H}{=} \lim_{x\to0^+}\frac{-1/x}{-1/x^2}$ (L-Hospital rule)

$= \lim_{x\to0^+} x = 0$

Since $\ln y \to 0$ as $x\to0^+$ so $y = e^0 = 1$ as $x\to0^+$

(C) From figure 1 we see that $f(x)$ has an absolute and local maximum

$\approx f(0.37)\approx 1.44$

Finding exact value $\rightarrow$

$f(x)=x^{-x}$

Let $y=x^{-x}\Rightarrow \ln y = -x\ln x$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = -x\frac{1}{x} - \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = -1 - \ln x$$

$$\frac{dy}{dx} = -y(1+\ln x) = -x^{-x}(1+\ln x)$$

$$f'(x) = -x^{-x}(1+\ln x)$$

$f'(x) = 0$ When $1 + \ln x = 0$

$$\Rightarrow \ln x = -1$$

$$\Rightarrow x = e^{-1} = \frac{1}{e}$$

Since $f'(x) < 0$ when $x < \frac{1}{e}$ and $f'(x) > 0$ when $x > \frac{1}{e}$

So $f(x)$ has maximum $f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{-1/e}$

$$= \boxed{e^{1/e} \approx 1.44}$$

(D) From part (C) we have $f'(x) = -x^{-x}(1+\ln x)$

Since $f(x) = x^{-x}$

So $f'(x) = -f(x)(1+\ln x)$

Then $f''(x) = -\left[f'(x)(1+\ln x) + f(x).\frac{1}{x}\right]$

$$= -\left[-x^{-x}(1+\ln x)^2 + \frac{x^{-x}}{x}\right]$$

$$= -\left[-x^{-x}(1+\ln x)^2 + \frac{1}{x^{x+1}}\right]$$

$$f''(x) = -\frac{1}{x^{x+1}}\left[1 - x(\ln x + 1)^2\right]$$

We sketch the curve $f''(x)$ in figure 2. We see that $f''(x)$ is changing from negative to positive at x = 1.0

So x – coordinate of the inflection point is $\boxed{x = 1}$

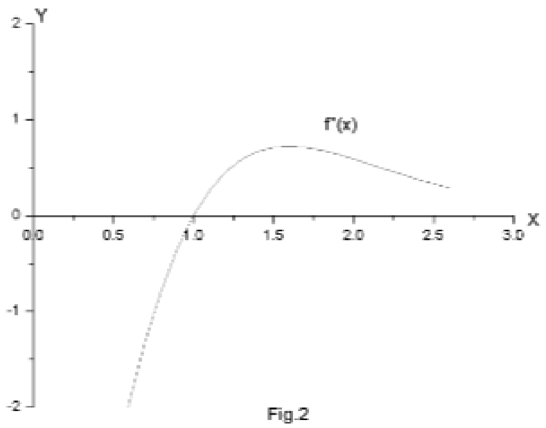


Fig.2

Answer 82E.

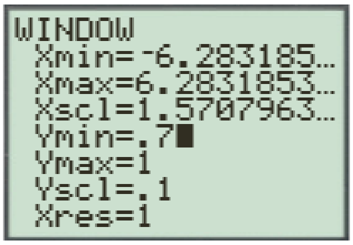
Consider the function $f(x)=(\sin x)^{\sin x}$.

(a)

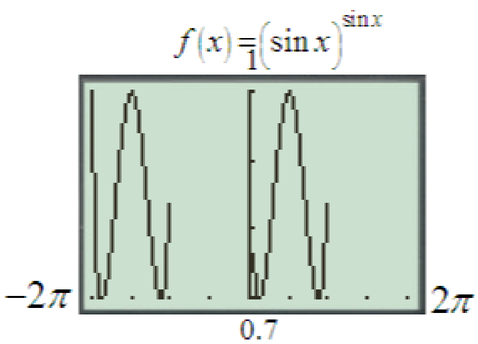
Sketch the graph of the function as follows:

First enter the function into the calculator as follows:

Set the following window:



Click on the GRAPH option, the display for the graph of the function as follows:



(b)

Find $\lim_{x\to 0^+}(\sin x)^{\sin x}$.

$$\lim_{x\to 0^+}(\sin x)^{\sin x}=(\sin 0)^{\sin 0}$$

Clear that as $x\to 0^+$, $(\sin x)^{\sin x}$ is indeterminate.

So find the limit by using L hospital's rule.

L hospital's rule says that,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Let $y=\lim_{x\to 0^+}(\sin x)^{\sin x}$.

Then,

$$\begin{aligned}\ln y &= \lim_{x\to 0^+}\ln(\sin x)^{\sin x}\\ &= \lim_{x\to 0^+}\sin x\cdot\ln(\sin x)\end{aligned}$$

$=\lim_{x\to 0^+}\dfrac{\ln(\sin x)}{\csc x}$ Since $\sin x=\dfrac{1}{\csc x}$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\sin x}(\cos x)}{-\csc x \cot x}$$ Apply L hospital's rule

$$= \lim_{x\to 0^+} \frac{\csc x(\cos x)}{-\csc x \cot x}$$

$$= -\lim_{x\to 0^+} \frac{\cos x}{\cot x}$$

$$= -\lim_{x\to 0^+} \frac{\cos x}{\left(\frac{\cos x}{\sin x}\right)}$$ Since $\cot x = \frac{\cos x}{\sin x}$

$$= -\lim_{x\to 0^+} \sin x$$

$$\ln y = 0$$
$$y = e^0$$
$$= 1$$

Therefore,

$$\lim_{x\to 0^+} (\sin x)^{\sin x} = \boxed{1}.$$

(c)

Need to find the local maximum and local minimum values of the function f.

The second derivative test state that:

If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c

If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c

Solve $f'(x) = 0$,

$$(\sin x)^{\sin x}(\cos x \ln(\sin x) + \cos x) = 0$$
$$\cos x \ln(\sin x) + \cos x = 0$$

Continuation to the above steps as follows:

$$\cos x(\ln(\sin x) + 1) = 0$$
$$\cos x = 0$$
$$x = \frac{\pi}{2}$$

Since $f'(x) < 0$ when $x < \frac{\pi}{2}$ and $f'(x) > 0$ when $x > \frac{\pi}{e}$

So $f(x)$ has maximum

$$f\left(\frac{\pi}{2}\right) = \left(\sin\left(\frac{\pi}{2}\right)\right)^{\sin\left(\frac{\pi}{2}\right)}$$
$$= 1$$

Therefore,

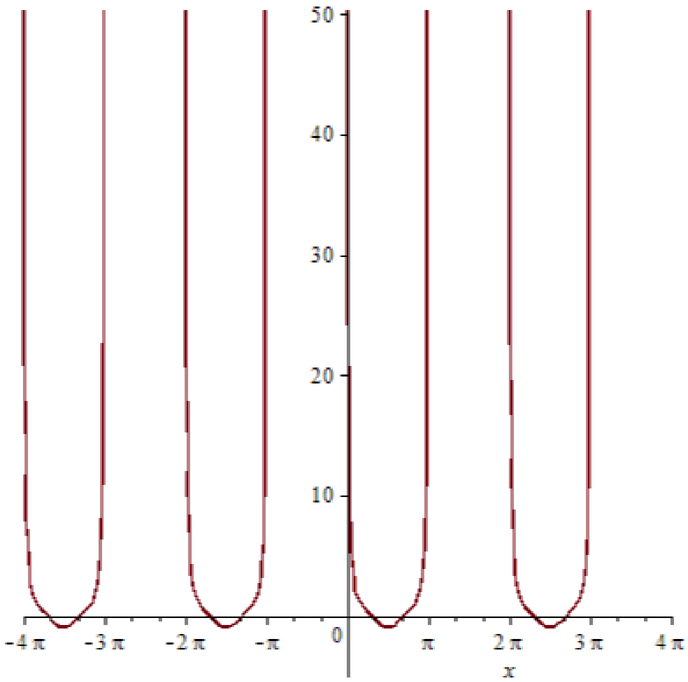
Maximum value of $f(x) = (\sin x)^{\sin x}$ is $\boxed{1}$.

(d)

To find the inflection points of the graph, differentiate $f'(x)$ with respect to x.

$$> \frac{d}{dx}\left(\sin(x)^{\sin(x)}\left(\cos(x)\ln(\sin(x)) + \cos(x)\right)\right)$$

$$\sin(x)^{\sin(x)}\left(\cos(x)\ln(\sin(x)) + \cos(x)\right)^2 + \sin(x)^{\sin(x)}\left(-\sin(x)\ln(\sin(x)) + \frac{\cos(x)^2}{\sin(x)} - \sin(x)\right)$$



By observing the graph, the curve changes its shape at multiples of $\frac{\pi}{2}$. So the inflection point occurs at these points.

From the graph, for $x > 0$, the inflection points occur at $\boxed{(4n-3)\frac{\pi}{2}}$.

And for $x < 0$, the inflection points occur at $\boxed{(4n-1)\frac{\pi}{2}}$.

Answer 83E.

(A) We sketch the curve $f(x) = x^{1/x}$ in figure 1

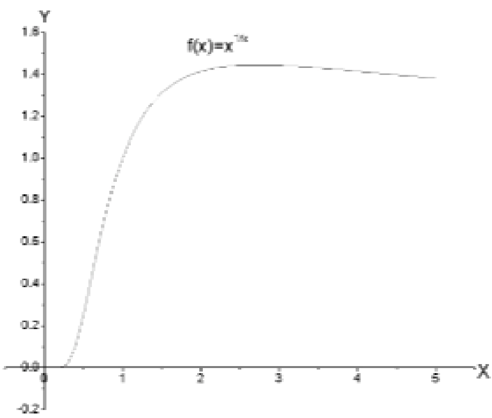


Fig.1

(B) We have $f(x)=x^{1/x}$

Taking logarithms

$$\ln\left(f(x)\right)=\frac{1}{x}\ln x$$

$$\lim_{x\to\infty}\ln\left(f(x)\right)=\lim_{x\to\infty}\frac{\ln x}{x}$$

$$=\lim_{x\to\infty}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x)} \qquad \text{[By L – hospitals rule]}$$

$$=\lim_{x\to\infty}\frac{1}{x}=0$$

So $\lim_{x\to\infty}\ln\left(f(x)\right)=0\Rightarrow\lim_{x\to\infty}e^{\ln(f(x))}=e^0$

$$\Rightarrow\boxed{\lim_{x\to\infty}f(x)=1}$$

Now $\ln\left(f(x)\right)=\frac{1}{x}\ln x$

Then $\lim_{x\to0^+}\ln\left(f(x)\right)=\lim_{x\to0^+}\frac{1}{x}(\ln x)=-\infty$

Then $\lim_{x\to0^+}\ln\left(f(x)\right)=-\infty$

$$\lim_{x\to0^+}e^{\ln(f(x))}=e^{-\infty}$$

$$\boxed{\lim_{x\to0^+}f(x)=0}$$

(C) From figure 1 we see that $f(x)$ has maximum

$$\approx f(e)\approx1.44$$

Finding the exact value

We have $f(x)=x^{1/x}$

We write $y=x^{1/x}$

Taking logarithm $\ln y=\frac{1}{x}\ln x$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx}=\left(\frac{1}{x}.\frac{1}{x}-\frac{1}{x^2}\ln x\right) \qquad \text{[By product rule]}$$

$$=\frac{1}{x^2}(1-\ln x)$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{x^2}(1-\ln x).y$$

$$\Rightarrow f'(x)=\frac{x^{1/x}}{x^2}(1-\ln x)$$

$f'(x)=0$ when $1-\ln x=0\Rightarrow\ln x=1$

$$\Rightarrow x=e$$

Since $f'(x)<0$ when x > e and $f'(x)>0$ when x < e

So $f(x)$ has maximum $\boxed{f(e)=e^{1/e}\approx1.44}$

(D) From part (C) we have

$$f'(x)=\frac{x^{1/x}}{x^2}(1-\ln x)$$

$$\Rightarrow f'(x)=\frac{f(x)}{x^2}(1-\ln x)$$

Differentiating with respect to x

$$f''(x)=\frac{1}{x^2}\left[f'(x)(1-\ln x)-\frac{1}{x}f(x)\right]-\frac{2}{x^3}f(x)(1-\ln x)$$

$$= \frac{1}{x^2}\left[\frac{x^{1/x}}{x^2}(1-\ln x)^2 - \frac{1}{x}x^{1/x}\right] - \frac{2x^{1/x}}{x^3}(1-\ln x)$$

$$= x^{1/x}\left[\frac{1}{x^4}(1-\ln x)^2 - \frac{1}{x^3} - \frac{2}{x^3}(1-\ln x)\right]$$

Or $\quad f''(x) = \frac{x^{1/x}}{x^4}\left[(1-\ln x)^2 - 2x(1-\ln x) - x\right]$

Now we sketch the graph of $f''(x)$ in figure 2

We see that $f''(x)$ is changing form positive to negative at $x \approx 0.58$ and $f''(x)$ is changing from negative to positive at $x \approx 4.4$

So x – coordinates of the inflection points are 0.58, 4.4

[Here second point of intersection can not be viewed clearly so for this point we zoom the scale of graph in (figure 3) and see that the x – coordinate of the point is about 4.4]

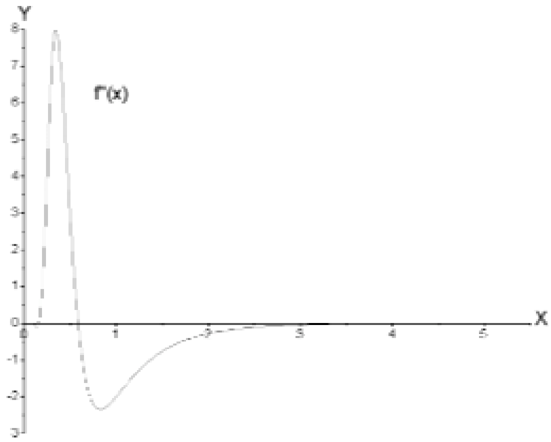


Fig.-2

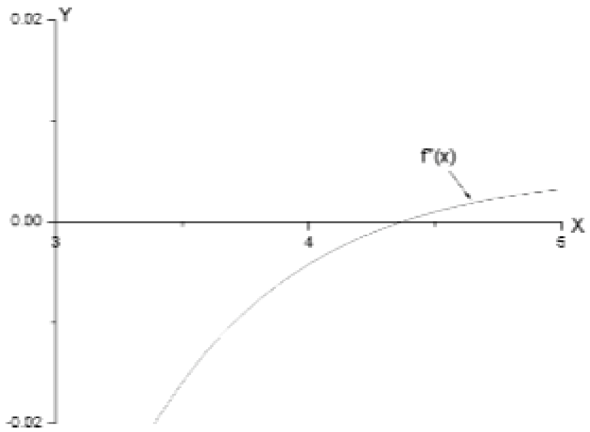


Fig.3

Answer 84E.

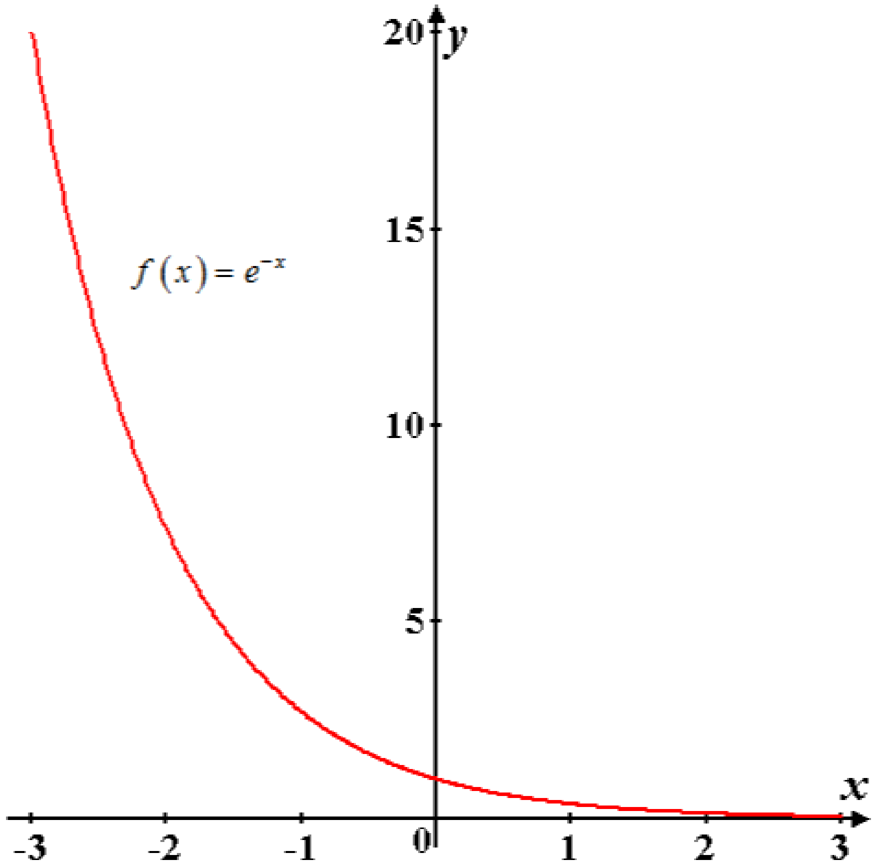
Consider the curve $f(x) = x^n e^{-x}$

Required to investigate the family of curves.

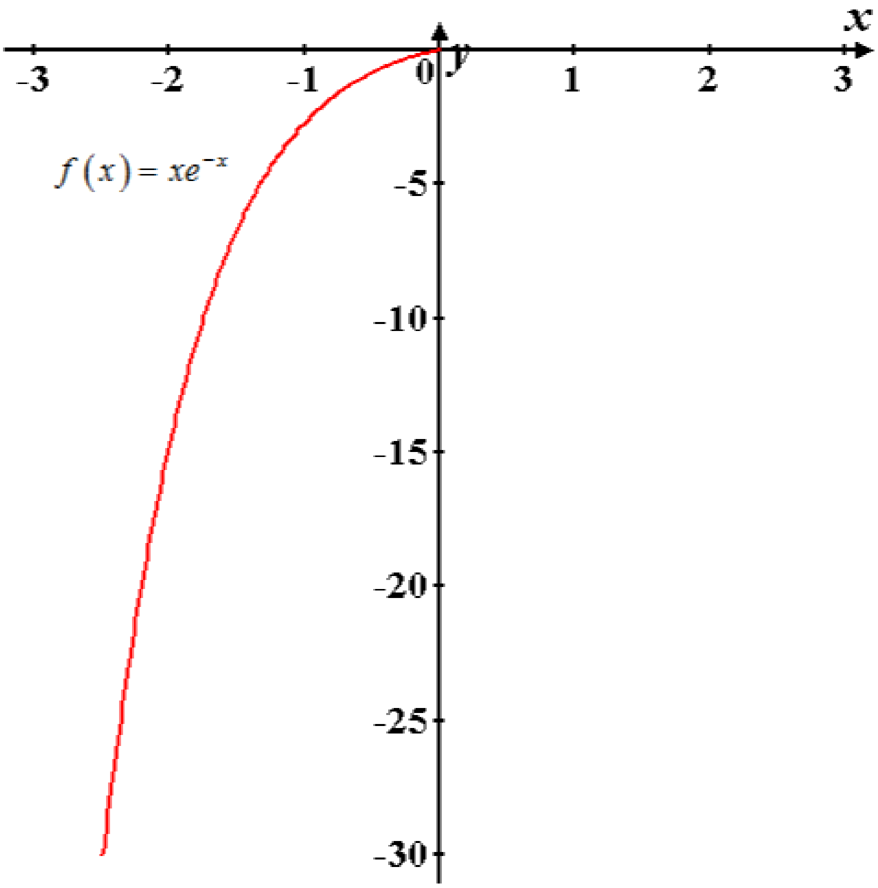
Graphs of $f(x)$ for different values of *n*.

So, $f(x)$ is an increasing function for even values of *n* ,decreasing for odd values of *n*.

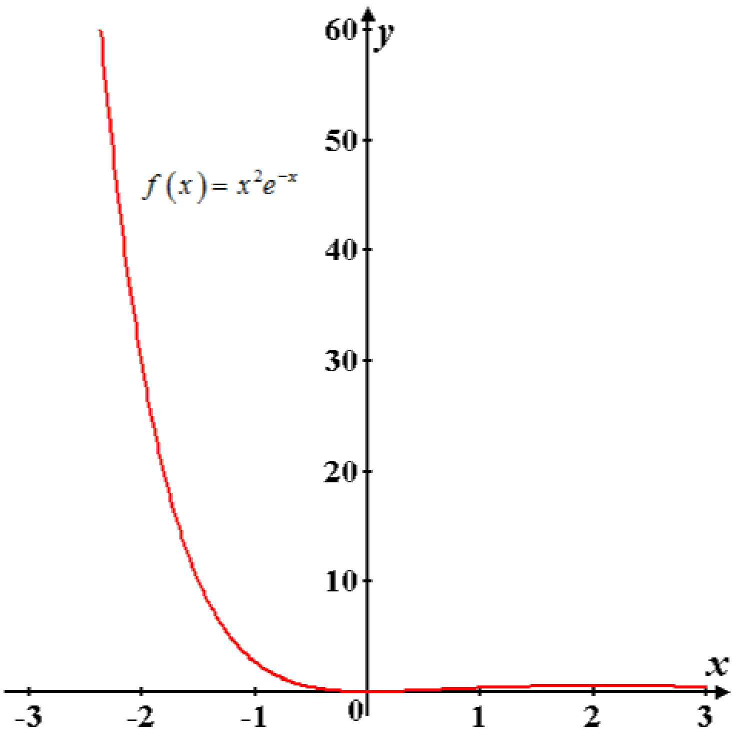
Graph of the curve for $n=0$



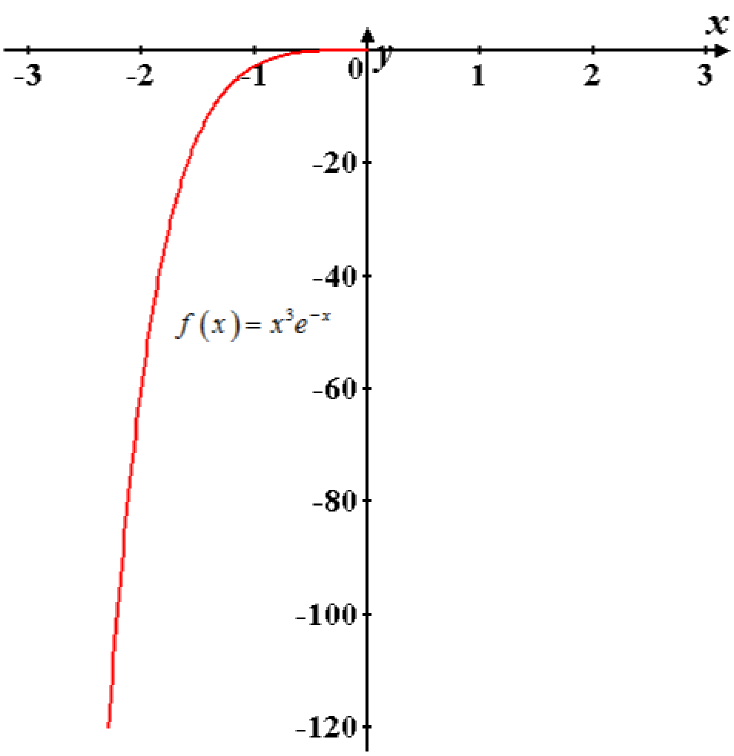
Graph of the curve for $n=1$



Graph of the curve for $n = 2$



Graph of the curve for $n = 3$



Find the limit as x tends to infinity

$$\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty}\frac{x^n}{e^x}$$
$$= \left\{\lim_{x\to\infty}\frac{x}{e^{x/n}}\right\}^n$$

Since the limit is of the form $\frac{\infty}{\infty}$,so use the L Hospitals rule to simplify the limit as

$$\left\{\lim_{x\to\infty}\frac{x}{e^{x/n}}\right\}^n = 0^n$$
$$= 0$$

Also, From the graph it is visible that $\lim_{x\to\infty} x^n e^{-x} = 0$

The family of curves $f(x) = x^n e^{-x}$ do not show any inflection points for any value of n.

The family of curves $f(x) = x^n e^{-x}$ will have a maximum only if n is odd, in this case the global as well as the local maximum occurs at $\boxed{x = n}$.

The family of curves $f(x) = x^n e^{-x}$ will have a minima if and only if n is even, in this case the global as well as the local minima occurs at $\boxed{x = 0}$.

Answer 85E.

$f(x) = xe^{-cx}$, putting $c = -2, -1, \frac{-1}{2}, \frac{-1}{3}, 0, \frac{1}{3}, \frac{1}{2}, 1, 2$

$f(x) = xe^{2x}, xe^{x}, xe^{\frac{x}{2}}, xe^{\frac{x}{3}}, x, xe^{\frac{-x}{3}}, xe^{\frac{-x}{2}}, xe^{-x}, xe^{-2x}$

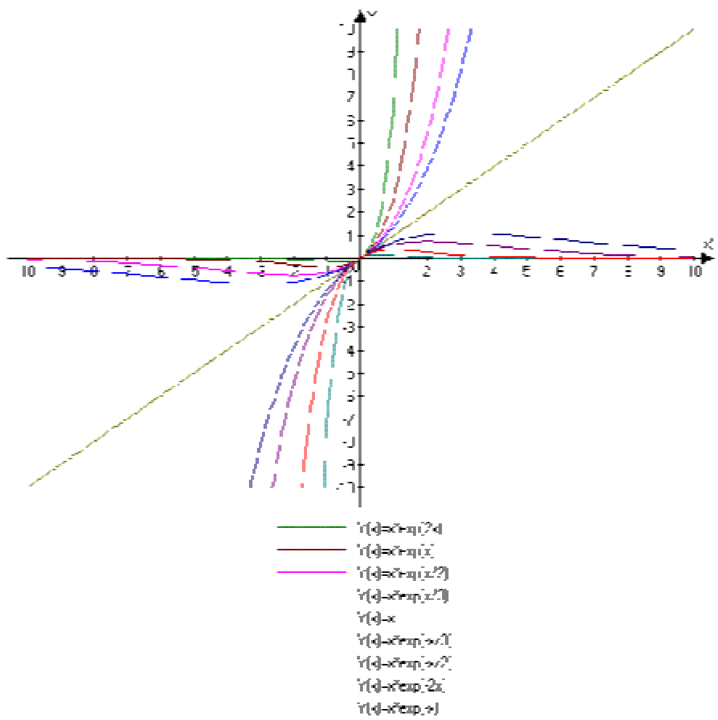
$f'(x) = (2x+1)e^{2x}, (x+1)e^{x}, \left(\frac{x}{2}+1\right)e^{\frac{x}{2}}, \left(\frac{x}{3}+1\right)e^{\frac{x}{3}},$

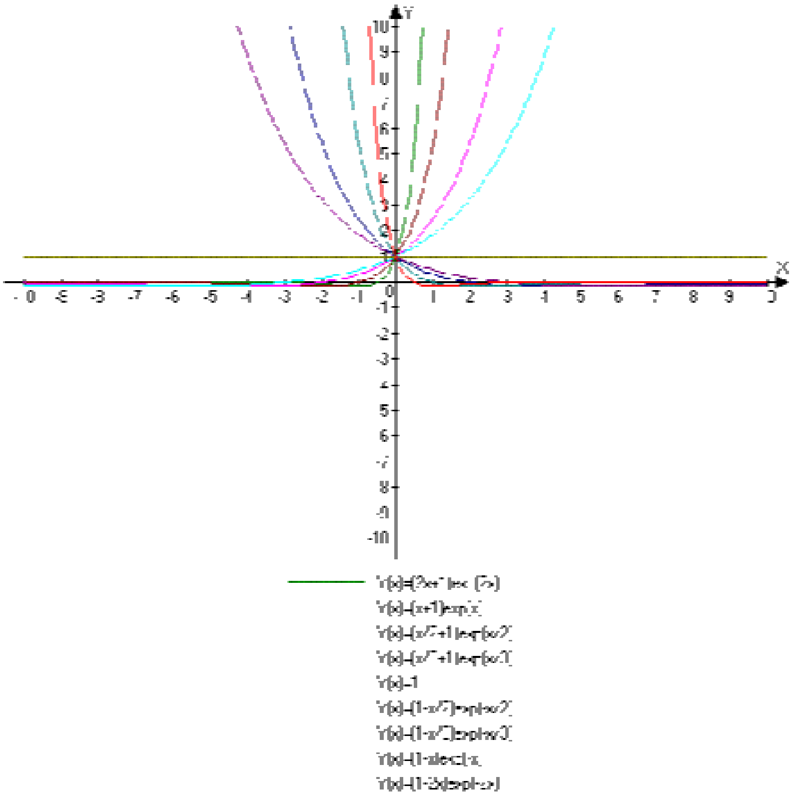
$1, \left(1-\frac{x}{2}\right)e^{\frac{-x}{2}}, \left(1-\frac{x}{3}\right)e^{\frac{-x}{3}}, (1-x)e^{-x}, (1-2x)e^{-2x}$

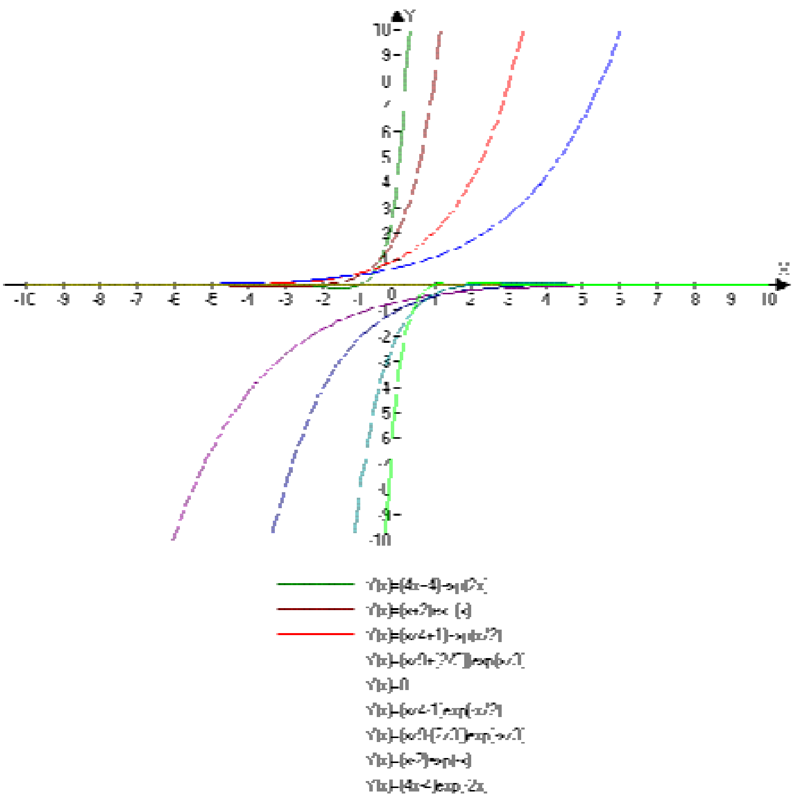
$f''(x) = (4x+4)e^{2x}, (x+2)e^{x}, \left(\frac{x}{4}+1\right)e^{\frac{x}{2}}, \left(\frac{x}{9}+\frac{2}{3}\right)e^{\frac{x}{3}},$

$0, \left(\frac{x}{4}-1\right)e^{\frac{-x}{2}}, \left(\frac{x}{9}-\frac{2}{3}\right)e^{\frac{-x}{3}}, (x-2)e^{-x}, (4-4x)e^{-2x}$

We graph the above and observe the required events.







see that the above graphs represent y, y', y'' in the order from above.

from these graphs we follow that

when $c > 0, \lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = 0$

when $c < 0, \lim_{x \to -\infty} f(x) = 0, \lim_{x \to \infty} f(x) = \infty$

on the other hand , either c decreases from 0 or c increases from 0 , the maximum and minimum points become close to the origin.

similarly from the graph of f " , the infliction points become close to the origin as c decreases from the origin or increases from the origin.

Answer 86E.

(A) We have $v = \frac{mg}{c}\left(1 - e^{-ct/m}\right)$

Then $\lim_{t\to\infty} v = \lim_{t\to\infty} \frac{mg}{c}\left(1 - e^{-ct/m}\right)$

$= \lim_{t\to\infty} \frac{mg}{c}\left(1 - \left(e^{-t}\right)^{c/m}\right)$

$= \frac{mg}{c}(1-0)$ since $\lim_{t\to\infty} e^{-t} = 0$

$= \boxed{\frac{mg}{c}}$

It means the object's speed approaches $\boxed{\frac{mg}{c}}$ as the time goes on

This is called limiting velocity

(B) Now again we have $v = \frac{mg}{c}\left(1 - e^{-ct/m}\right)$

Taking t as a constant, take the limit as $m \to \infty$

$\lim_{m\to\infty} v = \lim_{m\to\infty} \frac{mg}{c}\left(1 - e^{-ct/m}\right)$

$= \lim_{m\to\infty} \frac{g\left(1 - e^{-ct/m}\right)}{c/m}$ [0 / 0]

Applying L-Hospital's rule:

$\lim_{m\to\infty} v = \lim_{m\to\infty} \frac{g\left(-e^{-ct/m}\left(ct/m^2\right)\right)}{-c/m^2}$

$= \lim_{m\to\infty} gt\left(e^{-ct/m}\right)$

$= gt\left(e^0\right) = \boxed{gt}$

Thus, we can say that the velocity of a very heavy body is $\boxed{v = gt}$

Answer 87E.

Given $A = A_0\left(1 + \frac{r}{n}\right)^{nt}$

Then $\ln A = \ln\left(A_0\left(1+\frac{r}{n}\right)^{nt}\right)$

$= \ln A_0 + \ln\left(1+\frac{r}{n}\right)^{nt}$

$= \ln A_0 + nt\ln\left(1+\frac{r}{n}\right)$

Consider $\lim_{n\to\infty} \ln A = \lim_{n\to\infty}\left(\ln A_0 + nt\cdot\ln\left(1+\frac{r}{n}\right)\right)$

$= \ln A_0 + \lim_{n\to\infty} nt\ln\left(1+\frac{r}{n}\right)$

$= \ln A_0 + t\lim_{n\to\infty} \frac{\ln\left(1+\frac{r}{n}\right)}{\frac{1}{n}}$

$= \ln A_0 + t\lim_{n\to\infty}\left(\frac{\frac{1}{1+\frac{r}{n}}\left(\frac{-r}{n^2}\right)}{\frac{-1}{n^2}}\right)$ (applying L' Hospital's rule)

(Since f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a. Suppose that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$)

Continue to the above

$$= \ln A_0 + t\lim_{n\to\infty}\left[-r\cdot\frac{1}{1+\frac{r}{n}}\right]$$
$$= \ln A_0 + (-rt)$$
$$= \ln A_0 + \ln e^{-rt}$$
$$= \ln\left(A_0 e^{-rt}\right)$$

Hence $\boxed{\lim_{n\to\infty} A = A_0 e^{-rt}}$

Answer 88E.

The distance travelled by the ball in time t when projected in water is

$$s(t) = \frac{m}{c}\ln\cosh\sqrt{\frac{gc}{mt}}.$$

Here, m is the mass of the metal ball, c is a positive constant.

Need to find $\lim_{c\to 0^+} s(t)$.

Note that as $c \to 0^+$, $\frac{m}{c} \to 0$ and $\ln\cosh\sqrt{\frac{gc}{mt}} \to 0.$

This is an indeterminate form of type $\frac{0}{0}$.

Apply L'Hospital rule $\lim_{c\to a}\frac{f(c)}{g(c)} = \lim_{c\to a}\frac{f'(c)}{g'(c)}$ to $\lim_{c\to 0^+}\left[\frac{m\ln\cosh\sqrt{\frac{gc}{mt}}}{c}\right]$.

$$\lim_{c\to 0^+}\left[\frac{m\ln\cosh\sqrt{\frac{gc}{mt}}}{c}\right] = m\lim_{c\to 0^+}\left[\frac{\frac{d}{dc}\left(\ln\cosh\sqrt{\frac{gc}{mt}}\right)}{\frac{d}{dc}(c)}\right]$$ m is constant multiple

$$= m\lim_{c\to 0^+}\left[\frac{\frac{\sinh\left(\sqrt{\frac{gc}{mt}}\right)\frac{d}{dc}\left(\sqrt{\frac{gc}{mt}}\right)}{\cosh\left(\sqrt{\frac{gc}{mt}}\right)}}{1}\right]$$

$$= \cancel{m}\lim_{c\to 0^+}\left[\frac{g\sinh\left(\sqrt{\frac{gc}{mt}}\right)}{2\cancel{m}t\sqrt{\frac{gc}{mt}}\cosh\left(\sqrt{\frac{gc}{mt}}\right)}\right]$$

Rewrite the limit as follows:

$$\lim_{c\to 0^+}\left[\frac{m\ln\cosh\sqrt{\frac{gc}{mt}}}{c}\right]=\frac{g}{2t}\lim_{c\to 0^+}\left[\frac{\sinh\left(\sqrt{\frac{gc}{mt}}\right)\Big/\sqrt{\frac{gc}{mt}}}{\cosh\left(\sqrt{\frac{gc}{mt}}\right)}\right] \quad \frac{g}{2t}\text{ is constant multiple}$$

$$=\frac{g}{2t}\left[\frac{\lim_{c\to 0^+}\sinh\left(\sqrt{\frac{gc}{mt}}\right)\Big/\sqrt{\frac{gc}{mt}}}{\lim_{c\to 0^+}\cosh\left(\sqrt{\frac{gc}{mt}}\right)}\right]$$

$$=\frac{g}{2t}\left(\frac{1}{1}\right) \quad \text{since}\lim_{x\to 0}\frac{\sin x}{x}=1\text{ and }\lim_{x\to 0}\cos x=1$$

$$=\frac{g}{2t}$$

Therefore, the value of $\lim_{c\to 0^+}s(t)$ is $\boxed{\frac{g}{2t}}$.

Answer 89E.

Consider an electrostatic field E, acting upon a liquid or a gaseous polar dielectric.

Then, the net dipole moment P per unit volume is given by the following equation:

$$P(E)=\frac{e^E+e^{-E}}{e^E-e^{-E}}-\frac{1}{E}.$$

Show that $\lim_{E\to 0^+}P(E)=0$.

$$\lim_{E\to 0^+}P(E)=\lim_{E\to 0^+}\frac{e^E+e^{-E}}{e^E-e^{-E}}-\frac{1}{E}$$

$$=\lim_{E\to 0^+}\frac{E\left(e^E+e^{-E}\right)-1\left(e^E-e^{-E}\right)}{E\left(e^E-e^{-E}\right)}$$

$$=\lim_{E\to 0^+}\frac{Ee^E+Ee^{-E}-e^E+e^{-E}}{Ee^E-Ee^{-E}} \quad \text{Indeterminate form }\frac{0}{0}$$

$$=\lim_{E\to 0^+}\frac{Ee^E+e^E-Ee^{-E}+e^{-E}-e^E-e^{-E}}{Ee^E+e^E+Ee^{-E}-e^{-E}} \quad \text{L'Hospital'sRule}$$

Simplify further as shown below:

$$=\lim_{E\to 0^+}\frac{Ee^E-Ee^{-E}}{Ee^E+Ee^{-E}}$$

$$=\lim_{E\to 0^+}\frac{E\left(e^E-e^{-E}\right)}{E\left(e^E+e^{-E}\right)}$$

$$=\lim_{E\to 0^+}\frac{e^E-e^{-E}}{e^E+e^{-E}}$$

$$=\frac{e^0-e^{-0}}{e^0+e^{-0}}$$

$$=\frac{1-1}{1+1}$$

$$=0$$

Therefore, $\lim_{E\to 0^+}P(E)=0$.

Answer 90E.

(a) $\lim_{R\to r^+} v = \lim_{R\to r^+}\left(-c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)\right)$

Since $R\to r^+$ then $\frac{r}{R}\to 1$ so it equals to

$= -c(1)^2 \ln 1$

$= -c\cdot 1\cdot 0$

Since $\ln = 1$

Its mean if extension insulation R reaches to positive of radius of metal cable then velocity reaches to 0.

(b) $\lim_{R\to 0^+} v = \lim_{R\to 0^+}\left(-c\left(\frac{r}{R}\right)^2 \cdot \ln\left(\frac{r}{R}\right)\right)$

Where $r\to 0^+$ then $\frac{r}{R}\to 0^+$

And $\ln\left(\frac{r}{R}\right) \Rightarrow \ln\left(0^+\right)\to 0$

So $\lim_{R\to 0^+} v = 0$

Its mean where reaches of the cable reaches to positive zero then the velocity v reaches zero.

Answer 91E.

Consider the Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt$

Which arises in the study of the diffraction of light waves.

Evaluate the $\lim_{x\to 0} \frac{S(x)}{x^3}$

$$\lim_{x\to 0} \frac{S(x)}{x^3} = \lim_{x\to 0} \frac{\int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt}{x^3}$$

The above limit is a $\left[\frac{0}{0}\right]$ indeterminate form because

$$\int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt = \int_0^0 \sin\left(\frac{1}{2}\pi t^2\right) dt$$
$$= 0$$

And $\lim_{x\to 0} x^3 = 0$

So, apply the L'Hospital rule

$$\lim_{x\to 0} \frac{\int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt}{x^3} = \lim_{x\to 0} \frac{\frac{d}{dx}\left(\int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt\right)}{\frac{d}{dx}\left(x^3\right)}$$

Use the fundamental theorem of calculus part-I $\frac{d}{dx}\left(\int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt\right) = \sin\left(\frac{1}{2}\pi x^2\right)$

$$= \lim_{x\to 0} \frac{\sin\left(\frac{1}{2}\pi x^2\right)}{3x^2}$$

Again apply the L'Hospital rule

$$\lim_{x\to 0}\frac{S(x)}{x^3}=\lim_{x\to 0}\frac{\pi x\cos\left(\frac{1}{2}\pi x^2\right)}{6x}$$

$$=\lim_{x\to 0}\frac{\pi\cos\left(\frac{1}{2}\pi x^2\right)}{6}$$

$$=\frac{\pi\cos(0)}{6}$$

$$=\frac{\pi\cdot 1}{6}$$

$$=\boxed{\frac{\pi}{6}}$$

Answer 92E.

The temperature of the rod at the point x at time t is

$$T(x,t)=\frac{C}{a\sqrt{4\pi kt}}\int_0^a e^{-(x-u)^2/(4kt)}du$$

Then $\lim_{a\to 0}T(x,t)=\lim_{a\to 0}\frac{C}{a\sqrt{4\pi kt}}\int_0^a e^{-(x-u)^2/(4kt)}du$

Here only 'a' is variable so we treat C, k, t and x as the constant. And using L – Hospitals rule

$$\lim_{a\to 0}T(x,t)=\lim_{a\to 0}\frac{C}{\frac{d}{da}\left(a\sqrt{4\pi kt}\right)}\frac{d}{da}\int_0^a e^{-(x-u)^2/4kt}du$$

$$=\lim_{a\to 0}\frac{Ce^{-(x-a)^2/4kt}}{\sqrt{4\pi kt}} \qquad \text{(By F.T.C. - 1)} \left(\frac{d}{dx}\int_a^x f(t)dt=f(x)\right)$$

$$\boxed{\lim_{a\to 0}T(x,t)=\frac{e^{-x^2/4kt}}{\sqrt{4\pi kt}}}$$

Answer 93E.

We have $y=\dfrac{\sqrt{2a^3x-x^4}-a\sqrt[3]{aax}}{a-\sqrt[4]{ax^3}}$

Or $y=\dfrac{\left(2a^3x-x^4\right)^{1/2}-a\left(aax\right)^{1/3}}{a-\left(ax^3\right)^{1/4}}$

Taking limit as x tends to a

$$\lim_{x\to a}y=\lim_{x\to a}\frac{\left(2a^3x-x^4\right)^{1/2}-a\left(aax\right)^{1/3}}{a-\left(ax^3\right)^{1/4}} \qquad \text{[Form of 0/0]}$$

Using L-Hospital's rule

$$\lim_{x\to a} y = \lim_{x\to a}\left[\frac{\left\{\frac{1}{2}\left(2a^3x-x^4\right)^{-1/2}\left(2a^3-4x^3\right)-\frac{a}{3}(aax)^{-2/3}(aa)\right\}}{-\frac{1}{4}\left(ax^3\right)^{-3/4}\left(3ax^2\right)}\right]$$

$$=\frac{\left\{\frac{1}{2}\left(2a^3a-a^4\right)^{-1/2}\left(2a^3-4a^3\right)-\frac{a}{3}(aaa)^{-2/3}(aa)\right\}}{-\frac{1}{4}\left(aa^3\right)^{-3/4}\left(3aa^2\right)}$$

$$=\frac{\left\{\frac{1}{2}\left(a^4\right)^{-1/2}\left(-2a^3\right)-\frac{a}{3}(a)^{-2}\left(a^2\right)\right\}}{-\frac{1}{4}(a)^{-3}\left(3a^3\right)}$$

$$=\frac{\left\{-a-\frac{a}{3}\right\}}{-3/4}$$

$$=\frac{\left\{\frac{4a}{3}\right\}}{3/4}=\boxed{\frac{16}{9}a}$$

Answer 94E.

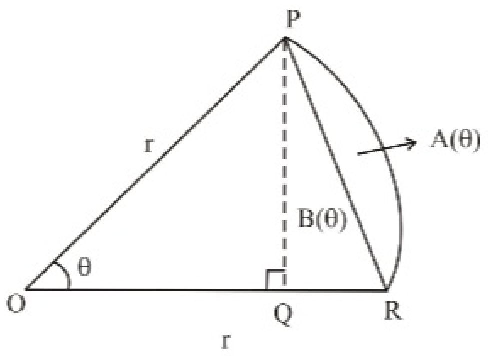


Fig. 1

Let the radius of the circle is r

Then the area of the sector is $=\frac{r^2\theta}{2}$

Area of the triangle is $=\frac{1}{2}|PQ|\times|OR|$

$$=\frac{1}{2}r\times r\sin\theta \qquad [PQ=r\sin\theta]$$

$$=\frac{1}{2}r^2\sin\theta$$

Then the area of segment

$$A(\theta)=\frac{r^2\theta}{2}-\frac{r^2}{2}\sin\theta$$

Or $$A(\theta)=\frac{r^2}{2}(\theta-\sin\theta) \qquad \text{--- (1)}$$

Now in triangle PQR (By the Pythagorean Theorem)

$$|OQ|^2=r^2-r^2\sin^2\theta$$

$$=|OP|^2-|PQ|^2$$

$$=r^2\left(1-\sin^2\theta\right)$$

$$=r^2\cos^2\theta \qquad \left[1-\sin^2\theta=\cos^2\theta\right]$$

Then $|OQ| = r\cos\theta$

Then $|QR| = r - r\cos\theta$

So the area of the triangle PQR is

$$B(\theta) = \frac{1}{2} r\sin\theta (r - r\cos\theta)$$

$$B(\theta) = \frac{1}{2} r^2 (\sin\theta - \sin\theta\cos\theta)$$

Or $$B(\theta) = \frac{1}{2} r^2 \left(\sin\theta - \frac{\sin 2\theta}{2}\right) \quad \text{--- (2)} \qquad [(\sin 2\theta = 2\sin\theta\cos\theta)]$$

Now from (1) and (2)

$$\frac{A(\theta)}{B(\theta)} = \frac{\frac{r^2}{2}(\theta - \sin\theta)}{\frac{1}{2} r^2 \left(\sin\theta - \frac{\sin 2\theta}{2}\right)}$$

Or $$\frac{A(\theta)}{B(\theta)} = \frac{\theta - \sin\theta}{\left(\sin\theta - \frac{\sin 2\theta}{2}\right)}$$

Taking limit as $\theta \to 0^+$

$$\lim_{\theta\to 0^+} \frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+} \frac{\theta - \sin\theta}{\left(\sin\theta - \frac{\sin 2\theta}{2}\right)}$$

Using L – Hospital rule

$$\lim_{\theta\to 0^+} \frac{A(\theta)}{B(\theta)} \overset{H}{=} \lim_{\theta\to 0^+} \frac{\frac{d}{d\theta}(\theta - \sin\theta)}{\frac{d}{d\theta}\left(\sin\theta - \frac{\sin 2\theta}{2}\right)}$$

$$= \lim_{\theta\to 0^+} \frac{(1-\cos\theta)}{(\cos\theta - \cos 2\theta)}$$

This is also form of $\frac{0}{0}$ so we use L – hospital rule again

$$\lim_{\theta\to 0^+} \frac{A(\theta)}{B(\theta)} \overset{H}{=} \lim_{\theta\to 0^+} \frac{\frac{d}{d\theta}(1-\cos\theta)}{\frac{d}{d\theta}(\cos\theta - \cos 2\theta)}$$

$$= \lim_{\theta\to 0^+} \frac{\sin\theta}{-\sin\theta + 2\sin 2\theta}$$

Again by L – hospital rule

$$\lim_{\theta\to 0^+} \frac{A(\theta)}{B(\theta)} \overset{H}{=} \lim_{\theta\to 0^+} \frac{\frac{d}{d\theta}(\sin\theta)}{\frac{d}{d\theta}(-\sin\theta + 2\sin 2\theta)}$$

$$= \lim_{\theta\to 0^+} \frac{\cos\theta}{-\cos\theta + 4\cos 2\theta}$$

$$= \frac{1}{-1+4.1}$$

$$= \frac{1}{-1+4}$$

So we have $\boxed{\lim_{\theta\to 0^+} \frac{A(\theta)}{B(\theta)} = \frac{1}{3}}$

Answer 95E.

Evaluate: $\lim_{x\to\infty}\left[x - x^2 \ln\left(\frac{1+x}{x}\right)\right]$

Make the substitution $x=\frac{1}{t}$ then $x\to\infty$ changes to $t\to 0$.

$$\lim_{x\to\infty}\left[x-x^2\ln\left(\frac{1+x}{x}\right)\right]=\lim_{t\to 0}\left[\frac{1}{t}-\frac{1}{t^2}\ln\left(\frac{1+\frac{1}{t}}{\frac{1}{t}}\right)\right] \quad \text{Substitute } \frac{1}{t} \text{ for } x.$$

$$=\lim_{t\to 0}\left[\frac{t-\ln(1+t)}{t^2}\right] \quad \text{Convert into single fraction}$$

$$=\left[\frac{0-\ln(1+0)}{0}\right] \quad \text{Apply the limit (Substitute 0 for } t.)$$

$$=\frac{0}{0} \quad \text{(Undefined form)}$$

So use the L hospital's rule to find the limit.

Now consider,

$$\lim_{t\to 0}\left[\frac{t-\ln(1+t)}{t^2}\right]=\lim_{t\to 0}\left[\frac{(t-\ln(1+t))'}{(t^2)'}\right] \quad \left\{\begin{array}{l}\text{Apply L hospital's rule}\\ \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right]\end{array}\right.$$

$$=\lim_{t\to 0}\left[\frac{\left(1-\frac{1}{1+t}\right)}{2t}\right] \quad \left\{\begin{array}{l}\text{If we apply the limit then we get } \frac{0}{0} \text{ form.}\\ \text{so again use the rule}\\ \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right]\end{array}\right.$$

$$=\lim_{t\to 0}\left[\frac{\left(1-\frac{1}{1+t}\right)'}{(2t)'}\right] \quad \left\{\begin{array}{l}\text{Apply L hospital's rule}\\ \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right]\end{array}\right.$$

$$=\lim_{t\to 0}\left[\frac{\left(\frac{1}{(1+t)^2}\right)}{2}\right] \quad \text{Simplify}$$

$$=\left[\frac{\left(\frac{1}{(1+0)^2}\right)}{2}\right] \quad \text{Apply the limit (Substitute 0 for } t.)$$

$$=\frac{1}{2}$$

Therefore $\lim_{x\to\infty}\left[x-x^2\ln\left(\frac{1+x}{x}\right)\right]=\boxed{\frac{1}{2}}$.

Answer 96E.

The objective is to show that if f is a positive function with $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$, then $\lim_{x\to a}\left[f(x)\right]^{g(x)}=0$.

That is it is required to show that for any $\varepsilon>0$ there exists a $\delta>0$ such that

$$0<|x-a|<\delta \Rightarrow 0<\left|\left[f(x)\right]^{g(x)}\right|<\varepsilon.$$

As f is a **positive** function it is equivalent to show that $0<|x-a|<\delta$ implies

$$0<\left[f(x)\right]^{g(x)}<\varepsilon.$$

Choose $\varepsilon > 0$, and for $\lim_{x\to a} f(x) = 0$ then by the definition of limit;

For any choice of $\varepsilon_1 > 0$ there exists a $\delta_1 > 0$ such that $0 < |x-a| < \delta_1$ implies

$0 < |f(x)| < \varepsilon_1$.

Choose ε_1 such that $0 < \varepsilon_1 < 1$ follows that $0 < |x-a| < \delta_1 \Rightarrow 0 < f(x) < \varepsilon_1 < 1$, since f is a positive function.

Also $\lim_{x\to a} g(x) = \infty$, then for any choice of a positive number $m > 0$ there exists a $\delta_2 > 0$ such that $0 < |x-a| < \delta_2$ implies $g(x) > m$.

Take $\delta = \min\{\delta_1, \delta_2\}$ and use the following results:

(1) If $a > b > 0$ and $0 < r < 1$ then $0 < r^a < r^b$.

(2) For any positive number k then there is a number m sufficiently large such that $0 < k^{\frac{1}{m}} < 1$.

Apply the result **(1)**, to get $0 < |x-a| < \delta \Rightarrow 0 < [f(x)]^{g(x)} < \varepsilon_1^m$.

But $0 < [f(x)]^{g(x)} < \varepsilon_1^m < \varepsilon$.

Based on the selection of ε_1 such that $\varepsilon_1 = \varepsilon^{\frac{1}{m}}$.

Choose *m* as large as possible, then by **(2)** $\varepsilon_1 = \varepsilon^{\frac{1}{m}} < 1$ for small and large value of the ε.

Thus, for $\varepsilon > 0$ there exists a $\delta = \text{minimum}(\delta_1, \delta_2)$, with *m* chosen sufficiently large and $\varepsilon_1 = \varepsilon^{\frac{1}{m}}$ then $0 < \varepsilon_1 < 1$ and $0 < |x-a| < \delta \Rightarrow 0 < [f(x)]^{g(x)} < \varepsilon_1^m < \varepsilon$

Therefore, by definition of limit conclude that $\boxed{\lim_{x\to a} [f(x)]^{g(x)} = 0}$.

Answer 97E.

We have $f(2) = 0$ and $f'(2) = 7$

We have to evaluate

$$\lim_{x\to 0} \frac{f(2+3x)+f(2+5x)}{x}$$

Since this is the form of $\frac{0}{0}$ so we can use L – hospital rule

$$\lim_{x\to 0} \frac{f(2+3x)+f(2+5x)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}[f(2+3x)+f(2+5x)]}{\frac{d}{dx}.x}$$

$$= \lim_{x\to 0} \frac{3f'(2+3x)+5f'(2+5x)}{1}$$

$$= \frac{3.f'(2+0)+5.f'(2+0)}{1}$$

$$= 3.f'(2)+5f'(2)$$

$$= 8.f'(2)$$

$$= 8\times 7$$

$$= 56$$

$$\boxed{\lim_{x\to 0} \frac{f(2+3x)+f(2+5x)}{x} = 56}$$

Answer 98E.

We have

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x^3}+a+\frac{b}{x^2}\right)=0$$

$$\Rightarrow \lim_{x\to 0}\frac{\sin 2x+bx}{x^3}+\lim_{x\to\infty} a=0$$

$$\Rightarrow \lim_{x\to 0}\frac{\sin 2x+bx}{x^3}=-a$$

Since left hand side of the equation is the from of $\frac{0}{0}$

So we can use L – hospital's rule

$$\lim_{x\to 0}\frac{\frac{d}{dx}(\sin 2x+bx)}{\frac{d}{dx}(x^3)}=-a$$

$$\Rightarrow \lim_{x\to 0}\frac{2\cos 2x+b}{3x^2}=-a$$

Since left hand side is $=\frac{2+b}{0}$ for making the form of $\frac{0}{0}$, b must be equal to -2 so

$(b=-2)$

Then $\Rightarrow \lim_{x\to 0}\frac{2\cos 2x-2}{3x^2}=-a$

Again using L – Hospital's rule

$$\lim_{x\to 0}\frac{-4\sin 2x}{6x}=-a$$

$$\Rightarrow -\frac{4}{3}\lim_{x\to 0}\frac{\sin 2x}{2x}=-a$$

$$\Rightarrow -\frac{4}{3}.(1)=-a \qquad \left[\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

$$\Rightarrow \boxed{a=\frac{4}{3}}$$

So we must have $\boxed{\left(a=\frac{4}{3}\right) and\ (b=-2)}$

Answer 99E.

We have to find

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$$

Since this is the form of $\frac{0}{0}$ so we can use L – Hospital rule, (Here we treat x as a constant)

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}=\lim_{h\to 0}\frac{\frac{d}{dh}\left[f(x+h)-f(x-h)\right]}{\frac{d}{dh}(2h)}$$

$$=\lim_{h\to 0}\frac{f'(x+h)+f'(x-h)}{2}$$

$$=\frac{f'(x+0)+f'(x-0)}{2}$$

$$=\frac{f'(x)+f'(x)}{2}$$

$$=\frac{\cancel{2}f'(x)}{\cancel{2}}$$

$$=f'(x)$$

So $\boxed{\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}=f'(x)}$

Since $\frac{f(x+h)-f(x-h)}{2h}$ denotes the slope of the secant line from $(x-h)$ to $(x+h)$ when $h\to 0$ this secant line approaches the tangent line at x that is $= f'(x)$

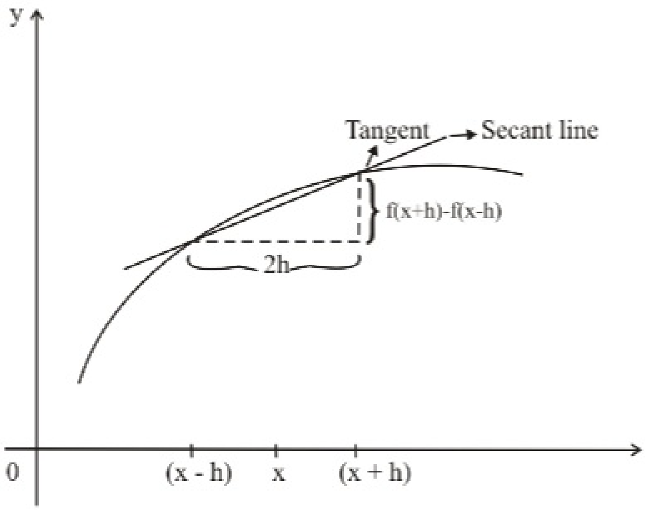


Fig. 1

Answer 100E.

We have to find

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}$$

This is form of $\frac{0}{0}$ so we can use the L – Hospital's rule. (We treat x as a constant)

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}=\lim_{h\to 0}\frac{\frac{d}{dh}\left[f(x+h)-2f(x)+f(x-h)\right]}{\frac{d}{dh}(h^2)}$$
$$=\lim_{h\to 0}\frac{f'(x+h)-0+f'(x-h)(-1)}{2h}$$
$$=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}$$

Again using L – hospital rule

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}=\lim_{h\to 0}\frac{\frac{d}{dh}\left[f'(x+h)-f'(x-h)\right]}{\frac{d}{dh}(2h)}$$
$$=\lim_{h\to 0}\frac{f''(x+h)-f''(x-h)(-1)}{2}$$
$$=\lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2}$$
$$=\frac{f''(x+0)+f''(x-0)}{2}$$
$$=\frac{2f''(x)}{2}$$

$$\boxed{\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}=f''(x)}$$

Answer 101E.

Consider the function,

$$f(x)=\begin{cases} e^{-1/x^2} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases} \quad \text{(a)}$$

Use the definition of derivative to compute $f'(0)$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Calculate this value when $x=0$.

And $x+h>0$ when $x=0$

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{e^{-1/h^2}-0}{h}$$ Substitute $x=h$ in the function $f(x)$

$$=\lim_{h\to 0}\frac{e^{-1/h^2}}{h}$$ Simplify

L'Hospital's Rule will not help at this point unless you factor *h* out of the limit first.

$$f'(0)=\lim_{h\to 0}h\cdot\lim_{h\to 0}\frac{e^{-1/h^2}}{h^2}$$

$$=0\cdot\lim_{h\to 0}\frac{1/h^2}{e^{1/h^2}}$$

The limit is now indeterminate of type $\frac{\infty}{\infty}$

So, use L'Hospital's Rule.

L'Hospital's Rule:

Suppose *f* and *g* are differentiable and $g'(x)\neq 0$ near *a* (except possibly at *a*).

If $\lim_{x\to a}f(x)=\infty$ and $\lim_{x\to a}g(x)=\infty$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

If the limit on the right exists (or is ∞ or $-\infty$)

Apply the L'Hospital's Rule.

Then,

$$f'(0)=0\cdot\lim_{h\to 0}\frac{-\frac{2}{h^3}}{-\frac{2}{h^3}e^{1/h^2}}$$

$$=0\cdot\lim_{h\to 0}e^{-1/h^2}$$

$$=0\cdot 0$$

$$=\boxed{0}$$

(b)

Use induction to show that there is a polynomial $p_n(x)$ and a nonnegative integer k_n such that $f^{(n)}(x)=\dfrac{p_n(x)f(x)}{x^{k_n}}$ for $x\neq 0$ ...... (i)

First prove this equation when $n=1$.

Calculate $f'(x)$ when $x\neq 0$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{e^{-1/(x+h)^2}-e^{-1/x^2}}{h}$$

Use L'Hospital's Rule.

$$f'(x)=\lim_{h\to 0}\frac{\frac{2}{(x+h)^3}e^{-1/(x+h)^2}-0}{1} \quad \text{Since } \lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$
$$=\frac{2}{(x+0)^3}e^{-1/(x+0)^2}$$
$$=\frac{2}{x^3}e^{-1/x^2}$$

Thus, $p_1(x)=2$ and $k_1=3$ since

$$f^{(1)}(x)=\frac{2}{x^3}e^{-1/x^2}$$
$$=\frac{2\cdot e^{-1/x^2}}{x^3}$$
$$=\frac{p_1(x)f(x)}{x^{k_1}}$$

For the induction step, suppose the equation is true when $n=m$ (i.e. $f^{(m)}(x)=p_m(x)f(x)/x^{k_m}$) and prove the statement when $n=m+1$.

$$f^{(m+1)}(x)=\lim_{h\to 0}\frac{f^{(m)}(x+h)-f^{(m)}(x)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{p_m(x+h)f(x+h)}{(x+h)^{k_m}}-\frac{p_m(x)f(x)}{(x)^{k_m}}}{h} \quad \text{From (i)}$$

Next use L'Hospital's Rule.

$$f^{(m+1)}(x)=\lim_{h\to 0}\frac{\frac{d}{dh}\left[\frac{p_m(x+h)e^{-1/(x+h)^2}}{(x+h)^{k_m}}\right]-0}{1}$$
$$=\lim_{h\to 0}\frac{(x+h)^{k_m}\cdot\frac{d}{dh}\left[p_m(x+h)e^{-1/(x+h)^2}\right]}{(x+h)^{2k_m}}$$
$$-\lim_{h\to 0}\frac{p_m(x+h)e^{-1/(x+h)^2}\cdot\frac{d}{dh}\left[(x+h)^{k_m}\right]}{(x+h)^{2k_m}}$$
$$=\lim_{h\to 0}\frac{p_m(x+h)\cdot\frac{2}{(x+h)^3}e^{-1/(x+h)^2}+p_m'(x+h)\cdot e^{-1/(x+h)^2}}{(x+h)^{k_m}}$$
$$-\lim_{h\to 0}\frac{p_m(x+h)e^{-1/(x+h)^2}\cdot k_m(x+h)^{k_m-1}}{(x+h)^{2k_m}}$$

Continuation to the above steps:

$$f^{(m+1)}(x)=\frac{\frac{2p_m(x)e^{-1/x^2}}{x^3}+p_m'(x)e^{-1/x^2}}{x^{k_m}}-\frac{k_m p_m(x)e^{-1/x^2}}{x^{k_m+1}}$$
$$=\frac{2p_m(x)e^{-1/x^2}+x^3p_m'(x)e^{-1/x^2}-x^2k_m p_m(x)e^{-1/x^2}}{x^{k_m+3}}$$
$$=\frac{\left[2p_m(x)+x^3p_m'(x)-x^2k_m p_m(x)\right]e^{-1/x^2}}{x^{k_m+3}}$$

The expression $2p_m(x)+x^3p_m'(x)-x^2k_m p_m(x)$ is a polynomial.

Let $p_{m+1}(x)=2p_m(x)+x^3p_m'(x)-x^2k_m p_m(x)$ and $k_{m+1}=k_m+3$.

Then

$$f^{(m+1)}(x)=\frac{\left[2p_m(x)+x^3p_m'(x)-x^2k_m p_m(x)\right]e^{-1/x^2}}{x^{k_m+3}}$$
$$=\frac{p_{m+1}(x)f(x)}{x^{k_{m+1}}}$$

Thus, by induction, $f^{(n)}(x)=\frac{p_n(x)f(x)}{x^{k_n}}$ for all positive values of *n*.

Therefore, *f* has derivatives of all orders that are defined on $\mathbb{R}$.

Answer 102E.

Consider:

$$f(x)=\begin{cases}|x|^x & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$$

(a)

Show that *f* is continuous at 0.

Here, $f(0)=1$.

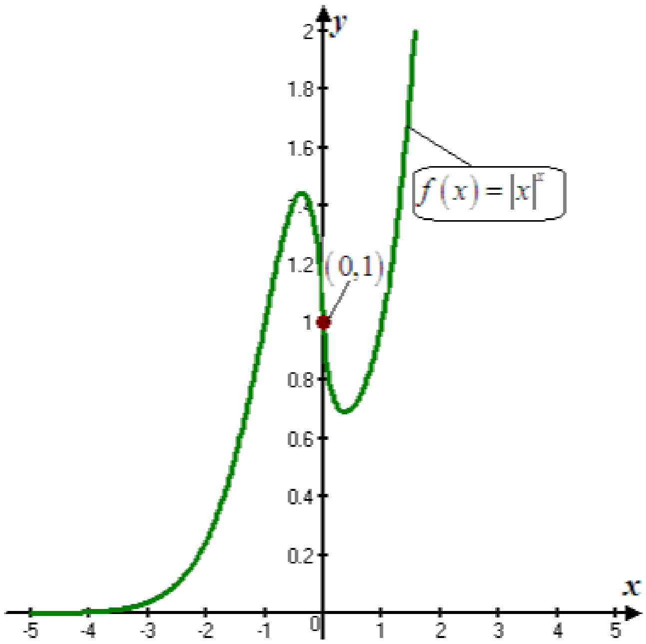
$$\lim_{x\to 0}f(x)=\lim_{x\to 0}|x|^x$$
$$=|0|^0$$
$$=1$$

$$\lim_{x\to 0}f(x)=f(0)$$

Therefore, *f* is continuous at 0.

(b)

The following is the graph of $f(x)=\begin{cases}|x|^x & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$.



From the above graph, $f(x)=\begin{cases}|x|^x & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$ is continuous at 0.

After zooming several times, towards, the point $f(x)=\begin{cases}|x|^x & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$ has vertical inflection point at $(0,1)$.

So $f(x)=\begin{cases}|x|^x & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$ is not differentiable at 0.

(c)

Show that $f(x)=\begin{cases}|x|^x & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$ is not differentiable at 0.

Now,

$$\begin{aligned} f'(0) &= \lim_{x\to 0}\frac{|x|^x-|0|^0}{x-0} \\ &= \frac{0}{0} \\ &= \infty \end{aligned}$$

Therefore, $f(x)=\begin{cases}|x|^x & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$ is not differentiable at 0.

From the graph in part(b), at point $(0,1)$, $f(x)=\begin{cases}|x|^x & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$ has a vertical tangent line.

Since, the vertical line has slope undefined, the $f(x)=\begin{cases}|x|^x & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$ is not differentiable at 0.