Exercise 11.9

Q1E

Let $f(x)=\sum_{n=0}^{\infty} c_n x^n$

Then $f'(x)=\sum_{n=1}^{\infty} n c_n x^{n-1}$

Since the radius of convergence of the series $\sum_{n=0}^{\infty} c_n x^n$ is 10, so it's derivative must have the same radius of convergence 10, therefore the radius of convergence of the series $\sum_{n=1}^{\infty} n c_n x^{n-1}$ is 10 (By the theorem)

Q2E

Suppose that the power series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x|<2$.

Notice that the series $\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$ is the term by term integration of the series $\sum_{n=0}^{\infty} b_n x^n$:

$$\int \sum_{n=0}^{\infty} b_n x^n = \int (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \ldots)$$

$$= \frac{b_0}{1}x + \frac{b_1}{2}x^2 + \frac{b_2}{3}x^3 + \frac{b_3}{4}x^4 + \frac{b_4}{5}x^5 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$$

Recall that the term by term integral of a power series exists and has the same radius of convergence as the original power series.

Since $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x|<2$, the radius of convergence of $\sum_{n=0}^{\infty} b_n x^n$ is at least 2, so that the radius of convergence of $\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$ is the same, $\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$ must have a radius of convergence of at least 2 as well.

Therefore, $\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$ also converges for $|x|<2$.

Q3E

Replace x by (−x) in the equation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1 \quad \ldots\ldots\ldots\ldots\ldots(1)$$

We have

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

And the radius of convergence is same as of (1) that is, $\boxed{R=1}$ and therefore the interval of convergence is $\boxed{(-1,1)}$.

Q4E

Consider the function

$$f(x)=\frac{5}{1-4x^2}.$$

The given function can be written as

$$f(x)=5\left(\frac{1}{1-4x^2}\right). \quad \text{...... (1)}$$

To write the power series representation for the given function, use the following equation:

$$\begin{aligned}\frac{1}{1-x} &= 1+x+x^2+4x^3+\cdots \\ &=\sum_{n=0}^{\infty}x^n,\ \text{for } |x|<1\end{aligned} \quad \text{...... (2)}$$

Substitute $4x^2$ instead of x in the equation (2), get

$$\begin{aligned}\frac{1}{1-4x^2} &= 1+4x^2+\left(4x^2\right)^2+4\left(4x^2\right)^3+\cdots \\ &=\sum_{n=0}^{\infty}\left(4x^2\right)^n,\ \text{for } \left|4x^2\right|<1\end{aligned} \quad \text{...... (3)}$$

Multiply equation (3) by 5.

$$\begin{aligned}\frac{5}{1-4x^2} &= 5+20x^2+5\left(4x^2\right)^2+20\left(4x^2\right)^3+\cdots \\ &=\sum_{n=0}^{\infty}5\left(4x^2\right)^n,\ \text{for } \left|4x^2\right|<1\end{aligned} \quad \text{...... (4)}$$

From equation (1) and (4), get

$$\begin{aligned}f(x)&=\frac{5}{1-4x^2} \\ &= 5+20x^2+5\left(4x^2\right)^2+20\left(4x^2\right)^3+\cdots,\ \text{for } \left|4x^2\right|<1 \\ &= 5+20x^2+80x^4+1280x^6+\cdots,\ \text{for } \left|4x^2\right|<1 \\ &=\sum_{n=0}^{\infty}5\left(4x^2\right)^n,\ \text{for } \left|4x^2\right|<1\end{aligned}$$

$$=\sum_{n=0}^{\infty}5(4)^n x^{2n},\ \text{for } \left|4x^2\right|<1$$

Hence the required power series representation for the given function $f(x)=\frac{5}{1-4x^2}$ is

$$\boxed{f(x)=\sum_{n=0}^{\infty}5(4)^n x^{2n}}.$$

This series converges when $|4x^2| < 1$, or, $|x^2| < \frac{1}{4}$, or $|x| < \frac{1}{2}$.

So the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Now test the interval of convergence at the endpoints.

At the endpoint $x = -\frac{1}{2}$, the series becomes

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} 5(4)^n \left(-\frac{1}{2}\right)^{2n} \\ &= \sum_{n=0}^{\infty} 5(4)^n (-1)^{2n} \left(\frac{1}{2^{2n}}\right) \\ &= \sum_{n=0}^{\infty} 5 \qquad \text{Since } (-1)^{2n} = 1 \text{ for any } n. \end{aligned}$$

Clearly, $\sum_{n=0}^{\infty} 5$ is a divergent series and therefore the given series is divergent at the endpoint $x = -\frac{1}{2}$.

At the endpoint $x = \frac{1}{2}$, the series becomes

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} 5(4)^n \left(\frac{1}{2}\right)^{2n} \\ &= \sum_{n=0}^{\infty} 5(4)^n \left(\frac{1}{2^{2n}}\right) \\ &= \sum_{n=0}^{\infty} 5 \end{aligned}$$

Clearly, $\sum_{n=0}^{\infty} 5$ is a divergent series and therefore the given series is divergent at the endpoint $x = \frac{1}{2}$.

Hence the interval of convergence of the given series $f(x) = \frac{5}{1-4x^2}$ is $\boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$.

Q5E

Consider the function $f(x)=\frac{2}{3-x}$

Recollect that:

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots$$
$$=\sum_{n=0}^{\infty}x^n \quad |x|<1$$

$$\frac{2}{3-x}=2\cdot\left[\frac{1}{3-x}\right]$$
$$=2\cdot\left[\frac{\frac{1}{3}}{1-\frac{x}{3}}\right]$$
$$=\frac{2}{3}\left[\frac{1}{1-\frac{x}{3}}\right]$$

So, $\frac{1}{1-\frac{x}{3}}=\sum_{n=0}^{\infty}\frac{x^n}{3^n}$

$$\frac{2}{3}\cdot\frac{1}{1-\frac{x}{3}}=\frac{2}{3}\sum_{n=0}^{\infty}\frac{x^n}{3^n}$$
$$=\frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n \qquad \text{provided } \left|\frac{x}{3}\right|<1$$
$$=2\sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}} \qquad \text{provided } |x|<3$$

Therefore the radius of convergence is 3.

To find the interval of convergence so the series converges at $x=3$

$$2\sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}} = 2\sum_{n=0}^{\infty}\frac{3^n}{3^{n+1}}$$
$$= \sum_{n=0}^{\infty}\frac{2}{3}$$

Which is diverges by using the divergence test

To find the interval of convergence so the series converges at $x=-3$

$$2\sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}} = 2\sum_{n=0}^{\infty}\frac{(-3)^n}{3^{n+1}}$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{2}{3}$$

Which is diverges by using the divergence test

Therefore the interval of convergence is $(-3,3)$

Q6E

Consider the function $f(x)=\frac{1}{x+10}$

Recollect that:

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots$$
$$= \sum_{n=0}^{\infty}x^n \quad |x|<1$$

$$\frac{1}{x+10} = \frac{1}{10+x}$$
$$= \frac{\frac{1}{10}}{1+\frac{x}{10}}$$
$$= \frac{1}{10}\cdot\frac{1}{1+\frac{x}{10}}$$
$$= \frac{1}{10}\cdot\frac{1}{1-\left(-\frac{x}{10}\right)}$$

So, $\dfrac{1}{1-\left(-\dfrac{x}{10}\right)}=\sum_{n=0}^{\infty}(-1)^n\dfrac{x^n}{10^n}$

$$\frac{1}{10}\cdot\frac{1}{1-\left(-\frac{x}{10}\right)}=\frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{10^n}$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{10^{n+1}}$$

Therefore,

$$\frac{1}{x+10}=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{10^{n+1}}$$

Therefore the above power series converges when $\left|-\dfrac{x}{10}\right|<1$ that implies $\left|\dfrac{x}{10}\right|<1$ so that

$|x|<10$

So the interval of convergence is $\boxed{(-10,10)}$

Q7E

We have $f(x)=\dfrac{x}{9+x^2}$

$$=\frac{x}{9\left(1+x^2/9\right)}$$

$$=\frac{x}{9\left[1-\left(-x^2/9\right)\right]}$$

Now replace x by $\left(-x^2/9\right)$ in the equation

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \qquad ,|x|<1$$

Then $f(x)=\dfrac{x}{9\left[1-\left(-x^2/9\right)\right]}$

$$=\frac{x}{9}\sum_{n=0}^{\infty}\left(-\frac{x^2}{9}\right)^n$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{9^{n+1}}$$

This series converges when $\left|-\dfrac{x^2}{9}\right|<1 \qquad \Rightarrow\left|x^2\right|<9$

$$\Rightarrow|x|<3$$

Thus the radius of convergence is $\boxed{R=3}$ and interval of convergence is $\boxed{(-3,3)}$

Q8E

Consider the function

$$f(x)=\frac{x}{2x^2+1}$$

The objective is to find the interval of convergence of the function.

The power series representation for this function $f(x)=\frac{x}{2x^2+1}$ is:

$$\begin{aligned}
f(x)&=\frac{x}{2x^2+1}\\
&=x\times\left\{\frac{1}{1-(-2x^2)}\right\}\\
&=x\times\left\{\sum_{n=0}^{\infty}(-2x^2)^n\right\} && \left[\text{Provided } |-2x^2|<1\right]\\
&=\sum_{n=0}^{\infty}(-1)^n 2^n x^{2n+1} && \left[\text{Provided } |x|<\frac{1}{\sqrt{2}}\right]
\end{aligned}$$

Check whether the series converges at $x=\pm\frac{1}{\sqrt{2}}$.

At $x=\frac{1}{\sqrt{2}}$, the series becomes

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n 2^n\left(\frac{1}{\sqrt{2}}\right)^{2n+1}&=\sum_{n=0}^{\infty}(-1)^n 2^n\left(\sqrt{2}\right)^{-2n-1}\\
&=\sum_{n=0}^{\infty}(-1)^n 2^n 2^{-2n\frac{1}{2}}2^{-1\frac{1}{2}}\\
&=\sum_{n=0}^{\infty}(-1)^n 2^n 2^{-n}2^{-\frac{1}{2}}\\
&=\sum_{n=0}^{\infty}(-1)^n\left(\sqrt{2}\right)^{-1}
\end{aligned}$$

By the geometric series test the series is diverges.

At $x=-\frac{1}{\sqrt{2}}$, the series is

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n 2^n\left(-\sqrt{2}\right)^{-2n-1}&=\sum_{n=0}^{\infty}(-1)^n 2^n\left(-2^{-2n\cdot\frac{1}{2}}\right)(-2)^{-\frac{1}{2}}\\
&=\sum_{n=0}^{\infty}(-1)^n 2^n\left(-2^{-n}\right)\left(-\sqrt{2}\right)^{-1}\\
&=\sum_{n=0}^{\infty}(-1)^n(-1)\left(-\sqrt{2}\right)^{-1}\\
&=\sum_{n=0}^{\infty}(-1)^n\left(\sqrt{2}\right)^{-1}
\end{aligned}$$

By the geometric series test the series is diverges.

Hence, the interval of convergence is $\boxed{\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)}$.

Q9E

Consider the function $f(x)$.

$$f(x)=\frac{1+x}{1-x}$$

Note that $\frac{1}{1-x}$ can be expanded as follows:

$$\frac{1}{1-x}=1+x+x^2+\cdots+x^n+\cdots=\sum_0^\infty x^n, \qquad |x|<1 \quad \cdots\cdots(1)$$

From equation (1), substitute $\sum_0^\infty x^n$ for $\frac{1}{1-x}$.

$$\begin{aligned} f(x)&=\frac{1+x}{1-x} \\ &=(1+x)\cdot\frac{1}{1-x} \\ &=(1+x)\cdot\sum_0^\infty x^n \\ &=\sum_0^\infty\left(x^n+x^{n+1}\right) \end{aligned}$$

Therefore, the series converges when $|x|<1$, that is, $-1<x<1$.

Thus, the interval of convergence is $(-1,1)$.

Q10E

We have $f(x)=\frac{x^2}{a^3-x^3}$

$$=\frac{x^2}{a^3\left(1-x^3/a^3\right)}=\frac{x^2}{a^3}\frac{1}{1-(x/a)^3}$$

Now change x to $(x/a)^3$ in the equation

$$\frac{1}{1-x}=\sum_{n=0}^\infty x^n \qquad , \quad |x|<1$$

Then $f(x)=\frac{x^2}{a^3}\left[\frac{1}{1-(x/a)^3}\right]$

$$=\frac{x^2}{a^3}\sum_{n=0}^\infty (x/a)^{3n}$$

$$=\sum_{n=0}^\infty \frac{x^{3n+2}}{a^{3n+3}}$$

This series converges when $\left|(x/a)^3\right|<1$

$$\Rightarrow \left|x^3\right|<a^3$$

$$\Rightarrow |x|<|a|$$

Thus the radius of convergence is $\boxed{R=|a|}$

and interval of convergence is $\boxed{(-|a|,\ |a|)}$

Q11E

We have $f(x) = \dfrac{3}{x^2 - x - 2}$

$= \dfrac{3}{x^2 - 2x + x - 2}$

$= \dfrac{3}{(x-2)(x+1)}$

Now $f(x) = \dfrac{A}{(x-2)} + \dfrac{B}{(x+1)} = \dfrac{3}{(x-2)(x+1)}$

$\Rightarrow A(x+1) + B(x-2) = 3$

Putting $x = -1 \Rightarrow B(-3) = 3 \Rightarrow B = -1$

Putting $x = 2 \Rightarrow A(3) = 3 \Rightarrow A = 1$

Thus $\dfrac{3}{x^2 + x - 2} = \dfrac{1}{x-2} - \dfrac{1}{x+1}$

$= \dfrac{1}{2(x/2-1)} - \dfrac{1}{x+1}$

$= -\dfrac{1}{2(1-x/2)} - \dfrac{1}{1-(-x)}$

$= -\dfrac{1}{2}\sum_{n=0}^{\infty}(x/2)^n - \sum_{n=0}^{\infty}(-x)^n$

$= \sum_{n=0}^{\infty}\left[-\dfrac{1}{2}\left(\dfrac{1}{2}\right)^n - (-1)^n\right]x^n$

$\Rightarrow f(x) = \sum_{n=0}^{\infty}\left[(-1)^{n+1} - \dfrac{1}{2^{n+1}}\right]x^n$

This power series is the sum of two geometrics series. The first series converges for $|x| < 1$ and second series converges for $\left|\dfrac{x}{2}\right| < 1 \Rightarrow |x| < 2$.

Then sum of these series converges for $|x| < 1$ so the interval of convergence is $\boxed{(-1,1)}$

Q12E

Given function is $f(x) = \dfrac{x+2}{2x^2 - x - 1}$

It can be written as $\dfrac{x+2}{(2x+1)(x-1)}$

Writing this as partial fractions, we get $f(x) = \dfrac{1}{x-1} - \dfrac{1}{2x+1}$ ...... (1)

We have $\dfrac{1}{x-1} = -\dfrac{1}{1-x} = -\{1 + x + x^2 + \dots + x^n + \dots\}$

$= -\sum_{n=0}^{\infty} x^n, |x| < 1$ ...... (2)

Also, $\dfrac{1}{2x+1}=\dfrac{1}{1-(-2x)}=1+(-2x)+(-2x)^2+...+(-2x)^n+...$

$$=\sum_{n=0}^{\infty}(-2x)^n, |-2x|<1 \text{ ...... (3)}$$

Further, $|-2x|<1 \Rightarrow |x|<\dfrac{1}{2}$ ...... (4)

Using (2), (3) in (1), we get $f(x)=-\sum_{n=0}^{\infty}x^n-\sum_{n=0}^{\infty}(-1)^n(2x)^n$

On simplification, we get $f(x)=-\sum_{n=0}^{\infty}(-1)^n 2^n x^n+x^n$

Or, $f(x)=\sum_{n=0}^{\infty}\left\{(-2)^{n+1}-1\right\}x^n$

To find the radius of convergence, we have take the intersection of the cases (2) and (4).

That is nothing but $|x|<\dfrac{1}{2}$

Thus, the radius of convergence is $\dfrac{1}{2}$ and the interval of convergence is $\left(\dfrac{-1}{2},\dfrac{1}{2}\right)$

Q13E

(A) First Replace x to −x in the equation

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \qquad |x|<1 \qquad ....(1)$$

Then $\dfrac{1}{1+x}=\sum_{n=0}^{\infty}(-x)^n, \qquad |x|<1$

$$=\sum_{n=0}^{\infty}(-1)^n x^n, \qquad |x|<1 \qquad ....(2)$$

Differentiate both sides with respect to x,

$$-\frac{1}{(1+x)^2}=\sum_{n=1}^{\infty}(-1)^n n x^{n-1}$$

So $\dfrac{1}{(1+x)^2}=\sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1}$

If we replace n by n+1, then

$$\frac{1}{(1+x)^2}=\sum_{n=0}^{\infty}(-1)^{n+2}(n+1)x^n$$

$$=\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$$

The radius of convergence is, R = 1.

(B) From part (A), the power series is

$$f(x)=\frac{1}{(1+x)^2}=\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\,, \qquad |x|<1$$

Again differentiate this series with respect to x

$$-2\frac{1}{(1+x)^3}=\sum_{n=0}^{\infty}(-1)^n(n+1)n\,x^{n-1}$$

So
$$\frac{1}{(1+x)^3}=-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)n\,x^{n-1}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}(n+1)n\,x^{n-1}$$

Changing n with n+1, we get

$$\frac{1}{(1+x)^3}=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+2}(n+2)(n+1)x^n$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$$

The radius of convergence is same as the power series of $\frac{1}{1+x}$ *i.e.* $\boxed{R=1.}$

(C) The power series for

$$\frac{1}{(1+x)^3}=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n \qquad |x|<1$$

Then
$$f(x)=\frac{x^2}{(1+x)^3}=x^2\,\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$$

$$=\frac{1}{2}\sum_{n=2}^{\infty}(-1)^n n(n-1)x^n \qquad \text{[Replace n by (n-2)]}$$

The radius of convergence is same as the power series of $\frac{1}{(1+x)^3}$ *i.e.* $\boxed{R=1.}$

Q14E

Consider the function,

$f(x)=\ln(1-x)$.

Use the fact that $\frac{1}{1-x}=1+x+x^2+x^3+x^4+...=\sum_{n=0}^{\infty}x^n$ for $|x|<1$ to find a power series representation for $f(x)=\ln(1-x)$.

Note that $\frac{d}{dx}\ln(1-x)=-\frac{1}{1-x}$.

So, $\ln(1-x)=-\int\frac{1}{1-x}$.

If a function has a power series representation, the power series representation of its integral is the term-by-term integral of its power series.

$$\ln(1-x)=-\int\frac{1}{1-x}$$

$$=-\int 1+x+x^2+x^3+x^4+\cdots \text{ Use } \frac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots$$

$$=-\left(x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5\right)$$

$$=-\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}$$

The radius of convergence of the integral is the same as the radius of convergence of the original power series.

Since $\frac{1}{1-x}=1+x+x^2+x^3+x^4+...=\sum_{n=0}^{\infty}x^n$ is true only when the ratio satisfies $|x|<1$, the radius of convergence is 1.

So, the integral of the power series has the same radius of convergence.

This implies that $\ln(1-x)=-\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}$ also has radius of convergence 1.

Hence, the power series representation of the function with radius of convergence is

$$\ln(1-x)=\boxed{-\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}}, R=\boxed{1}.$$

(b)

Use part (a), to find a power series for the function $f(x)=x\ln(1-x)$.

Since the power series representation of the function $\ln(1-x)$ is

$$\ln(1-x)=-\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}$$

Multiply both sides by x, and then the power series representation of the function is

$$x\ln(1-x)=-x\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}$$

$$=-\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+2}$$

The function $x\ln(1-x)=-x\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}$ converges only when the original series $\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}$ converges, so the radius of convergence is still 1.

Hence, the power series representation of the function with radius of convergence is

$$x\ln(1-x)=\boxed{-\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+2}}, R=\boxed{1}.$$

(c)

Use part (a) to express $\ln 2$ as the sum of an infinite series.

Substitute $x=\frac{1}{2}$ in the series representation of part (a) result, then

$$\ln(1-x)=-\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}, |x|<1$$

$$\ln\left(1-\frac{1}{2}\right)=-\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{1}{2}\right)^{n+1}$$

$$\ln\left(\frac{1}{2}\right)=-\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{1}{2}\right)^{n+1}$$ Use $\ln\left(\frac{1}{2}\right)=\ln(1)-\ln(2)$

$$-\ln(2)=-\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{1}{2}\right)^{n+1}$$

$$\ln(2)=\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{1}{2}\right)^{n+1}$$

Hence, $\ln(2)=\boxed{\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{1}{2}\right)^{n+1}}$.

Q15E

We have $f(x) = \ln(5-x)$

Then $f'(x) = -\frac{1}{5-x}$.

Since $-\frac{1}{5-x} = -\frac{1}{5[1-x/5]} = -\frac{1}{5}\sum_{n=0}^{\infty}(x/5)^n$. $\quad |x/5| < 1$

$$= -\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}, \qquad |x| < 5$$

Integrating both sides, we have

$$-\int \frac{1}{5-x}dx = -\int \sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}dx$$

$$= -\sum_{n=0}^{\infty}\frac{1}{5^{n+1}}\int x^n dx$$

$$\Rightarrow \ln(5-x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}(n+1)} + C$$

Putting $x = 0$ in this power series, we have $\ln(5) = C$

So $\ln(5-x) = -\sum_{n=0}^{\infty}\left(\frac{1}{5^{n+1}}\right)\frac{x^{n+1}}{n+1} + \ln 5$

Or $\ln(5-x) = \ln 5 - \sum_{n=0}^{\infty}\left(\frac{1}{5^{n+1}}\right)\frac{x^{n+1}}{n+1}$

We can write it as

$$\boxed{\ln(5-x) = \ln 5 - \sum_{n=1}^{\infty}\frac{x^n}{n5^n}}$$

The radius of convergence is $\boxed{R = 5}$.

Q16E

Consider the function.

$f(x) = x^2\tan^{-1}(x^3)$.

The objective is to find the power series representation of function, and, the radius of convergence.

Use the following equation to write the power series representation for the given function:

$$\frac{1}{1-x} = 1 + x + x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty}x^n, \text{ for } |x| < 1 \text{ ...... (1)}$$

Replace x with $-x^2$ in equation (1).

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$= 1 + (-x^2) + (-x^2)^2 + 4(-x^2)^3 + \cdots$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n, \text{ for } |-x^2| < 1$$

This series converges for $|-x^2| < 1$, that is, $x^2 < 1$, or $|x| < 1$.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n, \text{ for } |x| < 1 \text{ ...... (2)}$$

Integrate each side of equation (2) with respect to x.

$$\int \frac{1}{1+x^2}dx = \int\left[\sum_{n=0}^{\infty}\left(-x^2\right)^n\right]dx$$

$$\tan^{-1}x = \sum_{n=0}^{\infty}\int(-1)^n x^{2n}dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}+C$$

To find C, put $x=0$ in the equation.

$$C = \tan^{-1}0$$
$$= 0$$

Substitute 0 for C, then

$$\tan^{-1}x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \quad \text{for } |x|<1 \quad \cdots\cdots (3)$$

Substitute x^3 for x in the equation (3).

$$\tan^{-1}\left(x^3\right) = \sum_{n=0}^{\infty}\frac{(-1)^n \left(x^3\right)^{2n+1}}{2n+1} \quad \text{for } \left|x^3\right|<1$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{2n+1} \quad \text{for } \left|x^3\right|<1$$

This series converges for $\left|x^3\right|<1$ or $|x|<1$.

$$\tan^{-1}\left(x^3\right) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{2n+1}, \quad \text{for } |x|<1. \quad \cdots\cdots (4)$$

Multiply equation (4) with x^2 on both sides.

$$x^2\tan^{-1}\left(x^3\right) = x^2\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}\cdot x^2}{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3+2}}{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+5}}{2n+1}, \quad \text{for } |x|<1$$

Hence, the required power series representation for the given indefinite integral $x^2\tan^{-1}\left(x^3\right)$

is $\boxed{x^2\tan^{-1}\left(x^3\right) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+5}}{2n+1}, \quad \text{for } |x|<1}$.

The radius of convergence for this series is $\boxed{R=1}$.

Q17E

Consider the function

$$f(x)=\frac{x}{(1+4x)^2}.$$

The given function can be written as

$$f(x)=(x)\left(\frac{1}{(1+4x)^2}\right) \quad \text{...... (1)}$$

To write the power series representation for the given function, use the following equation:

$$\begin{aligned}\frac{1}{1-x} &= 1+x+x^2+4x^3+\cdots \\ &=\sum_{n=0}^{\infty} x^n,\ \text{for } |x|<1\end{aligned} \quad \text{...... (2)}$$

Differentiating each side of the equation (2), get

$$\begin{aligned}\frac{1}{(1-x)^2} &= 1+2x+3x^2+4x^3+\cdots \\ &=\sum_{n=1}^{\infty} nx^{n-1},\ \text{for } |x|<1\end{aligned} \quad \text{...... (3)}$$

Substitute $-4x$ instead of x in equation (3), get

$$\begin{aligned}\frac{1}{(1+4x)^2} &= \frac{1}{(1-(-4x))^2} \\ &= 1+2(-4x)+3(-4x)^2+4(-4x)^3+\cdots \\ &= 1-8x+48x^2-256x^3+\cdots \\ &=\sum_{n=1}^{\infty} n(-4x)^{n-1}, \qquad \text{for } |-4x|<1 \\ &=\sum_{n=1}^{\infty} n(-4)^{n-1}(x)^{n-1},\ \text{for } |-4x|<1\end{aligned}$$

The convergence of this series is $|-4x|<1$, that is $|4x|<1$ or $|x|<\frac{1}{4}$.

So, get $\frac{1}{(1+4x)^2}=\sum_{n=1}^{\infty} n(-4)^{n-1}(x)^{n-1}$, for $|x|<\frac{1}{4}$. ...... (4)

Multiply equation (4) by x on both sides, get

$$x\left(\frac{1}{(1+4x)^2}\right)=x\sum_{n=1}^{\infty}n(-4)^{n-1}(x)^{n-1}\ ,\ \text{for } |x|<\frac{1}{4}$$

$$=\sum_{n=1}^{\infty}n(-4)^{n-1}(x)^{n-1+1}\ ,\ \text{for } |x|<\frac{1}{4}$$

$$=\sum_{n=1}^{\infty}n(-4)^{n-1}x^{n}\ ,\qquad \text{for } |x|<\frac{1}{4}$$

So, get $x\left(\dfrac{1}{(1+4x)^2}\right)=\displaystyle\sum_{n=1}^{\infty}n(-4)^{n-1}x^{n}$, for $|x|<\dfrac{1}{4}$ ...... (5)

From equation (1) and (5), get

$$f(x)=(x)\left(\frac{1}{(1+4x)^2}\right)$$

$$=\sum_{n=1}^{\infty}n(-4)^{n-1}x^{n}$$

$$=\sum_{n+1=1}^{\infty}(n+1)(-4)^{n+1-1}x^{n+1}$$

$$=\sum_{n=0}^{\infty}(n+1)(-4)^{n}x^{n+1}$$

$$=\sum_{n=0}^{\infty}(n+1)(-1)^{n}(4)^{n}x^{n+1}\ ,\ \text{for } |x|<\frac{1}{4}$$

Hence the required power series representation for the given function $f(x)=\dfrac{x}{(1+4x)^2}$ is

$$\boxed{f(x)=\sum_{n=0}^{\infty}(n+1)(-1)^{n}(4)^{n}x^{n+1}\ ,\ \text{for } |x|<\frac{1}{4}}.$$

And the radius of convergence for this series is $\boxed{R=\dfrac{1}{4}}$.

Q18E

Consider the function

$$f(x)=\left(\frac{x}{2-x}\right)^3.$$

The given function can be written as

$$f(x)=\left(\frac{x}{2-x}\right)^3$$

$$=\frac{x^3}{(2-x)^3}$$

$$=\frac{x^3}{\left(2\left(1-\frac{x}{2}\right)\right)^3}$$

$$=\frac{x^3}{2^3\left(1-\frac{x}{2}\right)^3}$$

$$=\frac{x^3}{8}\left(\frac{1}{\left(1-\frac{x}{2}\right)^3}\right)$$

$$=\frac{x^3}{16}\left(\frac{2}{\left(1-\frac{x}{2}\right)^3}\right)$$

So the function $f(x)$ can be written as $f(x)=\frac{x^3}{16}\left(\frac{2}{\left(1-\frac{x}{2}\right)^3}\right)$ ...... (1)

To write the power series representation for the given function, use the following equation:

$$\frac{1}{1-x} = 1 + x + x^2 + 4x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} x^n, \text{ for } |x| < 1 \qquad \ldots\ldots (2)$$

Differentiating each side of the equation (2), get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$
$$= \sum_{n=1}^{\infty} nx^{n-1}, \text{ for } |x| < 1 \qquad \ldots\ldots (3)$$

Differentiating each side of the equation (3), get

$$\frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots$$
$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \text{ for } |x| < 1 \qquad \ldots\ldots (4)$$

Replace x with $\frac{x}{2}$ in the equation (4).

$$\frac{2}{\left(1-\frac{x}{2}\right)^3} = 2 + 6\left(\frac{x}{2}\right) + 12\left(\frac{x}{2}\right)^2 + \cdots$$
$$= \sum_{n=2}^{\infty} n(n-1)\left(\frac{x}{2}\right)^{n-2}, \text{ for } \left|\frac{x}{2}\right| < 1 \qquad \ldots\ldots (5)$$

Multiply equation (5) by $\frac{x^3}{4}$ on both sides.

$$\frac{x^3}{16}\left(\frac{2}{\left(1-\frac{x}{2}\right)^3}\right) = 2\left(\frac{x^3}{16}\right) + 6\left(\frac{x}{2}\right)\left(\frac{x^3}{16}\right) + 12\left(\frac{x}{2}\right)^2\left(\frac{x^3}{16}\right) + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)\left(\frac{x^3}{16}\right)\left(\frac{x}{2}\right)^{n-2} \qquad \text{for } \left|\frac{x}{2}\right| < 1$$

$$= \sum_{n=2}^{\infty} n(n-1)\left(\frac{x^3}{2^4}\right)\left(\frac{x^{n-2}}{2^{n-2}}\right) \qquad \text{for } |x| < 2$$

Continuation to the above.

$$= \sum_{n=2}^{\infty} n(n-1)\left(\frac{x^{3+n-2}}{2^{4+n-2}}\right) \qquad \text{for } |x| < 2$$

$$= \sum_{n=2}^{\infty} n(n-1)\left(\frac{x^{n+1}}{2^{n+2}}\right) \qquad \text{for } |x| < 2$$

$$= \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n+2}}\left(x^{n+1}\right) \qquad \text{for } |x| < 2$$

So, get $\frac{x^3}{16}\left(\frac{2}{\left(1-\frac{x}{2}\right)^3}\right) = \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n+2}}\left(x^{n+1}\right), \quad \text{for } |x| < 2$ ...... (6)

From equation (1) and (6), get

$$f(x) = \frac{x^3}{16}\left(\frac{2}{\left(1-\frac{x}{2}\right)^3}\right)$$

$$= \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n+2}}\left(x^{n+1}\right), \qquad \text{for } |x| < 2$$

Hence the required power series representation for the given function $f(x) = \left(\frac{x}{2-x}\right)^3$ is

$$\boxed{f(x) = \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n+2}}\left(x^{n+1}\right), \quad \text{for } |x| < 2}.$$

And the radius of convergence for this series is $\boxed{R = 2}$.

Q19E

Consider the following function:

$$f(x)=\frac{1+x}{(1-x)^2}$$

The objective is to find a power series representation for the function and determine the radius of convergence.

Rewrite the given function as,

$$f(x)=(1+x)\left(\frac{1}{(1-x)^2}\right)$$

The power series is,

$$f(x)=\frac{1}{1-x}$$

$$=1+x+x^2+x^3+x^4+\cdots \quad \forall|x|<1$$

$$=\sum_{n=0}^{\infty}x^n \quad \forall|x|<1$$

So, the series is converges when $|x|<1$ and the interval of convergence is $(-1,1)$ and its radius of convergence is $R=1$.

Take the function $f(x)=\frac{1}{(1-x)}$.

Find the derivative of $f(x)=\frac{1}{(1-x)}$

Differentiate the function $f(x)$ with respect to x.

$$f'(x)=\frac{d}{dx}\left(\frac{1}{(1-x)}\right)$$

$$=\frac{d}{dx}\left(\frac{1}{(1-x)}\right)$$

$$=\left(\frac{1}{(1-x)^2}\right)$$

Thus, $f'(x)=\frac{1}{(1-x)^2}$ ......(1)

Now, write the $f'(x)$ in terms of power series.

$$f'(x) = \frac{d}{dx}\frac{1}{(1-x)}$$
$$= \frac{d}{dx}(1-x)^{-1}$$
$$= \frac{d}{dx}\left(1+x+x^2+x^3+x^4+\ldots\right)$$
$$= 1+2x+3x^2+4x^3+\ldots$$
$$= \sum_{n=1}^{\infty} nx^{n-1}$$

$$f'(x) = \sum_{n=1}^{\infty} nx^{n-1} \qquad \ldots\ldots(2)$$

From (1) and (2).

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

Substitute the value of $\frac{1}{(1-x)^2}$ in the given function $f(x)$.

$$f(x) = \frac{1+x}{(1-x)^2}$$
$$= (1+x)\left(\sum_{n=1}^{\infty}\left(nx^{n-1}\right)\right)$$
$$= \sum_{n=1}^{\infty}\left(nx^{n-1}\right) + \sum_{n=1}^{\infty} nx^n$$

Continue the above simplification.

$$\frac{1+x}{(1-x)^2} = \sum_{n=1}^{\infty}\left(nx^{n-1}\right) + \sum_{n=1}^{\infty} nx^n$$
$$= \sum_{n+1=1}^{\infty}(n+1)x^{n+1-1} + \sum_{n=1}^{\infty} nx^n$$
$$= \sum_{n=0}^{\infty}(n+1)x^n + \sum_{n=1}^{\infty} nx^n$$
$$= (0+1)x^0 + \sum_{n=1}^{\infty}(n+1)x^n + \sum_{n=1}^{\infty} nx^n$$
$$= 1 + \sum_{n=1}^{\infty}\left(nx^n + (n+1)x^n\right)$$
$$= 1 + \sum_{n=1}^{\infty}\left((n+n+1)x^n\right)$$

Continue the above simplification.

$$\frac{1+x}{(1-x)^2} = 1 + \sum_{n=1}^{\infty}\left((2n+1)x^n\right)$$
$$= \sum_{n=0}^{\infty}(2n+1)x^n, \qquad \text{for } |x| < 1$$

Hence, the radius of convergence of the differentiated series is same as the radius of convergence of the original series namely $R = 1$.

Therefore, the radius of convergence is $\boxed{R = 1}$.

Q20E

Now write the function in power series representation as,

$$\frac{1}{(1-x)^3} = x^2\frac{1}{(1-x)^3} + x\cdot\frac{1}{(1-x)^3}$$
$$= \frac{1}{2}x^2\sum_{n=2}^{\infty} n(n-1)x^{n-2} + \frac{1}{2}x\sum_{n=2}^{\infty} n(n-1)x^{n-2}$$
$$= \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n} + \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-1}$$

Expand the series, and then the terms in the sequence are

$$\frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n} + \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-1}$$
$$= \frac{1}{2}\begin{pmatrix}\left(2\cdot 1x^2 + 3\cdot 2x^3 + 4\cdot 3x^4 + 5\cdot 4x^5 + \cdots\right) \\ +\left(2\cdot 1x + 3\cdot 2x^2 + 4\cdot 3x^3 + 5\cdot 4x^4 + \cdots\right)\end{pmatrix}$$
$$= \frac{1}{2}\left(2\cdot 1x + (2\cdot 1 + 3\cdot 2)x^2 + (3\cdot 2 + 4\cdot 3)x^3 + (4\cdot 3 + 5\cdot 4)x^4 + \cdots\right)$$
$$= \frac{1}{2}(1(0+2)x + 2(1+3)x^2 + 3(2+4)x^3 + 4(3+5)x^4 + \cdots)$$
$$= \frac{1}{2}\left(1(2)x + 2(4)x^2 + 3(6)x^3 + 4(8)x^4 + \cdots\right)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} n(2n)x^n$$
$$= \sum_{n=1}^{\infty} n^2x^n$$

Hence, the power series representation of the function is

$$f(x) = \frac{x^2+x}{(1-x)^3}$$
$$= \boxed{\sum_{n=1}^{\infty} n^2x^n}.$$

Here, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ is true only when the ratio satisfies $|x| < 1$, so the radius of convergence is 1.

Since the derivative of the power series has the same radius of convergence.

So $\frac{1}{(1-x)^3} = \frac{1}{2}\frac{d}{dx}\frac{d}{dx}\frac{1}{1-x} = \frac{1}{2}\frac{d}{dx}\frac{d}{dx}\sum_{n=0}^{\infty} x^n$ also has radius of convergence 1.

Hence, the radius of convergence of the function is $\boxed{1}$.

Q21E

Consider the function $f(x) = \frac{x}{x^2+16}$.

Rewrite the function as

$$f(x) = \frac{x}{x^2+16}$$

$$= \frac{x}{16\left(1+\frac{x^2}{16}\right)}$$

$$= \frac{x}{16\left(1+\frac{x^2}{16}\right)}$$

$$= \frac{x}{16}\left(\frac{1}{\left(1-\left(-\frac{x^2}{16}\right)\right)}\right)$$

$$= \frac{x}{16}\sum_{n=0}^{\infty}\left(-\frac{x^2}{16}\right)^n$$

$$= \frac{x}{16}\sum_{n=0}^{\infty}\frac{(-1)^n}{16^n}x^{2n}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{16^{n+1}}x^{2n+1}$$

And this series is converges only when $\left|-\frac{x^2}{16}\right| < 1$ or

$$\left|\frac{x^2}{16}\right| < 1$$

$$\left|x^2\right| < 16$$

$$|x| < 4$$

So the radius of the convergence is $R = 4$ centered at 0.

Therefore the required power series of the given function is

$$\boxed{f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{16^{n+1}}x^{2n+1}, R = 4}.$$

Here the *n*th partial sum is

$$s_n(x)=\frac{x}{16}-\frac{x^3}{256}+\frac{x^5}{4096}-\frac{x^7}{65536}+\frac{x^9}{1048576}+\cdots+\frac{(-1)^n}{16^{n+1}}x^{2n+1}$$

For $n=0,\ \ s_0(x)=\dfrac{x}{16}$

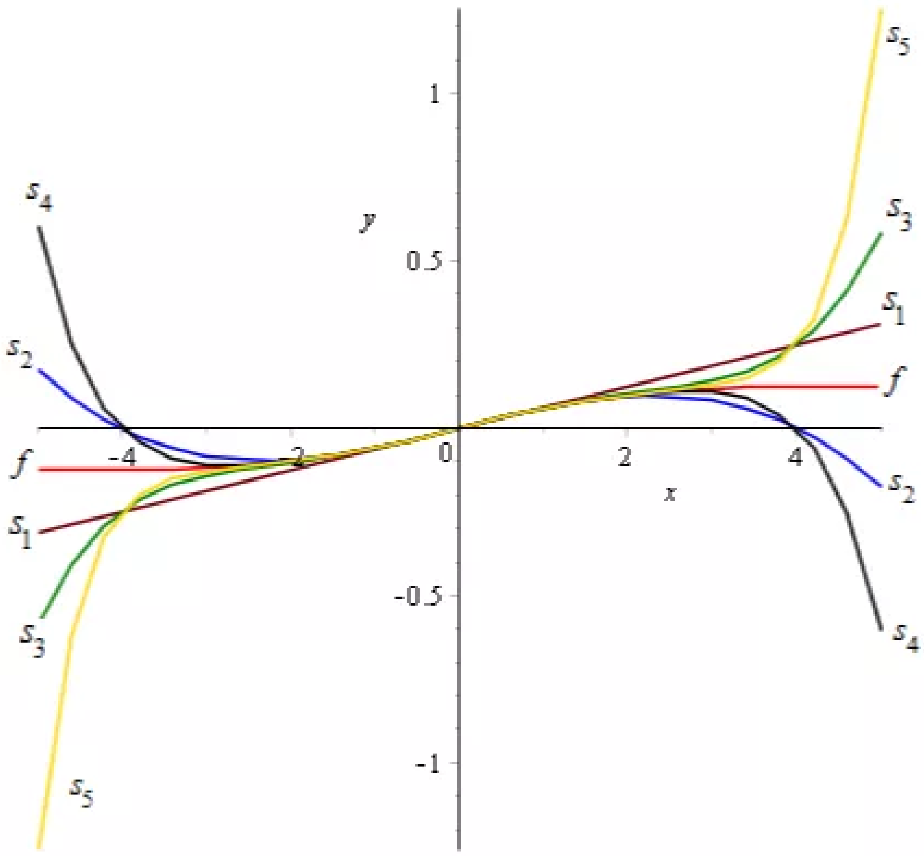
For $n=1,\ \ s_1(x)=\dfrac{x}{16}-\dfrac{x^3}{256}$

For $n=2,\ \ s_2(x)=\dfrac{x}{16}-\dfrac{x^3}{256}+\dfrac{x^5}{4096}$

For $n=3,\ \ s_3(x)=\dfrac{x}{16}-\dfrac{x^3}{256}+\dfrac{x^5}{4096}-\dfrac{x^7}{65536}$

For $n=4,\ \ s_4(x)=\dfrac{x}{16}-\dfrac{x^3}{256}+\dfrac{x^5}{4096}-\dfrac{x^7}{65536}+\dfrac{x^9}{1048576}$

The graph of the function and some partial sums are shown below:



From the graph notice that as n increases, $s_n(x)$ becomes a better approximation to $f(x)$.

Q22E

The sum of the terms up to infnite terms is called a series.

A convergent series is the one in which the sum tends to a limit or an arbitrarily small value, otherwise the series is divergent.

The radius of the largest disk in which the series converges is called as the radius of convergence of the power series.

The radius of convergence is a value that is a non-negative real number.

The function is shown below:

$$f(x)=\ln\left(x^2+4\right)$$

Consider the function:

$$g(x)=\int\frac{2x}{x^2+4}dx$$

The above function is in the form of $\int\frac{h'(x)}{h(x)}dx$, where $h(x)=x^2+4$.

$$\begin{aligned}g(x)&=\ln h(x)\\&=\ln\left(x^2+4\right)\\&=f(x)\end{aligned}$$

Consider the function $f(x)$:

$$\begin{aligned}f(x)&=\int\frac{2x}{x^2+4}dx\\&=\int\frac{2x}{4}\times\frac{1}{1-\left(\frac{-x^2}{4}\right)}dx\end{aligned}\qquad ;\left|\frac{-x^2}{4}\right|<1 \quad \cdots\cdots (1)$$

The expansion of $\frac{1}{1-y}$ is:

$$\begin{aligned}\frac{1}{1-y}&=1+y+y^2+\cdots+y^n+\cdots\\&=\sum_{n=0}^{\infty}y^n, |y|<1\end{aligned}$$

Substitute $y=\frac{-x^2}{4}$ in the equation (1):

$$f(x)=\int\frac{x}{2}\cdot\left\{1+\left(\frac{-x^2}{4}\right)+\left(\frac{-x^2}{4}\right)^2+\left(\frac{-x^2}{4}\right)^3+\ldots\right\}dx$$

$$=\int\frac{x}{2}\left\{\sum_{n=0}^{\infty}\left(\frac{-x^2}{2^2}\right)^n\right\}dx$$

$$=\sum_{n=0}^{\infty}\int(-1)^n\left(\frac{x}{2}\right)^{2n+1}dx$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{1}{\frac{1}{2}}\frac{\left(\frac{x}{2}\right)^{2n+2}}{2n+2}$$

So, the function becomes:

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\left(\frac{x}{2}\right)^{2n+2}$$

Hence, the required power series representation is $\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\left(\frac{x}{2}\right)^{2n+2}}$ and the series converges when $\left|\frac{-x^2}{4}\right|<1$.

$$\left|\frac{-x^2}{4}\right|<1$$

$$-2<x<2$$

Hence, the radius of convergence is 2 and the **interval of convergence** is $(-2,2)$.

As, the value of *n* increases in the expansion of the series, the series becomes convergent.

Further, replace $n=1,2,3,\cdots$ to give the various curves of the function.

$$f(x)=\ln(x+2)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\left(\frac{x}{2}\right)^{2n+2}$$

Determine the partial sum for $n=1$:

$$S_1=\sum_{n=0}^{1}\frac{(-1)^n}{n+1}\left(\frac{x}{2}\right)^{2n+2}$$

$$=\left(\frac{x}{2}\right)^2-\frac{1}{2}\left(\frac{x}{2}\right)^4 \quad \text{...... (2)}$$

$$=\frac{x^2}{4}-\frac{x^4}{32}$$

Determine the partial sum for $n=2$:

$$S_2 = \sum_{n=0}^{2} \frac{(-1)^n}{n+1}\left(\frac{x}{2}\right)^{2n+2}$$

$$= \sum_{n=0}^{2} \frac{(-1)^n}{n+1}\left(\frac{x}{2}\right)^{2n+2} \quad \text{...... (3)}$$

$$= \frac{x^2}{4} - \frac{x^4}{32} + \frac{1}{3}\left(\frac{x}{2}\right)^6$$

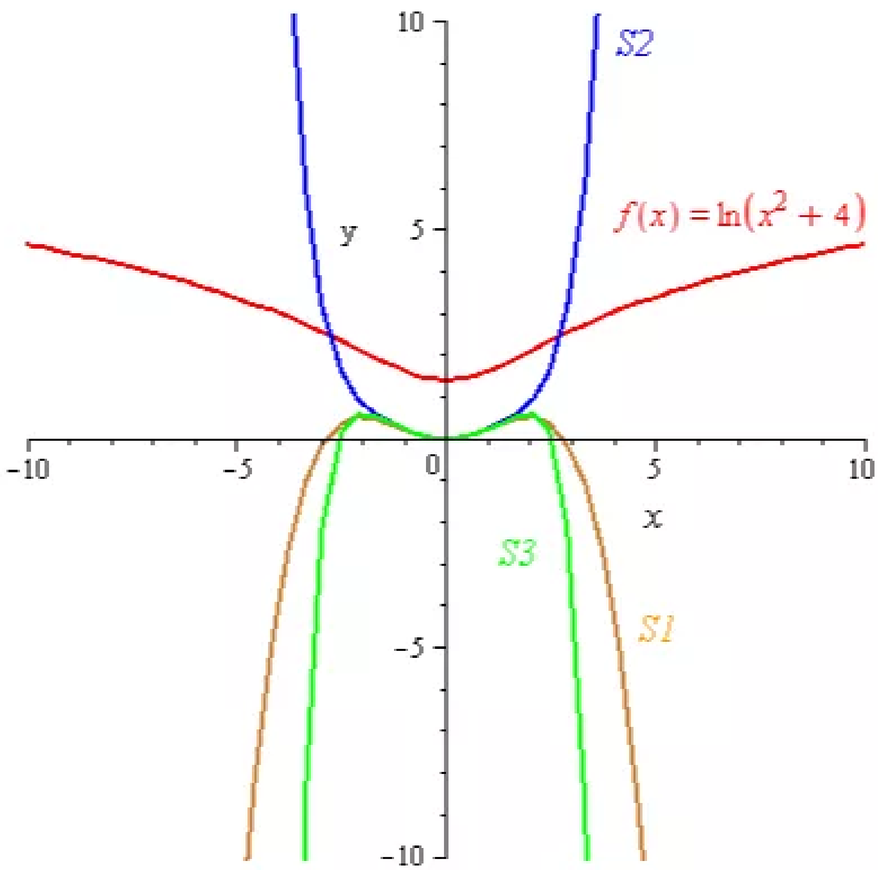
Determine the partial sum for $n=3$:

$$S_3 = \sum_{n=0}^{3} \frac{(-1)^n}{n+1}\left(\frac{x}{2}\right)^{2n+2}$$

$$= \frac{x^2}{4} - \frac{x^4}{32} + \frac{x^6}{192} - \frac{1}{4}\left(\frac{x}{2}\right)^8 \quad \text{...... (4)}$$

$$= \frac{x^2}{4} - \frac{x^4}{32} + \frac{x^6}{192} - \frac{x^8}{1024}$$

Consider the sketch of the function and the partial sums as shown below:



Q23E

We have $f(x) = \ln\left(\frac{1+x}{1-x}\right)$

$$= \ln(1+x) - \ln(1-x)$$

$$= \int \frac{1}{1+x}dx + \int \frac{1}{1-x}dx$$

$$= \int \left[\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n\right] dx$$

$$= \int \left[\left(1-x+x^2+x^4 ....\right) + \left(1+x+x^2+x^3+.......\right)\right]dx$$

$$= \int \left[2+2x^2+2x^4+2x^6 ....\right]dx$$

$$= \int \sum_{n=0}^{\infty} 2x^{2n} dx$$

$$= \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{(2n+1)} + C$$

So $\boxed{\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{(2n+1)}}$ Since $\begin{bmatrix} \text{for}, x=0 \\ f(0)=0 \\ \text{so } C=0 \end{bmatrix}$

The radius of convergence is $\boxed{R=1}$

Terms of series are $a_0 = 2x, a_1 = \frac{2x^3}{3}, a_2 = \frac{2x^5}{5}, a_3 = \frac{2x^7}{7}, a_4 = \frac{2x^9}{9}, a_5 = \frac{2x^{11}}{11}$

Now we sketch the curves $f(x)$ and $s_n(x)$ on the same screen and see that, as n increases, $s_n(x)$ becomes the better approximation.

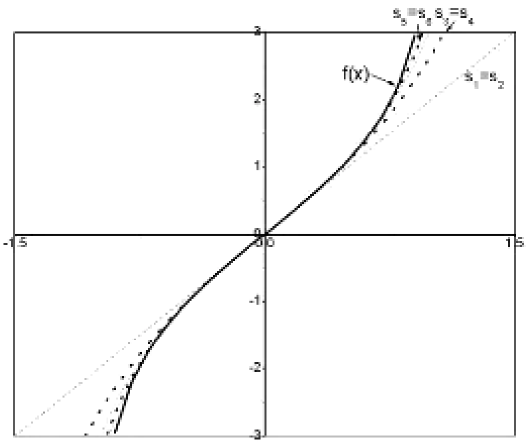


Fig.1

Q24E

We have $f(x) = \tan^{-1}(2x)$

$$\Rightarrow \tan^{-1}(2x) = \int \frac{2}{1+4x^2}dx$$

$$= \int \sum_{n=0}^{\infty} 2\left(-4x^2\right)^n dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n 2.2^{2n} x^{2n} dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} x^{2n} dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{(2n+1)} + C$$

For, $x=0, f(x)=0$ then $\boxed{0=C}$

So $\boxed{f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n+1}}{(2n+1)}}$

This series is converges when $|-4x^2|<1 \Rightarrow |x|<1/2$

So interval of convergence is $[-1/2,\ 1/2]$

Since at the end points of this interval the series converges by Alternating series test.

Terms of the series are

$$a_0=2x, a_1=\frac{-(2x)^3}{3} a_2=\frac{(2x)^5}{5}, a_3=\frac{-(2x)^7}{7}, a_4=\frac{(2x)^9}{9}, \ldots\ldots$$

Now we sketch the graph of f(x) and $s_n(x)$ on the same screen and see that as n increases $s_n(x)$ be the better approximation for f (x).

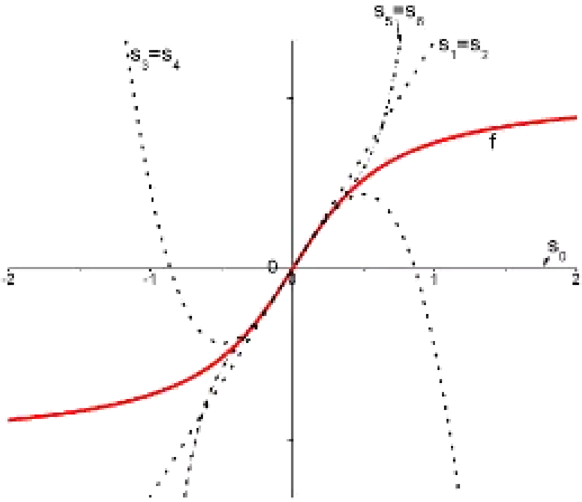


Fig.1

Q25E

Consider the indefinite integral,

$$\int \frac{t}{1-t^8}dt$$

To express this integral as a power series, use $(1-x)^{-1}=\sum_{n=0}^{\infty}x^n$

$$\frac{t}{1-t^8}=t\cdot\frac{1}{1-t^8}$$

$$=t\cdot(1-t^8)^{-1}$$

$$=t\sum_{n=0}^{\infty}(t^8)^n \qquad \text{Replace } x \text{ by } t^8 \text{ in } (1-x)^{-1}=\sum_{n=0}^{\infty}x^n$$

$$=t\sum_{n=0}^{\infty}t^{8n}$$

$$=\sum_{n=0}^{\infty}t^{8n+1}$$

Use $\frac{t}{1-t^8}=\sum_{n=0}^{\infty}t^{8n+1}$ and $\int t^n dt=\frac{t^{n+1}}{n+1}$, we get

$$\begin{aligned}\int\frac{t}{1-t^8}\,dt&=\int\sum_{n=0}^{\infty}t^{8n+1}dt\\&=\int\left[t+t^9+t^{17}+t^{25}+\cdots+t^{8n+1}+\cdots\right]dt\\&=\frac{t^2}{2}+\frac{t^{10}}{10}+\frac{t^{18}}{18}+\frac{t^{26}}{26}+\cdots+\frac{t^{8n+2}}{8n+2}+\cdots+C,\ \text{where } C \text{ is a constant}\\&=\sum_{n=0}^{\infty}\frac{t^{8n+2}}{(8n+2)}+C\end{aligned}$$

Therefore the power series representation of the integral $\int\frac{t}{1-t^8}\,dt$ is,

$$\int\frac{t}{1-t^8}\,dt=\boxed{\sum_{n=0}^{\infty}\frac{t^{8n+2}}{(8n+2)}+C}$$

Radius of convergence of $\int\frac{t}{1-t^8}\,dt$ will be same as that of $\frac{t}{1-t^8}$ or $\sum_{n=0}^{\infty}t^{8n+1}$

Now for the series $\sum_{n=0}^{\infty}t^{8n+1}$

The n^{th} term, $a_n=t^{8n+1}$ and, $(n+1)^{th}$ term, $a_{n+1}=t^{8(n+1)+1}=t^{8n+9}$

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right|&=\left|\frac{t^{8n+9}}{t^{8n+1}}\right|\\&=\left|t^8\right|\end{aligned}$$

And $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|t^8\right|=\left|t^8\right|$

By ratio test, this series will be convergent if $\left|t^8\right|<1\text{(or)}|t|<1$

Therefore, radius of convergence of series $\sum_{n=0}^{\infty}t^{8n+1}$ is 1.

And so radius of convergence of $\int\frac{t}{1-t^8}\,dt$ (or) $\sum_{n=0}^{\infty}\frac{t^{8n+2}}{(8n+2)}+C$ is $\boxed{R=1}$.

Q26E

We know that $\dfrac{1}{1+t^3} = 1 - t^3 + t^6 - t^9 + t^{12} - \ldots$, for $|t^3| < 1$ or $|t| < 1$.

Multiply both the sides by t.

$$\frac{t}{1+t^3} = t - t^4 + t^7 - t^{10} + t^{13} - \ldots$$

Now, integrate both the sides with respect to t.

$$\int \frac{t}{1+t^3}\,dt = \int \left(t - t^4 + t^7 - t^{10} + t^{13} - \ldots\right)dt$$

$$= C + \frac{t^2}{2} - \frac{t^5}{5} + \frac{t^8}{8} - \frac{t^{11}}{11} + \frac{t^{14}}{14} - \ldots$$

Thus, we get $\boxed{\int \frac{t}{1+t^3}\,dt = C + \frac{t^2}{2} - \frac{t^5}{5} + \frac{t^8}{8} - \frac{t^{11}}{11} + \frac{t^{14}}{14} - \ldots}$, for $|t| < 1$ and the radius of convergence is obtained as $\boxed{R = 1}$.

Q27E

Consider the indefinite integral,

$\int x^2 \ln(1+x)\,dx$.

The object is to evaluate the following indefinite integral as a power series.

If a function has a power series, the integral of that function has the same power series as integrating the power series of the function term by term. That is, if we find a power series for $x^2 \ln(1+x)$, we can get a power series for $\int x^2 \ln(1+x)dx$ by integrating it term by term.

First find a power series for $x^2 \ln(1+x)$.

Note that $\dfrac{d}{dx}\ln(1+x) = \dfrac{1}{1+x}$, so $\ln(1+x) = \int \dfrac{1}{1+x}dx$.

Use the integration strategy to find a power series for $\ln(1+x)$.

$$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

$= \sum_{n=0}^{\infty}(-x)^n$ By the geometric series formula

$$= \sum_{n=0}^{\infty}(-1)^n x^n$$

So integrate $\dfrac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$ to find a power series for $\ln(1+x)$.

$$\ln(1+x) = \int \frac{1}{1+x}dx$$

$$= \int \sum_{n=0}^{\infty}(-1)^n x^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} + C$$

Let's find the constant of integration.

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C$$

When $x = 0$, the term $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$ in the right hand side is 0, so the right hand side is just C. The left hand side is $\ln(1+0) = \ln(1) = 0$.

So we have $C = 0$ and thus $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$.

Because $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$, $x^2 \ln(1+x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+3}$.

So we can finally integrate term by term to find $\int x^2 \ln(1+x)dx$:

$$\int x^2 \ln(1+x)dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+3} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+4)} x^{n+3} + C$$

Note that $\frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$ only when $|-x| < 1$, since the geometric series formula $\frac{1}{1-a} = \sum_{n=0}^{\infty} a^n$ only holds when the ratio a satisfies $|a| < 1$. $|-x| < 1$ is the same as $|x| < 1$, so the radius of convergence of $\frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$ is 1.

The same Theorem 2 also tells us that the radius of convergence of the integral of a series has the same radius of convergence as the series.

So our power series expression $\ln(1+x) = \int \sum_{n=0}^{\infty} (-x)^n \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C$ has the same radius of convergence 1.

Then, the power series expression can be written as,

$$x^2 \ln(1+x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+3}$$

Also has the same radius of convergence 1 since it depends on the convergence of

$$\ln(1+x) = \int \sum_{n=0}^{\infty} (-x)^n \ln(1+x)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C$$

So, the integral $\int x^2 \ln(1+x)dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+3} dx$ has the same radius of convergence as

$$x^2 \ln(1+x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+3}$$

This is equal to 1.

Hence, $\int x^2 \ln(1+x)dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+4)} x^{n+4} + C$ with radius of convergence is 1.

Q28E

Consider the indefinite integral

$$\int \frac{\tan^{-1} x}{x} dx.$$

To write the power series representation for the given indefinite integral, use the following equation:

$$\frac{1}{1-x} = 1 + x + x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \text{ for } |x| < 1 \text{ ...... (1)}$$

Substitute $-x^2$ instead of x in the equation (1), get

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$
$$= 1 + (-x^2) + (-x^2)^2 + 4(-x^2)^3 + \cdots$$
$$= \sum_{n=0}^{\infty} (-x^2)^n, \text{ for } |-x^2| < 1$$

So, get $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n, \text{ for } |-x^2| < 1$

This series is converges for $|-x^2| < 1$ that is, $x^2 < 1$, or $|x| < 1$.

So, get $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n, \text{ for } |x| < 1$ ...... (2)

Integrating each side of the equation (2) with respect to x, get

$$\begin{aligned}\tan^{-1}x &= \int \frac{1}{1+x^2}\,dx \\ &= \int\left[\sum_{n=0}^{\infty}\left(-x^2\right)^n\right]dx \\ &= \sum_{n=0}^{\infty}\int(-1)^n x^{2n}\,dx \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}+C\end{aligned}$$

To find C, put $x=0$ and obtain

$$\begin{aligned}C &= \tan^{-1}0 \\ &= 0\end{aligned}$$

Therefore, get $\tan^{-1}x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$ for $|x|<1$ ...... (3)

Divide equation (3) by x, on both sides.

$$\begin{aligned}\frac{\tan^{-1}x}{x} &= \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1-1}}{2n+1} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2n+1}\end{aligned}$$

So, get $\frac{\tan^{-1}x}{x} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2n+1}$, for $|x|<1$ ...... (4)

Integrate equation (4) with respect to x on both sides.

$$\int \frac{\tan^{-1} x}{x} = \int \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1} \right] dx$$

$$= \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n}}{2n+1} dx$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)} + C_1 \right]$$

$$= C_1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2}$$

Hence the required power series representation for the given indefinite integral $\int \frac{\tan^{-1} x}{x}$ is

$$\boxed{\int \frac{\tan^{-1} x}{x} = C_1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2}, \quad \text{for } |x| < 1}.$$

This series converges when $|x| < 1$, and the radius of convergence for this series is same as the radius of convergence of the original series $\frac{1}{1-x}$, namely $\boxed{R = 1}$.

Q29E

We have to approximate $\int_0^{0.2} \frac{1}{1+x^5} dx$

First we find power series representation for $\frac{1}{1+x^5}$

$$\frac{1}{1+x^5} = \frac{1}{1-(-x^5)} = \sum_{n=0}^{\infty} (-x^5)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{5n}$$

Then $\int \frac{1}{1+x^5} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{5n} dx$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+1}}{(5n+1)} + C$$

Therefore

$$\int_0^{0.2} \frac{1}{1+x^5} dx = \left[x - \frac{x^6}{6} + \frac{x^{11}}{11} - \frac{x^{16}}{16} + \ldots \right]_0^{0.2}$$

$$= (0.2) - \frac{(0.2)^6}{6} + \frac{(0.2)^{11}}{11} - \frac{(0.2)^{16}}{16} + \ldots\ldots$$

We want approximation up to six decimal places

$$b_2 = \frac{(0.2)^{11}}{11} \approx 1.8618\times10^{-9}$$

By alternating series estimation theorem we have

$$|s - s_1| \le b_2 \approx 1.8618x10^{-09}$$

This error does not affect the sixth decimal places so we have

$$s \approx 0.2 - \frac{(0.2)^6}{6} \approx 0.199989$$

$$\Rightarrow \boxed{\int_0^{0.2} \frac{1}{1+x^5}\,dx \approx 0.199989}$$

Q30E

We have to approximate $\int_0^{0.4} \ln(1+x^4)\,dx$

First we find power series representation for ln $(1+x^4)$

$$\Rightarrow \ln(1+x^4) = \int \frac{4\,x^3}{(1+x^4)}\,dx$$

$$= 4\int x^3 \sum_{n=0}^{\infty} (-x^4)^n\,dx$$

$$= 4\int \sum_{n=0}^{\infty} (-1)^n\, x^{4n+3}\,dx$$

$$= 4\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+4}}{(4n+4)} + A$$

If we put x = 0 then we get A =0

So we have $\ln(1+x^4) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+4}}{(n+1)}$

Then $\int \ln(1+x^4)\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+4}}{(n+1)}\,dx$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{4n+5}}{(n+1)(4n+5)} + C$$

for $x = 0,\quad C = 0$

So $\int \ln(1+x^4)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{4n+5}}{(n+1)(4n+5)}$

Therefore $\int_0^{0.4} \ln(1+x^4)\,dx = \left[\frac{x^5}{5} - \frac{x^9}{18} + \frac{x^{13}}{39}..........\right]_0^{0.4}$

$$= \left[\frac{(0.4)^5}{5} - \frac{(0.4)^9}{18} + \frac{(0.4)^{13}}{39}.......\right]$$

Since $b_3 = \frac{(0.4)^{17}}{68} \approx 2.5\times10^{-9}$

By alternating series estimation theorem we have

$$|s - s_1| \le b_3 \approx 2.5\times10^{-9}$$

This error does not affect the sixth decimal place. So we have

$$s \approx \frac{(0.4)^5}{5} - \frac{(0.4)^9}{18} + \frac{(0.4)^{13}}{39}$$

$$\Rightarrow s \approx 0.002034$$

Then

$$\boxed{\int_0^{0.4} \ln(1+x^4)\,dx \approx 0.002034}$$

Q31E

Consider the definite integral $\int_0^{0.1} x\arctan(3x)dx$

Let $f(x) = \tan^{-1}(3x)$

Differentiating with respect to x

$$f'(x) = \frac{3}{1+9x^2}$$

The power series of $\frac{3}{1+9x^2}$ is $3\left(1-(9x^2)+(9x^2)^2-(9x^2)^3+\cdots\right)$

$$\tan^{-1}(3x) = \int \frac{3}{1+9x^2}dx$$

$$= 3\int\left(1-(9x^2)+(9x^2)^2-(9x^2)^3+\cdots\right)dx$$

$$= 3\left[x - 9\frac{x^3}{3} + 81\frac{x^5}{5} - 729\frac{x^7}{7} + \cdots\right]$$

Therefore,

$$\int_0^{0.1} x\arctan(3x)dx = 3\int_0^{0.1} x\left[x - 9\frac{x^3}{3} + 81\frac{x^5}{5} - 729\frac{x^7}{7} + \cdots\right]dx$$

$$= 3\int_0^{0.1}\left[x^2 - 9\frac{x^4}{3} + 81\frac{x^6}{5} - 729\frac{x^8}{7} + \cdots\right]dx$$

$$= 3\left[\frac{x^3}{3} - 9\frac{x^5}{3\times5} + 81\frac{x^7}{5\times7} - 729\frac{x^9}{7\times9} + \cdots\right]_0^{0.1}$$

$$= 3\left[\frac{(0.1)^3}{3} - 9\frac{(0.1)^5}{3\times5} + 81\frac{(0.1)^7}{5\times7} - 729\frac{(0.1)^9}{7\times9} + \cdots\right]$$

$$= 3\left[\frac{(0.1)^3}{3} - 9\frac{(0.1)^5}{3\times5} + 81\frac{(0.1)^7}{5\times7} - 729\frac{(0.1)^9}{7\times9} + \cdots + \frac{9^n(-1)^n(0.1)^{(2n+3)}}{(2n+1)(2n+3)}\right]$$

This infinite series is the exact value of the definite integral, but since it is an alternating series.

The above sum using by the alternating series Estimation theorem.

If we stop adding after the term with $n=3$, the error is smaller than the term with $n=4$.

$$6561\frac{(0.1)^{11}}{10\times 11}=5.964546$$

Therefore,

$$\int_0^{0.1} x\arctan(3x)dx = 3\left[\frac{(0.1)^3}{3}-9\frac{(0.1)^5}{3\times 5}+81\frac{(0.1)^7}{5\times 7}-729\frac{(0.1)^9}{7\times 9}\right]$$
$$=\boxed{0.0009823}$$

Q32E

Consider the definite integral:

$$\int_0^{0.3}\frac{x^2}{1+x^4}dx$$

Let

$$f(x)=\frac{x^2}{1+x^4}$$

Determine the power series of the above function:

$$f(x)=\frac{x^2}{1+x^4}$$
$$=x^2\left(1+x^4\right)^{-1}$$

Use binomial theorem;

$$f(x)=x^2\left(1-x^4+x^8-x^{12}+\cdots\right)$$
$$=x^2-x^6+x^{10}-x^{14}+\cdots$$

Now, evaluate the definite integral:

$$\int_0^{0.3}\frac{x^2}{1+x^4}dx=\int_0^{0.3}\left(x^2-x^6+x^{10}-x^{14}+\cdots\right)dx$$
$$=\left[\frac{x^3}{3}-\frac{x^7}{7}+\frac{x^{11}}{11}-\frac{x^{15}}{15}+\cdots\right]_0^{0.3}$$
$$=\left[\frac{(0.3)^3}{3}-\frac{(0.3)^7}{7}+\frac{(0.3)^{11}}{11}-\frac{(0.3)^{15}}{15}+\cdots+\frac{(-1)^{n+1}(0.3)^{2n+3}}{(2n+3)}+\cdots\right]$$

This infinite series is the exact value of the definite integral, but since it is an alternating series.

So, the above sum can be approximated by the alternating series Estimation theorem.

Now, note that for $n=3$, the error is smaller than for $n=4$:

$$\frac{(0.3)^{19}}{19}=6.1171656157$$

Therefore,

$$\int_0^{0.3}\frac{x^2}{1+x^4}dx=\frac{(0.3)^3}{3}-\frac{(0.3)^7}{7}+\frac{(0.3)^{11}}{11}-\frac{(0.3)^{15}}{15}$$
$$=0.008968$$

Hence, the required value of the integral is:

$$\boxed{\int_0^{0.3}\frac{x^2}{1+x^4}dx=0.008968}$$

Q33E

Consider the expression

$\arctan 0.2$.

To compute the value of the given expression correct to five decimal places, use the result given below:

$$\arctan(x)=\tan^{-1}(x)$$
$$=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \qquad \text{...... (1)}$$
$$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1},\quad \text{for } |x|<1$$

This is an alternating series, to find the sum of this series need to use Alternating Series Estimation Theorem.

Alternating Series Estimation Theorem states that

If $s=\sum(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

(i) $b_{n+1}<b_n$ and (ii) $\lim_{n\to\infty}b_n=0$

Then $|R_n|=|s-s_n|\le b_{n+1}$.

Substitute, $x = 0.2$ in the given result (1).

$$\arctan(0.2) = \tan^{-1}(0.2)$$
$$= \sum_{n=0}^{\infty}(-1)^n \frac{(0.2)^{2n+1}}{2n+1}$$
$$= (0.2) - \frac{(0.2)^3}{3} + \frac{(0.2)^5}{5} - \frac{(0.2)^7}{7} + \cdots$$
$$= (0.2) - \frac{0.008}{3} + \frac{0.00032}{5} - \frac{0.0000128}{7} + \cdots$$

Let $s = (0.2) - \frac{(0.2)^3}{3} + \frac{(0.2)^5}{5} - \frac{(0.2)^7}{7} + \cdots$

Notice that $b_3 = \frac{(0.2)^7}{7} = \frac{0.0000128}{7} < \frac{0.0000128}{5} = 0.00000256$

And

$$s_2 = (0.2) - \frac{(0.2)^3}{3} + \frac{(0.2)^5}{5}$$
$$\approx 0.1973973333$$
$$\approx 0.19739$$

By the Alternating Series Estimation Theorem,

$$|s - s_2| \le b_2 < 0.00000256$$

This error of less than 0.00000256 does not affect the fifth decimal place, so the required sum of the series is $s \approx 0.19739$ correct to five decimal places.

Therefore,

$$\arctan 0.2 = s$$
$$\approx 0.19739$$
$$\approx 0.19740$$

Hence the required value of the given expression is $\arctan 0.2 \approx \boxed{0.19740}$.

Q34E

We have $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

If it is a solution of the equation $f''(x) + f(x) = 0$

Then $f(x)$ will satisfy the equation $f''(x) + f(x) = 0$

Again we have $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Then $$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n\, x^{2n-1}}{(2n)!}$$

$$\Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$$

Then $f''(x)=\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)x^{2n-2}}{(2n-1)!}$

$$\Rightarrow f''(x)=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n-2)!}$$

Then $f''(x)+f(x)=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n-2)!}+\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

For making similar Sigma notation, we put n -1 in places of n in second sum

So $f''(x)+f(x)=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n-2)!}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1} x^{2n-2}}{(2n-2)!}$

$$\Rightarrow f''(x)+f(x)=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n-2)!}[1-1]=0$$

$$\Rightarrow f''(x)+f(x)=0$$

Hence f(x) is a solution of $f''(x)+f(x)=0$.

Q35E

(A) Bessel function of order 0 is

$$J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Then $J_0'(x)=\sum_{n=1}^{\infty}\frac{(-1)^n 2n(x)^{2n-1}}{2^{2n}(n!)^2}$

And $J_0''(x)=\sum_{n=1}^{\infty}\frac{(-1)^n(2n)(2n-1)x^{2n-2}}{2^{2n}(n!)^2}$

Then $x^2J_0''(x)+xJ_0'(x)+x^2J_0(x)$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2}+\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2}$$

Replace n by n-1 in third part of the sum

$$\Rightarrow x^2J_0''(x)+xJ_0'(x)+x^2J_0(x)=\sum_{n=1}^{\infty}\frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1} x^{2n}}{2^{2n-2}\left[(n-1)!\right]^2}$$

$$=\sum_{n=1}^{\infty}(-1)^n\left[\frac{2n(2n-1)}{2^{2n}(n!)^2}+\frac{2n}{2^{2n}(n!)^2}-\frac{1}{2^{2n-2}\left[(n-1)!\right]^2}\right]x^{2n}$$

$$=\sum_{n=1}^{\infty}(-1)^n\left[\frac{4n^2-2n+2n-4n^2}{2^{2n}(n!)^2}\right]x^{2n}=0$$

So $\boxed{x^2J_0''(x)+xJ_0'(x)+x^2J_0(x)=0}$

(B) Now $\int J_0(x)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}dx$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \quad x^{2n+1}}{(n!)^2 \; 2^{2n} \; (2n+1)} + c$$

Then $\int_0^1 J_0(x)\,dx = \left[x - \frac{x^3}{3.(4)} + \frac{x^5}{5.(64)} - \frac{x^7}{7(2304)} + ...\right]_0^1$

$$= \left[1 - \frac{1}{12} + \frac{1}{320} - \frac{1}{16128} + ---\right]$$

Since $\frac{1}{16128} \approx 0.000062$, this error does not affect the third decimal place (by the alternating series estimation theorem)

So $\int_0^1 J_0(x)\,dx \approx 1 - \frac{1}{12} + \frac{1}{320}$

$$\Rightarrow \boxed{\int_0^1 J_0(x) \approx 0.920}$$

Q36E

(A) We have Bessel function of order 1, as

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

Then $J_1'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{n! \; (n+1)! \; 2^{2n+1}}$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{n!(n+1)!2^{2n+1}}$$

$$\Rightarrow J_1''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n) x^{2n-1}}{n! \quad (n+1)! \; 2^{2n+1}}$$

Then $x^2 J_1''(x) + x J_1'(x) + (x^2-1) J_1(x)$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n)\, x^{2n+1}}{n!\,(n+1)!\, 2^{2n+1}} + \left(\frac{x}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)\, x^{2n+1}}{n!\,(n+1)!\, 2^{2n+1}} \right)$$

$$+ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!\,(n+1)!\, 2^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\, 2^{2n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n)\, x^{2n+1}}{n!\,(n+1)!\, 2^{2n+1}} + \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)\, x^{2n+1}}{n!\,(n+1)!\, 2^{2n+1}} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n+1}}{(n-1)!\, n!\, 2^{2n-1}}$$

$$- \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\, 2^{2n+1}} + \frac{x}{2} - \frac{x}{2}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,(n-1)!} \left[\frac{(2n+1)(2n)}{n(n+1)2^{2n+1}} + \frac{(2n+1)}{n(n+1)2^{2n+1}} - \frac{1}{2^{2n-1}} - \frac{1}{n(n+1)2^{2n+1}} \right] x^{2n+1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n-1)!} \left[\frac{4n^2+2n+2n+1-n(n+1)4-1}{n(n+1)2^{2n+1}} \right] x^{2n+1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,(n-1)!} \left[\frac{4n^2+4n+1-4n^2-4n-1}{n(n+1)2^{2n+1}} \right] x^{2n+1} = 0$$

So $\boxed{x^2 J_1''(x) + xJ_1'(x) + (x^2-1) J_1(x) = 0}$

(B) We have $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$

$$\Rightarrow J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n\, 2n\, x^{2n-1}}{2^{2n}\, (n!)^2}$$

Putting n + 1 in place of n

$$\Rightarrow J_0'(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\, 2(n+1)\, x^{2n+1}}{2^{2n+2} \left[(n+1)!\right]^2}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n\, 2(n+1)\, x^{2n+1}}{2\, 2^{2n+1}\, (n+1)!\, (n+1)!}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{2^{2n+1}\, n!\,(n+1)!}$$

$$= -J_1(x)$$

$$\Rightarrow \boxed{J_0'(x) = -J_1(x)}$$

Q37E

(A) We have $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$$\Rightarrow f'(x)=\sum_{n=1}^{\infty}\frac{n\ x^{n-1}}{n!}$$

$$\Rightarrow f'(x)=\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}$$

$$=\sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad \left[\text{putting } (n+1) \text{ in place of n}\right]$$

$$=f(x)$$

So $\boxed{f'(x)=f(x)}$

(B) From part (a) $f'(x)=f(x)$

Then $\frac{d}{dx}f(x)=f(x)$

Solution of differential equation of this form is $f(x)=Ae^x$

For $x=0$ $f(0)=1$

Then A=1

And so $\boxed{f(x)=e^x}$

Q38E

For the series $\sum f_n(x)$

n^{th} term $a_n=\frac{\sin nx}{n^2}$

Let us consider an auxiliary series $\sum b_n$ where $b_n=\frac{1}{n^2}$.

Here $|a_n|\le|b_n|$ since $\sin nx\le 1$

And $\sum b_n$ is a convergent p-series as $p=2>1$.

Therefore, by comparison test, the series $\sum\frac{\sin nx}{n^2}$ i. e. $\sum f_n(x)$ is convergent.

$$f_n'(x)=\frac{d}{dx}f_n(x)=\frac{d}{dx}\frac{\sin nx}{n^2}$$

$$=\frac{\cos nx}{n^2}n$$

$$=\frac{\cos nx}{n}$$

When $x=2m\pi$ then $\cos n(2m\pi)=1$.

Therefore, the series $\sum f_n'(x)=\sum\frac{1}{n}$.

Which is a harmonic series. So it is divergent. Thus for $x=2n\pi$, $\sum f_n'(x)$ is divergent, where n is an integer.

Also, $f_n''(x)=\frac{d}{dx}f_n'(x)=\frac{d}{dx}\frac{\cos nx}{n}$

$=-\frac{\sin nx}{n}.n=-\sin nx$

Thus, the series $\sum f_n''(x)$ is $\sum(-\sin nx)$

Or $\sum f_n''(x)=-\sum \sin nx$

If $x\neq k\pi$ (k is an integer), then $\lim_{n\to\infty}\sin nx$ does not tend to a unique finite value and oscillates between -1 to +1. So for $x\neq k\pi$ the series $-\sum \sin nx$ is divergent.

If $x=k\pi$ (k is an integer) then $\lim_{n\to\infty}\sin nx=0$ and also $\lim_{n\to\infty}s_n=0$(finite)

Therefore, the series $-\sum \sin nx$ i.e. $\sum f_n''(x)$ is convergent only when $x=k\pi$ where k is an integer.

Q39E

We have $f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}$

Here $a_n=\frac{x^n}{n^2}$

We use ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}.\frac{n^2}{(n+1)^2}\right|$$

$$=\lim_{n\to\infty}|x|\left(\frac{n}{n+1}\right)^2$$

$$=|x|.\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2$$

$$=|x|.\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^2=|x|$$

By test of convergent series converges when $|x|<1$

The radius of convergence R =1,

For $x=\pm 1$

$\sum_{n=1}^{\infty}\left|\frac{x^n}{n^2}\right|=\sum_{n=1}^{\infty}\frac{1}{n^2}$, this is convergent p-series with $p=2>1$

So interval of convergence is $\boxed{[-1,1]}$

Since radii of convergence of f' *and* f'' are also 1.

So we check the endpoints of the interval.

Since $f'(x)=\sum_{n=1}^{\infty}\frac{n\,x^{n-1}}{n^2}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}=\sum_{n=0}^{\infty}\frac{x^n}{n+1}$

For x = 1

$f'(x)=\sum_{n=0}^{\infty}\frac{1}{(n+1)}$, this is a divergent harmonic series

For x = -1

$$f'(x)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(n+1)}$$, this is an alternating series with $b_n=\frac{1}{1+n}$

Since $b_{n+1}\le b_n$

$$\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{1}{1+n}=\lim_{n\to\infty}\frac{1/n}{1/n+1}=0$$

So for x = -1, $f'(x)$ *is convergent*

Then interval of convergence is $\boxed{[-1,1)}$

Now $$f''(x)=\sum_{n=0}^{\infty}\frac{n\,x^{n-1}}{(n+1)}$$

For x = 1 $$f''(x)=\sum_{n=0}^{\infty}\frac{n}{(n+1)}$$

$$\lim_{n\to\infty}\frac{n}{n+1}=\lim_{n\to\infty}\frac{1}{1+1/n}-1\neq 0$$

So $f''(x)$ diverges for $x=1$, by the test for divergence

For $x=-1$, $$f''(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n-1}n}{n+1}$$

This is an alternating series with $b_n=\frac{n}{n+1}$

But $\lim_{n\to\infty}b_n=1\neq 0$ so $f''(x)$ diverges for $x=-1$

So $f''(x)$ diverges for $x=\pm 1$

Then interval of convergence is $\boxed{(-1,\ 1)}$

Q40E

(A) We have

$$\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\ldots\ldots\ldots.\infty$$

$$=\frac{1}{1-x}\ \text{since}\ |x|<1$$

i.e. $$\sum_{n=0}^{\infty}x^n=\frac{1}{(1-x)}\qquad |x|<1$$

Differentiating both sides with respect to x,

$$\frac{d}{dx}\sum_{n=0}^{\infty}x^n=\frac{d}{dx}\left(\frac{1}{1-x}\right)\qquad |x|<1$$

$$\Rightarrow\quad \sum_{n=1}^{\infty}n\,x^{n-1}=-\frac{1}{(1-x)^2}.(-1)\qquad |x|<1$$

$$\Rightarrow\quad \sum_{n=1}^{\infty}n\,x^{n-1}=\frac{1}{(1-x)^2}\qquad |x|<1$$

(B) (i) From part (a), we have

$$\sum_{n=1}^{\infty} n\,x^{n-1} = \frac{1}{(1-x)^2} \qquad |x|<1$$

Multiplying both sides by x,

$$x\sum_{n=1}^{\infty} n\,x^{n-1} = \frac{x}{(1-x)^2} \qquad ,|x|<1$$

Or $$\sum_{n=1}^{\infty} n\,x\,x^{n-1} = \frac{x}{(1-x)^2} \qquad |x|<1$$

Or $$\sum_{n=1}^{\infty} n\,x^{n} = \frac{x}{(1-x)^2} \qquad |x|<1$$

(ii) We have in part (i)

$$\sum_{n=1}^{\infty} n\,x^{n} = \frac{x}{(1-x)^2} \qquad |x|<1$$

Putting $x = \frac{1}{2}$, we get,

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n} = \frac{1/2}{(1-1/2)^2}$$

$$\Rightarrow \qquad \sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{1/4} = 2$$

$$\Rightarrow \qquad \sum_{n=1}^{\infty} \frac{n}{2^n} = 2$$

Hence,

$$\boxed{\sum_{n=1}^{\infty} \frac{n}{2^n} = 2}$$

(C) From part (b) we have

$$\sum_{n=1}^{\infty} n\,x^{n-1} = \frac{1}{(1-x)^2} \qquad |x|<1$$

Differentiating both sides with respect to x,

$$\frac{d}{dx}\sum_{n=1}^{\infty} n\,x^{n-1} = \frac{d}{dx}\frac{1}{(1-x)^2}$$

$$\Rightarrow \qquad \sum_{n=2}^{\infty} n(n-1)\,x^{n-2} = \frac{-2}{(1-x)^3}.(-1)$$

$$= \frac{2}{(1-x)^3} \qquad |x|<1$$

Multiplying both sides by x^2, we get

$$x^2\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2x^2}{(1-x)^3} \qquad |x|<1$$

$$\Rightarrow \qquad \sum_{n=2}^{\infty} n(n-1)x^2.x^{n-2} = \frac{2x^2}{(1-x)^3} \qquad |x|<1$$

$$\Rightarrow \qquad \sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3} \qquad |x|<1$$

Thus $\boxed{\sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}} \qquad |x|<1$

(ii) From part (i), we have

$$\sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}$$

Putting $x = \frac{1}{2}$, we get,

$$\sum_{n=2}^{\infty} (n^2-n)\left(\frac{1}{2}\right)^n = \frac{2(1/2)^2}{\left(1-\frac{1}{2}\right)^3}$$

$$\Rightarrow \qquad \sum_{n=2}^{\infty} \frac{(n^2-n)}{2^n} = \frac{2(1/2)^2}{(1/2)^3}$$

$$= \frac{2}{(1/2)} = 4$$

Hence,

$$\boxed{\sum_{n=2}^{\infty} \frac{n^2-n}{2^n} = 4}$$

(iii) From part (b) (ii) we have,

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$$

And from part (c), (ii) we have,

$$\sum_{n=2}^{\infty} \frac{n^2 - n}{2^n} = 4$$

Now, $\displaystyle \sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{\left(n^2 - n + n\right)}{2^n}$

$$= \sum_{n=1}^{\infty} \frac{n^2 - n}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n}$$

$$= \frac{1^2 - 1}{2^1} + \sum_{n=2}^{\infty} \frac{n^2 - n}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n}$$

$$= 0 + 4 + 2$$

$$= 6$$

Hence,

$$\boxed{\sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6}$$

Q41E

We have $\displaystyle \tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ --(1)

We know that $\tan^{-1}(\tan x) = x$ for $-\pi/2 < x < \pi/2$

So we put $x = \tan \dfrac{\pi}{6}$ in equation (1)

$$\tan^{-1}\left(\tan(\pi/6)\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\tan(\pi/6)\right)^{2n+1}}{(2n+1)}$$

Since $\tan(\pi/6) = 1/\sqrt{3}$

$$\Rightarrow \frac{\pi}{6} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(1/\sqrt{3}\right)^{2n+1}}{(2n+1)}$$

$$\Rightarrow \frac{\pi}{6} = \sum_{n=0}^{\infty} \frac{(-1)^n \frac{1}{3^n} \cdot \frac{1}{\sqrt{3}}}{(2n+1)}$$

$$\Rightarrow \frac{\pi}{6} = \sum_{n=0}^{\infty} \frac{(-1)^n 1}{(2n+1)\sqrt{3}.3^n}$$

$$\Rightarrow \pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

$$\Rightarrow \boxed{\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}}$$

Q42E

(A) $$\int_0^{1/2} \frac{dx}{x^2 - x + 1} = \int_0^{1/2} \frac{dx}{x^2 - x + \frac{1}{4} - \frac{1}{4} + 1}$$

$$= \int_0^{1/2} \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \int_0^{1/2} \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \left[\frac{1}{\sqrt{3}/2} \tan^{-1} \frac{\left(x - \frac{1}{2}\right)}{\sqrt{3}/2}\right]_0^{1/2} \qquad \left[\text{Since} \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}\right.$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{1}{2} - \frac{1}{2}\right)\frac{2}{\sqrt{3}} - \tan^{-1}\left(0 - \frac{1}{2}\right)\frac{2}{\sqrt{3}}\right]$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1} 0 - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{2}{\sqrt{3}}\left[0 + \tan^{-1} \frac{1}{\sqrt{3}}\right]$$

$$= \frac{2}{\sqrt{3}}\left[\frac{\pi}{6}\right]$$

$$= \frac{\pi}{3\sqrt{3}}$$

Hence

$$\boxed{\int_0^{1/2} \frac{dx}{x^2 - x + 1} = \frac{\pi}{3\sqrt{3}}}$$

(B) We have $x^3 + 1 = (x+1)\left(x^2 - x + 1\right)$

Now,

$$\frac{1}{\left(x^2 - x + 1\right)} = \frac{1+x}{1+x^3} = \frac{(1+x)}{1 - \left(-x^3\right)}$$

$$= (1+x)\sum_{n=0}^{\infty}\left(-x^3\right)^n$$

$$= (1+x)\sum_{n=0}^{\infty}(-1)^n x^{3n}$$

$$= \sum_{n=0}^{\infty}(-1)^n x^{3n} + \sum_{n=0}^{\infty}(-1)^n x^{3n+1} \text{ for } |x| < 1$$

Therefore

$$\int \frac{1}{x^2 - x + 1} dx = \int \left[\sum_{n=0}^{\infty} (-1)^n x^{3n} + \sum_{n=0}^{\infty} (-1)^n x^{3n+1} \right] dx$$

$$= c + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{(3n+2)} \qquad \text{for } |x| < 1$$

And

$$\int_0^{1/2} \frac{1}{x^2 - x + 1} dx = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{3n+1}}{(3n+1)} + \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{3n+2}}{(3n+2)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n+1}(3n+1)} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n+2}(3n+2)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2.2^{3n}(3n+1)} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^2.2^{3n}(3n+2)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{4.2^{3n}} \left[\frac{2}{(3n+1)} + \frac{1}{(3n+2)} \right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{4.8^n} \left[\frac{2}{3n+1} + \frac{1}{3n+2} \right]$$

Now, from part (a) , $\int_0^{1/2} \frac{dx}{x^2 - x + 1} = \frac{\pi}{3\sqrt{3}}$

Therefore, we have,

$$\frac{\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4.8^n} \left[\frac{2}{3n+1} + \frac{1}{3n+2} \right]$$

$$\Rightarrow \qquad \pi = 3\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{4.8^n} \left[\frac{2}{3n+1} + \frac{1}{3n+2} \right]$$

Or

$$\pi = \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \left(\frac{2}{3n+1} + \frac{1}{3n+2} \right)$$