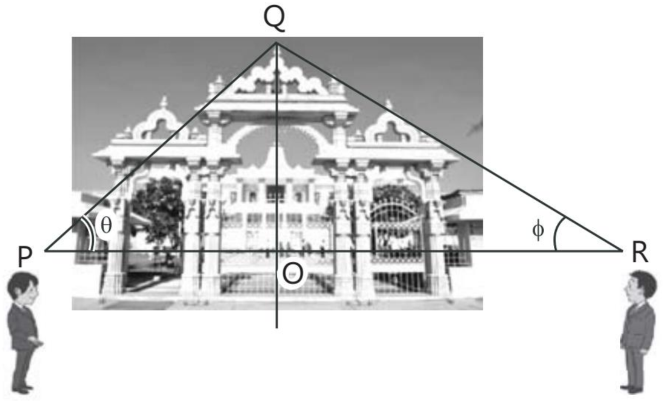
Inverse Trigonometric Functions

Case Study Based Questions

Case Study 1

Two persons on either side of a temple of 60√3 m height observes its top at the angles of elevations and Φ respectively (as shown in the given figure). The distance between the two persons is 240 m and the distance between the first-person P and the temple is 60 m.



Based on the above information, solve the following questions:

Q 1. $\angle$**RPQ** $= \theta =$

a. $\sin^{-1}\left(\frac{2}{\sqrt{3}}\right)$

b. $\sin^{-1}\left(\frac{1}{2}\right)$

c. $\sin^{-1}(2)$

d. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Q 2. $\angle$**RPQ** $= \theta =$

a. $\cos^{-1}\left(\frac{1}{2}\right)$

b. $\cos^{-1}\left(\frac{2}{5}\right)$

c. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

d. $\cos^{-1}\left(\frac{4}{5}\right)$

Q 3. $\angle$**QRP** $= \phi =$

a. $\tan^{-1}\left(\frac{1}{2}\right)$

b. $\tan^{-1}(2)$

c. $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

d. $\tan^{-1}(\sqrt{3})$

Q 4. $\angle$**PQR** $=$

a. $\frac{\pi}{4}$

b. $\frac{\pi}{6}$

c. $\frac{\pi}{2}$

d. $\frac{\pi}{3}$

Q 5. Domain and range of $\tan^{-1} x =$

a. $R, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

b. $R, (0, \pi)$

c. $R-(-1, 1), [0, \pi]-\left\{\frac{\pi}{2}\right\}$

d. $R-(-1, 1), \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$

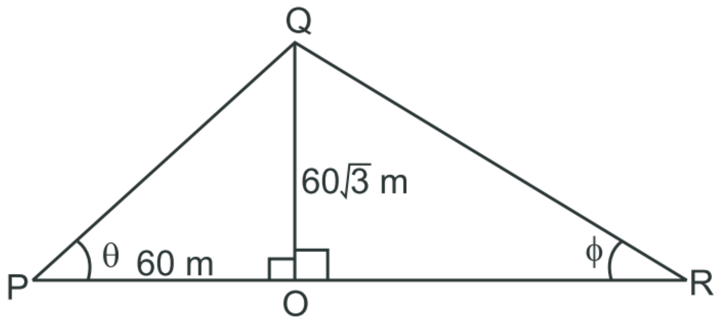
Solutions

1. Given that,

Height of the temple, $OQ = 60\sqrt{3}$ m

Distance between two persons, $PR = 240$ m

and the distance between the first person P and the temple, $OP = 60$ m



Now, in right-angled ΔQOP,

$$\tan \theta = \frac{OQ}{OP} = \frac{60\sqrt{3}}{60}$$

$$\Rightarrow \quad \tan \theta = \sqrt{3} = \tan 60^\circ$$

$$\therefore \quad \theta = 60^\circ$$

So, option (d) is correct.

2. $\cos\theta = \cos 60° \Rightarrow \cos\theta = \frac{1}{2}$ $\quad [\because \theta = 60°]$

$\therefore \quad \theta = \cos^{-1}\left(\frac{1}{2}\right)$

So, option (a) is correct.

3. $\because$ $PR = 240$ m and $OP = 60$ m

$\therefore$ $OR = PR - OP = 240 - 60 = 180$ m

Now, in right-angled ΔQOR,

$$\tan\phi = \frac{OQ}{OR} = \frac{60\sqrt{3}}{180}$$

$$\Rightarrow \quad \tan\phi = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

So, option (c) is correct.

4.

In ΔPQR, $\angle RPQ + \angle QRP + \angle PQR = 180°$

$\therefore \quad \theta + \phi + \angle PQR = 180°$

$\Rightarrow \quad \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \angle PQR = 180°$

$\Rightarrow \sin^{-1}(\sin 60°) + \tan^{-1}(\tan 30°) + \angle PQR = 180°$

$\Rightarrow \quad 60° + 30° + \angle PQR = 180°$

$\Rightarrow \quad \angle PQR = 180° - 90° = 90°$

$\therefore \quad \angle PQR = 90° \times \frac{\pi}{180°} = \frac{\pi}{2}$

So, option (c) is correct.

5. Domain of $\tan^{-1} x$ is R.

and range of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

So, option (a) is correct.

Case Study 2

The value of an inverse trigonometric function which lies in its principal value branch is called the principal value of that inverse trigonometric function.

For example: $\sin^{-1}\left\{\sin\left(\frac{-\pi}{3}\right)\right\} = \frac{-\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Based on the above information, solve the following questions:

Q 1. The value of $\sin^{-1}\left\{\sin\left(\frac{2\pi}{3}\right)\right\}+\cos^{-1}\left\{\cos\left(\frac{4\pi}{3}\right)\right\}$ is:

a. π b. $\frac{\pi}{2}$ c. $\frac{\pi}{3}$ d. $\frac{2\pi}{3}$

Q 2. The value of $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ is:

a. $\frac{-\pi}{3}$ b. $\frac{\pi}{3}$ c. $\frac{\pi}{6}$ d. $\frac{-\pi}{6}$

Q 3. $\sin^{-1}\left\{\sin\left(\frac{4\pi}{5}\right)\right\}$ is equal to:

a. $\frac{4\pi}{5}$ b. $\frac{3\pi}{5}$ c. $\frac{\pi}{5}$ d. $\frac{-\pi}{5}$

Q 4. $\tan^{-1}\left\{\tan\left(\frac{5\pi}{6}\right)\right\}$ is equal to:

a. $\frac{-\pi}{6}$ b. $\frac{5\pi}{6}$ c. $\frac{\pi}{6}$ d. $\frac{7\pi}{6}$

Q 5. The value of $\sin^{-1}\left(\frac{-1}{2}\right)-\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is:

a. $\frac{\pi}{3}$ b. $\frac{2\pi}{3}$ c. $\frac{-\pi}{3}$ d. $\frac{4\pi}{3}$

Solutions

1. $\sin^{-1}\left\{\sin\left(\frac{2\pi}{3}\right)\right\}+\cos^{-1}\left\{\cos\left(\frac{4\pi}{3}\right)\right\}$

$$=\sin^{-1}\left[\sin\left(\pi-\frac{\pi}{3}\right)\right]+\cos^{-1}\left[\cos\left(2\pi-\frac{2\pi}{3}\right)\right]$$

$$=\sin^{-1}\left[\sin\left(\frac{\pi}{3}\right)\right]+\cos^{-1}\left[\cos\left(\frac{2\pi}{3}\right)\right]$$

$$=\frac{\pi}{3}+\frac{2\pi}{3}=\frac{3\pi}{3}=\pi \qquad \left[\because \frac{2\pi}{3}\in[0,\ \pi]\right]$$

So, option (a) is correct.

2. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\sin^{-1}\sin\left(-\frac{\pi}{3}\right)=-\frac{\pi}{3}$

So, option (a) is correct.

3. $\sin^{-1}\left[\sin\left(\frac{4\pi}{5}\right)\right]=\sin^{-1}\left[\sin\left(\pi-\frac{\pi}{5}\right)\right]$

$$=\sin^{-1}\left[\sin\left(\frac{\pi}{5}\right)\right]=\frac{\pi}{5}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

So, option (c) is correct.

4. $\tan^{-1}\left\{\tan\left(\frac{5\pi}{6}\right)\right\}=\tan^{-1}\left\{\tan\left(\pi-\frac{\pi}{6}\right)\right\}=\tan^{-1}\left(-\tan\frac{\pi}{6}\right)$

$$=\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]=-\frac{\pi}{6}\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$$

So, option (a) is correct.

5. $\sin^{-1}\left(-\frac{1}{2}\right)-\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\sin^{-1}\left(-\sin\frac{\pi}{6}\right)-\cos^{-1}\left(\cos\frac{\pi}{6}\right)$

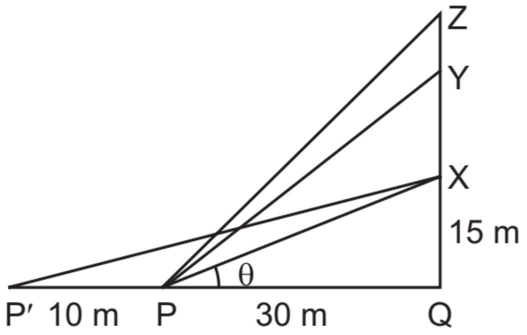
$$=\sin^{-1}\sin\left(-\frac{\pi}{6}\right)-\cos^{-1}\cos\left(\frac{\pi}{6}\right)$$

$$=-\frac{\pi}{6}-\frac{\pi}{6}=-\frac{2\pi}{6}=-\frac{\pi}{3}$$

So, option (c) is correct.

Case Study 3

The Government of India is planning to fix a hoarding at the face of a tower side by the road of a busy place for awareness on COVID-19 protocol. Sanjeev, Rohit and Alok are three engineers who are working on this project. 'P' is considered to be a person viewing the hoarding 30 metres away from the tower, standing at the edge of a pathway nearby. Sanjeev, Rohit and Alok suggested to the firm to place the hoarding at three different locations namely X, Y and Z. 'X' is at the height of 15 metres from the ground level. For the viewer P, the angle of elevation of 'Y' is double the angle of elevation of 'X'. The angle of elevation of 'Z is triple the angle of elevation of 'X' for the same viewer.



Look at the given figure and based on the given information, solve the following questions:

Q 1. Find the measure of ∠XPQ.

Or

Find the measure of ∠XP′ Q

Q 2. Find the measure of ∠YPQ.

Q 3. Find the measure of ∠ZPQ.

Solutions

1. Given that, $XQ = 15$ m and $PQ = 30$ m

Now in right-angled ΔPQX,

$$\tan\theta = \frac{XQ}{PQ} \qquad [\because \text{ say } \angle XPQ = \theta]$$

$$\Rightarrow \qquad \tan\theta = \frac{15}{30} = \frac{1}{2}$$

$$\therefore \qquad \angle XPQ = \theta = \tan^{-1}\left(\frac{1}{2}\right)$$

Or

Given, $XQ = 15$ m, $PQ = 30$ m and $PP' = 10$ m

$\therefore \quad P'Q = PP' + PQ = 10 + 30 = 40$ m

Now in right-angled $\Delta XQP'$,

$$\tan\angle XP'Q = \frac{XQ}{P'Q} = \frac{15}{40} = \frac{3}{8}$$

$$\therefore \qquad \angle XP'Q = \tan^{-1}\left(\frac{3}{8}\right)$$

2. According to the question, $\angle YPQ = 2 \times \angle XPQ = 2\theta$

Now in right-angled ΔYQP, $\tan \angle YPQ = \tan 2\theta$

$$= \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2 \times \frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}$$

$$\left[\because \tan\theta = \frac{1}{2} \text{ and } \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right]$$

$$= \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

$$\therefore \qquad \angle YPQ = \tan^{-1}\left(\frac{4}{3}\right)$$

3. According to the question,

$$\angle ZPQ = 3 \times \angle XPQ = 3\theta$$

Now in right-angled ΔZQP,

$$\tan \angle ZPQ = \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

$$\left[\because \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right]$$

$$= \frac{3 \times \frac{1}{2} - \left(\frac{1}{2}\right)^3}{1-3\left(\frac{1}{2}\right)^2} \qquad \left[\because \tan\theta = \frac{1}{2}\right]$$

$$= \frac{\frac{3}{2} - \frac{1}{8}}{1-3\times\frac{1}{4}} = \frac{12-1}{8} \times \frac{4}{1} = \frac{11}{2}$$

$\therefore \qquad \angle ZPQ = \tan^{-1}\left(\frac{11}{2}\right)$

Case Study 4

The value of an inverse trigonometric function which lies in its principal value branch is called the principal value of that inverse trigonometric function.

When we refer to function $\sin^{-1}$, we take it as the function whose domain is [-1, 1] and range is

$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. The branch with range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called the principal value branch.

Based on the given information, solve the following questions:

Q 1. Find the principal value of $\sin^{-1}\left(\frac{-1}{2}\right)$.

Q 2. Find the principal value of $\cos^{-1}\left(\frac{-1}{2}\right)$.

Q 3. Find the principal value of $\tan^{-1}(-\sqrt{3})$.

Or

Find the principal value of $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$.

Solutions

1. $\sin^{-1}\left(-\frac{1}{2}\right) = \sin^{-1}\left(-\sin\frac{\pi}{6}\right)$

$$= \sin^{-1}\sin\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

2. $\cos^{-1}\left(-\frac{1}{2}\right) = \cos^{-1}\left[-\cos\left(\frac{\pi}{3}\right)\right] = \cos^{-1}\left[\cos\left(\pi - \frac{\pi}{3}\right)\right]$

$$= \cos^{-1}\cos\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} \in [0, \pi]$$

3. $\tan^{-1}(-\sqrt{3}) = \tan^{-1}\left[\tan\left(-\frac{\pi}{3}\right)\right]$

$$= -\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Or

$$\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \cot^{-1}\left[-\cot\left(\frac{\pi}{3}\right)\right] = \cot^{-1}\left[\cot\left(\pi - \frac{\pi}{3}\right)\right]$$

$$= \cot^{-1}\cot\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} \in (0, \ \pi)$$