EXERCISE 7.11

QNo 1 $I = \int_0^{\pi/2} \cos^2 x \, dx$

Sol. $I = \int_0^{\pi/2} \cos^2\left(\frac{\pi}{2} - x\right) dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \int_0^{\pi/2} \sin^2 x \, dx \qquad \left[\because \cos\left(\frac{\pi}{2} - x\right) = \sin x\right]$

on adding we get $2I = \int_0^{\pi/2} (\cos^2 x + \sin^2 x)\, dx.$

$2I = \int_0^{\pi/2} 1 \cdot dx = \left[x\right]_0^{\pi/2} = \left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2}$

$\therefore I = \frac{\pi}{4}.$

QNo 2. $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

Sol. $I = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx. \qquad \left[\because \sin\left(\frac{\pi}{2} - x\right) = \cos x\right]$

On Adding we get $2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_0^{\pi/2} 1 \cdot dx.$

$= \left[x\right]_0^{\pi/2} = \left(\pi/2 - 0\right) = \pi/2$

$\therefore I = \pi/4.$

QNo 3 $I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx.$

Sol.: $I = \int_0^{\pi/2} \frac{\sin^{3/2}\left(\frac{\pi}{2} - x\right)}{\sin^{3/2}\left(\frac{\pi}{2} - x\right) + \cos^{3/2}\left(\pi/2 - x\right)} dx \qquad$ (same as above)

on Adding we get $2I = \int_0^{\pi/2} \frac{\sin^{3/2} x + \cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx.$

$$\therefore \; 2I = \int_0^{\pi/2} 1 \cdot dx = [x]_0^{\pi/2} = (\pi/2 - 0) = \pi/2$$

$$\therefore \; I = \pi/4.$$

QNo 4. $\int_0^{\pi/2} \frac{\cos^5 x \, dx}{\sin^5 x + \cos^5 x} dx$ (Same as QNo. 2 and 3)

QNo 5. $\int_{-5}^{5} |x+2| \, dx.$

For $x+2 \geq 0$ or $x \geq -2$ $|x+2| = x+2$

and $x+2 \leq 0$ or $x \leq -2$ $|x+2| = -(x+2)$

Since $-5 < -2 < 5$

$$\therefore I = \int_{-5}^{5} |x+2| \, dx = \int_{-5}^{-2} |x+2| \, dx + \int_{-2}^{5} |x+2| \, dx \quad \left[\because \text{If } c \in [a,b] \; \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right.$$

$$= \int_{-5}^{-2} -(x+2) \, dx + \int_{-2}^{5} +(x+2) \, dx$$

$$= -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5)\right] + \left[\frac{5^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2)\right]$$

$$= -\left[2 - 4 - \frac{25}{2} + 10\right] + \left[\frac{25}{2} + 10 - 2 + 4\right]$$

$$= -\left[8 - \frac{25}{2}\right] + \left[\frac{25}{2} + 12\right]$$

$$= \frac{-16 + 25 + 25 + 24}{2} = \frac{58}{2} = 29.$$

Q No 6 $I = \int_2^8 |x-5| \, dx$

Sol: When $x-5 \geq 0$ ie $x \geq 5$ $|x-5| = x-5$

When $x-5 \leq 0$ ie $x \leq 5$ $|x-5| = -(x-5)$

$$\therefore I = \int_2^8 |x-5| \, dx = \int_2^5 |x-5| \, dx + \int_5^8 |x-5| \, dx$$

$$= \int_2^5 -(x-5) \, dx + \int_5^8 (x-5) \, dx.$$

$$= -\left[\left(\frac{x^2}{2} - 5x\right)\right]_2^5 + \left[\frac{x^2}{2} - 5x\right]_5^8$$

$$= -\left[\left(\frac{25}{2} - 25\right) - \left(\frac{4}{2} - 10\right)\right] + \left[\left(\frac{64}{2} - 40\right) - \left(\frac{25}{2} - 25\right)\right]$$

$$= -\left[\frac{25}{2} - 25 - 2 + 10\right] + \left[32 - 40 - \frac{25}{2} + 25\right]$$

$$= -\left[\frac{25}{2} - 17\right] + \left[17 - \frac{25}{2}\right]$$

$$= -\frac{25}{2} + 17 + 17 - \frac{25}{2} = 34 - 25 = 9$$

Q No 7 $I = \int_0^1 x(1-x)^n \, dx$

Sol. $I = \int_0^1 (1-x)\left(1-(1-x)\right)^n dx$ $\quad \left| \because \int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx \right.$

$$= \int_0^1 (1-x)(x)^n \, dx. = \int_0^1 \left(x^n - x^{n+1}\right) dx.$$

$$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1 = \left[\frac{1}{n+1} - \frac{1}{n+2}\right] - 0.$$

$$= \frac{n+2-n-1}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}$$

Q No. 8 : $I = \int_0^{\pi/4} \log(1+\tan x)\,dx$

Sol Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. We get

$$I = \int_0^{\pi/4} \log\left(1+\tan\left(\tfrac{\pi}{4}-x\right)\right)dx.$$

$$= \int_0^{\pi/4} \log\left\{1+\frac{1-\tan x}{1+\tan x}\right\}dx \qquad \left[\because \tan\left(\frac{\pi}{4}-x\right) = \frac{1-\tan x}{1+\tan x}\right]$$

$$= \int_0^{\pi/4} \log\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right)dx = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} \left(\log 2 - \log(1+\tan x)\right)dx$$

$$= \int_0^{\pi/4} \log 2\,dx - \int_0^{\pi/4} \log(1+\tan x)\,dx$$

$$= \int_0^{\pi/4} \log 2\,dx - I$$

$$\Rightarrow \quad 2I = \int_0^{\pi/4} \log 2\,dx = \log 2\left[x\right]_0^{\pi/4} = \left(\frac{\pi}{4}-0\right)\log 2$$

$$\therefore \quad I = \frac{\pi}{8}\log 2.$$

Q No. 9 : $I = \int_0^2 x\sqrt{2-x}\,dx.$

Sol.

Using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ We get

$$I = \int_0^2 (2-x)\sqrt{2-(2-x}\;dx = \int_0^2 (2-x)\sqrt{x}\,dx$$

$$= \int_0^2 2x^{1/2}dx - \int_0^2 x^{3/2}dx$$

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^2 - \left[\frac{x^{5/2}}{5/2}\right]_0^2$$

$$= 2\times\frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] - \frac{2}{5}\left[(2)^{5/2} - (0)^{5/2}\right]$$

$$= \frac{4}{3}\times 2\sqrt{2} - \frac{2}{5}\times 4\sqrt{2} = \frac{8}{3}\sqrt{2} - \frac{8}{5}\sqrt{2}$$

$$= \frac{40\sqrt{2} - 24\sqrt{2}}{15} = \frac{16}{15}\sqrt{2}$$

QNo. 10 $\quad I = \int_0^{\pi/2} (2\log\sin x - \log\sin 2x)\, dx.$

Sol $\quad I = \int_0^{\frac{\pi}{2}} \left(2\log\sin x - \log(2\sin x\cos x)\right) dx \quad \left[\because \sin 2x = 2\sin x\cos x\right]$

$$= \int_0^{\pi/2} 2\log\sin x - \int_0^{\frac{\pi}{2}} \log 2\, dx - \int_0^{\pi/2} \log\sin x - \int_0^{\pi/2} \log\cos x\, dx$$

$$= \int_0^{\pi/2} \log\sin x\, dx - \int_0^{\frac{\pi}{2}} \log\cos x - \log 2\int_0^{\pi/2} dx$$

$$= \int_0^{\pi/2} \log\sin x\, dx - \int_0^{\pi/2} \log\cos(\pi/2 - x)\, dx - \log 2\,[x]_0^{\pi/2}$$

$$= \int_0^{\pi/2} \log\sin x\, dx - \int_0^{\pi/2} \log\sin x\, dx - \log 2\,[\pi/2 - 0]$$

$$= -\frac{\pi}{2}\log 2.$$

QNo 11. $\quad I = \int_{-\pi/2}^{\pi/2} \sin^2 x\, dx$

Sol. Now since $\sin^2(-x) = (-\sin x)^2 = \sin^2 x$ i.e $f(-x) = f(x)$

$\therefore \sin^2 x$ is even function

$$\therefore \int_{-\pi/2}^{\pi/2} \sin^2 x\, dx = 2\int_0^{\pi/2} \sin^2 x\, dx \quad \left[\because \text{If } f(x) \text{ is even fn. } \int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx.\right]$$

$$= 2\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 2x}{2}\right)dx. = \left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) - (0+0) = \frac{\pi}{2}$$

Q No 12. $I = \int_0^{\pi} \frac{x\,dx}{1+\sin x} \quad -(i)$

Sol. $I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin(\pi - x)}\,dx \quad (ii) = \int_0^{\pi} \frac{\pi - x}{1+\sin x}\cdot dx \quad -(ii)$

Adding (i) and (ii) We get.

$$2I = \int_0^{\pi} \frac{x+\pi-x}{1+\sin x}\,dx = \int_0^{\pi}\frac{\pi}{1+\sin x}\cdot dx.$$

$$2I = \pi\int_0^{\pi}\frac{1}{1+\sin x}\cdot dx. = 2\pi\int_0^{\pi/2}\frac{1}{1+\sin x}\,dx \quad \left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx. \text{ When } f(2a-x) = f(x)\right.$$

$$I = \frac{2\pi}{2}\int_0^{\pi/2}\frac{1}{1+\sin x}\,dx.$$

Put $\tan\frac{x}{2} = t \Rightarrow x = 2\tan^{-1}(t)$ and $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$

$\Rightarrow \sec^2\frac{x}{2}\cdot\frac{1}{2}\,dx = dt \Rightarrow dx = \frac{2dt}{\sec^2\frac{x}{2}} = \frac{2dt}{1+\tan^2\frac{x}{2}} = \frac{2dt}{1+t^2}$

When $x = 0$, $t = 0$ and when $x = \pi/2$, $t = 1$

$$\therefore I = \pi\int_0^1 \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}} = 2\pi\int_0^1\frac{dt}{(1+t)^2}$$

$$= 2\pi\left[\frac{(1+t)^{-2+1}}{-2+1}\right]_0^1 = 2\pi\left[-\frac{1}{1+t}\right]_0^1$$

$$= 2\pi\left[-\frac{1}{2} - \left(-\frac{1}{1}\right)\right] = 2\pi\left[-\frac{1}{2}+1\right] = 2\pi\times\frac{1}{2} = \pi$$

Q No 13. $I = \int_{-\pi/2}^{\pi/2} \sin^7 x \, dx.$

Sol. Let $f(x) = \sin^7 x = (\sin x)^7$

$\therefore f(-x) = [\sin(-x)]^7 = (-\sin x)^7 = -\sin^7 x = -f(x)$

$\therefore f(x) = \sin^7 x$ is odd function.

$\therefore I = \int_{-\pi/2}^{\pi/2} \sin^7 x \, dx = 0$ $\left[\because \text{If } f(x) \text{ is odd } \int_{-a}^{a} f(x)\,dx = 0\right.$

Q No. 14. $I = \int_0^{2\pi} \cos^5 x \, dx$

Sol. Using $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ when $f(2a-x) = f(x)$

$I = \int_0^{2\pi} \cos^5 x \, dx = 2\int_0^{\pi} \cos^5 x \, dx.$

Now $\because \int_0^{2a} f(x)\,dx = 0$ if $f(2a-x) = -f(x)$

Here $\cos^5(\pi - x) = -\cos^5 x$

$\therefore I = 2 \times 0 = 0$

Q No 15: $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx.$

Sol. Using $\int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx$, we get

$$I = \int_0^{\pi/2} \frac{\sin(\pi/2 - x) - \cos(\pi/2 - x)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}$$

$$= \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx = -\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx.$$

Q No. 16 $I = \int_0^{\pi} \log(1+\cos x)\,dx \quad -(1)$

Sol. - $I = \int_0^{\pi} \log(1+\cos(\pi - x))\,dx$

$= \int_0^{\pi} \log(1-\cos x)\,dx \quad -(11)$

Adding (1) and (2) we get

$$2I = \int_0^{\pi} \left(\log(1+\cos x) + \log(1-\cos x)\right) dx$$

$$= \int_0^{\pi} \log[(1+\cos x)(1-\cos x)]\,dx = \int_0^{\pi} \log(1-\cos^2 x)\,dx$$

$$= \int_0^{\pi} \log\cdot \sin^2 x\,dx = \int_0^{\pi} 2\log(\sin x)\,dx = 2\int_0^{\pi} \log(\sin x)\,dx$$

$$= 2\times 2\int_0^{\pi/2} \log \sin x\,dx \quad \left[\because \log\sin(\pi - x) = \log \sin x.\right.$$

$$= 4\int_0^{\pi/2} \log(\sin x)\,dx$$

Now Let $I_1 = \int_0^{\frac{\pi}{2}} \log(\sin x)\,dx$

$$\therefore I_1 = \int_0^{\frac{\pi}{2}} \log \sin(\pi/2 - x)\,dx = \int_0^{\pi/2} \log.\cos x\,dx.$$

$$\therefore \quad 2I_1 = \int_0^{\pi/2} (\log \sin x + \log \cos x)\,dx.$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\,dx = \int_0^{\pi/2} \log\left(\frac{2\sin x \cos x}{2}\right) dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx = \int_0^{\pi/2} \log \sin 2x - \int_0^{\pi/2} \log 2\,dx.$$

$$\therefore 2I_1 = I_2 - I_3 = I_2 - I_3 \quad (\text{say})$$

Now $I_3 = \int_0^{\pi/2} \log 2\, dx = \log 2 \int_0^{\pi/2} dx = \log 2 \left[x\right]_0^{\pi/2} = \frac{\pi}{2} \log 2.$

and $I_2 = \int_0^{\pi/2} \log \sin 2x\, dx.$

Put $2x = t \Rightarrow 2dx = dt$

When $x = 0,\ t = 0$

$x = \pi/2,\ t = \pi$

$\therefore\ I_2 = \int_0^{\pi} \log \sin t \cdot dt/2 = \frac{1}{2}\int_0^{\pi} \log \sin t\, dt$

$= \frac{1}{2}\int_0^{\pi} \log \sin x \cdot dx \qquad \left[\because \int_a^b f(x)\, dx = \int_a^b f(t)\, dt\right.$

$= \frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin x\, dx \qquad \left[\because \int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx \right.$ when $\left. f(2a-x) = f(x)\right]$

$= \int_0^{\pi/2} \log \sin x\, dx$

$= I_1.$

$\therefore\ 2I_1 = I_2 - I_3$

$\Rightarrow\ 2I_1 = I_1 - I_3$

$\Rightarrow\ 2I_1 - I_1 = -I_3$

$\Rightarrow\ I_1 = -I_3 = -\frac{\pi}{2} \log 2.$

$\therefore\ 2I = 4I_1 = 4\left(-\frac{\pi}{2} \log 2\right)$

$\therefore\ I = 2\left(-\pi/2 \log 2\right)$

$I. = -\pi \log 2$

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QNo 17. $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx \quad -(i)$

Sol. Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{a-(a-x)}}\,dx = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx \quad -(ii)$$

Adding (i) and (ii)

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\,dx = \int_0^a 1.dx = [x]_0^a = a-0=a$$

$$\therefore I = \frac{1}{2}a.$$

QNo. 18: $I = \int_0^4 |x-1|\,dx$

Sol. for $x-1 \le 0$ ie $x \le 1$, $|x-1| = -(x-1)$

and $x-1 \geq 0$ ie $x \geq 1$, $|x-1| = x-1$

$$\therefore \int_0^4 |x-1|\,dx = \int_0^1 |x-1|\,dx + \int_1^4 |x-1|\,dx$$

$$= \int_0^1 -(x-1)\,dx + \int_1^4 (x-1)\,dx$$

$$= -\left[\frac{x^2}{2}-x\right]_0^1 + \left[\frac{x^2}{2}-x\right]_1^4$$

$$= -\left\{\left(\frac{(1)^2}{2}-1\right)-0\right\} + \left\{\left(\frac{(4)^2}{2}-4\right)-\left(\frac{(1)^2}{2}-1\right)\right\}$$

$$= -\left(-\frac{1}{2}\right) + \left(4-\left(-\frac{1}{2}\right)\right)$$

$$= \frac{1}{2}+4+\frac{1}{2}$$

$$= 5$$

Q.No 19 Show that $\int_0^a f(x)g(x)\,dx = 2\int_0^a f(x)\,dx$, If $f(x)$ and $g(x)$ are defined $f(x) = f(a-x)$ and $g(x)+g(a-x) = 4$

Sol.: Let $I = \int_0^a f(x)g(x)\,dx$. — (1)

then $I = \int_0^a f(a-x)g(a-x)\,dx$. $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right.$

or. $I = \int_0^a f(x)[4-g(x)]\,dx$ $\left[\because g(x)+g(a-x) = 4 \Rightarrow g(a-x) = 4-g(x)\right.$

$= \int_0^a [4f(x) - f(x)g(x)]\,dx$. — (2)

Adding (1) and (2)

$2I = \int_0^a 4f(x)\,dx \Rightarrow I = 2\int_0^a f(x)\,dx$.

$\therefore \int_0^a f(x)g(x)\,dx = 2\int_0^a f(x)\,dx$.

Choose the correct answer in Exercises 20 and 21

Q.No 20 The value of $\int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x + 1)\,dx$ is

(A) 0 (B) 2 (C) π (D) 1

Soln. $I = \int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x) + \int_{-\pi/2}^{\pi/2} 1.\,dx$.

Now since $x^3 + x\cos x + \tan^5 x$ is an odd function

$\therefore \int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x)\,dx = 0$

$$\therefore \quad I = \int_{-\pi/2}^{\pi/2} 1\, dx = [x]_{-\pi/2}^{\pi/2} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

So (C) is correct option.

Q.No. 21. The value of $\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ is

(A) 2 (B) $3/4$ (C) 0 (D) -2

Soln:

$$f(x) = \log\left(\frac{4+3\sin x}{4+3\cos x}\right)$$

$$f\left(\frac{\pi}{2}-x\right) = \log\left(\frac{4+3\sin(\pi/2 - x)}{4+3\cos(\pi/2 - x)}\right) = \log\left(\frac{4+3\cos x}{4+3\sin x}\right)$$

$$= \log\left(\frac{4+3\sin x}{4+3\cos x}\right)^{-1} = -\log\left(\frac{4+3\sin x}{4+3\cos x}\right)$$

$$= -f(x)$$

Now

$$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$$

$$= \int_0^a -f(x)\, dx$$

$$= -\int_0^a f(x)\, dx$$

$$\Rightarrow 2\int_0^a f(x)\, dx = 0 \Rightarrow \int_0^a f(x)\, dx = 0$$

(C) is correct option.

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