$$U = \frac{3}{2}k_BT \times N_A = \frac{3}{2}RT \qquad (13.27)$$

The molar specific heat at constant volume, C_v, is

$$C_v \text{ (monatomic gas)} = \frac{dU}{dT} = \frac{3}{2}RT \qquad (13.28)$$

For an ideal gas,

$$C_p - C_v = R \qquad (13.29)$$

where C_p is the molar specific heat at constant pressure. Thus,

$$C_p = \frac{5}{2}R \qquad (13.30)$$

The ratio of specific heats $\gamma = \frac{C_p}{C_v} = \frac{5}{3}$ (13.31)

13.6.2 Diatomic Gases

As explained earlier, a diatomic molecule treated as a rigid rotator like a dumbbell has 5 degrees of freedom : 3 translational and 2 rotational. Using the law of equipartition of energy, the total internal energy of a mole of such a gas is

$$U = \frac{5}{2}k_BT \times N_A = \frac{5}{2}RT \qquad (13.32)$$

The molar specific heats are then given by

$$C_v \text{ (rigid diatomic)} = \frac{5}{2}R,\ C_p = \frac{7}{2}R \qquad (13.33)$$

$$\gamma \text{ (rigid diatomic)} = \frac{7}{5} \qquad (13.34)$$

If the diatomic molecule is not rigid but has in addition a vibrational mode

$$U = \left(\frac{5}{2}k_BT + k_BT\right)N_A = \frac{7}{2}RT$$

$$C_v = \frac{7}{2}R,\ C_p = \frac{9}{2}R,\ \gamma = \frac{9}{7}R \qquad (13.35)$$

13.6.3 Polyatomic Gases

In general a polyatomic molecule has 3 translational, 3 rotational degrees of freedom and a certain number (f) of vibrational modes. According to the law of equipartition of energy, it is easily seen that one mole of such a gas has

$$U = (\frac{3}{2}\ k_BT + \frac{3}{2}\ k_BT + f k_BT)\ N_A$$

i.e. $C_v = (3 + f)\ R$, $C_p = (4 + f)\ R$,

$$\gamma = \frac{(4+f)}{(3+f)} \qquad (13.36)$$

Note that $C_p - C_v = R$ is true for any ideal gas, whether mono, di or polyatomic.

Table 13.1 summarises the theoretical predictions for specific heats of gases ignoring any vibrational modes of motion. The values are in good agreement with experimental values of specific heats of several gases given in Table 13.2. Of course, there are discrepancies between predicted and actual values of specific heats of several other gases (not shown in the table), such as Cl_2, C_2H_6 and many other polyatomic gases. Usually, the experimental values for specific heats of these gases are greater than the predicted values given in Table13.1 suggesting that the agreement can be improved by including vibrational modes of motion in the calculation. The law of equipartition of energy is thus well

Table 13.1 **Predicted values of specific heat capacities of gases (ignoring vibrational modes),**

Nature of Gas	C_v (J mol^{-1} K^{-1})	C_p (J mol^{-1} K^{-1})	$C_p - C_v$ (J mol^{-1} K^{-1})	γ
Monatomic	12.5	20.8	8.31	1.67
Diatomic	20.8	29.1	8.31	1.40
Triatomic	24.93	33.24	8.31	1.33

Table13.2 **Measured values of specific heat capacities of some gases**

Nature of gas	Gas	C_v (J mol^{-1} K^{-1})	C_p (J mol^{-1} K^{-1})	$C_p - C_v$ (J mol^{-1} K^{-1})	γ
Monatomic	He	12.5	20.8	8.30	1.66
Monatomic	Ne	12.7	20.8	8.12	1.64
Monatomic	Ar	12.5	20.8	8.30	1.67
Diatomic	H_2	20.4	28.8	8.45	1.41
Diatomic	O_2	21.0	29.3	8.32	1.40
Diatomic	N_2	20.8	29.1	8.32	1.40
Triatomic	H_2O	27.0	35.4	8.35	1.31
Polyatomic	CH_4	27.1	35.4	8.36	1.31

verified experimentally at ordinary temperatures.

Example 13.8 A cylinder of fixed capacity 44.8 litres contains helium gas at standard temperature and pressure. What is the amount of heat needed to raise the temperature of the gas in the cylinder by 15.0 °C ? (R = 8.31 J mol^{-1} K^{-1}).

Answer Using the gas law $PV = \mu RT$, you can easily show that 1 mol of any (ideal) gas at standard temperature (273 K) and pressure (1 atm = 1.01 × 10^5 Pa) occupies a volume of 22.4 litres. This universal volume is called molar volume. Thus the cylinder in this example contains 2 mol of helium. Further, since helium is monatomic, its predicted (and observed) molar specific heat at constant volume, C_v = (3/2) R, and molar specific heat at constant pressure, C_p = (3/2) $R + R$ = (5/2) R. Since the volume of the cylinder is fixed, the heat required is determined by C_v. Therefore,
Heat required = no. of moles × molar specific heat × rise in temperature

$= 2 \times 1.5\, R \times 15.0 = 45\, R$

$= 45 \times 8.31 = 374$ J.

13.6.4 Specific Heat Capacity of Solids

We can use the law of equipartition of energy to determine specific heats of solids. Consider a solid of N atoms, each vibrating about its mean position. An oscillation in one dimension has average energy of $2 \times ½\, k_BT = k_BT$. In three dimensions, the average energy is 3 k_BT. For a mole of solid, $N = N_A$, and the total energy is

$U = 3\ k_BT \times N_A = 3\ RT$

Now at constant pressure $\Delta Q = \Delta U + P\Delta V = \Delta U$, since for a solid ΔV is negligible. Hence,

$$C = \frac{\Delta Q}{\Delta T} = \frac{\Delta U}{\Delta T} = 3R \qquad (13.37)$$

Table 13.3 **Specific Heat Capacity of some solids at room temperature and atmospheric pressure**

Substance	Specific heat (J kg^{-1} K^{-1})	Molar specific Heat(J mol^{-1} K^{-1})
Aluminium	900.0	24.4
Carbon	506.5	6.1
Copper	386.4	24.5
Lead	127.7	26.5
Silver	236.1	25.5
Tungsten	134.4	24.9

As Table 13.3 shows the prediction generally agrees with experimental values at ordinary temperature (Carbon is an exception).

13.6.5 Specific Heat Capacity of Water

We treat water like a solid. For each atom average energy is $3k_BT$. Water molecule has three atoms, two hydrogen and one oxygen. So it has

$U = 3 \times 3\ k_BT \times N_A = 9\ RT$

and $C = \Delta Q/\Delta T = \Delta U/\Delta T = 9R$.

This is the value observed and the agreement is very good. In the calorie, gram, degree units, water is defined to have unit specific heat. As 1 calorie = 4.179 joules and one mole of water is 18 grams, the heat capacity per mole is ~ 75 J mol^{-1} K^{-1} ~ 9R. However with more complex molecules like alcohol or acetone the arguments, based on degrees of freedom, become more complicated.

Lastly, we should note an important aspect of the predictions of specific heats, based on the classical law of equipartition of energy. The predicted specific heats are independent of temperature. As we go to low temperatures, however, there is a marked departure from this prediction. Specific heats of all substances approach zero as $T \rightarrow 0$. This is related to the fact that degrees of freedom get frozen and ineffective at low temperatures. According to classical physics degrees of freedom must remain unchanged at all times. The behaviour of specific heats at low temperatures shows the inadequacy of classical physics and can be explained only by invoking quantum considerations, as was first shown by Einstein. Quantum mechanics requires a minimum, nonzero amount of energy before a degree of freedom comes into play. This is also the reason why vibrational degrees of freedom come into play only in some cases.

13.7 MEAN FREE PATH

Molecules in a gas have rather large speeds of the order of the speed of sound. Yet a gas leaking from a cylinder in a kitchen takes considerable time to diffuse to the other corners of the room. The top of a cloud of smoke holds together for hours. This happens because molecules in a gas have a finite though small size, so they are bound to undergo collisions. As a result, they cannot

Seeing is Believing

Can one see atoms rushing about. Almost but not quite. One can see pollen grains of a flower being pushed around by molecules of water. The size of the grain is ~ 10^{-5} m. In 1827, a Scottish botanist Robert Brown, while examining, under a microscope, pollen grains of a flower suspended in water noticed that they continuously moved about in a zigzag, random fashion.

Kinetic theory provides a simple explanation of the phenomenon. Any object suspended in water is continuously bombarded from all sides by the water molecules. Since the motion of molecules is random, the number of molecules hitting the object in any direction is about the same as the number hitting in the opposite direction. The small difference between these molecular hits is negligible compared to the total number of hits for an object of ordinary size, and we do not notice any movement of the object.

When the object is sufficiently small but still visible under a microscope, the difference in molecular hits from different directions is not altogether negligible, i.e. the impulses and the torques given to the suspended object through continuous bombardment by the molecules of the medium (water or some other fluid) do not exactly sum to zero. There is a net impulse and torque in this or that direction. The suspended object thus, moves about in a zigzag manner and tumbles about randomly. This motion called now 'Brownian motion' is a visible proof of molecular activity. In the last 50 years or so molecules have been seen by scanning tunneling and other special microscopes.

In 1987 Ahmed Zewail, an Egyptian scientist working in USA was able to observe not only the molecules but also their detailed interactions. He did this by illuminating them with flashes of laser light for very short durations, of the order of tens of femtoseconds and photographing them. (1 femto-second = 10^{-15} s). One could study even the formation and breaking of chemical bonds. That is really seeing !

move straight unhindered; their paths keep getting incessantly deflected.

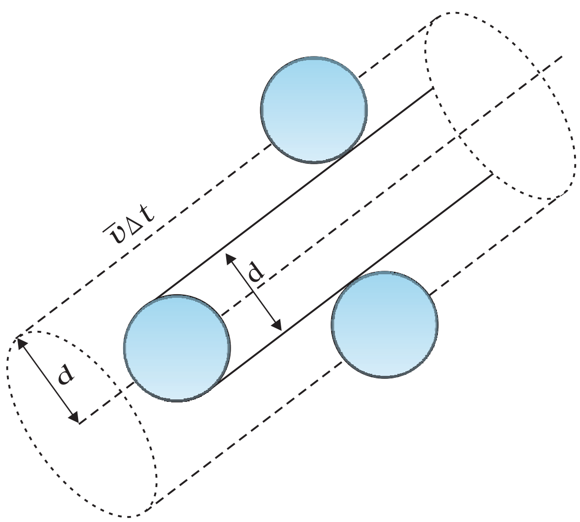


Fig. 13.7 *The volume swept by a molecule in time Δt in which any molecule will collide with it.*

Suppose the molecules of a gas are spheres of diameter d. Focus on a single molecule with the average speed $\langle v\rangle$. It will suffer collision with any molecule that comes within a distance d between the centres. In time Δt, it sweeps a volume $\pi d^2 \langle v\rangle \Delta t$ wherein any other molecule will collide with it (see Fig. 13.7). If n is the number of molecules per unit volume, the molecule suffers $n\pi d^2 \langle v\rangle \Delta t$ collisions in time Δt. Thus the rate of collisions is $n\pi d^2 \langle v\rangle$ or the time between two successive collisions is on the average,

$$\tau = 1/(n\pi \langle v\rangle d^2) \quad (13.38)$$

The average distance between two successive collisions, called the mean free path l, is :

$$l = \langle v\rangle \tau = 1/(n\pi d^2) \quad (13.39)$$

In this derivation, we imagined the other molecules to be at rest. But actually all molecules are moving and the collision rate is determined by the average relative velocity of the molecules. Thus we need to replace $\langle v\rangle$ by $\langle v_r\rangle$ in Eq. (13.38). A more exact treatment gives

$$l = 1/\left(\sqrt{2}\, n\pi d^2\right) \quad (13.40)$$

Let us estimate l and τ for air molecules with average speeds $\langle v\rangle$ = (485m/s). At STP

$$n = \frac{(0.02\times10^{23})}{(22.4\times10^{-3})}$$

$= 2.7 \times 10^{25}$ m^{-3}.

Taking, $d = 2 \times 10^{-10}$ m,

$\tau = 6.1 \times 10^{-10}$ s

and $l = 2.9 \times 10^{-7}$ m $\approx 1500d$ (13.41)

As expected, the mean free path given by Eq. (13.40) depends inversely on the number density and the size of the molecules. In a highly evacuated tube n is rather small and the mean free path can be as large as the length of the tube.

Example 13.9 Estimate the mean free path for a water molecule in water vapour at 373 K. Use information from Exercises 13.1 and Eq. (13.41) above.

Answer The d for water vapour is same as that of air. The number density is inversely proportional to absolute temperature.

So $n = 2.7 \times 10^{25} \times \frac{273}{373} = 2 \times 10^{25}\, m^{-3}$

Hence, mean free path $l = 4 \times 10^{-7}$ m

Note that the mean free path is 100 times the interatomic distance ~ 40 Å = 4×10^{-9} m calculated earlier. It is this large value of mean free path that leads to the typical gaseous behaviour. Gases can not be confined without a container.

Using, the kinetic theory of gases, the bulk measurable properties like viscosity, heat conductivity and diffusion can be related to the microscopic parameters like molecular size. It is through such relations that the molecular sizes were first estimated.

SUMMARY

1. The ideal gas equation connecting pressure (P), volume (V) and absolute temperature (T) is

$$PV = \mu RT = k_B NT$$

where μ is the number of moles and N is the number of molecules. R and k_B are universal constants.

$$R = 8.314 \text{ J mol}^{-1} \text{ K}^{-1}, \quad k_B = \frac{R}{N_A} = 1.38 \times 10^{-23} \text{ J K}^{-1}$$

Real gases satisfy the ideal gas equation only approximately, more so at low pressures and high temperatures.

2. Kinetic theory of an ideal gas gives the relation

$$P = \frac{1}{3} n m \overline{v^2}$$

where n is number density of molecules, m the mass of the molecule and $\overline{v^2}$ is the mean of squared speed. Combined with the ideal gas equation it yields a kinetic interpretation of temperature.

$$\frac{1}{2} m \overline{v^2} = \frac{3}{2} k_B T, \quad v_{rms} = \left(\overline{v^2}\right)^{1/2} = \sqrt{\frac{3k_B T}{m}}$$

This tells us that the temperature of a gas is a measure of the average kinetic energy of a molecule, *independent of the nature of the gas or molecule*. In a mixture of gases at a fixed temperature the heavier molecule has the lower average speed.

3. The translational kinetic energy

$$E = \frac{3}{2} k_B NT.$$

This leads to a relation

$$PV = \frac{2}{3} E$$

4. The law of equipartition of energy states that if a system is in equilibrium at absolute temperature T, the total energy is distributed equally in different energy modes of

absorption, the energy in each mode being equal to ½ k_B T. Each translational and rotational degree of freedom corresponds to one energy mode of absorption and has energy ½ k_B T. Each vibrational frequency has two modes of energy (kinetic and potential) with corresponding energy equal to
2 × ½ k_B T = k_B T.

5. Using the law of equipartition of energy, the molar specific heats of gases can be determined and the values are in agreement with the experimental values of specific heats of several gases. The agreement can be improved by including vibrational modes of motion.

6. The mean free path l is the average distance covered by a molecule between two successive collisions :

$$\bar{\ell} = \frac{1}{\sqrt{2}\, n \pi d^2}$$

where n is the number density and d the diameter of the molecule.

POINTS TO PONDER

1. Pressure of a fluid is not only exerted on the wall. Pressure exists everywhere in a fluid. Any layer of gas inside the volume of a container is in equilibrium because the pressure is the same on both sides of the layer.

2. We should not have an exaggerated idea of the intermolecular distance in a gas. At ordinary pressures and temperatures, this is only 10 times or so the interatomic distance in solids and liquids. What is different is the mean free path which in a gas is 100 times the interatomic distance and 1000 times the size of the molecule.

3. The law of equipartition of energy is stated thus: the energy for each degree of freedom in thermal equilibrium is ½ k_B T. Each quadratic term in the total energy expression of a molecule is to be counted as a degree of freedom. Thus, each vibrational mode gives 2 (not 1) degrees of freedom (kinetic and potential energy modes), corresponding to the energy 2 × ½ k_B T = k_B T.

4. Molecules of air in a room do not all fall and settle on the ground (due to gravity) because of their high speeds and incessant collisions. In equilibrium, there is a very slight increase in density at lower heights (like in the atmosphere). The effect is small since the potential energy (mgh) for ordinary heights is much less than the average kinetic energy ½ mv^2 of the molecules.

5. $< v^2 >$ is not always equal to $(< v >)^2$. The average of a squared quantity is not necessarily the square of the average. Can you find examples for this statement.

EXERCISES

13.1 Estimate the fraction of molecular volume to the actual volume occupied by oxygen gas at STP. Take the diameter of an oxygen molecule to be 3 Å.

13.2 Molar volume is the volume occupied by 1 mol of any (ideal) gas at standard temperature and pressure (STP : 1 atmospheric pressure, 0 °C). Show that it is 22.4 litres.

13.3 Figure 13.8 shows plot of PV/T versus P for 1.00×10^{-3} kg of oxygen gas at two different temperatures.

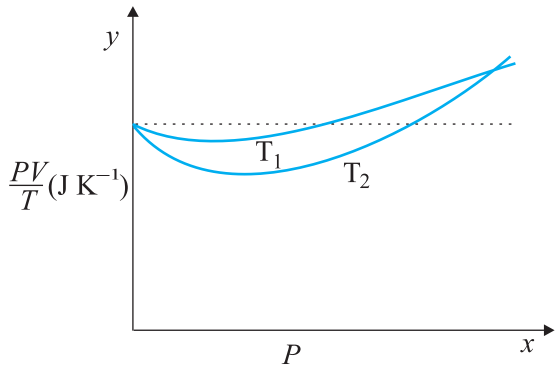


Fig. 13.8

(a) What does the dotted plot signify?
(b) Which is true: $T_1 > T_2$ or $T_1 < T_2$?
(c) What is the value of PV/T where the curves meet on the y-axis?
(d) If we obtained similar plots for 1.00×10^{-3} kg of hydrogen, would we get the same value of PV/T at the point where the curves meet on the y-axis? If not, what mass of hydrogen yields the same value of PV/T (for low pressurehigh temperature region of the plot) ? (Molecular mass of H_2 = 2.02 u, of O_2 = 32.0 u, R = 8.31 J mo1^{-1} K^{-1}.)

13.4 An oxygen cylinder of volume 30 litres has an initial gauge pressure of 15 atm and a temperature of 27 °C. After some oxygen is withdrawn from the cylinder, the gauge pressure drops to 11 atm and its temperature drops to 17 °C. Estimate the mass of oxygen taken out of the cylinder (R = 8.31 J mol^{-1} K^{-1}, molecular mass of O_2 = 32 u).

13.5 An air bubble of volume 1.0 cm^3 rises from the bottom of a lake 40 m deep at a temperature of 12 °C. To what volume does it grow when it reaches the surface, which is at a temperature of 35 °C ?

13.6 Estimate the total number of air molecules (inclusive of oxygen, nitrogen, water vapour and other constituents) in a room of capacity 25.0 m^3 at a temperature of 27 °C and 1 atm pressure.

13.7 Estimate the average thermal energy of a helium atom at (i) room temperature (27 °C), (ii) the temperature on the surface of the Sun (6000 K), (iii) the temperature of 10 million kelvin (the typical core temperature in the case of a star).

13.8 Three vessels of equal capacity have gases at the same temperature and pressure. The first vessel contains neon (monatomic), the second contains chlorine (diatomic), and the third contains uranium hexafluoride (polyatomic). Do the vessels contain equal number of respective molecules ? Is the root mean square speed of molecules the same in the three cases? If not, in which case is v_{rms} the largest ?

13.9 At what temperature is the root mean square speed of an atom in an argon gas cylinder equal to the rms speed of a helium gas atom at – 20 °C ? (atomic mass of Ar = 39.9 u, of He = 4.0 u).

13.10 Estimate the mean free path and collision frequency of a nitrogen molecule in a cylinder containing nitrogen at 2.0 atm and temperature 17 ^{0}C. Take the radius of a nitrogen molecule to be roughly 1.0 Å. Compare the collision time with the time the molecule moves freely between two successive collisions (Molecular mass of N_2 = 28.0 u).

Additional Exercises

13.11 A metre long narrow bore held horizontally (and closed at one end) contains a 76 cm long mercury thread, which traps a 15 cm column of air. What happens if the tube is held vertically with the open end at the bottom ?

13.12 From a certain apparatus, the diffusion rate of hydrogen has an average value of 28.7 $cm^3 s^{-1}$. The diffusion of another gas under the same conditions is measured to have an average rate of 7.2 $cm^3 s^{-1}$. Identify the gas.
[**Hint** : Use Graham's law of diffusion: $R_1/R_2 = (M_2/M_1)^{1/2}$, where R_1, R_2 are diffusion rates of gases 1 and 2, and M_1 and M_2 their respective molecular masses. The law is a simple consequence of kinetic theory.]

13.13 A gas in equilibrium has uniform density and pressure throughout its volume. This is strictly true only if there are no external influences. A gas column under gravity, for example, does not have uniform density (and pressure). As you might expect, its density decreases with height. The precise dependence is given by the so-called law of atmospheres

$$n_2 = n_1 \exp\left[-mg\,(h_2 - h_1)/\,k_B T\right]$$

where n_2, n_1 refer to number density at heights h_2 and h_1 respectively. Use this relation to derive the equation for sedimentation equilibrium of a suspension in a liquid column:

$$n_2 = n_1 \exp\left[-mg\,N_A\,(\rho - P')\,(h_2 - h_1)/\,(\rho\,RT)\right]$$

where ρ is the density of the suspended particle, and ρ' that of surrounding medium. [N_A is Avogadro's number, and R the universal gas constant.] [**Hint** : Use Archimedes principle to find the apparent weight of the suspended particle.]

13.14 Given below are densities of some solids and liquids. Give rough estimates of the size of their atoms :

Substance	Atomic Mass (u)	Density (10^3 Kg m^{-3})
Carbon (diamond)	12.01	2.22
Gold	197.00	19.32
Nitrogen (liquid)	14.01	1.00
Lithium	6.94	0.53
Fluorine (liquid)	19.00	1.14

[**Hint** : Assume the atoms to be 'tightly packed' in a solid or liquid phase, and use the known value of Avogadro's number. You should, however, not take the actual numbers you obtain for various atomic sizes too literally. Because of the crudeness of the tight packing approximation, the results only indicate that atomic sizes are in the range of a few Å].

CHAPTER FOURTEEN

OSCILLATIONS

14.1 INTRODUCTION

In our daily life we come across various kinds of motions. You have already learnt about some of them, e.g. rectilinear motion and motion of a projectile. Both these motions are non-repetitive. We have also learnt about uniform circular motion and orbital motion of planets in the solar system. In these cases, the motion is repeated after a certain interval of time, that is, it is periodic. In your childhood you must have enjoyed rocking in a cradle or swinging on a swing. Both these motions are repetitive in nature but different from the periodic motion of a planet. Here, the object moves to and fro about a mean position. The pendulum of a wall clock executes a similar motion. Examples of such periodic to and fro motion abound : a boat tossing up and down in a river, the piston in a steam engine going back and forth, etc. Such a motion is termed as oscillatory motion. In this chapter we study this motion.

The study of oscillatory motion is basic to physics; its concepts are required for the understanding of many physical phenomena. In musical instruments like the sitar, the guitar or the violin, we come across vibrating strings that produce pleasing sounds. The membranes in drums and diaphragms in telephone and speaker systems vibrate to and fro about their mean positions. The vibrations of air molecules make the propagation of sound possible. In a solid, the atoms vibrate about their equilibrium positions, the average energy of vibrations being proportional to temperature. AC power supply give voltage that oscillates alternately going positive and negative about the mean value (zero).

The description of a periodic motion in general, and oscillatory motion in particular, requires some fundamental concepts like period, frequency, displacement, amplitude and phase. These concepts are developed in the next section.

14.2 PERIODIC AND OSCILLATORY MOTIONS

Fig. 14.1 shows some periodic motions. Suppose an insect climbs up a ramp and falls down it comes back to the initial point and repeats the process identically. If you draw a graph of its height above the ground versus time, it would look something like Fig. 14.1 (a). If a child climbs up a step, comes down, and repeats the process, its height above the ground would look like that in Fig. 14.1 (b). When you play the game of bouncing a ball off the ground, between your palm and the ground, its height versus time graph would look like the one in Fig. 14.1 (c). Note that both the curved parts in Fig. 14.1 (c) are sections of a parabola given by the Newton's equation of motion (see section 3.6),

$$h = ut + \frac{1}{2}gt^2 \text{ for downward motion, and}$$

$$h = ut - \frac{1}{2}gt^2 \text{ for upward motion,}$$

with different values of u in each case. These are examples of periodic motion. Thus, a motion that repeats itself at regular intervals of time is called **periodic motion.**

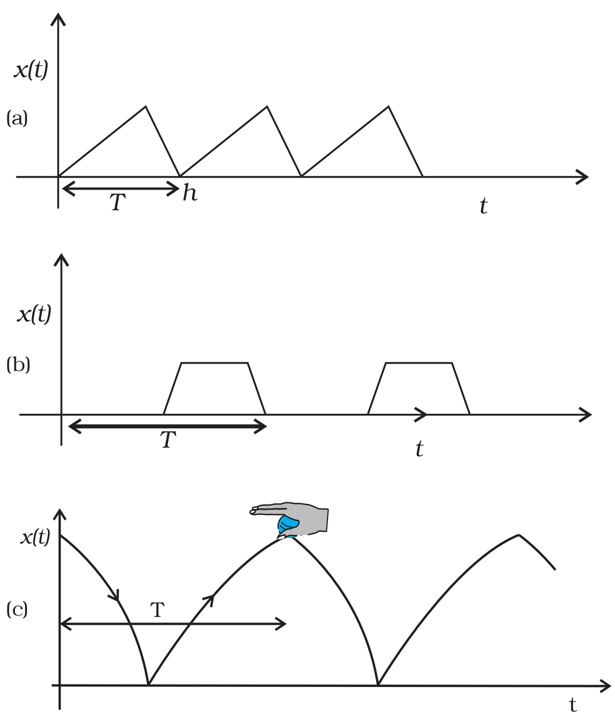


Fig. 14.1 Examples of periodic motion. The period T is shown in each case.

Very often the body undergoing periodic motion has an equilibrium position somewhere inside its path. When the body is at this position no net external force acts on it. Therefore, if it is left there at rest, it remains there forever. If the body is given a small displacement from the position, a force comes into play which tries to bring the body back to the equilibrium point, giving rise to **oscillations** or **vibrations**. For example, a ball placed in a bowl will be in equilibrium at the bottom. If displaced a little from the point, it will perform oscillations in the bowl. Every oscillatory motion is periodic, but every periodic motion need not be oscillatory. Circular motion is a periodic motion, but it is not oscillatory.

There is no significant difference between oscillations and vibrations. It seems that when the frequency is small, we call it oscillation (like the oscillation of a branch of a tree), while when the frequency is high, we call it vibration (like the vibration of a string of a musical instrument).

Simple harmonic motion is the simplest form of oscillatory motion. This motion arises when the force on the oscillating body is directly proportional to its displacement from the mean position, which is also the equilibrium position. Further, at any point in its oscillation, this force is directed towards the mean position.

In practice, oscillating bodies eventually come to rest at their equilibrium positions, because of the damping due to friction and other dissipative causes. However, they can be forced to remain oscillating by means of some external periodic agency. We discuss the phenomena of damped and forced oscillations later in the chapter.

Any material medium can be pictured as a collection of a large number of coupled oscillators. The collective oscillations of the constituents of a medium manifest themselves as waves. Examples of waves include water waves, seismic waves, electromagnetic waves. We shall study the wave phenomenon in the next chapter.

14.2.1 Period and frequency

We have seen that any motion that repeats itself at regular intervals of time is called **periodic motion**. **The smallest interval of time after which the motion is repeated is called its period**. Let us denote the period by the symbol *T*. Its S.I. unit is second. For periodic motions, which are either too fast or too slow on the scale of seconds, other convenient units of time are used. The period of vibrations of a quartz crystal is expressed in units of microseconds (10^{-6} s) abbreviated as μs. On the other hand, the orbital period of the planet Mercury is 88 earth days. The Halley's comet appears after every 76 years.

The reciprocal of T gives the number of repetitions that occur per unit time. This quantity is called the **frequency of the periodic motion**. It is represented by the symbol ν. The relation between ν and T is

$$\nu = 1/T \qquad (14.1)$$

The unit of ν is thus s^{-1}. After the discoverer of radio waves, Heinrich Rudolph Hertz (1857-1894), a special name has been given to the unit of frequency. It is called hertz (abbreviated as Hz). Thus,

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ oscillation per second} = 1 s^{-1} \qquad (14.2)$$

Note, that the frequency, ν, is not necessarily an integer.

Example 14.1 On an average a human heart is found to beat 75 times in a minute. Calculate its frequency and period.

Answer The beat frequency of heart = 75/(1 min)
= 75/(60 s)
= 1.25 s^{-1}
= 1.25 Hz

The time period T = 1/(1.25 s^{-1})
= 0.8 s

14.2.2 Displacement

In section 4.2, we defined displacement of a particle as the change in its position vector. In this chapter, we use the term displacement in a more general sense. It refers to change with time of any physical property under consideration. For example, in case of rectilinear motion of a steel ball on a surface, the distance from the starting point as a function of time is its position displacement. The choice of origin is a matter of convenience. Consider a block attached to a spring, the other end of which is fixed to a rigid wall [see Fig.14.2(a)]. Generally it is convenient to measure displacement of the body from its equilibrium position. For an oscillating simple pendulum, the angle from the vertical as a function of time may be regarded as a displacement variable [see Fig.14.2(b)]. The term displacement is not always to be referred

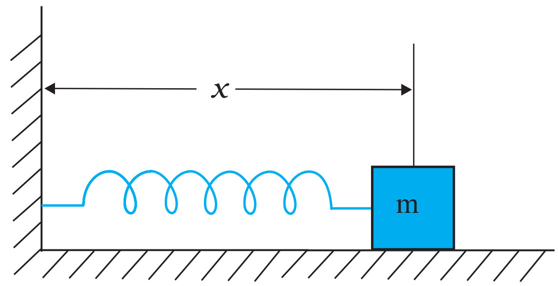


Fig. 14.2(a) A block attached to a spring, the other end of which is fixed to a rigid wall. The block moves on a frictionless surface. The motion of the block can be described in terms of its distance or displacement x from the wall.

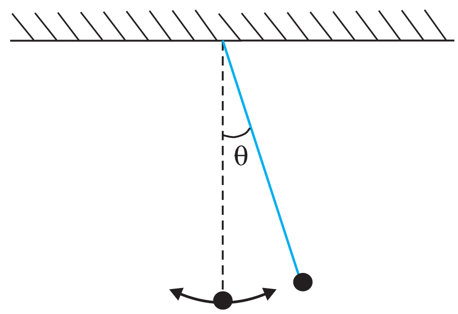


Fig.14.2(b) An oscillating simple pendulum; its motion can be described in terms of angular displacement θ from the vertical.

in the context of position only. There can be many other kinds of displacement variables. The voltage across a capacitor, changing with time in an a.c. circuit, is also a displacement variable. In the same way, pressure variations in time in the propagation of sound wave, the changing electric and magnetic fields in a light wave are examples of displacement in different contexts. The displacement variable may take both positive and negative values. In experiments on oscillations, the displacement is measured for different times.

The displacement can be represented by a mathematical function of time. In case of periodic motion, this function is periodic in time. One of the simplest periodic functions is given by

$$f(t) = A \cos \omega t \qquad (14.3a)$$

If the argument of this function, ωt, is increased by an integral multiple of 2π radians, the value of the function remains the same. The function $f(t)$ is then periodic and its period, T, is given by

$$T = \frac{2\pi}{\omega} \tag{14.3b}$$

Thus, the function $f(t)$ is periodic with period T,

$$f(t) = f(t+T)$$

The same result is obviously correct if we consider a sine function, $f(t) = A \sin \omega t$. Further, a linear combination of sine and cosine functions like,

$$f(t) = A \sin \omega t + B \cos \omega t \tag{14.3c}$$

is also a periodic function with the same period T. Taking,

$$A = D \cos \phi \ \ and \ \ B = D \sin \phi$$

Eq. (14.3c) can be written as,

$$f(t) = D \sin (\omega t + \phi) \ , \tag{14.3d}$$

Here D and ϕ are constant given by

$$D = \sqrt{A^2 + B^2} \text{ and } \phi = \tan^{-1}\left(\frac{B}{A}\right)$$

The great importance of periodic sine and cosine functions is due to a remarkable result proved by the French mathematician, Jean Baptiste Joseph Fourier (1768-1830): **Any periodic function can be expressed as a superposition of sine and cosine functions of different time periods with suitable coefficients**.

Example 14.2 Which of the following functions of time represent (a) periodic and (b) non-periodic motion? Give the period for each case of periodic motion [ω is any positive constant].

(i) $\sin \omega t + \cos \omega t$
(ii) $\sin \omega t + \cos 2\omega t + \sin 4\omega t$
(iii) $e^{-\omega t}$
(iv) $\log (\omega t)$

Answer

(i) $\sin \omega t + \cos \omega t$ is a periodic function, it can also be written as $\sqrt{2} \sin (\omega t + \pi/4)$.

Now $\sqrt{2} \sin (\omega t + \pi/4) = \sqrt{2} \sin (\omega t + \pi/4 + 2\pi)$

$= \sqrt{2} \sin [\omega (t + 2\pi/\omega) + \pi/4]$

The periodic time of the function is $2\pi/\omega$.

(ii) This is an example of a periodic motion. It can be noted that each term represents a periodic function with a different angular frequency. Since period is the least interval of time after which a function repeats its value, $\sin \omega t$ has a period $T_0 = 2\pi/\omega$; $\cos 2\omega t$ has a period $\pi/\omega = T_0/2$; and $\sin 4\omega t$ has a period $2\pi/4\omega = T_0/4$. The period of the first term is a multiple of the periods of the last two terms. Therefore, the smallest interval of time after which the sum of the three terms repeats is T_0, and thus the sum is a periodic function with a period $2\pi/\omega$.

(iii) The function $e^{-\omega t}$ is not periodic, it decreases monotonically with increasing time and tends to zero as $t \to \infty$ and thus, never repeats its value.

(iv) The function $\log(\omega t)$ increases monotonically with time t. It, therefore, never repeats its value and is a non-periodic function. It may be noted that as $t \to \infty$, $\log(\omega t)$ diverges to ∞. It, therefore, cannot represent any kind of physical displacement. ◀

14.3 SIMPLE HARMONIC MOTION

Consider a particle oscillating back and forth about the origin of an x-axis between the limits $+A$ and $-A$ as shown in Fig. 14.3. This oscillatory motion is said to be simple harmonic if the displacement x of the particle from the origin varies with time as :

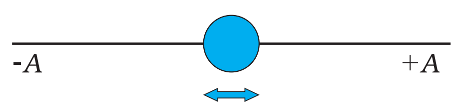


Fig. 14.3 *A particle vibrating back and forth about the origin of x-axis, between the limits +A and –A.*

$$x(t) = A \cos (\omega t + \phi) \tag{14.4}$$

where A, ω and ϕ are constants.

Thus, simple harmonic motion (SHM) is not any periodic motion but one in which displacement is a sinusoidal function of time. Fig. 14.4 shows what the positions of a particle executing SHM are at discrete value of time, each interval of time being $T/4$ where T is the period

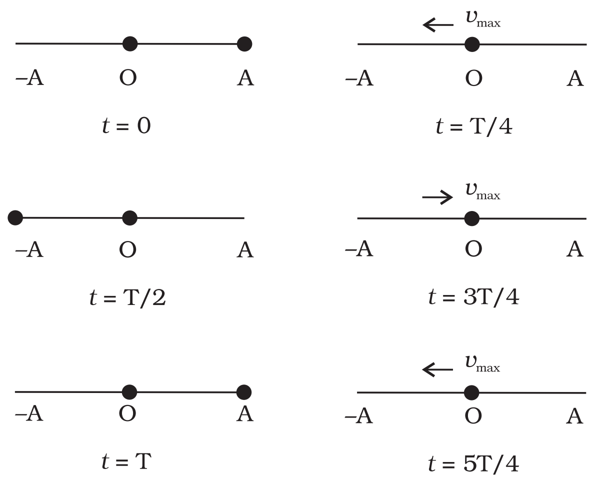


Fig. 14.4 *The location of the particle in SHM at the discrete values t = 0, T/4, T/2, 3T/4, T, 5T/4. The time after which motion repeats itself is T. T will remain fixed, no matter what location you choose as the initial (t = 0) location. The speed is maximum for zero displacement (at x = 0) and zero at the extremes of motion.*

of motion. Fig. 14.5 plots the graph of x versus t, which gives the values of displacement as a continuous function of time. The quantities A, ω and ϕ which characterize a given SHM have standard names, as summarised in Fig. 14.6. Let us understand these quantities.

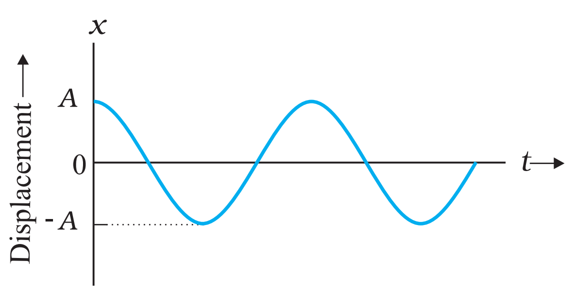


Fig. 14.5 *Displacement as a continuous function of time for simple harmonic motion.*

$x(t)$	:	displacement x as a function of time t
A	:	amplitude
ω	:	angular frequency
$\omega t + \phi$	:	phase (time-dependent)
ϕ	:	phase constant

Fig. 14.6 *The meaning of standard symbols in Eq. (14.4)*

The amplitutde A of SHM is the magnitude of maximum displacement of the particle. [Note, A can be taken to be positive without any loss of generality]. As the cosine function of time varies from +1 to –1, the displacement varies between the extremes A and $-A$. Two simple harmonic motions may have same ω and ϕ but different amplitudes A and B, as shown in Fig. 14.7 (a).

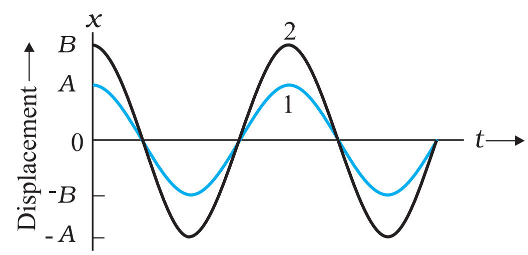


Fig. 14.7 (a) *A plot of displacement as a function of time as obtained from Eq. (14.4) with ϕ = 0. The curves 1 and 2 are for two different amplitudes A and B.*

While the amplitude A is fixed for a given SHM, the state of motion (position and velocity) of the particle at any time t is determined by the argument $(\omega t + \phi)$ in the cosine function. This time-dependent quantity, $(\omega t + \phi)$ is called the *phase* of the motion. The value of plase at $t = 0$ is ϕ and is called the *phase constant* (or *phase angle*). If the amplitude is known, ϕ can be determined from the displacement at $t = 0$. Two simple harmonic motions may have the same A and ω but different phase angle ϕ, as shown in Fig. 14.7 (b).

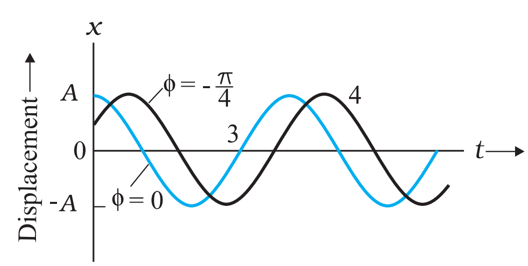


Fig. 14.7 (b) *A plot obtained from Eq. (14.4). The curves 3 and 4 are for ϕ = 0 and $-\pi/4$ respectively. The amplitude A is same for both the plots.*

Finally, the quantity ω can be seen to be related to the period of motion T. Taking, for simplicity, $\phi = 0$ in Eq. (14.4), we have

$$x(t) = A \cos \omega t \tag{14.5}$$

Since the motion has a period T, $x(t)$ is equal to $x(t + T)$. That is,

$$A \cos \omega t = A \cos \omega (t + T) \tag{14.6}$$

Now the cosine function is periodic with period 2π, i.e., it first repeats itself when the argument changes by 2π. Therefore,

$$\omega(t + T) = \omega t + 2\pi$$

that is $\omega = 2\pi/ T$ (14.7)

ω is called the angular frequency of SHM. Its S.I. unit is radians per second. Since the frequency of oscillations is simply $1/T$, ω is 2π times the frequency of oscillation. Two simple harmonic motions may have the same A and ϕ, but different ω, as seen in Fig. 14.8. In this plot the curve (b) has half the period and twice the frequency of the curve (a).

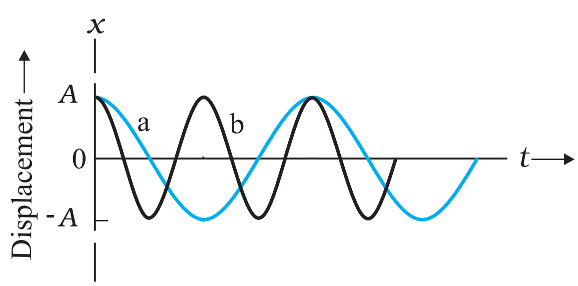


Fig. 14.8 *Plots of Eq. (14.4) for $\phi = 0$ for two different periods.*

> **Example 14.3** Which of the following functions of time represent (a) simple harmonic motion and (b) periodic but not simple harmonic? Give the period for each case.
> (1) $\sin \omega t - \cos \omega t$
> (2) $\sin^2 \omega t$

Answer

(a) $\sin \omega t - \cos \omega t$

$= \sin \omega t - \sin (\pi/2 - \omega t)$

$= 2 \cos (\pi/4) \sin (\omega t - \pi/4)$

$= \sqrt{2} \sin (\omega t - \pi/4)$

This function represents a simple harmonic motion having a period $T = 2\pi/\omega$ and a phase angle $(-\pi/4)$ or $(7\pi/4)$

(b) $\sin^2 \omega t$

$= ½ - ½ \cos 2 \omega t$

The function is periodic having a period $T = \pi/\omega$. It also represents a harmonic motion with the point of equilibrium occurring at ½ instead of zero. ◀

14.4 SIMPLE HARMONIC MOTION AND UNIFORM CIRCULAR MOTION

In this section we show that the projection of uniform circular motion on a diameter of the circle follows simple harmonic motion. A simple experiment (Fig. 14.9) helps us visualize this connection. Tie a ball to the end of a string and make it move in a horizontal plane about a fixed point with a constant angular speed. The ball would then perform a uniform circular motion in the horizontal plane. Observe the ball sideways or from the front, fixing your attention in the plane of motion. The ball will appear to execute to and fro motion along a horizontal line with the point of rotation as the midpoint. You could alternatively observe the shadow of the ball on a wall which is perpendicular to the plane of the circle. In this process what we are observing is the motion of the ball on a diameter of the circle normal to the direction of viewing.

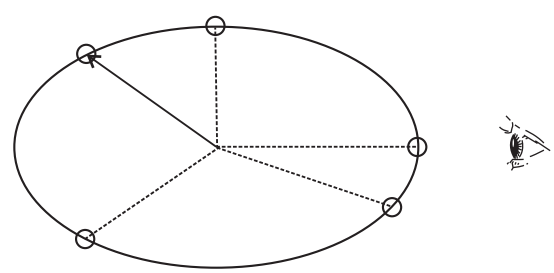


Fig. 14.9 *Circular motion of a ball in a plane viewed edge-on is SHM.*

Fig. 14.10 describes the same situation mathematically. Suppose a particle P is moving uniformly on a circle of radius A with angular speed ω. The sense of rotation is anticlockwise. The initial position vector of the particle, i.e., the vector $\overline{OP}$ at $t = 0$ makes an angle of ϕ with

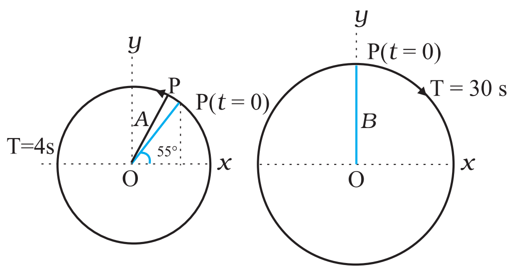


Fig. 14.10

the positive direction of x-axis. In time t, it will cover a further angle ωt and its position vector will make an angle of $\omega t + \phi$ with the +ve x-axis. Next consider the projection of the position vector OP on the x-axis. This will be OP′. The position of P′ on the x-axis, as the particle P moves on the circle, is given by

$$x(t) = A \cos (\omega t + \phi)$$

which is the defining equation of SHM. This shows that if P moves uniformly on a circle, its projection P′ on a diameter of the circle executes SHM. The particle P and the circle on which it moves arc sometimes referred to as the *reference particle* and the *reference circle* respectively.

We can take projection of the motion of P on any diameter, say the y-axis. In that case, the displacement $y(t)$ of P′ on the y-axis is given by

$$y = A \sin (\omega t + \phi)$$

which is also an SHM of the same amplitude as that of the projection on x-axis, but differing by a phase of $\pi/2$.

In spite of this connection between circular motion and SHM, the force acting on a particle in linear simple harmonic motion is very different from the centripetal force needed to keep a particle in uniform circular motion.

Example 14.4 Fig. 14.10 depicts two circular motions. The radius of the circle, the period of revolution, the initial position and the sense of revolution are indicated on the figures. Obtain the simple harmonic motions of the x-projection of the radius vector of the rotating particle P in each case.

Answer

(a) At $t = 0$, OP makes an angle of $45^\circ = \pi/4$ rad with the (positive direction of) x-axis. After time t, it covers an angle $\frac{2\pi}{T}t$ in the anticlockwise sense, and makes an angle of $\frac{2\pi}{T}t + \frac{\pi}{4}$ with the x-axis.

The projection of OP on the x-axis at time t is given by,

$$x(t) = A \cos \left(\frac{2\pi}{T}t + \frac{\pi}{4}\right)$$

For $T = 4$ s,

$$x(t) = A \cos \left(\frac{2\pi}{4}t + \frac{\pi}{4}\right)$$

which is a SHM of amplitude A, period 4 s, and an initial phase* = $\frac{\pi}{4}$.

(b) In this case at $t = 0$, OP makes an angle of $90^\circ = \frac{\pi}{2}$ with the x-axis. After a time t, it covers an angle of $\frac{2\pi}{T}t$ in the clockwise

* *The natural unit of angle is radian, defined through the ratio of arc to radius. Angle is a dimensionless quantity. Therefore it is not always necessary to mention the unit 'radian' when we use π, its multiples or submultiples. The conversion between radian and degree is not similar to that between metre and centimetre or mile. If the argument of a trigonometric function is stated without units, it is understood that the unit is radian. On the other hand, if degree is to be used as the unit of angle, then it must be shown explicitly. For example, sin(15°) means sine of 15 degree, but sin(15) means sine of 15 radians. Hereafter, we will often drop 'rad' as the unit, and it should be understood that whenever angle is mentioned as a numerical value, without units, it is to be taken as radians.*

sense and makes an angle of $\left(\frac{\pi}{2} - \frac{2\pi}{T}t\right)$ with the x-axis. The projection of OP on the x-axis at time t is given by

$$x(t) = B\cos\left(\frac{\pi}{2} - \frac{2\pi}{T}t\right)$$

$$= B\sin\left(\frac{2\pi}{T}t\right)$$

For $T = 30$ s,

$$x(t) = B\sin\left(\frac{\pi}{15}t\right)$$

Writing this as $x(t) = B\cos\left(\frac{\pi}{15}t - \frac{\pi}{2}\right)$, and comparing with Eq. (14.4). We find that this represents a SHM of amplitude B, period 30 s, and an initial phase of $-\frac{\pi}{2}$. ◀

14.5 VELOCITY AND ACCELERATION IN SIMPLE HARMONIC MOTION

The speed of a particle v in uniform circular motion is its angular speed ω times the radius of the circle A.

$$v = \omega A \tag{14.8}$$

The direction of velocity $\bar{v}$ at a time t is along the tangent to the circle at the point where the particle is located at that instant. From the geometry of Fig. 14.11, it is clear that the velocity of the projection particle P′ at time t is

$$v(t) = -\omega A\sin(\omega t + \phi) \tag{14.9}$$

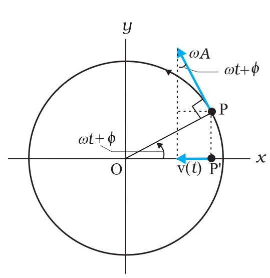


Fig. 14.11 *The velocity, $v(t)$, of the particle P′ is the projection of the velocity $\bar{v}$ of the reference particle, P.*

where the negative sign shows that $v(t)$ has a direction opposite to the positive direction of x-axis. Eq. (14.9) gives the instantaneous velocity of a particle executing SHM, where displacement is given by Eq. (14.4). We can, of course, obtain this equation without using geometrical argument, directly by differentiating (Eq. 14.4) with respect of t:

$$v(t) = \frac{d}{dt}x(t) \tag{14.10}$$

The method of reference circle can be similarly used for obtaining instantaneous acceleration of a particle undergoing SHM. We know that the centripetal acceleration of a particle P in uniform circular motion has a magnitude v^2/A or $\omega^2 A$, and it is directed towards the centre i.e., the direction is along PO. The instantaneous acceleration of the projection particle P′ is then (See Fig. 14.12)

$$a(t) = -\omega^2 A\cos(\omega t + \phi)$$

$$= -\omega^2 x(t) \tag{14.11}$$

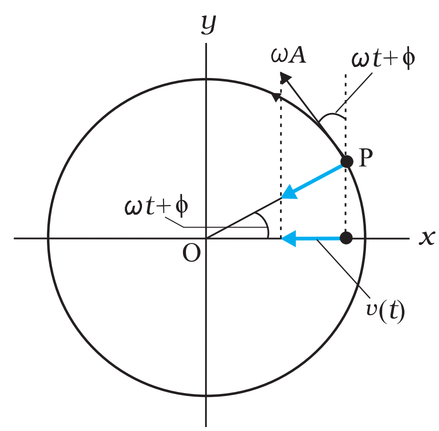


Fig. 14.12 *The acceleration, $a(t)$, of the particle P′ is the projection of the acceleration **a** of the reference particle P.*

Eq. (14.11) gives the acceleration of a particle in SHM. The same equation can again be obtained directly by differentiating velocity $v(t)$ given by Eq. (14.9) with respect to time:

$$a(t) = \frac{d}{dt}v(t) \tag{14.12}$$

We note from Eq. (14.11) the important property that acceleration of a particle in SHM is proportional to displacement. For $x(t) > 0$, $a(t) < 0$ and for $x(t) < 0$, $a(t) > 0$. Thus, whatever the value of x between $-A$ and A, the acceleration $a(t)$ is always directed towards the centre.

For simplicity, let us put $\phi = 0$ and write the expression for $x(t)$, $v(t)$ and $a(t)$

$x(t) = A\cos\omega t$, $v(t) = -\omega A\sin\omega t$, $a(t) = -\omega^2 A\cos\omega t$

The corresponding plots are shown in Fig. 14.13. All quantitites vary sinusoidally with time; only their maxima differ and the different plots differ in phase. x varies between $-A$ to A; $v(t)$ varies from $-\omega A$ to ωA and $a(t)$ from $-\omega^2 A$ to $\omega^2 A$. With respect to displacement plot, velocity plot has a phase difference of $\pi/2$ and acceleration plot has a phase difference of π.

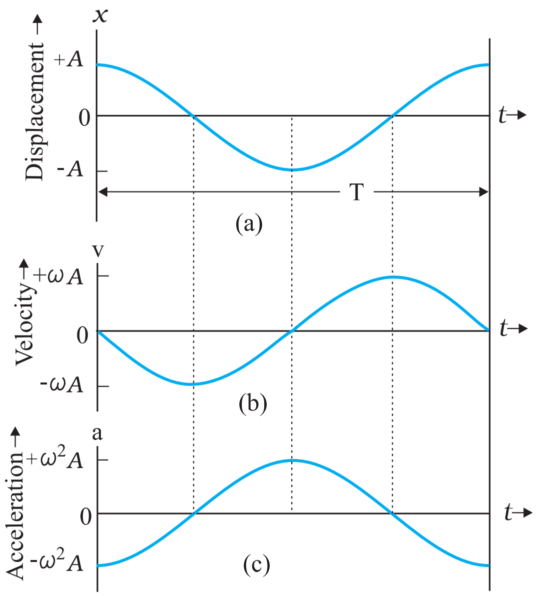


Fig. 14.13 *Displacement, velocity and acceleration of a particle in simple harmonic motion have the same period T, but they differ in phase*

Example 14.5 A body oscillates with SHM according to the equation (in SI units),

$x = 5\cos[2\pi t + \pi/4]$.

At $t = 1.5$ s, calculate the (a) displacement, (b) speed and (c) acceleration of the body.

Answer The angular frequency ω of the body $= 2\pi\ s^{-1}$ and its time period $T = 1$ s.

At $t = 1.5$ s

(a) displacement $= (5.0\ m)\cos[(2\pi\ s^{-1}) \times 1.5\ s + \pi/4]$

$= (5.0\ m)\cos[(3\pi + \pi/4)]$

$= -5.0 \times 0.707\ m$

$= -3.535\ m$

(b) Using Eq. (14.9), the speed of the body

$= -(5.0\ m)(2\pi\ s^{-1})\sin[(2\pi\ s^{-1}) \times 1.5\ s + \pi/4]$

$= -(5.0\ m)(2\pi\ s^{-1})\sin[(3\pi + \pi/4)]$

$= 10\pi \times 0.707\ m\ s^{-1}$

$= 22\ m\ s^{-1}$

(c) Using Eq.(14.10), the acceleration of the body

$= -(2\pi\ s^{-1})^2 \times$ displacement

$= -(2\pi\ s^{-1})^2 \times (-3.535\ m)$

$= 140\ m\ s^{-2}$ ◀

14.6 FORCE LAW FOR SIMPLE HARMONIC MOTION

Using Newton's second law of motion, and the expression for acceleration of a particle undergoing SHM (Eq. 14.11), the force acting on a particle of mass m in SHM is

$$F(t) = ma$$

$$= -m\omega^2 x(t)$$

i.e., $$F(t) = -k\,x(t) \qquad (14.13)$$

where $$k = m\omega^2 \qquad (14.14a)$$

or $$\omega = \sqrt{\frac{k}{m}} \qquad (14.14b)$$

Like acceleration, force is always directed towards the mean position - hence it is sometimes called the restoring force in SHM. To summarize the discussion so far, simple harmonic motion can be defined in two equivalent ways, either by Eq. (14.4) for displacement or by Eq. (14.13) that gives its force law. Going from Eq. (14.4) to Eq. (14.13) required us to differentiate two times. Likewise by integrating the force law Eq. (14.13) two times, we can get back Eq. (14.4).

Note that the force in Eq. (14.13) is linearly proportional to $x(t)$. A particle oscillating under such a force is, therefore, calling a linear harmonic oscillator. In the real world, the force may contain small additional terms proportional to x^2, x^3, etc. These then are called non-linear oscillators.

Example 14.6 Two identical springs of spring constant k are attached to a block of mass m and to fixed supports as shown in Fig. 14.14. Show that when the mass is displaced from its equilibrium position on either side, it executes a simple harmonic motion. Find the period of oscillations.

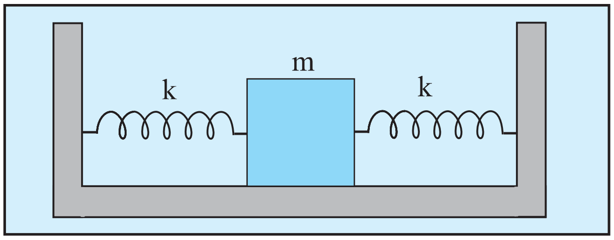


Fig. 14.14

Answer Let the mass be displaced by a small distance x to the right side of the equilibrium position, as shown in Fig. 14.15. Under this situation the spring on the left side gets

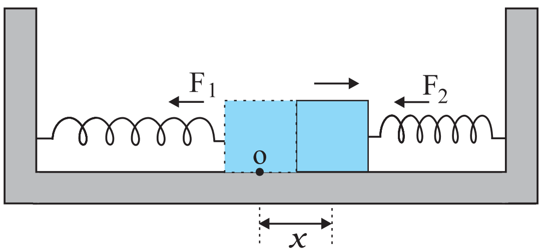


Fig. 14.15

elongated by a length equal to x and that on the right side gets compressed by the same length. The forces acting on the mass are then,

$F_1 = -k\,x$ (force exerted by the spring on the left side, trying to pull the mass towards the mean position)

$F_2 = -k\,x$ (force exerted by the spring on the right side, trying to push the mass towards the mean position)

The net force, F, acting on the mass is then given by,

$$F = -2kx$$

Hence the force acting on the mass is proportional to the displacement and is directed towards the mean position; therefore, the motion executed by the mass is simple harmonic. The time period of oscillations is,

$$T = 2\pi\sqrt{\frac{m}{2k}}$$

14.7 ENERGY IN SIMPLE HARMONIC MOTION

Both kinetic and potential energies of a particle in SHM vary between zero and their maximum values.

In section14.5 we have seen that the velocity of a particle executing SHM, is a periodic function of time. It is zero at the extreme positions of displacement. Therefore, the kinetic energy (K) of such a particle, which is defined as

$$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\,\omega^2 A^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2}k\,A^2 \sin^2(\omega t + \phi) \qquad (14.15)$$

is also a periodic function of time, being zero when the displacement is maximum and maximum when the particle is at the mean position. Note, since the sign of v is immaterial in K, the period of K is $T/2$.

What is the potential energy (U) of a particle executing simple harmonic motion? In Chapter 6, we have seen that the concept of potential energy is possible only for conservative forces. The spring force $F = -kx$ is a conservative force, with associated potential energy

$$U = \frac{1}{2}k\,x^2 \qquad (14.16)$$

Hence the potential energy of a particle executing simple harmonic motion is,

$$U(x) = \frac{1}{2}k\,x^2$$

$$= \frac{1}{2}k\,A^2 \cos^2(\omega t + \phi) \qquad (14.17)$$

Thus, the potential energy of a particle executing simple harmonic motion is also periodic, with period $T/2$, being zero at the mean position and maximum at the extreme displacements.

It follows from Eqs. (14.15) and (14.17) that the total energy, E, of the system is,

$$E = U + K$$

$$= \frac{1}{2} k A^2 \cos^2(\omega t + \phi) + \frac{1}{2} k A^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2} k A^2 \left[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right]$$

Using the familiar trigonometric identity, the value of the expression in the brackets is unity. Thus,

$$E = \frac{1}{2} k A^2 \qquad (14.18)$$

The total mechanical energy of a harmonic oscillator is thus independent of time as expected for motion under any conservative force. The time and displacement dependence of the potential and kinetic energies of a linear simple harmonic oscillator are shown in Fig. 14.16.

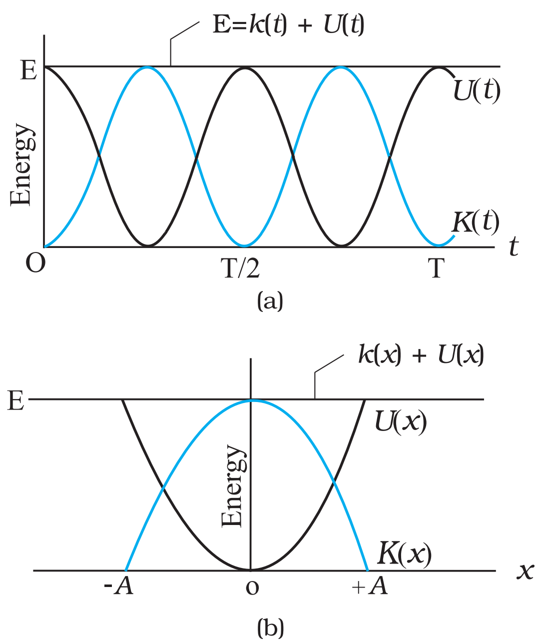


Fig. 14.16 *Kinetic energy, potential energy and total energy as a function of time [shown in (a)] and displacement [shown in (b)] of a particle in SHM. The kinetic energy and potential energy both repeat after a period T/2. The total energy remains constant at all t or x.*

Observe that both kinetic energy and potential energy in SHM are seen to be always positive in Fig. 14.16. Kinetic energy can, of course, be never negative, since it is proportional to the square of speed. Potential energy is positive by choice of the undermined constant in potential energy. Both kinetic energy and potential energy peak twice during each period of SHM. For $x = 0$, the energy is kinetic; at the extremes $x = \pm A$, it is all potential energy. In the course of motion between these limits, kinetic energy increases at the expense of potential energy or vice-versa.

Example 14.7 A block whose mass is 1 kg is fastened to a spring. The spring has a spring constant of 50 N m^{-1}. The block is pulled to a distance x = 10 cm from its equilibrium position at $x = 0$ on a frictionless surface from rest at t = 0. Calculate the kinetic, potential and total energies of the block when it is 5 cm away from the mean position.

Answer The block executes SHM, its angular frequency, as given by Eq. (14.14b), is

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{50 \text{ N m}^{-1}}{1 \text{kg}}}$$

$$= 7.07 \text{ rad s}^{-1}$$

Its displacement at any time t is then given by,

$$x(t) = 0.1 \cos(7.07t)$$

Therefore, when the particle is 5 cm away from the mean position, we have

$$0.05 = 0.1 \cos(7.07t)$$

Or $\cos(7.07t) = 0.5$ and hence

$$\sin(7.07t) = \frac{\sqrt{3}}{2} = 0.866$$

Then, the velocity of the block at x = 5 cm is

$= 0.1 \times 7.07 \times 0.866 \text{ m s}^{-1}$

$= 0.61 \text{ m s}^{-1}$

Hence the K.E. of the block,

$$= \frac{1}{2} m v^2$$

$= ½[1\text{kg} \times (0.6123 \text{ m s}^{-1})^2]$

$= 0.19$ J

The P.E. of the block,

$$= \frac{1}{2} k x^2$$

$= ½(50 \text{ N m}^{-1} \times 0.05 \text{ m} \times 0.05 \text{ m})$

$= 0.0625$ J

The total energy of the block at x = 5 cm,

= K.E. + P.E.

= 0.25 J

we also know that at maximum displacement, K.E. is zero and hence the total energy of the system is equal to the P.E. Therefore, the total energy of the system,

$= ½(50 \text{ N m}^{-1} \times 0.1 \text{ m} \times 0.1 \text{ m})$

$= 0.25$ J

which is same as the sum of the two energies at a displacement of 5 cm. This is in conformity with the principle of conservation of energy. ◀

14.8 SOME SYSTEMS EXECUTING SIMPLE HARMONIC MOTION

There are no physical examples of absolutely pure **simple harmonic motion**. In practice we come across systems that execute simple harmonic motion approximately under certain conditions. In the subsequent part of this section, we discuss the motion executed by some such systems.

14.8.1 Oscillations due to a Spring

The simplest observable example of simple harmonic motion is the small oscillations of a block of mass m fixed to a spring, which in turn is fixed to a rigid wall as shown in Fig. 14.17. The block is placed on a frictionless horizontal surface. If the block is pulled on one side and is released, it then executes a to and fro motion about a mean position. Let x = 0, indicate the position of the centre of the block when the

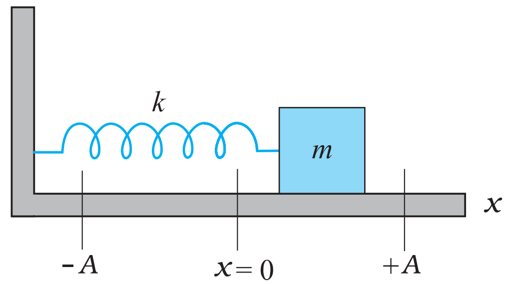


Fig. 14.17 *A linear simple harmonic oscillator consisting of a block of mass m attached to a spring. The block moves over a frictionless surface. The box, when pulled or pushed and released, executes simple harmonic motion.*

spring is in equilibrium. The positions marked as $-A$ and $+A$ indicate the maximum displacements to the left and the right of the mean position. We have already learnt that springs have special properties, which were first discovered by the English physicist Robert Hooke. He had shown that such a system when deformed, is subject to a restoring force, the magnitude of which is proportional to the deformation or the displacement and acts in opposite direction. This is known as Hooke's law (Chapter 9). It holds good for displacements small in comparison to the length of the spring. At any time t, if the displacement of the block from its mean position is x, the restoring force F acting on the block is,

$$F(x) = -k\,x \tag{14.19}$$

The constant of proportionality, k, is called the spring constant, its value is governed by the elastic properties of the spring. A stiff spring has large k and a soft spring has small k. Equation (14.19) is same as the force law for SHM and therefore the system executes a simple harmonic motion. From Eq. (14.14) we have,

$$\omega = \sqrt{\frac{k}{m}} \tag{14.20}$$

and the period, T, of the oscillator is given by,

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{14.21}$$

Stiff springs have high value of k (spring constant). A block of small mass m attached to a stiff spring will have, according to Eq. (14.20), large oscillation frequency, as expected physically.

> **Example 14.8** A 5 kg collar is attached to a spring of spring constant 500 N m^{-1}. It slides without friction over a horizontal rod. The collar is displaced from its equilibrium position by 10.0 cm and released. Calculate
> (a) the period of oscillation,
> (b) the maximum speed and
> (c) maximum acceleration of the collar.

Answer (a) The period of oscillation as given by Eq. (14.21) is,

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{5.0\,\text{kg}}{500\,\text{N m}^{-1}}}$$
$$= (2\pi/10)\ \text{s}$$
$$= 0.63\ \text{s}$$

(b) The velocity of the collar executing SHM is given by,

$$v(t) = -A\omega \sin(\omega t + \phi)$$

The maximum speed is given by,

$$v_m = A\omega$$
$$= 0.1 \times \sqrt{\frac{k}{m}}$$
$$= 0.1 \times \sqrt{\frac{500\,\text{N m}^{-1}}{5\,\text{kg}}}$$
$$= 1\ \text{m s}^{-1}$$

and it occurs at $x = 0$

(c) The acceleration of the collar at the displacement $x(t)$ from the equilibrium is given by,

$$a(t) = -\omega^2 x(t)$$
$$= -\frac{k}{m}\,x(t)$$

Therefore the maximum acceleration is,

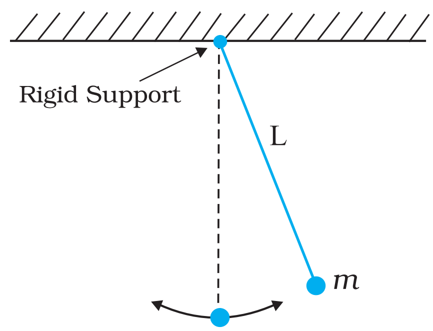
$$a_{max} = \omega^2 A$$
$$= \frac{500\ \text{N m}^{-1}}{5\ \text{kg}} \times 0.1\ \text{m}$$
$$= 10\ \text{m s}^{-2}$$

and it occurs at the extremities. ◀

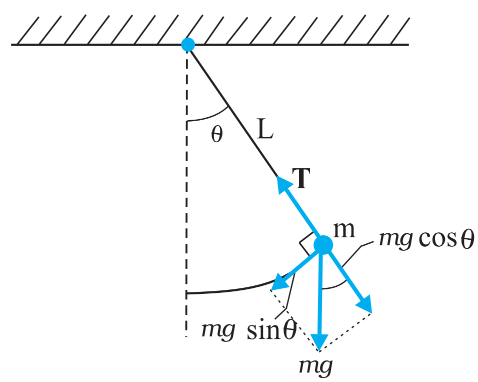
14.8.2 The Simple Pendulum

It is said that Galileo measured the periods of a swinging chandelier in a church by his pulse beats. He observed that the motion of the chandelier was periodic. The system is a kind of pendulum. You can also make your own pendulum by tying a piece of stone to a long unstretchable thread, approximately 100 cm long. Suspend your pendulum from a suitable support so that it is free to oscillate. Displace the stone to one side by a small distance and let it go. The stone executes a to and fro motion, it is periodic with a period of about two seconds.

We shall show that this periodic motion is simple harmonic for small displacements from the mean position. Consider simple pendulum — a small bob of mass m tied to an inextensible mass less string of length L. The other end of the string is fixed to a support in the ceiling. The bob oscillates in a plane about the vertical line through the support. Fig. 14.18(a) shows this system. Fig. 14.18(b) is a kind of 'free-body' diagram of the simple pendulum showing the forces acting on the bob.



(a)



(b)

Fig. 14.18 *(a) A bob oscillating about its mean position. (b) The radial force T-mg cosθ provides centripetal force but no torque about the support. The tangential force mgsinθ provides the restoring torque.*