Chapter

3

Pair of Straight Lines

Contents

Gabriel Lame

The general equation of second degree

$$ax^2 + by^2 + 2gx + 2fy + c = 0$$

represents pair of straight line; if $\Delta = 0$ and $ab - h^2 \leq 0$

Clairaut (1729 A.D.) was the first to gave the distance formulae although in clumsy form. He also gave the intercept form of the linear equation.

In 1818, Gabriel Lame a civil engineer gave mE + mE' = 0 as the curve passing through the point of intersection of two loci E = 0 and E' = 0.

Pair of Straight Lines

3.1 Equation of Pair of Straight lines

Let the equation of two lines be

$a'x + b'y + c' = 0$ .....(i) and $a''x + b''y + c'' = 0$ .....(ii)

Hence $(a'x + b'y + c')(a''x + b''y + c'') = 0$ is called the joint equation of lines (i) and (ii) and conversely, if joint equation of two lines be $(a'x + b'y + c')$ $(a''x + b''y + c'') = 0$ then their separate equation will be $a'x + b'y + c' = 0$ and $a''x + b''y + c'' = 0$.

(1) **Equation of a pair of straight lines passing through origin :** The equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight line passing through the origin where a, h, b are constants.

Let the lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1 x = 0$ and $y - m_2 x = 0$

where, $m_1 = \frac{-h + \sqrt{h^2 - ab}}{b}$ and $m_2 = \frac{-h - \sqrt{h^2 - ab}}{b}$ then, $m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$

Then, two straight lines represented by $ax^2 + 2hxy + by^2 = 0$ are $ax + hy + y\sqrt{h^2 - ab} = 0$ and $ax + hy - y\sqrt{h^2 - ab} = 0$.

Note :
- The lines are real and distinct if $h^2 - ab > 0$
- The lines are real and coincident if $h^2 - ab = 0$
- The lines are imaginary if $h^2 - ab < 0$
- If the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ and $a'x^2 + 2h'xy + b'y^2 = 0$ should have one line common, then $(ab' - a'b)^2 = 4(ah' - a'h)(hb' - h'b)$.
- The equation of the pair of straight lines passing through origin and perpendicular to the pair of straight lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by $bx^2 - 2hxy + ay^2 = 0$
- If the slope of one of the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ be the square of the other, then $a^2b + ab^2 - 6abh + 8h^3 = 0$.
- If the slope of one of the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ be λ times that of the other, then $4\lambda h^2 = ab(1 + \lambda)^2$.

(2) **General equation of a pair of straight lines :** An equation of the form,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

where a, b, c, f, g, h are constants, is said to be a general equation of second degree in x and y.

The necessary and sufficient condition for $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines is that $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(3) **Separate equations from joint equation:** The general equation of second degree be $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. To find the lines represented by this equation we proceed as follows :

Step I : Factorize the homogeneous part $ax^2 + 2hxy + by^2$ into two linear factors. Let the linear factors be $a'x + b'y$ and $a''x + b''y$.

Step II: Add constants c' and c'' in the factors obtained in step I to obtain $a'x + b'y + c'$ and $a''x + b''y + c''$. Let the lines be $a'x + b'y + c' = 0$ and $a''x + b''y + c'' = 0$.

Step III : Obtain the joint equation of the lines in step II and compare the coefficients of x, y and constant terms to obtain equations in c' and c''.

Step IV : Solve the equations in c' and c'' to obtain the values of c' and c''.

Step V : Substitute the values of c' and c'' in lines in step II to obtain the required lines.

Example: 1 If the sum of the slopes of the lines given by $x^2 - 2cxy - 7y^2 = 0$ is four times their product. Then c has the value

[AIEEE 2004]

(a) – 2 (b) – 1 (c) 2 (d) 1

Solution: (c) We know that, $m_1 + m_2 = \frac{-2h}{b}$ and $m_1 m_2 = \frac{a}{b}$.

Given, $m_1 + m_2 = 4m_1 m_2 \Rightarrow \frac{-2c}{7} = 4\left(\frac{1}{-7}\right) \Rightarrow c = 2$

Example: 2 If one of the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ be $y = mx$, then

(a) $bm^2 + 2hm + a = 0$ (b) $bm^2 + 2hm - a = 0$ (c) $am^2 + 2hm + b = 0$ (d) $bm^2 - 2hm + a = 0$

Solution: (a) Substituting the value of y in the equation $ax^2 + 2hxy + by^2 = 0$

$\Rightarrow$ $ax^2 + 2hx(mx) + b(mx)^2 = 0 \Rightarrow a + 2hm + bm^2 = 0$

Example: 3 If the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + K = 0$ represent two straight lines, then the value of K is **[MP PET 20**

(a) 1 (b) 2 (c) 0 (d) 3

Solution: (b) Condition for pair of lines, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, Here $a = 12, h = -5, b = 2, g = 11/2, f = -5/2, c = K$

Then, $12 \times 2 \times K + 2 \times \frac{-5}{2} \times \frac{11}{2} - 12 \times \left(\frac{-5}{2}\right)^2 - 2 \times \left(\frac{11}{2}\right)^2 - K(-5)^2 = 0$. On solving, we get K= 2.

3.2 Angle between the Pair of Lines

(1) The angle θ between the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$

(i) The lines are coincident if the angle between them is zero.

$\therefore$ Lines are coincident *i.e.*, $\theta = 0 \Rightarrow \tan\theta = 0 \Rightarrow \frac{2\sqrt{h^2 - ab}}{a+b} = 0 \Rightarrow h^2 - ab = 0 \Rightarrow h^2 = ab$

Hence, the lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident, $iff\ h^2 = ab$

(ii) The lines are perpendicular if the angle between them is $\pi/2$.

$\therefore \theta = \frac{\pi}{2} \Rightarrow \cot\theta = \cot\frac{\pi}{2} \Rightarrow \cot\theta = 0 \Rightarrow \frac{a+b}{2\sqrt{h^2 - ab}} = 0 \Rightarrow a + b = 0 \Rightarrow$ coeff. of x^2 + coeff. of $y^2 = 0$

Thus, the lines represented by $ax^2 + 2hxy + by^2 = 0$ are perpendicular *iff* $a + b = 0$ *i.e.*, coeff. of x^2 + coeff. of $y^2 = 0$.

(2) The angle between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is given by

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| \Rightarrow \theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

(i) The lines are parallel if the angle between them is zero. Thus, the lines are parallel *iff*

$\theta = 0 \Rightarrow \tan\theta = 0 \Rightarrow \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = 0 \Rightarrow h^2 = ab$.

Hence, the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are parallel *iff* $h^2 = ab$ and $af^2 = bg^2$ or $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$.

(ii) The lines are perpendicular if the angle between them is $\pi/2$.

Thus, the lines are perpendicular *i.e.*, $\theta = \pi/2 \Rightarrow \cot\theta = 0 \Rightarrow \frac{a+b}{2\sqrt{h^2 - ab}} = 0$

$\Rightarrow a + b = 0 \Rightarrow$ coeff. of x^2 + coeff. of $y^2 = 0$

Hence, the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are perpendicular *iff* $a + b = 0$

i.e., coeff. of x^2 + coeff. of $y^2 = 0$.

(iii) The lines are coincident, if $g^2 = ac$.

Example: 4 The angle between the lines $x^2 - xy - 6y^2 - 7x + 31y - 18 = 0$ is **[Karnataka CET 2003]**

(a) 45^o (b) 60^o (c) 90^o (d) 30^o

Solution: (b) Angle between the lines is $\theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \tan^{-1}\left|\frac{2\sqrt{\left(\frac{-1}{2}\right)^2 - 1 \times (-6)}}{1 + (-6)}\right| = \tan^{-1}\left|\frac{2\sqrt{\frac{1}{4} + 6}}{1 + (-6)}\right| = \tan^{-1}|-1| = \tan^{-1}(1) = \frac{\pi}{4}$, 45^o

Example: 5 If the angle between the pair of straight lines represented by the equation $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$ is $\tan^{-1}\left(\frac{1}{3}\right)$, where λ is a non- negative real number, then λ is

(a) 2 (b) 0 (c) 3 (d) 1

Solution: (a) Given that $\theta = \tan^{-1}\left(\frac{1}{3}\right) \Rightarrow \tan\theta = \frac{1}{3}$

Now, since $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| \Rightarrow \frac{1}{3} = \left|\frac{2\sqrt{\left(\frac{-3}{2}\right)^2 - \lambda}}{\lambda + 1}\right| \Rightarrow (\lambda+1)^2 = 9(9-4\lambda) \Rightarrow \lambda^2 + 38\lambda - 80 = 0$

$\Rightarrow \lambda^2 + 40\lambda - 2\lambda - 80 = 0 \Rightarrow \lambda(\lambda+40) - 2(\lambda+40) = 0 \Rightarrow (\lambda-2)(\lambda+40) = 0 \Rightarrow \lambda = 2$ or – 40, but λ is a non-negative real number. Hence $\lambda = 2$.

Example: 6 The angle between the pair of straight lines represented by $2x^2 - 7xy + 3y^2 = 0$ is **[Kurukshetra CEE 2002]**

(a) 60^o (b) 45^o (c) $\tan^{-1}(7/6)$ (d) 30^o

Solution: (b) Angle between the lines is , $\theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \tan^{-1}\left|\frac{2\sqrt{\left(-\frac{7}{2}\right)^2 - (2)(3)}}{2+3}\right| \Rightarrow \theta = \tan^{-1}\left(\frac{2}{5}.\frac{5}{2}\right) = \tan^{-1}(1) \Rightarrow$

$\theta = 45^o$

3.3 Bisectors of the Angles between the Lines

(1) The joint equation of the bisectors of the angles between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a-b} = \frac{xy}{h}$ .....(i)

$\Rightarrow \quad hx^2 - (a-b)xy - hy^2 = 0$

Here, coefficient of x^2 + coefficient of $y^2 = 0$. Hence, the bisectors of the angles between the lines are perpendicular to each other. The bisector lines will pass through origin also.

Note : ❑ If $a = b$, the bisectors are $x^2 - y^2 = 0$ *i.e.,* $x - y = 0, x + y = 0$

❑ If $h = 0$, the bisectors are $xy = 0$ *i.e.,* $x = 0, y = 0$.

❑ If bisectors of the angles between lines represented by $ax^2 + 2hxy + by^2 = 0$ and $a'x^2 + 2h'xy + b'y^2 = 0$ are same, then $\frac{h'}{h} = \frac{a'-b'}{a-b}$.

❑ If the equation $ax^2 + 2hxy + by^2 = 0$ has one line as the bisector of the angle between the coordinate axes, then $4h^2 = (a+b)^2$.

(2) The equation of the bisectors of the angles between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are given by $\frac{(x-\alpha)^2 - (y-\beta)^2}{a-b} = \frac{(x-\alpha)(y-\beta)}{h}$, where α, β is the point of intersection of the lines represented by the given equation.

Example: 7 The equation of the bisectors of the angles between the lines represented by $x^2 + 2xy\cot\theta + y^2 = 0$ is

(a) $x^2 - y^2 = 0$ (b) $x^2 - y^2 = xy$ (c) $(x^2 - y^2)\cot\theta = 2xy$ (d) None of these

Solution: (a) Equation of bisectors is given by $\frac{x^2 - y^2}{a-b} = \frac{xy}{h}$ or $\frac{x^2 - y^2}{0} = \frac{xy}{\cot\theta} \Rightarrow x^2 - y^2 = 0$

Example: 8 If the bisectors of the lines $x^2 - 2pxy - y^2 = 0$ be $x^2 - 2qxy - y^2 = 0$, then

[MP PET 1993; DCE 1999; Rajasthan PET 2003; AIEEE 2003]

(a) $pq + 1 = 0$ (b) $pq - 1 = 0$ (c) $p + q = 0$ (d) $p - q = 0$

Solution: (a) Bisectors of the angle between the lines $x^2 - 2pxy - y^2 = 0$ is $\frac{x^2 - y^2}{xy} = \frac{1-(-1)}{-p} \Rightarrow px^2 + 2xy - py^2 = 0$

But it is represented by $x^2 - 2qxy - y^2 = 0$. Therefore $\frac{p}{1} = \frac{2}{-2q} \Rightarrow pq = -1 \Rightarrow pq + 1 = 0$

3.4 Point of Intersection of Lines represented by $ax^2+2hxy+by^2+2gx+2fy+c = 0$

Let $\phi \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\frac{\partial \phi}{\partial x} = 2ax + 2hy + 2g = 0$ (Keeping y as constant)

and $\frac{\partial \phi}{\partial y} = 2hx + 2by + 2f = 0$ (Keeping x as constant)

For point of intersection $\frac{\partial \phi}{\partial x} = 0$ and $\frac{\partial \phi}{\partial y} = 0$

We obtain, $ax + hy + g = 0$ and $hx + by + f = 0$

On solving these equations, we get $\frac{x}{fh - bg} = \frac{y}{gh - af} = \frac{1}{ab - h^2}$ *i.e.* $(x, y) = \left(\frac{bg - fh}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$

Also, since $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, from first two rows

$a \quad h \quad g \quad \Rightarrow ax + hy + g = 0$ and

$h \quad b \quad f \quad \Rightarrow hx + by + f = 0$ and then solve, we get the point of intersection.

Note : ❑ The point of intersection of lines represented by $ax^2 + 2hxy + by^2 = 0$ is (0, 0).

Example: 9 The point of intersection of the lines represented by the equation $2x^2 + 3y^2 + 7xy + 8x + 14y + 8 = 0$ is

(a) $(0,2)$ (b) $(1,2)$ (c) $(-2,0)$ (d) $(-2,1)$

Solution: (c) Let $\phi \equiv 2x^2 + 3y^2 + 7xy + 8x + 14y + 8 = 0$

$\frac{\partial \phi}{\partial x} = 4x + 7y + 8 = 0$ and $\frac{\partial \phi}{\partial y} = 6y + 7x + 14 = 0$

On solving these equations, we get $x = -2,\ y = 0$

Trick : If the equation is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

The points of intersection are given by $\left\{\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right\}$. Hence point is (– 2, 0)

Example: 10 If the pair of straight lines $xy - x - y + 1 = 0$ and line $ax + 2y - 3 = 0$ are concurrent, then a =

(a) – 1 (b) 0 (c) 3 (d) 1

Solution: (d) Given that equation of pair of straight lines $xy - x - y + 1 = 0$

$\Rightarrow \quad (x-1)(y-1)=0 \quad \Rightarrow \quad x-1=0$ or $y-1=0$

The intersection point of $x-1=0, y-1=0$ is (1,1)

$\therefore$ Lines $x-1=0, y-1=0$ and $ax+2y-3=0$ are concurrent.

$\therefore$ The intersecting points of first two lines satisfy the third line.

Hence, $a+2-3=0 \Rightarrow a=1$

3.5 Equation of the Lines joining the Origin to the Points of Intersection of a given Line and a given Curve

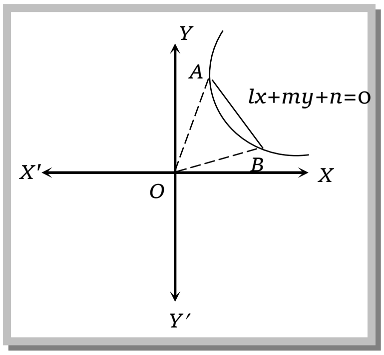
The equation of the lines which joins origin to the point of intersection of the line $lx+my+n=0$ and curve $ax^2+2hxy+by^2+2gx+2fy+c=0$, can be obtained by making the curve homogeneous with the help of line $lx+my+n=0$, which is

$$ax^2+2hxy+by^2+2(gx+fy)\left(\frac{lx+my}{-n}\right)+c\left(\frac{lx+my}{-n}\right)^2=0$$

We have $ax^2+2hxy+by^2+2gx+2fy+c=0$ ......(i)

and $lx+my+n=0$ .....(ii)

Suppose the line (ii) intersects the curve (i) at two points *A* and *B*. We wish to find the combined equation of the straight lines *OA* and *OB*. Clearly *OA* and *OB* pass through the origin, so their joint equation is a homogeneous equation of second degree in *x* and *y*.



From equation (ii), $lx+my=-n \Rightarrow \frac{lx+my}{-n}=1$ ......(iii)

Now, consider the equation

$$ax^2+2hxy+by^2+2gx\left(\frac{lx+my}{-n}\right)+2fy\left(\frac{lx+my}{-n}\right)+c\left(\frac{lx+my}{-n}\right)^2=0$$ .....(iv)

Clearly, this equation is a homogeneous equation of second degree. So, it represents a pair of straight lines passing through the origin. Moreover, it is satisfied by the points *A* and *B*.

Hence (iv) represents a pair of straight lines *OA* and *OB* through the origin *O* and the points *A* and *B* which are points of intersection of (i) and (ii).

Example: 11 The lines joining the origin to the point of intersection of the circle $x^2+y^2=3$ and the line $x+y=2$ are

(a) $y-(3+2\sqrt{2})x=0$ (b) $x-(3+2\sqrt{2})y=0$ (c) $x-(3-2\sqrt{2})y=0$ (d) $y-(3-2\sqrt{2})x=0$

Solution: (a,b,c,d) Make homogenous the equation of circle, we get $x^2-6xy+y^2=0$

$$\Rightarrow \quad x=\frac{6y\pm\sqrt{(36-4)y^2}}{2}=\frac{6y\pm 4\sqrt{2}y}{2}=3y\pm 2\sqrt{2}y$$

Hence, the equation are $x=(3+2\sqrt{2})y$ and $x=(3-2\sqrt{2})y$

Also after rationalizing these equations becomes $y-(3+2\sqrt{2})x=0$ and $y-(3-2\sqrt{2})x=0$.

Example: 12 The pair of straight lines joining the origin to the points of intersection of the line $y = 2\sqrt{2}x + c$ and the circle $x^2 + y^2 = 2$ are at right angles, if
[MP PET 1996]

(a) $c^2 - 4 = 0$ (b) $c^2 - 8 = 0$ (c) $c^2 - 9 = 0$ (d) $c^2 - 10 = 0$

Solution: (c) Pair of straight lines joining the origin to the points of intersection of the line $y = 2\sqrt{2}x + c$ and the circle $x^2 + y^2 = 2$ are

$$\Rightarrow \quad x^2 + y^2 + (-2)\left(\frac{2\sqrt{2}x - y}{-c}\right)^2 = 0 \Rightarrow x^2 + y^2 - \frac{2}{c^2}\left(8x^2 + y^2 - 4\sqrt{2}xy\right) = 0 \Rightarrow x^2\left(1 - \frac{16}{c^2}\right) + y^2\left(1 - \frac{2}{c^2}\right) + \frac{8\sqrt{2}xy}{c^2} = 0$$

If these lines are perpendicular, $1 - \frac{16}{c^2} + 1 - \frac{2}{c^2} = 0$

$$\Rightarrow \quad \frac{2c^2 - 18}{c^2} = 0 \Rightarrow c^2 - 9 = 0.$$

3.6 Removal of First degree Terms

Let point of intersection of lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ......(i) is (α, β).

Here $(\alpha, \beta) = \left(\frac{bg - fh}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$

For removal of first degree terms, shift the origin to (α, β) *i.e.*, replacing *x* by $(X + \alpha)$ and *y* be $(Y + \beta)$ in (i).

Alternative Method : Direct equation after removal of first degree terms is

$$aX^2 + 2hXY + bY^2 + (g\alpha + f\beta + c) = 0$$

Where $\alpha = \frac{bg - fh}{h^2 - ab}$ and $\beta = \frac{af - gh}{h^2 - ab}$

3.7 Removal of the Term *xy* from *f* (*x*, *y*) = $ax^2 + 2hxy + by^2$ without changing the Origin

Clearly, $h \neq 0$. Rotating the axes through an angle θ, we have,

$x = X\cos\theta - Y\sin\theta$ and $y = X\sin\theta + Y\cos\theta$

$\therefore \quad f(x, y) = ax^2 + 2hxy + by^2$

After rotation, new equation is

$$F(X,Y) = (a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)X^2 + 2\{(b-a)\cos\theta\sin\theta + h(\cos^2\theta - \sin^2\theta)XY + (a\sin^2\theta - 2h\cos\theta\sin\theta + b\cos^2\theta)Y^2$$

Now coefficient of XY = 0. Then we get $\cot 2\theta = \frac{a - b}{2h}$

Note : ❑ Usually, we use the formula, $\tan 2\theta = \frac{2h}{a - b}$ for finding the angle of rotation, θ. However, if $a = b$, we use $\cot 2\theta = \frac{a - b}{2h}$ as in this case $\tan 2\theta$ is not defined.

Example: 13 The new equation of curve $12x^2 + 7xy - 12y^2 - 17x - 31y - 7 = 0$ after removing the first degree terms

(a) $12X^2 - 7XY - 12Y^2 = 0$ (b) $12X^2 + 7XY + 12Y^2 = 0$

(c) $12X^2+7XY-12Y^2=0$ (d) None of these

Solution: (c) Let $\phi \equiv 12x^2+7xy-12y^2-17x-31y-7=0$ .....(i)

$\therefore \frac{\partial\phi}{\partial x} \equiv 24x+7y-17=0$ and $\frac{\partial\phi}{\partial y} \equiv 7x-24y-31=0$

Their point of intersection is $(x, y) \equiv (1, -1)$

Here $\alpha=1, \beta=-1$

Shift the origin to (1, –1) then replacing $x=X+1$ and $y=Y-1$ in (i), the required equation is

$12(X+1)^2+7(X+1)(Y-1)-12(Y-1)^2-17(X+1)-31(Y-1)-7=0$ *i.e.*, $12X^2+7XY-12Y^2=0$

Alternative Method : Here $\alpha=1$ and $\beta=-1$ and $g=-17/2, f=-31/2, c=-7$

$\therefore \quad g\alpha+f\beta+c=-\frac{17}{2}\times 1-\frac{31}{2}\times -1-7=0$

$\therefore$ Removed equation is $aX^2+2hXY+bY^2+(g\alpha+f\beta+c)=0$

i.e., $12X^2+7XY-12Y^2+0=0 \Rightarrow 12X^2+7XY-12Y^2=0$.

Example: 14 Mixed term *xy* is to be removed from the general equation $ax^2+by^2+2hxy+2gx+2fy+c=0$, one should rotate the axes through an angle θ given by $\tan 2\theta=$

(a) $\frac{a-b}{2h}$ (b) $\frac{2h}{a+b}$ (c) $\frac{a+b}{2h}$ (d) $\frac{2h}{a-b}$

Solution: (d) Let (x', y') be the coordinates on new axes, then put $x=x'\cos\theta-y'\sin\theta,\ y=x'\sin\theta+y'\cos\theta$ in the equation, then the coefficient of *xy* in the transformed equation is 0.

So, $2(b-a)\sin\theta.\cos\theta+2h\cos 2\theta=0 \Rightarrow \tan 2\theta=\frac{2h}{a-b}$

3.8 Distance between the Pair of parallel Straight lines

If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represent a pair of parallel straight lines, then the distance between them is given by $2\sqrt{\frac{g^2-ac}{a(a+b)}}$ or $2\sqrt{\frac{f^2-bc}{b(a+b)}}$

Example: 15 Distance between the pair of lines represented by the equation $x^2-6xy+9y^2+3x-9y-4=0$ **[Kerala (Engg.) 20**

(a) $\frac{15}{\sqrt{10}}$ (b) $\frac{1}{2}$ (c) $\sqrt{\frac{5}{2}}$ (d) $\frac{1}{\sqrt{10}}$

Solution: (c) The distance between the pair of straight lines given by

$ax^2+2hxy+by^2+2gx+2fy+c=0$ is $2\sqrt{\frac{g^2-ac}{a(a+b)}}$, Here $a=1, b=9, c=4,\ g=\frac{3}{2}=2\times\sqrt{\frac{\frac{9}{4}-(-4)}{1(1+9)}}=2\times\sqrt{\frac{\frac{25}{4}}{10}}=\sqrt{\frac{5}{2}}$

Example: 16 Distance between the lines represented by the equation $x^2+2\sqrt{3}xy+3y^2-3x-3\sqrt{3}y-4=0$ is **[Roorkee 1989]**

(a) 5/2 (b) 5/4 (c) 5 (d) 0

Solution: (a) First check for parallel lines *i.e.*, $\frac{a}{h}=\frac{h}{b}=\frac{g}{f} \Rightarrow \frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}=\frac{\frac{-3}{2}}{\frac{-3\sqrt{3}}{2}}$

which is true, hence lines are parallel. $\therefore$ Distance between them is $2\sqrt{\frac{g^2-ac}{a(a+b)}}=2\sqrt{\frac{(-3/2)^2-1(-4)}{1(1+3)}}$

$=5/2$

3.9 Some Important Results

(1) The lines joining the origin to the points of intersection of the curves $ax^2+2hxy+by^2+2gx=0$ and $a'x^2+2h'xy+b'y^2+2g'x=0$ will be mutually perpendicular, if $g(a'+b')=g'(a+b)$.

(2) If the equation $hxy+gx+fy+c=0$ represents a pair of straight lines, then $fg=ch$.

(3) The pair of lines $(a^2-3b^2)x^2+8abxy+(b^2-3a^2)\ y^2=0$ with the line $ax+by+c=0$ form an equilateral triangle.

(4) The area of a triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ is given by $\dfrac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}$

(5) The lines joining the origin to the points of intersection of line $y=mx+c$ and the circle $x^2+y^2=a^2$ will be mutually perpendicular, if $a^2(m^2+1)=2c^2$.

(6) If the distance of two lines passing through origin from the point (x_1,y_1) is *d,* then the equation of lines is $(xy_1-yx_1)^2=d^2(x^2+y^2)$

(7) The lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ will be equidistant from the origin, if $f^4-g^4=c(bf^2-ag^2)$

(8) The product of the perpendiculars drawn from (x_1,y_1) on the lines $ax^2+2hxy+by^2=0$ is given by

$$\frac{ax_1^2+2hx_1y_1+by_1^2}{\sqrt{(a-b)^2+4h^2}}$$

(9) The product of the perpendiculars drawn from origin on the lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ is

$$\frac{c}{\sqrt{(a-b)^2+4h^2}}$$

(10) If the lines represented by the general equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ are perpendicular, then the square of distance between the point of intersection and origin is $\dfrac{f^2+g^2}{h^2+b^2}$

(11) The square of distance between the point of intersection of the lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ and origin is $\dfrac{c(a+b)-f^2-g^2}{ab-h^2}$

Example: 17 The area of the triangle formed by the lines $4x^2-9xy-9y^2=0$ and $x=2$ is **[Roorkee 2000]**

(a) 2 (b) 3 (c) $\frac{10}{3}$ (d) $\frac{20}{3}$

Solution: (c) The area of triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ is given by $\left|\dfrac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}\right|$

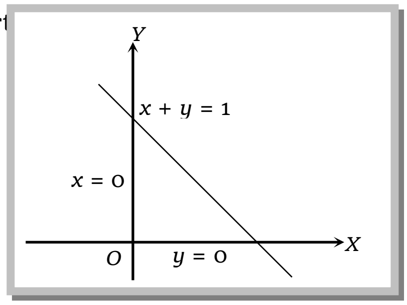
Here $a=4,b=-9,h=-\frac{9}{2},l=1,m=0,n=-2$, then area of triangle

$$= \left| \frac{(-2)^2\sqrt{\left(\frac{-9}{2}\right)^2 - 4\times\frac{-9}{2}}}{-9\times(1)^2} \right| = \left| \frac{4\sqrt{\frac{81}{4}+\frac{36}{2}}}{-9} \right| = \left|\frac{-30}{9}\right| = \frac{10}{3}$$

Example: 18 The orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 1$ is **[IIT 1995]**

(a) (0, 0) (b) $\left(\frac{1}{2},\frac{1}{2}\right)$ (c) $\left(\frac{1}{3},\frac{1}{3}\right)$ (d) $\left(\frac{1}{4},\frac{1}{4}\right)$

Solution: (a) Lines represented by $xy = 0$ is $x = 0$, $y = 0$. Then the triangle formed is right angled triangle at O(0, 0), therefore O(0, 0) is its ort



Example: 19 If the pair of straight lines given by $Ax^2 + 2Hxy + By^2 = 0, (H^2 > AB)$ forms an equilateral triangle with line $ax + by + c = 0$ then $(A + 3B)(3A + B)$ is **[EAMCET 2003]**

(a) H^2 (b) $-H$ (c) $2H^2$ (d) $4H^2$

Solution: (d) We know that the pair of lines $(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$ with the line $ax + by + c = 0$ form an equilateral triangle. Hence comparing with $Ax^2 + 2Hxy + By^2 = 0$ then $A = a^2 - 3b^2$, $B = b^2 - 3a^2$, $2H = 8ab$

Now $(A + 3B)(3A + B) = (-8a^2)(-8b^2) \Rightarrow (8ab)^2 = (2H)^2 = 4H^2$.

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Assignment

Equation of Pair of Straight lines

Basic Level

1. The values of h for which the equation $3x^2 + 2hxy - 3y^2 - 40x + 30y - 75 = 0$ represents a pair of straight lines, are**[MP PE**

(a) 4, 4 (b) 4, 6 (c) 4, -4 (d) 0, 4

2. Which of the following second degree equation represents a pair of straight lines **[MP PET 1990]**

(a) $x^2 - xy - y^2 = 1$ (b) $-x^2 + xy - y^2 = 1$ (c) $4x^2 - 4xy + y^2 = 4$ (d) $x^2 + y^2 = 4$

3. The equation $2y^2 - xy - x^2 + 6x - 8 = 0$ represents **[MP PET 1992]**

(a) A pair of straight lines (b) A circle (c) An ellipse (d)

4. One of the lines represented by the equation $x^2 + 6xy = 0$ is

(a) Parallel to x-axis (b) Parallel to y-axis (c) x-axis (d) y-axis

5. The equation $x^2 - 7xy + 12y^2 = 0$ represents a **[Ranchi BIT 1991]**

(a) Circle (b) Pair of parallel straight lines

(c) Pair of perpendicular straight lines (d) Pair of non-perpendicular intersecting straight lines

6. The equation $y^2 - x^2 + 2x - 1 = 0$ represents **[MNR 1991]**

(a) A pair of straight lines (b) A circle (c) A parabola (d)

7. If the equation $\lambda x^2 + 2y^2 - 5xy + 5x - 7y + 3 = 0$ represents two straight lines, then the value of λ will be

(a) 3 (b) 2 (c) 8 (d) – 8

8. The joint equation of the straight lines $x + y = 1$ and $x - y = 4$ is

(a) $x^2 - y^2 = -4$ (b) $x^2 - y^2 = 4$ (c) $(x + y - 1)(x - y - 4) = 0$ (d) $(x + y + 1)(x - y + 4) = 0$

9. The value of λ for which the equation $x^2 - \lambda xy + 2y^2 + 3x - 5y + 2 = 0$ may represent a pair of straight lines is **[Kurukshetra CEE 1996]**

(a) 2 (b) 3 (c) 4 (d) 1

10. $2x^2 + 7xy + 3y^2 + 8x + 14y + \lambda = 0$ will represent a pair of straight lines, when $\lambda =$ **[MP PET 1996]**

(a) 2 (b) 4 (c) 6 (d) 8

11. If $Lx^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight line, then L is **[MP PET 2001]**

(a) 1 (b) 2 (c) 3 (d) -1

12. Separate equations of lines, for a pair of lines, whose equation is $x^2 + xy - 12y^2 = 0$, are

(a) $x + 4y = 0$ and $x + 3y = 0$ (b) $2x - 3y = 0$ and $x - 4y = 0$

(c) $x - 6y = 0$ and $x - 3y = 0$ (d) $x + 4y = 0$ and $x - 3y = 0$

13. If the equation $2x^2 + 7xy + 3y^2 - 9x - 7y + k = 0$ represents a pair of lines, then k is equal to

(a) 4 (b) 2 (c) 1 (d) – 4

14. If equation $3x^2 + xy - y^2 - 3x + 6y + k = 0$ represents a pair of lines, then k is equal to **[Karnataka CET 2002]**

(a) 9 (b) 1 (c) 0 (d) – 9

15. Equation $3x^2+7xy+2y^2+5x+3y+2=0$ represents **[UPSEAT 2002]**

(a) Pair of straight lines (b) Ellipse (c) Hyperbola (d) None of these

16. For what value of 'p' , $y^2+xy+px^2-x-2y=0$ represents two straight lines **[UPSEAT 2002]**

(a) 2 (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

17. If $6x^2+11xy-10y^2+x+31y+k=0$ represents a pair of straight lines, then k = **[MP PET 1991]**

(a) –15 (b) 6 (c) –10 (d) –4

18. If the equation $x^2+y^2+2gx+2fy+1=0$ represents a pair of lines, then **[Karnataka CET 1999]**

(a) $g^2-f^2=1$ (b) $f^2-g^2=1$ (c) $g^2+f^2=1$ (d) $f^2+g^2=1/2$

19. The equation $x^2+kxy+y^2-5x-7y+6=0$ represents a pair of straight lines, then k is

(a) $\frac{5}{3}$ (b) $\frac{10}{3}$ (c) $\frac{3}{2}$ (d) $\frac{3}{10}$

20. The equation $2x^2+4xy-ky^2+4x+2y-1=0$ represents a pair of lines. The value of k is

(a) $-\frac{5}{3}$ (b) $\frac{5}{3}$ (c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

21. The equation $4x^2-24xy+11y^2=0$ represents **[Orissa JEE 2003]**

(a) Two parallel lines (b) Two perpendicular lines (c) Two lines through the origin (d) A circle

22. The value of k so that the equation $2x^2+5xy+3y^2+6x+7y+k=0$ represents a pair of straight lines, is

(a) 4 (b) 6 (c) 0 (d) 8

23. The equation to the pair of straight lines through the origin which are perpendicular to the lines $2x^2-5xy+y^2=0$, is **[MP PET 1990]**

(a) $2x^2+5xy+y^2=0$ (b) $x^2+2y^2+5xy=0$ (c) $x^2-5xy+2y^2=0$ (d) $2x^2+y^2-5xy=0$

24. The equation $xy+a^2=a(x+y)$ represents **[MP PET 1991]**

(a) A parabola (b) A pair of straight lines (c) An ellipse (d) Two parallel straight lines

25. If the equation $Ax^2+2Bxy+Cy^2+Dx+Ey+F=0$ represents a pair of straight lines, then B^2-AC

(a) < 0 (b) = 0 (c) > 0 (d) None of these

26. The equation of pair of straight lines perpendicular to the pair $ax^2+2hxy+by^2=0$ is **[MP PET 1989]**

(a) $ax^2-2hxy+by^2=0$ (b) $bx^2+2hxy+ay^2=0$ (c) $ay^2-2hxy+bx^2=0$ (d) $ay^2-bx^2=0$

27. If the equation $ax^2+2hxy+by^2=0$ represents two lines $y=m_1x$ and $y=m_2x$, then **[Kurukshetra CEE 1993; MP PET 1988]**

(a) $m_1+m_2=\frac{-2h}{b}$ and $m_1m_2=\frac{a}{b}$ (b) $m_1+m_2=\frac{2h}{b}$ and $m_1m_2=\frac{-a}{b}$

(c) $m_1+m_2=\frac{2h}{b}$ and $m_1m_2=\frac{a}{b}$ (d) $m_1+m_2=\frac{2h}{b}$ and $m_1m_2=-ab$

28. Difference of slopes of the lines represented by equation $x^2(\sec^2\theta-\sin^2\theta)-2xy\tan\theta+y^2\sin^2\theta=0$ is

(a) 4 (b) 3 (c) 2 (d) None of these

29. If the ratio of gradients of the lines represented by $ax^2+2hxy+by^2=0$ is 1 : 3, then the value of the ratio $h^2:ab$ is **[MP PET 1998]**

(a) $\frac{1}{3}$ (b) $\frac{3}{4}$ (c) $\frac{4}{3}$ (d) 1

30. If the sum of slopes of the pair of lines represented by $4x^2+2hxy-7y^2=0$ is equal to the product of the slopes, then the value of h is

(a) –6 (b) –2 (c) –4 (d) 4

31. The gradient of one of the lines of $ax^2+2hxy+by^2=0$ is twice that of the other, then **[MP PET 2000]**

(a) $h^2 = ab$ (b) $h = a+b$ (c) $8h^2 = 9ab$ (d) $9h^2 = 8ab$

32. If the slope of one line of the pair of lines represented by $ax^2 + 4xy + y^2 = 0$ is 3 times the slope of the other line, then a is **[DCE 1999]**

(a) 1 (b) 2 (c) 3 (d) 4

33. If the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is 5 times the other, then

(a) $5h^2 = ab$ (b) $5h^2 = 9ab$ (c) $9h^2 = 5ab$ (d) $h^2 = ab$

34. The value of k such that $3x^2 - 11xy + 10y^2 - 7x + 13y + k = 0$ may represent a pair of straight lines, is

(a) 3 (b) 4 (c) 6 (d) 8

35. If $x^2 - kxy + y^2 + 2y + 2 = 0$ denotes a pair of straight lines, then k =

(a) 2 (b) $\frac{1}{\sqrt{2}}$ (c) $2\sqrt{2}$ (d) $\sqrt{2}$

36. The equation $4x^2 + mxy - 3y^2 = 0$ represents a pair of real and distinct lines if

(a) $m \in R$ (b) $m \in (3,4)$ (c) $m \in (-3,4)$ (d) $m > 4$

37. Lines represented by $9x^2 + y^2 + 6xy - 4 = 0$ are **[EAMCET 1988]**

(a) Coincident (b) Parallel but not coincident (c) Not parallel (d) Perpendicular

38. If $kx^2 + 10xy + 3y^2 - 15x - 21y + 18 = 0$ represents a pair of straight lines, then k = **[Kurukshetra CEE 1982]**

(a) 3 (b) 4 (c) –3 (d) None of these

39. Equation of pair of straight lines drawn through (1, 1) and perpendicular to the pair of lines $3x^2 - 7xy - 2y^2 = 0$ is

[Roorkee 1984: MNR 1988]

(a) $2x^2 + 7xy - 11x + 6 = 0$ (b) $2(x-1)^2 + 7(x-1)(y-1) - 3y^2 = 0$

(c) $2(x-1)^2 + 7(x-1)(y-1) + 3(y-1)^2 = 0$ (d) None of these

40. If the lines represented by the equation $2x^2 - 3xy + y^2 = 0$ make angles α and β with x-axis, then $\cot^2\alpha + \cot^2\beta =$

(a) 0 (b) $\frac{3}{2}$ (c) $\frac{7}{4}$ (d) $\frac{5}{4}$

41. If one of the lines given by $6x^2 - xy + 4cy^2 = 0$ is $3x + 4y = 0$, then c equals **[AIEEE 2004]**

(a) –3 (b) –1 (c) 3 (d) 1

42. If $ax^2 - y^2 + 4x - y = 0$ represents a pair of lines, then a = **[Karnataka CET 2004]**

(a) –16 (b) 16 (c) 4 (d) – 4

43. The value of λ, for which the equation $x^2 - y^2 - x + \lambda y - 2 = 0$ represent a pair of straight lines, are

(a) 3, –3 (b) –3, 1 (c) 3, 1 (d) –1, 1

Advance Level

44. The equation $\sqrt{(x-2)^2 + y^2} + \sqrt{(x+2)^2 + y^2} = 4$ represents a

(a) Circle (b) Pair of straight lines (c) Parabola (d) Ellipse

45. The locus of the point $P(x,y)$ satisfying the relation $\sqrt{(x-3)^2 + (y-1)^2} + \sqrt{(x+3)^2 + (y-1)^2} = 6$ is a

(a) Straight line (b) Pair of straight lines (c) Circle (d) Ellipse

46. If the equation $12x^2 + 7xy - py^2 - 18x + qy + 6 = 0$ represents a pair of perpendicular straight lines, then

(a) $p = 12,\ q = 1$ (b) $p = 1, q = 12$ (c) $p = -1, q = 12$ (d) $p = 1, q = -12$

47. The equation of the pair of straight lines parallel to x-axis and touching the circle $x^2 + y^2 - 6x - 4y - 12 = 0$ is **[Kerala (En**

(a) $y^2 - 4y - 21 = 0$ (b) $y^2 + 4y - 21 = 0$ (c) $y^2 - 4y + 21 = 0$ (d) $y^2 + 4y + 21 = 0$

48. Two pairs of straight lines have the equations $y^2 + xy - 12x^2 = 0$ and $ax^2 + 2hxy + by^2 = 0$. One line will be common among them if

(a) $a = -3(2h + 3b)$ (b) $a = 8(h - 2b)$ (c) $a = 2(b + h)$ (d) $a = -3(b + h)$

49. If $u \equiv a_1x + b_1y + c_1 = 0$, $v \equiv a_2x + b_2y + c_2 = 0$ and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then curve $u + kv = 0$ is **[MNR 1987]**

(a) A line represented by u (b) A different line (c) Not a line (d)

50. If one of the line represented by the equation $ax^2 + 2hxy + by^2 = 0$ is coincident with one of the line represented by $a'x^2 + 2h'xy + b'y^2 = 0$, then

(a) $(ab' - a'b)^2 = 4(ah' - a'h)(hb' - h'b)$ (b) $(ab' + a'b)^2 = 4(ah' - a'h)(hb' - h'b)$

(c) $(ab' - a'b)^2 = (ah' - a'h)(hb' - h'b)$ (d) None of these

Angle between the Pair of Lines

Basic Level

51. The angle between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ is given by

(a) $\tan\theta = \frac{2(h^2 - ab)}{(a+b)}$ (b) $\tan\theta = \frac{2\sqrt{(h^2 - ab)}}{(a+b)}$ (c) $\tan\theta = \frac{2(h^2 - ab)}{\sqrt{a+b}}$ (d) $\tan\theta = \frac{2\sqrt{h^2 + ab}}{(a+b)}$

52. The angle between the pair of straight lines $x^2 - y^2 - 2y - 1 = 0$, is

(a) 90^o (b) 60^o (c) 75^o (d) 36^o

53. If the angle 2θ is acute, then the acute angle between $x^2(\cos\theta - \sin\theta) + 2xy\cos\theta + y^2(\cos\theta + \sin\theta) = 0$ is **[EAMCET 2002]**

(a) 2θ (b) $\frac{\theta}{3}$ (c) θ (d) $\frac{\theta}{2}$

54. The angle between the pair of lines $2x^2 + 5xy + 2y^2 + 3x + 3y + 1 = 0$ is **[EAMCET 1994]**

(a) $\cos^{-1}\left(\frac{4}{5}\right)$ (b) $\tan^{-1}\left(\frac{4}{5}\right)$ (c) 0 (d) $\frac{\pi}{2}$

55. The equation $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$ when λ is a real number, represents a pair of straight lines. If θ is the angle between the lines, then $\text{cosec}^2\theta =$

(a) 3 (b) 9 (c) 10 (d) 100

56. The equation $12x^2 + 7xy + ay^2 + 13x - y + 3 = 0$ represents a pair of perpendicular lines. Then the value of 'a' is **[Karnataka CET**

(a) $\frac{7}{2}$ (b) –19 (c) –12 (d) 12

57. The angle between the lines $x^2 + 4xy + y^2 = 0$ is **[Karnataka CET 2001]**

(a) 60^o (b) 15^o (c) 30^o (d) 45^o

58. If the angle between the two lines represented by $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ is $\tan^{-1} m$, then $m =$ **[MNR 1993]**

(a) $\frac{1}{5}$ (b) 1 (c) $\frac{7}{5}$ (d) 7

59. Pair of straight lines perpendicular to each other represented by

(a) $2x^2 = 2y(2x + y)$ (b) $x^2 + y^2 + 3 = 0$ (c) $2x^2 = y(2x + y)$ (d) $x^2 = 2(x - y)$

60. The angle between the pair of straight lines $x^2 + 4y^2 - 7xy = 0$, is

[MNR 1983; Kurukshetra CEE 1999]

(a) $\tan^{-1}\left(\frac{1}{3}\right)$ (b) $\tan^{-1}(3)$ (c) $\tan^{-1}\left(\frac{\sqrt{33}}{5}\right)$ (d) $\tan^{-1}\left(\frac{5}{\sqrt{33}}\right)$

61. The angle between the pair of straight lines $y^2\sin^2\theta - xy\sin^2\theta + x^2(\cos^2\theta - 1) = 1$, is **[MNR 1985; UPSEAT 2000]**

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{2\pi}{3}$ (d) None of these

62. The angle between the pair of lines given by equation $x^2 + 2xy - y^2 = 0$, is **[MNR 1990]**

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{2}$ (d) 0

63. Acute angle between the lines represented by $(x^2+y^2)\sqrt{3} = 4xy$ is

(a) $\pi/6$ (b) $\pi/4$ (c) $\pi/3$ (d) None of these

64. The angle between the lines given by $x^2 - y^2 = 0$ is **[MP PET 1999]**

(a) 15^o (b) 45^o (c) 75^o (d) 90^o

65. The angle between the lines $xy = 0$ is **[MP PET 1990, 92]**

(a) 45^o (b) 60^o (c) 90^o (d) 180^o

66. The angle between the lines represented by the equation $4x^2 - 24xy + 11y^2 = 0$ are

(a) $\tan^{-1}\left(\frac{3}{4}\right), \tan^{-1}\left(-\frac{3}{4}\right)$ (b) $\tan^{-1}\left(\frac{1}{3}\right), \tan^{-1}\left(\frac{-1}{3}\right)$ (c) $\tan^{-1}\left(\frac{4}{3}\right), \tan^{-1}\left(-\frac{4}{3}\right)$ (d) $\tan^{-1}\left(\frac{1}{2}\right), \tan^{-1}\left(-\frac{1}{2}\right)$

67. Condition that the two lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ to be perpendicular is **[Kurukshetra CEE 1998; MP PET 2001]**

(a) $ab = -1$ (b) $a = -b$ (c) $a = b$ (d) $ab = 1$

68. The straight lines represented by the equation $9x^2 - 12xy + 4y^2 = 0$ are

(a) Coincident (b) Perpendicular (c) Parallel (d) Inclined at an angle of 45^o

69. The nature of straight lines represented by the equation $4x^2 + 12xy + 9y^2 = 0$ is **[MP PET 1988]**

(a) Real and coincident (b) Real and different (c) Imaginary and different (d) None of these

70. The equation $x^2 + ky^2 + 4xy = 0$ represents two coincident lines, if $k =$

(a) 0 (b) 1 (c) 4 (d) 16

71. The straight lines joining the origin to the points of intersection of the line $2x + y = 1$ and curve $3x^2 + 4xy - 4x + 1 = 0$ include an angle

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

72. If the acute angles between the pairs of lines $3x^2 - 7xy + 4y^2 = 0$ and $6x^2 - 5xy + y^2 = 0$ be θ_1 and θ_2 respectively, then

(a) $\theta_1 = \theta_2$ (b) $\theta_1 = 2\theta_2$ (c) $2\theta_1 = \theta_2$ (d) None of these

73. The point of lines represented by $3ax^2 + 5xy + (a^2 - 2)y^2 = 0$ and perpendicular to each other for

(a) Two values of *a* (b) For all values of *a* (c) For one value of *a* (d) For no values of *a*

Advance Level

74. The figure formed by the lines $x^2 + 4xy + y^2 = 0$ and $x - y = 4$, is

(a) A right angled triangle (b) An isosceles triangle (c) An equilateral triangle(d)

75. The equation of the pair of straight lines, each of which makes an angle α with the line $y = x$, is

(a) $x^2 + 2xy \sec 2\alpha + y^2 = 0$ (b) $x^2 + 2xy \operatorname{cosec} 2\alpha + y^2 = 0$

(c) $x^2 - 2xy \operatorname{cosec} 2\alpha + y^2 = 0$ (d) $x^2 - 2xy \sec 2\alpha + y^2 = 0$

76. The combined equation of the lines l_1, l_2 is $2x^2 + 6xy + y^2 = 0$ and that of the lines m_1, m_2 is $4x^2 + 18xy + y^2 = 0$. If the angle between l_1 and m_2 be α then the angle between l_2 and m_1 will be

(a) $\frac{\pi}{2} - \alpha$ (b) 2α (c) $\frac{\pi}{4} + \alpha$ (d) α

77. If θ_1 and θ_2 are the angles which the lines $x^2(\tan^2\theta + \cos^2\theta) - 2xy \tan\theta + y^2 \sin^2\theta = 0$ make with the axis of x, then $\tan\theta_1 - \tan\theta_2$ is equal to

(a) $\cos 2\theta$ (b) $2\cos\theta\sin\theta$ (c) 2 (d) 1

Bisectors of the Angles between the Lines

Basic Level

78. The combined equation of bisectors of angles between coordinate axes, is

(a) $x^2 + y^2 = 0$ (b) $x^2 - y^2 = 0$ (c) $xy = 0$ (d) $x + y = 0$

79. The equation of the bisectors of the angle between the lines represented by the equation $x^2 - y^2 = 0$, is

(a) $x = 0$ (b) $y = 0$ (c) $xy = 0$ (d) None of these

80. If $y = mx$ be one of the bisectors of the angle between the lines $ax^2 - 2hxy + by^2 = 0$, then

(a) $h(1+m^2) + m(a-b) = 0$ (b) $h(1-m^2) + m(a+b) = 0$ (c) $h(1-m^2) + m(a-b) = 0$ (d) $h(1+m^2) + m(a+b) = 0$

81. The combined equation of the bisectors of the angle between the lines represented by $(x^2 + y^2)\sqrt{3} = 4xy$ is **[MP PET 1992**

(a) $y^2 - x^2 = 0$ (b) $xy = 0$ (c) $x^2 + y^2 = 2xy$ (d) $\frac{x^2 - y^2}{\sqrt{3}} = \frac{xy}{2}$

82. One bisector of the angle between the lines given by $a(x-1)^2 + 2h(x-1)y + by^2 = 0$ is $2x + y - 2 = 0$. The other bisector is

(a) $x - 2y + 1 = 0$ (b) $2x + y - 1 = 0$ (c) $x + 2y - 1 = 0$ (d) $x - 2y - 1 = 0$

Advance Level

83. If the equation $ax^2 + 2hxy + by^2 = 0$ has the one line as the bisector of angle between the coordinate axes, then **[Bihar CEE 1990; Roorkee 1992]**

(a) $(a-b)^2 = h^2$ (b) $(a+b)^2 = h^2$ (c) $(a-b)^2 = 4h^2$ (d) $(a+b)^2 = 4h^2$

84. If the bisectors of the angles between the pairs of lines given by the equation $ax^2 + 2hxy + by^2 = 0$ and $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ be coincident, then $\lambda =$

(a) a (b) b (c) h (d) Any real number

85. If the bisectors of the angles of the lines represented by $3x^2 - 4xy + 5y^2 = 0$ and $5x^2 + 4xy + 3y^2 = 0$ are same, then the angle made by the lines represented by first with the second, is

(a) 30^o (b) 60^o (c) 45^o (d) 90^o

86. If pairs of straight lines $x^2 - 2mxy - y^2 = 0$ and $x^2 - 2nxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, then $mn =$ **[MP PET 1991;UPSEAT 2001]**

(a) 1 (b) –1 (c) 0 (d) $-\frac{1}{2}$

87. If the lines represented by $x^2 - 2pxy - y^2 = 0$ are rotated about the origin through an angle θ, one in clockwise direction and other in anti –clockwise direction , then the equation of the bisectors of the angle between the lines in the new position is

(a) $px^2 + 2xy - py^2 = 0$ (b) $px^2 + 2xy + py^2 = 0$ (c) $x^2 - 2pxy + y^2 = 0$ (d) None of these

88. If $r(1-m^2) + m(p-q) = 0,$ then a bisector of the angle between the lines represented by the equation $px^2 - 2rxy + qy^2 = 0$ is

(a) $y = x$ (b) $y = -x$ (c) $y = mx$ (d) $my = x$

Point of intersection of the Lines

Basic Level

89. If the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on the y-axis, then **[AIEEE 2002]**

(a) $2fgh = bg^2 + ch^2$ (b) $bg^2 \neq ch^2$ (c) $abc = 2fgh$ (d) None of these

90. The point of intersection of the lines represented by equation $2(x+2)^2 + 3(x+2)(y-2) - 2(y-2)^2 = 0$ is

(a) $(2, 2)$ (b) $(-2, -2)$ (c) $(-2, 2)$ (d) $(2, -2)$

Advance Level

91. The equations to a pair of opposite sides of a parallelogram are $x^2 - 5x + 6 = 0$ and $y^2 - 6y + 5 = 0$. The equations to its diagonals are

(a) $x + 4y = 13$ and $y = 4x - 7$ (b) $4x + y = 13$ and $4y = x - 7$

(c) $4x + y = 13$ and $y = 4x - 7$ (d) $y - 4x = 13$ and $y + 4x = 7$

92. The circumcentre of the triangle formed by the lines $xy + 2x + 2y + 4 = 0$ and $x + y + 2 = 0$ is

(a) $(0, 0)$ (b) $(-2, -2)$ (c) $(-1, -1)$ (d) $(-1, -2)$

93. If the equations of opposite sides of a parallelogram are $x^2 - 7x + 6 = 0$ and $y^2 - 14y + 40 = 0,$ then the equation of its one diagonal is

(a) $6x + 5y + 14 = 0$ (b) $6x - 5y + 14 = 0$ (c) $5x + 6y + 14 = 0$ (d) $5x - 6y + 14 = 0$

94. The limiting position of the point of intersection of the straight lines $3x + 5y = 1$ and $(2+c)x + 5c^2y = 1$ as $c \to 1$ is

(a) $\left(\frac{2}{5}, \frac{-1}{25}\right)$ (b) $\left(\frac{1}{2}, -\frac{1}{10}\right)$ (c) $\left(\frac{3}{8}, \frac{-1}{40}\right)$ (d) None of these

95. If two sides of a triangle are represented by $x^2 - 7xy + 6y^2 = 0$ and the centroid is (1, 0), then the equation of third side is

(a) $2x + 7y + 3 = 0$ (b) $2x - 7y + 3 = 0$ (c) $2x + 7y - 3 = 0$ (d) $2x - 7y - 3 = 0$

96. If the lines $ax^2 + 2hxy + by^2 = 0$ represents the adjacent sides of a parallelogram, then the equation of second diagonal if one is $lx + my = 1$, will be

(a) $(am + hl)x = (bl + hm)y$ (b) $(am - hl)x = (bl - hm)y$ (c) $(am - hl)x = (bl + hm)y$ (d) None of these

Equation of lines joining the origin to the point of intersection of a curve and a Line, Distance between the

Basic Level

97. The lines joining the origin to the points of intersection of the line $3x-2y=1$ and the curve $3x^2+5xy-3y^2+2x+3y=0$, are

(a) Parallel to each other (b) Perpendicular to each other

(c) Inclined at 45^o to each other (d) None of these

98. The distance between the parallel lines $9x^2-6xy+y^2+18x-6y+8=0$ is **[EAMCET 1994]**

(a) $\frac{1}{\sqrt{10}}$ (b) $\frac{2}{\sqrt{10}}$ (c) $\frac{4}{\sqrt{10}}$ (d) $\sqrt{10}$

99. The equation $8x^2+8xy+2y^2+26x+13y+15=0$ represents a pair of straight lines. The distance between them is **[UPSEAT 2001]**

(a) $\frac{7}{\sqrt{5}}$ (b) $\frac{7}{2\sqrt{5}}$ (c) $\frac{\sqrt{7}}{5}$ (d) None of these

100. The equation of second degree $x^2+2\sqrt{2}xy+2y^2+4x+4\sqrt{2}y+1=0$ represents a pair of straight lines. The distance between them is

(a) 4 (b) $\frac{4}{\sqrt{3}}$ (c) 2 (d) $2\sqrt{3}$

101. If the straight lines joining origin to the points of intersections of the line $x+y=1$ with the curve $x^2+y^2+x-2y-m=0$ are perpendicular to each other, then the value of m should be

(a) 0 (b) 1/2 (c) 1 (d) – 1

102. The lines joining the points of intersection of the curve $(x-h)^2+(y-k)^2-c^2=0$ and the line $kx+hy=2hk$ to the origin are perpendicular, then

(a) $c=h\pm k$ (b) $c^2=h^2+k^2$ (c) $c^2=(h+k)^2$ (d) $4c^2=h^2+k^2$

103. The equation of pair of lines joining origin to the points of intersection of $x^2+y^2=9$ and $x+y=3$ is

(a) $(x+y)^2=9$ (b) $x^2+(3-x)^2=9$ (c) $xy=0$ (d) $(3-x)^2+y^2=9$

104. The acute angle formed between the lines joining the origin to the points of intersection of the curves $x^2+y^2-2x-1=0$ and $x+y=1$, is

(a) $\tan^{-1}\left(-\frac{1}{2}\right)$ (b) $\tan^{-1}(2)$ (c) $\tan^{-1}\left(\frac{1}{2}\right)$ (d) 60^o

105. The lines joining the origin to the points of intersection of the line $y=mx+c$ and the circle $x^2+y^2=a^2$ will be mutually perpendicular, if

(a) $a^2(m^2+1)=c^2$ (b) $a^2(m^2-1)=c^2$ (c) $a^2(m^2+1)=2c^2$ (d) $a^2(m^2-1)=2c^2$

106. The angle between lines joining the origin to the points of intersection of the line $x\sqrt{3}+y=2$ and the curve $x^2+y^2=4$ is

[Roorkee 1998]

(a) $\pi/6$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/2$

Advance Level

107. The pair of lines joining the origin to the points of intersection of the curves $ax^2+2hxy+by^2+2gx=0$ and $a'x^2+2h'xy+b'y^2+2g'x=0$ will be at right angles to one another if

(a) $g(a'+b')=g'(a+b)$ (b) $g(a+b)=g'(a'+b')$ (c) $gg'=(a+b)(a'+b')$ (d) None of these

108. The square of distance between the point of intersection of the lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ and origin, is

(a) $\frac{c(a+b)-f^2-g^2}{ab-h^2}$ (b) $\frac{c(a-b)+f^2+g^2}{\sqrt{ab-h^2}}$ (c) $\frac{c(a+b)-f^2-g^2}{ab+h^2}$ (d) None of these

109. If the portion of the line $lx+my=1$ falling inside the circle $x^2+y^2=a^2$ subtends an angle of 45^o at the origin, then

(a) $4[a^2(l^2+m^2)-1]=a^2(l^2+m^2)$ (b) $4[a^2(l^2+m^2)-1]=a^2(l^2+m^2)-2$

(c) $4[a^2(l^2+m^2)-1]=[a^2(l^2+m^2)-2]^2$ (d) None of these

Miscellaneous problems

Basic Level

110. The product of perpendiculars drawn from the origin to the lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ will be

(a) $\frac{ab}{\sqrt{a^2-b^2+4h^2}}$ (b) $\frac{bc}{\sqrt{a^2-b^2+4h^2}}$ (c) $\frac{ca}{\sqrt{(a^2+b^2)+4h^2}}$ (d) $\frac{c}{\sqrt{(a-b)^2+4h^2}}$

111. A curve with equation of the form $y=ax^4+bx^3+cx+d$ has zero gradient at the point (0,1) and also touches the x-axis at the point (–1,0). Then the values of x for which the curve has negative gradients are

(a) $x>-1$ (b) $x<1$ (c) $x<-1$ (d) $-1\le x\le 1$

Advance Level

112. Two of the lines represented by the equation $ay^4+bxy^3+cx^2y^2+dx^3y+ex^4=0$ will be perpendicular, then **[Kurukshetra CE**

(a) $(b+d)(ad+be)+(e-a)^2(a+c+e)=0$ (b) $(b+d)(ad+be)+(e+a)^2(a+c+e)=0$

(c) $(b-d)(ad-be)+(e-a)^2(a+c+e)=0$ (d) $(b-d)(ad-be)+(e+a)^2(a+c+e)=0$

113. Let PQR be a right angled isosceles triangle, right angled at $P(2,1)$. If the equation of the line QR is $2x+y=3$, then the equation representing the pair of lines PQ and PR is

(a) $3x^2-3y^2+8xy+20x+10y+25=0$ (b) $3x^2-3y^2+8xy-20x-10y+25=0$

(c) $3x^2-3y^2+8xy+10x+15y+20=0$ (d) $3x^2-3y^2-8xy-10x-15y-20=0$

114. The area (in square units) of the quadrilateral formed by the two pairs of lines $l^2x^2-m^2y^2-n(lx+my)=0$ and $l^2x^2-m^2y^2+n(lx-my)=0$ is **[EAMCET 2003]**

(a) $\frac{n^2}{2|lm|}$ (b) $\frac{n^2}{|lm|}$ (c) $\frac{n}{2|lm|}$ (d) $\frac{n^2}{4|lm|}$

115. Two lines represented by the equation $x^2-y^2-2x+1=0$ are rotated about the point (1, 0), the line making the bigger angle with the positive direction of the x-axis being turned by 45^o in the clockwise sense and the other line being turned by 15^o in the anticlockwise sense. The combined equation of the pair of lines in their new positions is

(a) $\sqrt{3}x^2-xy+2\sqrt{3}x-y+\sqrt{3}=0$ (b) $\sqrt{3}x^2-xy-2\sqrt{3}x+y+\sqrt{3}=0$

(c) $\sqrt{3}x^2-xy-2\sqrt{3}x+\sqrt{3}=0$ (d) None of these

116. The combined equation of three sides of a triangle is $(x^2 - y^2)(2x + 3y - 6) = 0$. If $(-2, a)$ is an interior point and $(b, 1)$ is an exterior point of the triangle, then

(a) $2 < a < \frac{10}{3}$ (b) $-2 < a < \frac{10}{3}$ (c) $-1 < b < \frac{9}{2}$ (d) $-1 < b < 1$

117. The diagonals of a square are along the pair of lines whose equation is $2x^2 - 3xy - 2y^2 = 0$. If (2, 1) is a vertex of the square, then another vertex consecutive to it can be

(a) $(1, -2)$ (b) $(1, 4)$ (c) $(-1, 2)$ (d) $(-1, -4)$

118. The equation $x^3 - 6x^2y + 11xy^2 - 6y^3 = 0$ represent three straight lines passing through the origin, the slopes of which form an

(a) A.P. (b) G.P. (c) H.P. (d) None of these

119. If P_1, P_2 denote the length of the perpendiculars from the point $(2, 3)$ on the lines given by $15x^2 + 31xy + 14y^2 = 0$ then

(a) $P_1 + P_2 = \frac{31}{14}$ (b) $|P_1 - P_2| = \frac{31}{\sqrt{74}} - \frac{12}{\sqrt{13}}$ (c) $P_1 P_2 = \frac{372}{\sqrt{962}}$ (d) $P_1 P_2 = \frac{15}{14}$

120. The equation of the locus of feet of perpendicular drawn from the origin to the line passing through a fixed point (a, b) is

(a) $x^2 + y^2 - ax - by = 0$ (b) $x^2 + y^2 + ax + by = 0$ (c) $x^2 + y^2 - 2ax - 2by = 0$ (d) None of these

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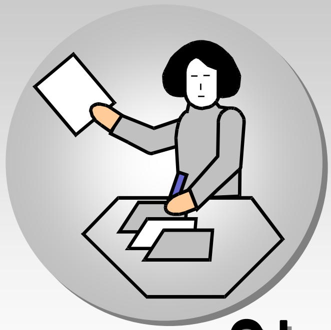

Answer Sheet

Pair of Straight Lines

Assignment (Basic and Advance Level)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a	c	a	d	d	a	b	c	b	d	b	d	a	d	a	c	a	c	b	a
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
c	a	b	b	d	c	a	c	c	b	c	c	b	b	d	a	b	a	d	d
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
a	b	a	b	b	a	a	a,b	a	a	b	a	c	a	c	c	a	a	a	c
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
d	c	a	d	c	c	b	a	a	c	a	a	a	c	d	d	c	b	c	c
81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
a	d	d	d	d	b	a	c	a	c	c	c	b	a	d	b	b	b	b	c
101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
b	b	c	b	c	c	a	a	c	d	c	a	b	a	b	a,d	a,c	c	b,c	a