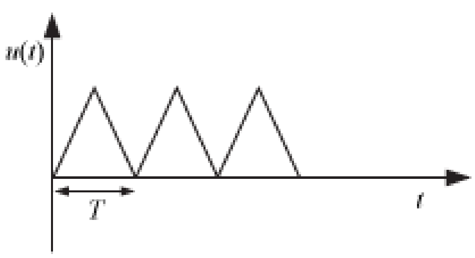
Oscillations

Periodic and Oscillatory Motions

- Periodic motion: A motion which repeats itself after a fixed interval of time
- Examples:
- Motion of the moon around the earth
- Motion of the hands of a clock
- Oscillatory motion: A body in oscillatory motion moves to and fro about its mean position in a fixed time interval.

  Examples:
- Motion of the pendulum of a wall clock
- Motion of the liquid contained in a U-tube when one of its limbs is compressed.
- Period (*T*): It is the interval of time after which a motion is repeated. Its unit is seconds (s).

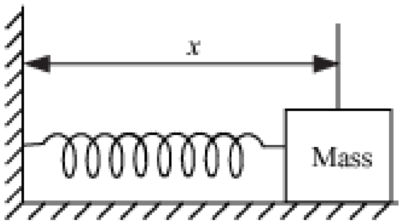


- Frequency (ν): Number of repetitions that occur per unit time

$$\nu = \frac{1}{T}$$

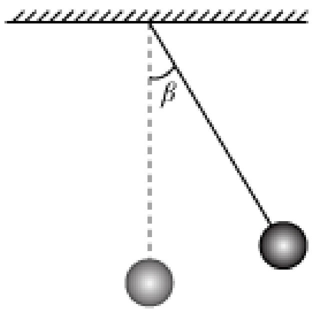
Its unit is $(\text{second})^{-1}$ or Hertz.

- Displacement: Change in position



The figure shows a block attached to a spring.

Here, displacement is *x*.



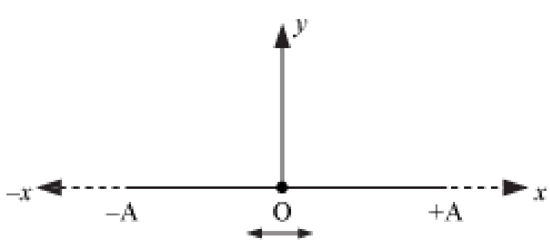
An oscillating simple pendulum's angular displacement is*β*.

- Displacement variable may take negative values.
- Periodic functions can be expressed as a superposition of the sine and cosine functions.

Simple Harmonic Motion and Uniform Circular Motion

- An oscillatory motion is said to be simple harmonic, when the displacement (*x*) of the particle from origin varies with time given as,

$$x(t) = A\cos(\omega t + \phi)$$



Displacement is sinusoidal function of time.

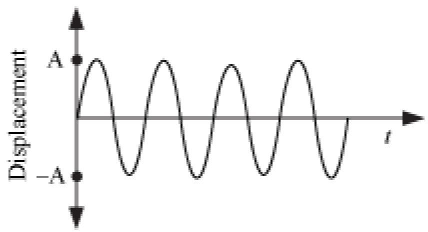
Where, *x* (*t*) → Displacement *x* as function of time *t*

A → Amplitude

ω → Angular frequency

t → Time

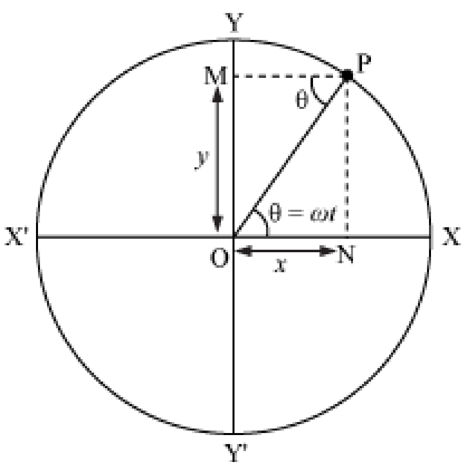
Φ → Phase constant



Displacement − A continuous function of time for SHM

- Non-harmonic oscillation is a combination of two or more harmonic oscillation.

Geometrical interpretation of simple harmonic motion



- Particle P(reference particle) goes around the circle and completes one revolution. The projection M moves to and fro about the centre O along diameter $Y'OY$.
- Motion of projection M on a diameter $Y'OY$ is called SHM.
- SHM is defined as the projection of uniform circular motion on the diameter of a circle of reference.

Where,

a → Radius of circle

ω → Angular velocity

θ → Angle

t → Time

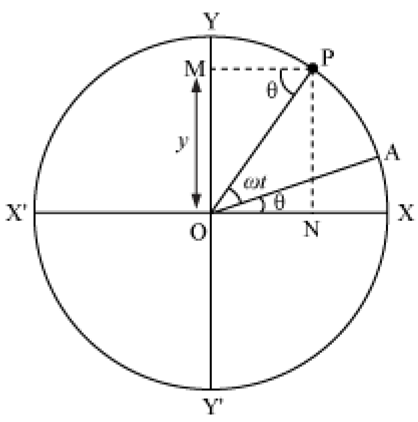
y → Displacement

In Δ*O*PM,

$$\sin\theta = \frac{OM}{OP} = \frac{y}{a}$$

$$\boxed{\therefore y = a\sin\theta = a\sin\omega t}$$

- Consider the particle has some phase (Φ_0) initially.



Here, $\theta = \omega t + \Phi_0$

$$\boxed{\therefore y = a\sin(\omega t + \phi_0)}$$

- Amplitude − Maximum displacement on either side of the mean position

Maximum value of *y* is *a*.

Velocity and Acceleration in Simple Harmonic Motion and Force Law for SHM

Velocity

- Velocity is obtained by directly differentiating displacement with respect to time (*t*).

  $y(t) = A \sin(\omega t + \Phi_0)$ …(i)

  A = Amplitude

$$v(t) = \frac{d}{dt}(y(t))$$

$$\boxed{v(t) = \omega A \cos(\omega t + \phi_0)}$$

Acceleration

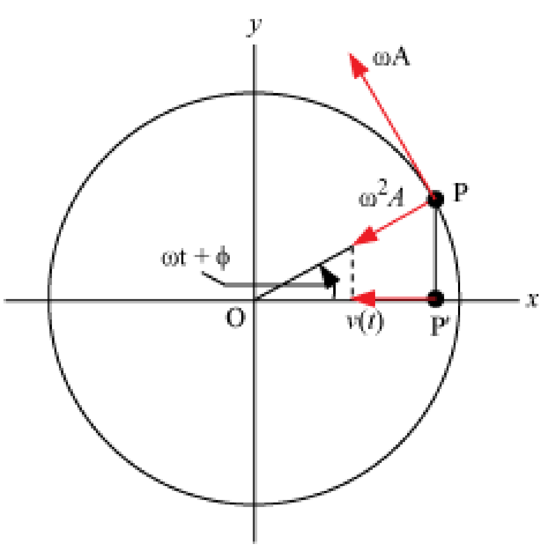
- Acceleration is obtained by differentiating velocity [$v(t)$] with respect to time (*t*).

$$a(t) = \frac{d}{dt}v(t)$$

$$a(t) = \frac{d}{dt}(\omega A \cos(\omega t + \phi_0))$$

$$a(t) = -\omega^2 A \sin(\omega t + \phi)$$

$$a(t) = -\omega^2 y(t) \qquad \text{[From (i)]}$$



SHM can be defined as the periodic motion in which acceleration is directly proportional to the displacement from the mean position; it is always directed towards the mean position.

Force Law for SHM

- According to Newton's second law of motion, Force, $F(t) = ma$

m = Mass

a = Acceleration $[-\omega^2 y(t)]$

$\therefore F(t) = -m\omega^2 y(t)$

$= -k\, y\,(t)$

$\therefore k = m\omega^2$

$$\omega = \sqrt{\frac{k}{m}}$$

Here, k is the spring constant.

- Force acting in simple harmonic motion is proportional to displacement and is always directed towards the centre of motion.

Maximum and Minimum Displacement

The displacement of a particle in SHM is a function of time that is given by
$x = a\sin(\omega t + \Phi_0)$

- At mean position, $t = 0$ and $\Phi_0 = 0$.
  $\therefore \ \omega t + \phi_0 = 0$
  And,
  $x = 0$
- At extreme positions, t = 0 and $x = \mp a$.
  $t = 0$ and $\alpha = \frac{\pi}{2}, \frac{3\pi}{2}$
  $\therefore \ \omega t + \alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$
  And,
  $x = a\sin\frac{\pi}{2} = a \times 1 = a$
  $\Rightarrow x = a\sin\frac{3\pi}{2} = a \times (-1) = -a$

Maximum and Minimum Velocity

A particle performing SHM will have velocity, $v = \pm\omega\sqrt{a^2 - x^2}$.

- At mean position $x = 0$, $v = \pm\omega\sqrt{a^2 - 0^2} = \pm\omega\sqrt{a^2} = \pm a\omega$.
  - The velocity of the particle in SHM will be maximum at the mean position. Thus, $v_{\max} = \pm a\omega$.
- At extreme position $x = \pm a$, $v = \pm\omega\sqrt{a^2 - a^2} = 0$.
  - The velocity of the particle in SHM will be minimum at the extreme position. Thus, $v_{\min} = 0$.

Maximum and Minimum Acceleration

Acceleration of a particle performing SHM is given by

Acceleration$= -\omega^2 x$

- It is always opposite to displacement in direction and has magnitude $\omega^2 x$.
- At mean position $x = 0$, acceleration = 0. Here, acceleration has the minimum value.
- At extreme position $x = \pm\, a$, acceleration = $\pm\omega^2 a$. Here, acceleration has the maximum value.

Phase of SHM

Amplitude of SHM

The instantaneous displacement of a particle in SHM is given by

$x = A\sin(\omega t + \Phi_0)$

Quantity *A* in the above expression is known as the amplitude of motion.

- It is a positive constant that represents the magnitude of the maximum displacement of the particle in either direction.
- For linear SHM, the amplitude has the units same as that of the length, that is, metres and dimensions $[M^0L^1T^0]$.

Oscillation

- The to-and-fro motion of a particle about a certain point is known as oscillation.
- Oscillations are performed by a particle periodically in SHM.

Period of SHM

- It is the time taken by a particle in SHM to complete one oscillation.
- It is the smallest interval of time after which the motion of a particle in SHM is repeated.

If a particle starts from the mean position, the displacement is given by

$x = a\sin(\omega t + \Phi_0)$

The period is the minimum value of time after which the motion repeats.

Let *T* be the time period of the given SHM.

$\therefore a\sin(\omega(t + T) + \Phi_0) = a\sin(\omega t + \Phi_0)$

If the argument of this function $(\omega t + \Phi_0)$ is increased by an integral multiple of 2π radians, then the value of $a\sin(\omega t + \Phi_0)$ will remain the same.

That is,

$\omega T = 2\pi$ or $T = \frac{2\pi}{\omega}$

The particle takes the time $T = \frac{2\pi}{\omega}$ to return to the extreme position from where the motion is started.

$\because\ \omega = \sqrt{\frac{k}{m}}$

$\therefore\ T = \frac{2\pi}{\sqrt{\frac{k}{m}}} = \frac{2\pi}{\sqrt{\frac{F}{mx}}}$

According to Newton's second law, $\frac{F}{m} = a$.

$\therefore T = \frac{2\pi}{\sqrt{\frac{a}{x}}} = \frac{2\pi}{\sqrt{Acceleration\ per\ unit\ displacement}}$

Frequency of SHM

- The number of oscillations performed by a particle in SHM per unit time is known as the frequency of SHM.
- It is the reciprocal of the time period.
- Frequency is given by $f = \frac{1}{T} = \frac{\omega}{2\pi}$.
- $\therefore f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$
- The dimensions of the frequency are $[M^0L^0T^{-1}]$.

Phase of SHM

- The physical quantity that describes the state of oscillation is known as the phase of SHM.
- The time-varying quantity, $(\omega t + \Phi_0)$, in $x = A\sin(\omega t + \Phi_0)$ is called the phase of the motion.
- It describes the state of motion at a given time.
- By knowledge of phase, the magnitude and the direction of the displacement of a particle at a given instant can be calculated.

Epoch of SHM

- It is the physical quantity that describes the state of oscillation of a particle performing SHM at the beginning of the motion.
- The term Φ_0 in $(\omega t + \Phi_0)$ is known as the epoch of the SHM.
- It is also known as the phase constant.

Energy in Simple Harmonic Motion

- Kinetic energy (E_k) of a particle is given as

$$E_k = \frac{1}{2}mv^2$$

$$\because v = \omega A\cos(\omega t + \phi_0)$$

$$E_k = \frac{1}{2}m\omega^2 A^2\cos^2(\omega t + \phi_0)$$

$$= \frac{1}{2}kA^2\cos^2(\omega t + \phi_0)$$

- Potential energy (E_p) of a particle is given as

$$E_p = \frac{1}{2}ky(t)^2$$

$$= \frac{1}{2}kA^2\sin^2(\omega t + \phi_0)$$

- Total energy of the system is given as

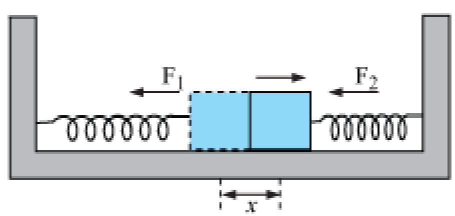
  $E = E_k + E_p$

$$= \frac{1}{2}kA^2\cos^2(\omega t + \phi_0) + \frac{1}{2}kA^2\sin^2(\omega t + \phi_0)$$

$$\boxed{E = \frac{1}{2}kA^2}$$

- Total mechanical energy of a harmonic oscillator is independent of time.

Oscillations Due to a Spring



- Oscillations of a block of mass, *m* fixed to a spring, which is in turn fixed to a rigid wall, are shown in the figure.

- The block is pulled and released so that it executes to and fro motion (SHM).

  Here,

m = Mass of the block

$+A, -A$ = Maximum displacement

($x = 0$) = Position of the centre of the block at the equilibrium of the spring

- When the block is pushed to the right side, one spring is compressed while the other is elongated hence the block is subjected to a restoring force of $F(x)$, which is proportional to the displacement, x (in the opposite direction).

  As the block feels twice of restoring force because of two spring system,

  $\therefore F(x) = -2kx$ …(*i*) Where k is the spring constant (depends on the property of the spring)

  Using Newton's law of motion, the force applied to pull the spring is

  $F = ma$(which must be equal and opposite to the restoring force)

  Since acceleration in SHM $= -\omega^2 x$

  $F = -m\omega^2 x$

  On comparing it with equation (*i*)

  $2k = m\omega^2$

  Where ω is the angular speed of the spring

$$\therefore \omega = \sqrt{\frac{2k}{m}}$$

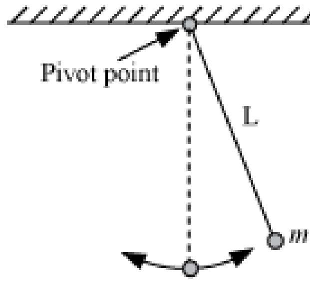
- Time period (T) of the oscillator is,

$$\therefore T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{2k}}$$

Simple Pendulum

- A simple pendulum is a heavy point mass suspended by a weightless, inextensible, flexible string attached to a rigid support from where it moves freely.

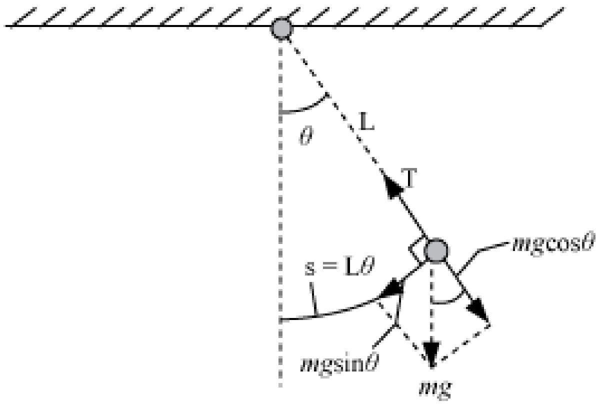
- The periodic motion of a simple pendulum for small displacements is simple harmonic.



m − Mass of the bob

L − Length of the massless string

- Given below is a free body diagram to show the forces acting on the bob.



θ − Angle made by the string with the vertical

T − Tension along the string

g − Acceleration due to gravity

Radial acceleration = $\omega^2 L$

Net radial force = $T - mg \cos \theta$

Tangential acceleration is provided by $mg \sin \theta$.

Torque, $\tau = -L\ (mg \sin \theta)$

- According to Newton's law of rotational motion,

$\tau = I\alpha$

Here,

I − Moment of inertia

α − Angular acceleration

∴ $I\alpha = -mg \sin\theta \, L$

If θ is very small, then

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \pm \ldots$$

Ignoring the higher powers of θ, we get

$$\sin\theta \approx \theta$$

$$\therefore \alpha = \frac{mgL}{I}\theta$$

$$\because \alpha = \omega^2\theta$$

$$\therefore \omega^2 = \frac{mgL}{I}$$

Angular frequency, $\omega = \sqrt{\frac{mgL}{I}}$

Time period, $T = \frac{2\pi}{\omega}$

Angular frequency, $\omega = \sqrt{\frac{mgL}{I}}$

Time period, $T = \frac{2\pi}{\omega}$

$$T = 2\pi\sqrt{\frac{I}{mgL}}$$

$$I = mL^2$$

$$\boxed{\therefore T = 2\pi\sqrt{\frac{L}{g}}}$$

Laws of Simple Pendulum

From the derived equation of the time period of a simple pendulum, we get some basic laws of a simple pendulum. They are as follows:

- Law of length: Time period of the pendulum is directly proportional to the square root of its length.
- Law of acceleration due to gravity: Time period of the pendulum is inversely proportional to the square root of the acceleration due to gravity of the place.
- Law of mass: Time period of the pendulum is independent of the mass of the bob.
- Law of isochronous: Time period of the pendulum does not depend upon the amplitude of oscillations.

Seconds Pendulum

- It is a simple pendulum that has time period equal to 2 seconds.
- Period of a simple pendulum is given by $T = 2\pi\sqrt{\frac{L}{g}}$ For a seconds pendulum, the time period is given by *T* = 2 seconds.

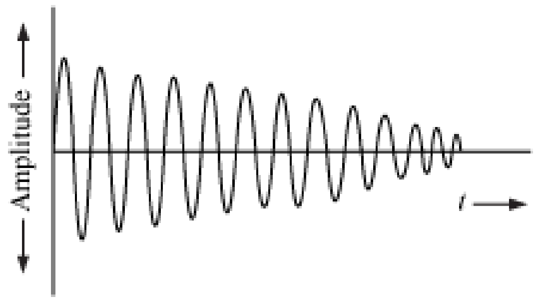
$$\therefore 2 = 2\pi\sqrt{\frac{L}{g}}$$

Length of the seconds pendulum = $L = \frac{g}{\pi^2}$

- It can be seen from the above relation that the length of the seconds pendulum depends on the acceleration due to gravity *g*. Thus, at different places on the Earth, the length of the seconds pendulum has different values.
- The length of a seconds pendulum is small at the Equator and large at the poles of the Earth because the value of *g* is greater at the Equator than that at the poles.

Damped Simple Harmonic Motion

- A simple harmonic system that oscillates with decreasing amplitude with time is called damped simple harmonic oscillation.
- Energy of the system is dissipated; dissipating forces are frictional forces



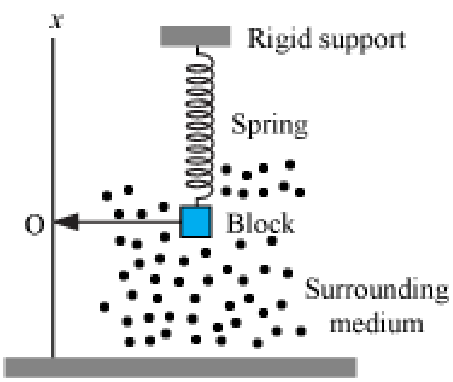
- Damping force (F_d) depends on the nature of the surrounding medium; it is proportional to the velocity (v) of the bob, and acts opposite to the direction of velocity.

$$F_d \propto -v$$
$$\therefore F_d = -bv$$

Where,

b = Positive constant (depends on characteristics of the medium such as viscosity, the shape and size of the bob)

- When a mass (m) attached to a spring is released, it settles to a height. When the mass is pulled up/down, the restoring force (F_s) on the spring is proportional to the displacement (x) from its equilibrium position.



$$\therefore F_s \propto -x$$
$$F_s = -kx$$

According to Newton's law of motion,

$$ma(t) = -k\,x(t) - b\,v(t),$$
$$\Rightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$
The solution of the equation is found to be,
$$x(t) = Ae^{-bt/2m}\cos(\omega' t + \phi)$$

Where,

A = Amplitude

- ω' = Angular frequency of the damped oscillator

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

- $x(t)$ is not periodic because $e^{-bt/2m}$ decreases continuously with time.
- Mechanical energy is represented as,

$$E = \frac{1}{2}kA^2e^{-bt/m}$$

- For small damping,

$$\frac{b}{\sqrt{2km}} << 1$$

Forced Oscillations and Resonance

- **Natural frequency**: If no external force acts on a system, the system will execute oscillations of frequency, v_0, called natural frequency.
- Oscillations produced under the effect of an external periodic force of frequency other than the natural frequency of the oscillator are called forced oscillations.
- If an external force $F(t)$ is applied to a damped oscillator,

$F(t) = F_0 \cos\omega_d t$ ……(i)

Here,

F_0 = Amplitude of external force

ω_d = Driven frequency

The motion of a particle under the combined action of a linear restoring force, a damping force and a time-dependent driving force, as represented by equation (*i*), is given by

$ma(t) = -kx(t) - bv(t) + F_0 \cos\omega_d t$

$$\therefore \frac{md^2x}{dt^2} + \frac{bdx}{dt} + kx = F_0 \cos \omega_d t \text{ ......}(ii)$$

The displacement after the natural oscillation dies out,

$x(t) = A \cos(\omega_d t + \Phi)$

Amplitude, *A,* is the function of the forced frequency (ω_d) and the natural frequency, ω. Analysis shows that it is given by

$$A = \frac{F_0}{\{m^2(\omega^2 - \omega_d^2)^2 + \omega_d^2 b^2\}^{\frac{1}{2}}} \qquad \text{......}(iii)$$

$$\tan\phi = \frac{-v_0}{\omega_d x_0}$$

Here,

m = Mass of the particle

v_0 = Velocity of the particle (at $t = 0$)

x_0 = Displacement of the particle (at $t = 0$)

ω = Natural frequency

Case 1: Small damping; driving frequency far from natural frequency

$$\omega_d b << m(\omega^2 - \omega_d^2)$$

$$\therefore A = \frac{F_0}{m\left(\omega^2 - \omega_d^2\right)} \qquad \text{[From } (iii)]$$

Case 2: Driving frequency close to natural frequency

ω_d is very close to ω.

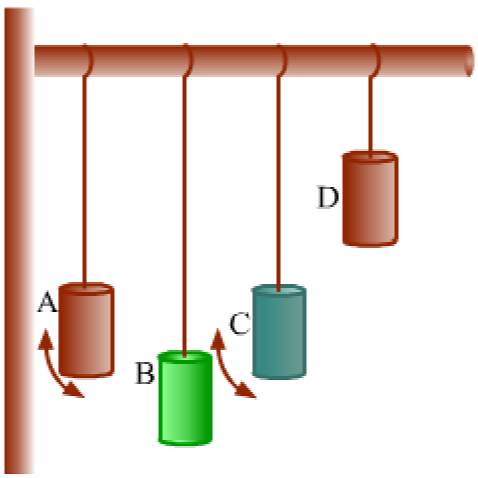
$$m(\omega^2 - \omega_d^2) << \omega_d b$$

$$\therefore A = \frac{F_0}{\omega_d b} \qquad \text{[From } (iii)\text{]}$$

- **Resonance**

  The increase in the amplitude, when the driving force is close to the natural frequency of the oscillation, is called resonance.

  Examples of Resonance



- When pendulums of different lengths are suspended from a horizontal support: If pendulum A is given some displacement, it is set into oscillation and the other pendulums begin to oscillate due to this.

  However, pendulum C, the length of which is approximate to the length of A, oscillates with a much greater amplitude than pendulums B and D, the lengths of which are much different from that of pendulum A. Pendulum D moves the way it does because its natural frequency is nearly the same as the driving frequency associated with pendulum A.

- Electrical resonance is provided by tuning the radio receiver. Frequency of the oscillatory circuit is made equal to that of signals from a radio station. Thus, only the signals of the selected frequency are amplified by the receiver and the other frequencies are rejected.

  Applications of Resonance

- The unknown frequency of a vibrating tuning fork can be determined using resonance.

- A radio receiver can be tuned to a desired frequency by using the principle of resonance.

- It is used to increase the intensity of sound in musical instruments.

- The analysis of a musical instrument is also done by resonance.

Disadvantages of resonance

- In an auditorium, if the frequency of clapping of hands by the audience becomes equal to the natural frequency of the roof, the roof may fall down due to resonance.

- If the frequency of the steps of soldiers marching on a bridge becomes equal to the natural frequency of oscillation of the bridge, it may collapse due to resonance.

- In a rough sea, if the natural frequency of the swinging of a ship matches the frequency of sea waves, then due to resonance, the amplitude of the swinging of the ship increases and crosses the safety limit, which is dangerous.

Musical Instruments

There are three types of musical instruments :
(A) String instruments
(B) Wind instruments
(C) Percussion instruments

String Instruments

- In these instruments, sound is produced by vibrating the strings.
- The strings in these instruments are vibrated by plucking them.
- Sitar, veena, guitar and tanpura are examples of string instruments.

Wind Instruments

- In these instruments, sound is produced due to the vibration of air columns.
- Flute, bugle, bassoon and harmonium are examples of wind instruments.
- **Flute** is an instrument that consists of a cylindrical pipe, which is closed at one end. Air blown at the narrow open end of the flute vibrates the air column inside the tube, causing standing waves to be formed by the incident wave and the wave reflected at the closed end. When any of the holes in the flute is closed by the player, the flute acts as a pipe open at both the ends and various sounds can be produced.

- **Harmonium** is a reed instrument without a pipe. It has a keyboard in which air is set into vibration by the means of thin metal reeds.

- **Basson** is a pipe instrument without reeds.

Percussion Instruments

- These are instruments that produce sound due to vibrations produced in a stretched membrane.
- Mridangam, tabla and drums are examples of percussion instruments.