Numerical Methods

LU Factorisation Method

$$A = LU$$

eg
$$\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

Consider $(u_{11} = 1, \; u_{22} = 1)$ or either $l_{11} = l_{22} = 1$

$$\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11} u_{12} \\ l_{21} & l_{21} u_{12} + l_{22} \end{bmatrix}$$

$l_{11} = 2$ $\quad$ $l_{11} u_{12} = 2$ $\quad$ $l_{21} u_{12} + l_{22} = 9$

$l_{21} = 4$ $\quad$ $2 u_{12} = 2$ $\quad$ $(4)(1) + l_{22} = 9$

$u_{12} = 1$ $\quad$ $l_{22} = 5$

$$L = \begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Gauss Elimination Method

Q: The second iterative value of z By Gauss elimination method is

where
$$8x - y + z = 18$$
$$2x + 5y - 2z = 3$$
$$x + y - 3z = -6$$

Sol^n

$$x = \frac{18 + y - z}{8} \quad -①$$

$$y = \frac{3 - 2x + 2z}{5} \quad -②$$

$$z = \frac{6 + x + y}{3} \quad -③$$

I iterative

Put $y=0, z=0$ in eq ① $\quad x = 2.25$

Put $x=0, z=0$ in eq ② $\quad y = 0.6$

Put $x=0, y=0$ in eq ③ $\quad z = 2$

II iterative

Put $y=0.6, z=2$ in eq ① $\quad x = 2.075$

Put $x=2.25, z=2$ in eq ② $\quad y = 0.5$

Put $x=2.25, y=0.6$ in eq ③ $\quad z = 2.95$

So 2nd iterative value of

$$z = 2.95$$ Ans

Gauss seidal Method

1 The second iterative value of x_3 by Gauss seidal method is where

$$3x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + 3x_2 + x_3 = 1$$
$$x_1 + 2x_2 + 3x_3 = 5$$

Ans

$$x_1 = \frac{1}{3}(3 - 2x_2 - x_3) \quad -(1)$$
$$x_2 = \frac{1}{3}(1 - 2x_1 - x_3) \quad -(2)$$
$$x_3 = \frac{1}{3}(5 - x_1 - 2x_2) \quad -(3)$$

(I) Put $x_2 = 0$, $x_3 = 0$ in eq(1) $\quad x_1 = 1$

Put $x_1 = 1$, $x_3 = 0$ in eq(2) $\quad x_2 = -\frac{1}{3} = -0.333$

Put $x_1 = 1$, $x_2 = -\frac{1}{3}$ in eq(3) $\quad x_3 = 1.55$

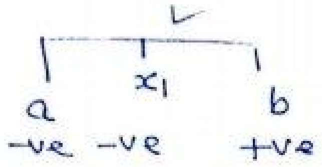
(II) Put $x_2 = -0.333$, $x_3 = 1.55$ in eq(1) $\quad x_1 = 0.703$

Put $x_1 = 0.703$, $x_3 = 1.55$ in eq(2) $\quad x_2 = -0.652$

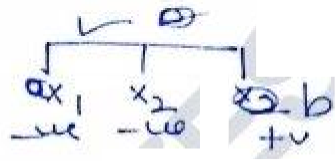
Put $x_1 = 0.703$ $\quad x_2 = -0.652$ in eq(3) $\quad x_3 = 1.866$

Ans

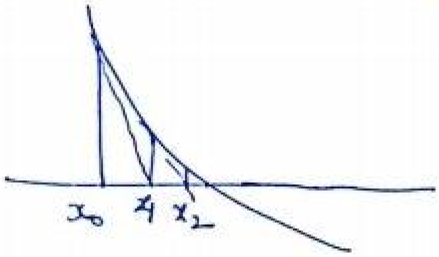
Non linear eq^n



$x_1 = \frac{a+b}{2}$



$x_2 = \frac{x_1+b}{2}$



Newton Bisection Method :-

The iteration formula is $x = \frac{a+b}{2}$, where

$f(a).f(b) < 0$

* The minimum number of iteration of this method are given by $\frac{|b-a|}{2^n} < \epsilon$

where a : lower limit of the method

b : upper limit of the interval

ϵ : Error of approximation

n : no of iteration

False-Position Method (Regula-Falsi Method)

The iteration formula is

$$x = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} \quad \text{where } f(a) \cdot f(b) < 0$$

Secant Method :-

The iteration formula is

$$x_{i+1} = \frac{x_{i-1}\,f(x_i) - x_i\,f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Newton-Raphson Method :

The iteration formula is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

i.e.) $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Method	Speed	Convergence	Order of Convergence	No. of Guesses	type
Newton-Bisection	Slower	Yes	1	two	Closed ended
False-position	slower	Yes	1	two	Closed ended
Secant	Medium	may/May not	1.62	two	open ended
Newton-Raphson	Faster	may/may not	2	one	open ended.

Error at present iteration is square of error at previous iteration.

Question The third approximation of $x^3 - 4x - 9 = 0$ in [2, 3] by bisection method

$$f(x) = x^3 - 4x - 9$$

$$f(2) = \underline{-9}, \quad f(3) = \underline{6}$$

$$\text{F.A.} = x_1 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$$

$$f(2.5) = (2.5)^3 - 4(2.5) - 9 = \underline{-3.375}$$

$x_2 =$

2	2.5	3
$-ve$	$\underline{-ve}$	$\underline{+ve}$

root lies b/w 2.5 & 3

second approximation S.A. $= x_2 = \dfrac{2.5+3}{2} = 2.75$

$$F(2.75) = 0.796$$

root lies b/w 2.5 & 2.75

$$T.A. = x_3 = \frac{2.5+2.75}{2} = 2.625$$

Third Approximation $x_3 = 2.625$ Ans

<u>Question</u> The second approximation of $2x - \log_{10} x - 7 = 0$ in $[3,4]$ by False - position method.

<u>Solⁿ</u>

$$F(x) = 2x - \log_{10} x - 7$$

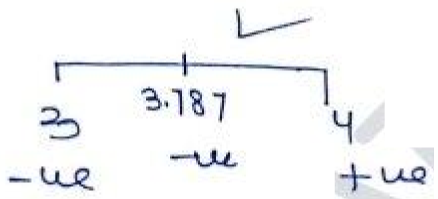
$$F(a) = F(3) = 2 \times 3 - \log_{10} 3 - 7 = -1.477$$

$$F(b) = F(4) = 2 \times 4 - \log_{10} 4 - 7 = 0.3979$$

First Approximation $= x_1 = \dfrac{aF(b) - b(F(a)}{F(b) - F(a)}$

$$x_1 = \frac{3(0.3979) - 4(-1.477)}{0.3979 + 1.477} = 3.787$$

$$F(3.787) = -4.29 \times 10^{-3} = -0.00429$$



$$x_2 = \frac{(3.787)(0.3979) - 4(-0.00429)}{0.3979 + 0.00429}$$

$$x_2 = 3.789$$

Second approximation

Q. 24
P.g = 41

$$x^2 - 4x + 4 = 0 \quad , \quad x_0 = 3$$

$$f(x) = x^2 - 4x + 4$$

$$f'(x) = 2x - 4$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 3 - \frac{9-12+4}{6-4}$$

$$x_1 = 3 - \frac{1}{2} = \frac{5}{2}$$

$$x_0 = 3 \quad , x_1 = 2.5$$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{3(0.25) - 2.5(1)}{0.25 - 1}$$

$$x_2 = \frac{0.75 - 2.5}{-0.75} = \frac{1.75}{0.75} = \frac{7}{3} = 2.33$$

$$x_2 = 2.33$$

Q.17 $\quad x^2 + 3x - 7 = 0 \quad , x_0 = 1$

$$f(x) = x^2 + 3x - 7 \quad , f(1) = 1+3-7 = -3$$

$$f'(x) = 3x^2 + 3 \quad , f(1) = 3+3 = 6$$

$$x_1 = 1 - \frac{(-3)}{6} = 1 + \frac{1}{2}$$

$$x_0 = \quad x_1 = 3/2$$

Q.2

$$f(x) = \frac{1}{x} - a$$

$$f'(x) = \frac{-1}{x^2}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$x_{k+1} = x_k - \frac{\frac{1}{x_k} - a}{-\frac{1}{x_k^2}} = x_k + \frac{(1 - ax_k)}{x_k} \times \frac{x_k^2}{1}$$

$$x_{k+1} = x_k + x_k - ax_k^2$$

$$x_{k+1} = 2x_k - ax_k^2$$

Q.3

$$x^3 + 4x - 9 = 0$$

$$f(x) = x^3 + 4x - 9$$

$$f'(x) = 3x^2 + 4$$

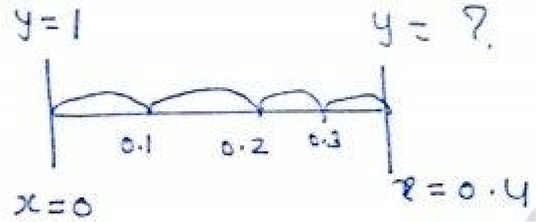
$$x_{k+1} = x_k - \frac{(x_k^3 + 4x_k - 9)}{(3x_k^2 + 4)}$$

$$x_{k+1} = \frac{3x_k^3 + \cancel{4x_k} - x_k^3 - \cancel{4x_k} + 9}{3x_k^2 + 4}$$

$$x_k = \frac{2x_k^2 + 9}{3x_k^2 + 4}$$

Numerical solution of D.E.

$$\frac{dy}{dx} = f(x,y) \quad , y(x_0) = y_0 \quad \text{step size } h$$



Eular's Method :-

The iteration formula is

$$y_{i+1} = y_i + h f(x_i, y_i)$$

$$x_{i+1} = x_i + h$$

i.e.) $y_1 = y_0 + h(x_0, y_0)$ $\quad x_1 = x_0 + h$

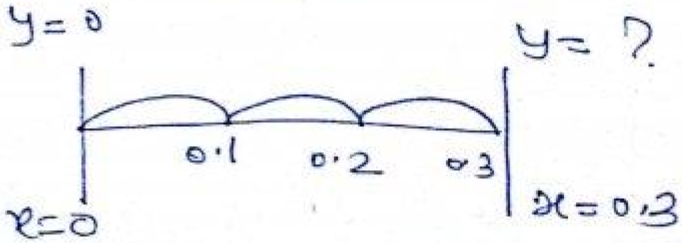
$y_2 = y_1 + h(x_1, y_1)$ $\quad x_2 = x_1 + h$

$y_3 = y_2 + h(x_2, y_2)$ $\quad x_3 = x_2 + h$

Question $\frac{dy}{dx} = x + y \quad , \quad y(0) = 0$

By eulars method with step side of 0.1 the value of $y(0.3)$ is –

Sol



$x_0 = 0 \quad , y_0 = 0 \quad , f(x,y) = x+y$

(I)

$y_1 = y_0 + h\, f(x_0, y_0) \quad , \quad x_1 = x_0 + h$

$y_1 = 0 + 0.1\,(0+0) \qquad x_1 = 0 + 0.1$

$y_1 = 0 \qquad x_1 = 0.1$

(II)

$y_2 = 0 + 0.1\,(0+0.1) \qquad x_2 = 0.1 + 0.1$

$x_2 = 0.2$

$y_2 = 0.01$

(III)

$y_3 = 0.01 + 0.1\,(0.2 + 0.01) \qquad x_3 = 0.2 + 0.1$

$x_3 = 0.3$

$y_3 = 0.01 + 0.1\,(0.21)$

$y_3 = 0.031$

$$y(0.3) = 0.031 \quad \underline{\text{Ans}}$$

$0.20 + 0.01 = 0.21$

$0.021 + 0.010 = 0.031$

Modified Eulers Method (Ranga kutta 2nd order method)

The iteration formula is ④ $y_1 = y_0 + k$

③ $x_1 = x_0 + h$

where $k = \dfrac{k_1 + k_2}{2}$

① $k_1 = h f(x_0, y_0)$

② $k_2 = h f(x_0 + h, y_0 + k_1)$

Q Consider $\frac{dy}{dx} = x - y$, $y(0) = 0$ by R-K 2^{nd} order method with step size 0.1, the value of $y(0.1)$ is ___

$\Rightarrow$ $f(x) = x - y$, $h = 0.1$

$x_0 = 0$, $y_0 = 0$

$k_1 = 0.1(0-0) \Rightarrow k_1 = 0$

$k_2 = 0.1(0.1 - 0) = 0.01$

$k = \frac{k_1 + k_2}{2} = \frac{0.01 + 0}{2} = 0.005$

$y_1 = y_0 + k = 0 + 0.005$

$\cancel{y_1 = 0.005 \text{ at } x_1 = 0.1}$

$x_1 = x_0 + h = 0.1$

$y_1 = 0 + 0.005$ $\quad$ $x_1 = 0.1$

$y(0.1) = 0.005$ Ans

Runga kutta 4th order Method:-

The iteration formula is ⑦ $y_1 = y_0 + k$ &

⑥ $x_1 = x_0 + h$

where ⑤ $k = \dfrac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

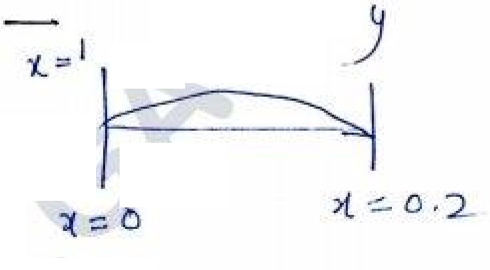
① $k_1 = h\{f(x_0, y_0)$

② $k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

③ $k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

④ $k_4 = h\ f\left(\underline{x_0 + h}, \underline{y_0 + k_3}\right)$

Ques $\frac{dy}{dx} = x+y$ $y(0) = 1$ R-k 4th order method stepsize of 0.2 the value of $y(0.2)$ is —



$f(x) = x+y$ $x_0 = 0$, $y_0 = 1$

① $\cancel{k_2} = k_1 = h\ (x_0 + y_0) = 0.2(0+1) = 0.2$

② $k_2 = 0.2\left\{0 + \frac{0.2}{2} + 1 + \frac{0.2}{2}\right\} = \cancel{2\cdot4}$ 0.24

③ $k_3 = 0.2\left\{0 + \frac{0.2}{2} + 1 + \frac{0.24}{2}\right\}$

$k_3 = 0.244$

④ $k_4 = 0.2\left\{0 + 0.2 + 1 + 0.244\right\}$

$k_4 = 0.2888$

⑤ $k. = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

$k = \frac{0.2 + 0.48 + 0.488 + 0.2888}{6}$

$k = 0.2428$

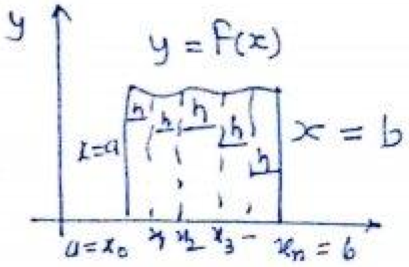
6 $x_1 = 0 + 0.2 \Rightarrow x_1 = 0.2$

$y_1 = 1 + 0.2428$

$y = 1.2428$

Numerical Integration :-

$h = \frac{b-a}{n}$



Trapezoidal:-

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left\{(y_0+y_n)+2(y_1+y_2+y_3 - - - + y_{n-1})\right.$$

Simpson's (1/3) Rule

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left\{(y_0+y_n)+4(y_1+y_3 - - -)+2(y_2+y_4+ - -)\right\}$$

Simpson's (3/8) Rule:-

$$\int_{x_0}^{x_n} y\,dx = \frac{3}{8}h\left\{(y_0+y_n)+3(y_1+y_2+y_4+y_5 - -)+2(\underbrace{y_3+y_6+y_9 -}_{\text{multiple of } 3})\right\}$$

Ques The value of $\int_0^6 \frac{1}{1+x^2} dx$ when evaluated by

1. Trapezoidal
2. Simpson's 1/3 Rule
3. Simpson's 3/8 Rule

on 6 subintervals of length 1 equal

Sol^n $\int_0^6 \frac{1}{1+x^2} dx$

$y = \frac{1}{1+x^2}$

$x_0 = 0 \quad , x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6$

$y_0 = 1 \quad , \quad y_1 = \frac{1}{1+1} = \frac{1}{2} \; , \; y_2 = \frac{1}{5} \; , \; y_3 = \frac{1}{10}$

$y_4 = \frac{1}{17} \quad , \quad y_5 = \frac{1}{26} \quad , \quad y_6 = \frac{1}{37}$

trapezoidal

$$\int y\,dx = \frac{h}{2}\left((y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right)$$

$$= \frac{1}{2}\left\{\left(1+\frac{1}{37}\right) + 2\left\{\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\frac{1}{26}\right\}\right\}$$

$$\int_0^6 \frac{1}{1+x^2} dx = 1.409$$

Simphon's $\left(\frac{1}{3}\right)$

$$\int_0^6 \frac{1}{1+x^6}\,dx = \frac{h}{3}\left\{(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right\}$$

$$= \frac{1}{3}\left\{\left(1+\frac{1}{37}\right)+4\left(\frac{1}{2}+\frac{1}{10}+\frac{1}{26}\right)+2\left(\frac{1}{5}+\frac{1}{17}\right)\right\}$$

$$= 1.365$$

$$\int_0^6 \frac{1}{1+x^6}\,dx = \frac{3h}{8}\left\{(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right\}$$

$$= \frac{3\times 1}{8}\left\{\left(1+\frac{1}{37}\right)+3\left(\frac{1}{2}+\frac{1}{5}+\frac{1}{17}+\frac{1}{26}\right)+2\left(\frac{1}{17}\right)\right\}$$

$$= 1.356$$

Remeber :- The accuracy of

(i) Trapzoidal rule in quadrature is $O(h^2)$

(ii) Simpson's $\frac{1}{3}$ rule in qua-drature is $O(h^4)$

(iii) Simpson's $\frac{3}{8}$ rule in quadrature is $O(h^5)$

Error palynomial

tropezoidal $= \frac{h^3}{12} \cdot \frac{b-a}{h} \cdot \cancel{(b-a)} \max_{a \le x \le b}\{f''(x)\}$