EXERCISE 5.8

QNo 1 Verify Rolle's theorem for function $f(x) = x^2+2x-8$; $x \in [-4,2]$

Sol: Given function is $f(x) = x^2+2x-8$; $x \in [-4,2]$

Note that $f(x)$ is polynomial function. Since a poly fn. is derivable on R

$\therefore$ (i) $f(x)$ is continuous on $[-4,2]$

(ii) $f(x)$ is differentiable on $(-4,2)$

Also $f(-4) = (-4)^2+2(-4)-8 = 16-8-8 = 0$

$f(2) = (2)^2+2\times2+8 = 4+4-8 = 0$

$\therefore$ $f(-4) = f(2)$

$\Rightarrow$ All the conditions of Rolle's thm are satisfied by $f(x)$ in $[-4,2]$

$\therefore$ $\exists$ at least on real No. $c(-4,2)$ such that $f'(c) = 0$

Now $f'(x) = 2x+2$

$\therefore$ $f'(c) = 0 \Rightarrow 2c+2 = 0 \Rightarrow c = -1 \in (-4,2)$

$\therefore$ Rolle's thm is verified with $c = -1$.

QNo 2 Examine if Rolle's theorem is applicable to any of the following functions. Can you say something about the converse of Rolle's theorem from these examples?

(i) $f(x) = [x]$ for $x \in [5,9]$

Given function is $f(x) = [x]$; $x \in [5,9]$.

We know that $f(x)$ is neither Cont. nor. differentiable at $5,6,7,8,9 \Rightarrow f(x)$ is not Cont. in $[5,9]$

and $f(x)$ is not diff. on $(5,9)$

Also $f(5) = 5$ and $f(7) = 7 \Rightarrow f(5) \neq f(7)$

Hence Rolle's theorem is not applicable to $f(x)$ in $[5,9]$

Note that $f'(x) = 0$ for all not integral points in $(5,9)$

$\Rightarrow$ Converse of Rolle's theorem is not true.

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

Same as above.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

Sol: $f(x) = x^2 - 1$ which is polynomial function and hence Cont. and diff $\forall\, x \in R$

$\therefore$ $f(x)$ is Cont in $[1,2]$

$f(x)$ is diff in $(1,2)$

But $f(1) = 1^2 - 1 = 0$ and $f(2) = 2^2 - 1 = 3$

$\therefore$ $f(1) \neq f(2)$

$\Rightarrow$ Rolle's theorem is not applicable in given interval.

Also $f'(x) = 2x \neq 0$ for any $x \in (1,2)$

CONCLUSION: From the above examples, we conclude that converse of Rolle's theorem is not true.

This means that if Conditions of Rolle's theorem are not satisfied by a function $f(x)$ on $[a,b]$ then $f'(x)$ may or may not vanish at some point in (a,b)

QNo 3: If $f: [-5, 5] \to R$ is differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$

Sol: Suppose $f(-5) = f(5)$

Now Since f is differentiable function and differentiable function is always continuous

$\Rightarrow$ f satisfies all the three conditions of Rolle's theorem

$\Rightarrow$ $f'(x) = 0$ atleast at some point c in $(-5, 5)$

But $f'(x)$ does not vanish anywhere

$\Rightarrow$ Our supposition is wrong.

$\Rightarrow$ $f(-5) \neq f(5)$

Q.No 4 Verify Mean Value theorem, $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$ where $a = 1$ and $b = 4$.

Sol. Here $f(x) = x^2 - 4x - 3$; $x \in [1, 4]$

Which being polynomial function is continuous and derivable $\forall\ x \in R$

$\therefore$ (i) $f(x)$ is Cont. in $[1, 4]$

(ii) $f(x)$ is diff. in $(1, 4)$

$\therefore$ Conditions of Lagrange's MV Thm are satisfied in $[1, 4]$

Hence there exsists ($\exists$) at least one real no. $c \in (1, 4)$ s.t.

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

ie $$2c - 4 = \frac{(4^2 - 4\times4 - 3) - (1^2 - 4\times1 - 3)}{3} \qquad \left[\because f'(x) = 2x - 4\right]$$

ie $$2c - 4 = \frac{(-3) - (-6)}{3} = \frac{-3+6}{3} = \frac{3}{3} = 1$$

ie $$2c - 4 = 1 \Rightarrow c = \frac{5}{2} \in (1, 4)$$

$\therefore$ Mean Value thm is verified in $[1, 4]$

Q.No 5: Verify MV. Thm if $f(x) = x^3 - 5x^2 - 3x$ in interval $[a, b]$, where $a = 1$, $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$

Sol. Given $f(x) = x^3 - 5x^2 - 3x$, which is polynomial fn.

Hence $f(x)$ continuous and differentiable $\forall\ x \in R$

$\therefore$ (i) $f(x)$ is Cont. in $[1, 3]$

(ii) $f(x)$ is diff. in $(1, 3)$

and. $f'(x) = 3x^2 - 10x - 3$

$\therefore$ Conditions of LMV Thm are satisfied on $[1, 3]$

$\therefore$ $\exists$ at least one $c \in (1, 3)$ so that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} \quad \text{ie} \quad 3c^2 - 10c - 3 = \frac{(3^3 - 5.3^2 - 3.3) - (1^3 - 5.1^2 - 3.1)}{3 - 1}$$

or. $3c^2-10c-3=-10$ ie $3c^2-10c+7=0$

$$\Rightarrow c = \frac{-(-10)\pm\sqrt{(-10)^2-4\times3\times7}}{2\times3} = \frac{10\pm\sqrt{100-84}}{6} = \frac{10\pm\sqrt{16}}{6} = \frac{10\pm4}{6}$$

$$= 1, \frac{7}{3}$$

Note that $\frac{7}{3} \in (1,3)$.

Hence LMV Thm is verified.

Now $f'(c)=0 \Longrightarrow 3c^2-10c-3=0$

$$\Rightarrow c = \frac{-(-10)\pm\sqrt{(-10)^2-4\times3\times(-3)}}{6} = \frac{10\pm\sqrt{100+36}}{6} = \frac{10\pm\sqrt{136}}{6}.$$

QNo 6: Examine the applicability of MV Thm for all fns given in QNo 2.

Sol. (i) and (ii) MV Thm is not applicable as $f(x)=[x]$ is neither Cont. nor diff. at every point on interval $(-2,2)$ and $(5,9)$

(iii) Here $f(x)=x^2-1$. which is polynomial fn.

$\therefore$ $f(x)$ is Cont. in $[1,2]$

and $f(x)$ is diff in $(1,2)$

$\Rightarrow$ $f(x)$ satisfies conditions of MVT in $[1,2]$

$\therefore$ $\exists$ at least one $c \in R$ and hence $c \in (1,2)$ so that

$$f'(c) = \frac{f(2)-f(1)}{2-1} \quad \text{ie} \quad 2c = \frac{(2^2-1)-(1^2-1)}{1}$$

ie $2c = \frac{3-0}{1}$ ie $2c=3 \Rightarrow c=\frac{3}{2} \in (1,2)$

$\therefore$ MTV is verified for given fn. in given interval.

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