Exercise 10.2

Q1E

Consider the following parametric equations:

$x = t\sin t, \qquad y = t^2 + t$

Find $\frac{dy}{dx}$.

Recollect that, $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}, \quad \frac{dx}{dt} \neq 0$.

Differentiate the parametric equations with respect to *t*.

$x = t\sin t$

Use Product Rule of differentiation.

$\frac{dx}{dt} = t\cos t + \sin t$

and, $y = t^2 + t$.

Then,

$\frac{dy}{dt} = 2t + 1$.

Therefore,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{2t+1}{t\cos t + \sin t}$$

Thus, $\frac{dy}{dx} = \boxed{\frac{2t+1}{t\cos t + \sin t}}$.

Q2E

Consider the following parametric equations:

$x = \frac{1}{t}, \quad y = \sqrt{t}e^{-t}$

The objective is to find $\frac{dy}{dx}$.

Differentiate $x = \frac{1}{t}$ with respect to t to get as follows:

$$\frac{dx}{dt} = -\frac{1}{t^2}$$

Differentiate $y = \sqrt{t}e^{-t}$ with respect to t to get as follows:

$$\begin{aligned}\frac{dy}{dt} &= \sqrt{t}\left(-e^{-t}\right) + \frac{1}{2\sqrt{t}} \cdot e^{-t} \\ &= \left(-\sqrt{t} + \frac{1}{2\sqrt{t}}\right)e^{-t} \\ &= \left(\frac{-2t+1}{2\sqrt{t}}\right)e^{-t}\end{aligned}$$

Therefore, obtain the following result:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{\left(\frac{-2t+1}{2\sqrt{t}}\right)e^{-t}}{\left(-\frac{1}{t^2}\right)} \\ &= -t^2\left(\frac{-2t+1}{2\sqrt{t}}\right)e^{-t} \\ &= \boxed{t^{3/2}\left(\frac{2t-1}{2}\right)e^{-t}}\end{aligned}$$

Q3E

We are required to find the equation of the tangent to the parametric curve
$x = 1 + 4t - t^2, y = 2 - t^3$ at $t = 1$

$$\frac{dx}{dt} = 4 - 2t, \qquad \frac{dy}{dt} = -3t^2$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3t^2}{4-2t} = \frac{3t^2}{2t-4}$$

when $t = 1$, $\frac{dy}{dx} = \frac{3}{2-4} = -\frac{3}{2}$

Therefore the slope of the tangent at $t = 1$ is $-\frac{3}{2}$

Further, when $t = 1$, from the given parametric equations, we get

$x = 1 + 4 \cdot 1 - 1^2 = 4,\ y = 2 - 1^3 = 1$

$\therefore$ The required tangent is the line passing through the point (4, 1) and having the slope $-\dfrac{3}{2}$

The equation is $y - 1 = -\dfrac{3}{2}(x - 4)$

$$\boxed{y = -\frac{3}{2}x + 7}$$

Q4E

Consider the parametric equations

$x = t - t^{-1}, \qquad y = 1 + t^2$

Differentiating with respect to t to obtain that

$\dfrac{dx}{dt} = 1 + \dfrac{1}{t^2}$ and $\dfrac{dy}{dt} = 2t$

Thus

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{2t}{1 + \frac{1}{t^2}} \\ &= \frac{2t^3}{t^2 + 1}\end{aligned}$$

When the parameter $t = 1$.

$x = 1 - (1)^{-1} = 0$ and $y = 1 + (1)^2 = 2$

The slope of the tangent at $t = 1$ is

$$\begin{aligned}\left(\frac{dy}{dx}\right)_{t=1} &= \frac{2(1)^3}{(1)^2 + 1} \\ &= \frac{2}{2} \\ &= 1\end{aligned}$$

Recollect that, the equation of the tangent line at (x_1, y_1) with slope *m* is

$$y - y_1 = m(x - x_1)$$

The equation of the tangent at $(0,2)$ is

$$y - 2 = 1(x - 0)$$
$$x - y + 2 = 0$$

Thus the equation of the tangent to the curve at point corresponding to the parameter $t = 1$ is $\boxed{x - y + 2 = 0}$.

Q5E

Given parametric equations are $x = t\cos t,\ y = t\sin t$

We are required to find the tangent to this curve at $t = \pi$

$$\frac{dx}{dt} = t(-\sin t) + \cos t,\ \frac{dy}{dt} = t\cos t + \sin t$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t\cos t + \sin t}{-t\sin t + \cos t}$$

$$\text{when } t = \pi, \frac{dy}{dx} = \frac{\pi\cos\pi + \sin\pi}{-\pi\sin\pi + \cos\pi}$$
$$= \frac{\pi(-1) + 0}{-\pi(0) + (-1)}$$
$$= \pi$$

On the other hand, from the parametric equations,

$$\text{when } t = \pi, \quad x = \pi\cos\pi = -\pi, y = \pi\sin\pi = 0$$

Now, the equation of the tangent to the parametric curve at $t = \pi$, having the slope $\frac{dy}{dx} = \pi$

passing through the point ($x = -\pi, y = 0$) is $y - y_1 = \frac{dy}{dx}(x - x_1)$

i.e., $y - 0 = \pi(x + \pi)$

$$\boxed{y = \pi x + \pi^2}$$

Q6E

Given parametric equation is $x = \sin^3\theta,\ y = \cos^3\theta$

We are required to find the equation of the tangent line to this curve at $\theta = \frac{\pi}{6}$

$$\frac{dx}{d\theta} = 3\sin^2\theta\cos\theta,\ \frac{dy}{d\theta} = 3\cos^2\theta(-\sin\theta)$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos^2\theta(-\sin\theta)}{3\sin^2\theta\cos\theta} = -\cot\theta$$

So, the slope of the tangent to the given curve at $\theta = \frac{\pi}{6}$ is $\frac{dy}{dx}_{at\,t=\frac{\pi}{6}} = -\cot\left(\frac{\pi}{6}\right) = -\sqrt{3}$

Further, at $\theta = \frac{\pi}{6}, x = \sin^3\frac{\pi}{6} = \frac{1}{8}, y = \cos^3\frac{\pi}{6} = \frac{3\sqrt{3}}{8}$

Therefore the tangent line to the parametric equation at $\theta = \frac{\pi}{6}$ is passing through

$\left(\frac{1}{8}, \frac{3\sqrt{3}}{8}\right)$ and having the slope $\frac{dy}{dx} = -\sqrt{3}$ is $y - y_1 = \frac{dy}{dx}(x - x_1)$

$$y - \frac{3\sqrt{3}}{8} = -\sqrt{3}\left(x - \frac{1}{8}\right)$$

$$\boxed{y = -\sqrt{3}x + \frac{\sqrt{3}}{2}}$$

Q7E

Consider the parametric equations

$$x = 1 + \ln t, \qquad y = t^2 + 2$$

(a).

Here the objective is to find the tangent line without eliminating the parameter.

Differentiate the parametric equations with respect to *t*.

$$\frac{dx}{dt} = \frac{1}{t} \quad \text{and} \quad \frac{dy}{dt} = 2t$$

Now, write $\frac{dy}{dt}$ by using the chain rule.

$$\frac{dy}{dt} = \left(\frac{dy}{dx}\right)\cdot\left(\frac{dx}{dt}\right)$$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right)$$

$$= \frac{2t}{\frac{1}{t}}$$

$$= 2t^2$$

(b).

Here the objective is to find the tangent line by eliminating the parameter.

Consider the parametric equations

$$x = 1 + \ln t, \qquad y = t^2 + 2$$

Eliminate the parameter 't':

$$\begin{aligned} x &= 1 + \ln t \\ \ln t &= x - 1 \\ t &= e^{x-1} \end{aligned}$$

Substitute this value of t in $y = t^2 + 2$ to get

$$\begin{aligned} y &= \left(e^{x-1}\right)^2 + 2 \\ y &= e^{2x-2} + 2 \end{aligned}$$

Differentiate with respect to x to get,

$$\frac{dy}{dx} = 2e^{2x-2}$$

The slope of the tangent at $(1,3)$ is

$$\begin{aligned} \left(\frac{dy}{dx}\right)_{(1,3)} &= 2e^{2-2} \\ &= 2 \end{aligned}$$

The equation of the tangent at $(1,3)$ is

$$\begin{aligned} y - 3 &= 2(x-1) \\ y - 3 &= 2x - 2 \\ 2x - y + 1 &= 0 \end{aligned}$$

Thus the equation of the tangent to the curve at point $(1,3)$ is $\boxed{2x - y + 1 = 0}$.

Q8E

(a). Given parametric curve is $x = 1 + \sqrt{t}, y = e^{t^2}$

We are required to find the tangent line to this curve at $(2, e)$ without eliminating the parameter.

Consider $\frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \frac{dy}{dt} = e^{t^2} 2t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{t^2} 2t}{\frac{1}{2\sqrt{t}}} = 4e^{t^2} t^{3/2}$$

Further, substituting $(2,e)$ in the parametric equations, we get

$$x = 1 + \sqrt{t} = 2 \Rightarrow \sqrt{t} = 1 \Rightarrow t = 1$$
$$y = e^{t^2} = e \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$$

Comparing these equations, we observe that we are required to find the tangent at $t = 1$

Now, slope of the tangent at $t = 1$ is $\frac{dy}{dx}_{at\, t=1} = 4e^{t^2} t^{\frac{3}{2}}{}_{at\, t=1} = 4e$

Therefore the slope of the tangent is $4e$

The tangent equation is

$$y - e = 4e(x-2)$$
$$\boxed{y = 4ex - 7e}$$

(b). in this case, we find the equation of the tangent line to the given parametric curve by first eliminating the parameter.

$$x = 1 + \sqrt{t} \Rightarrow \sqrt{t} = x - 1 \Rightarrow t^2 = (x-1)^4$$
$$y = e^{t^2} \Rightarrow \ln y = t^2$$

comparing these equations, we get $\ln y = (x-1)^4$

$$\frac{1}{y}\frac{dy}{dx} = 4(x-1)^3$$
$$\Rightarrow \frac{dy}{dx} = 4y(x-1)^3$$

$$\frac{dy}{dx}_{at\,(2,e)} = 4e(2-1)^3 = 4e$$

So, the required equation of the tangent passing through $(2,e)$ and having the slope $\frac{dy}{dx}$=4e is

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

i.e., $y - e = 4e(x-2)$

$$\boxed{y = 4ex - 7e}$$

Q9E

Consider the parametric equations

$x = 6\sin t, \qquad y = t^2 + t$

Need to find the tangent to the curve at $(0,0)$.

Differentiate the parametric equations with respect to *t* to obtain that

$$\frac{dx}{dt} = 6\cos t \quad \text{and} \quad \frac{dy}{dt} = 2t + 1$$

Thus

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+1}{6\cos t}$$

From parametric equations, $t = 0$ when the point $(x, y) = (0,0)$.

The slope of the tangent at $t = 0$ is

$$\left(\frac{dy}{dx}\right)_{t=1} = \frac{2(0)+1}{6\cos 0} = \frac{1}{6}$$

Recollect that, the equation of the tangent line at (x_1, y_1) with slope *m* is

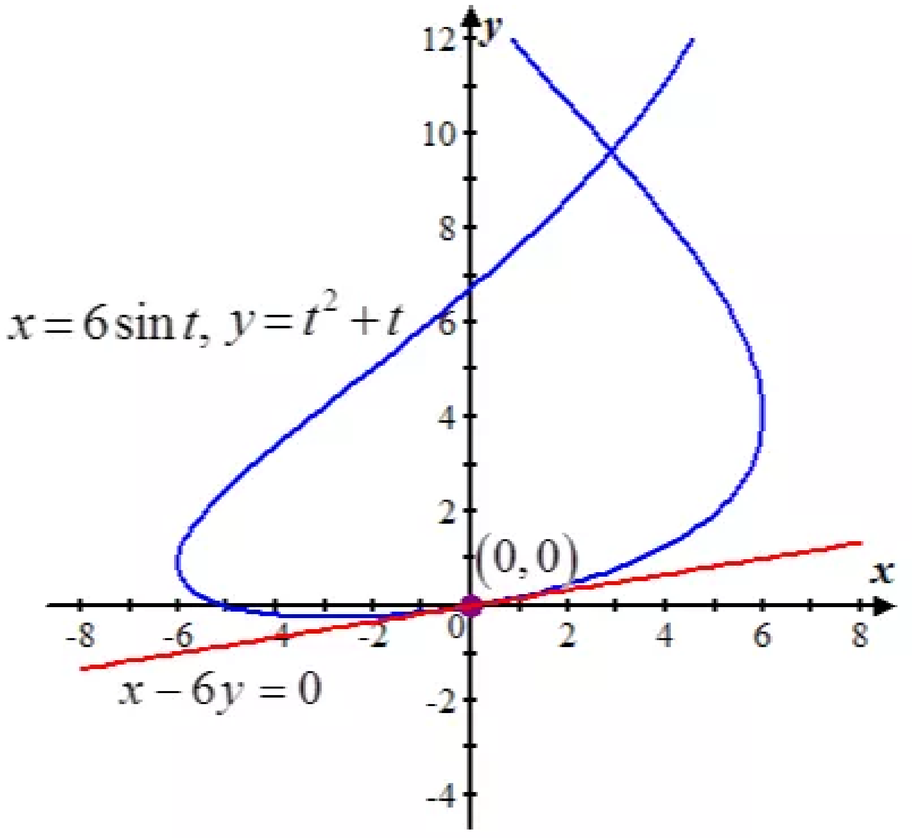
$$y - y_1 = m(x - x_1)$$

The equation of the tangent at $(0,0)$ is

$$y - 0 = \frac{1}{6}(x - 0)$$
$$6y = x$$
$$x - 6y = 0$$

Thus the equation of the tangent to the curve at point $(0,0)$ is $\boxed{x - 6y = 0}$.

Sketch the graph of the curve and the tangent line is shown below:



Q10E

Consider the parametric equations

$$x=\cos t+\cos 2t,\quad y=\sin t+\sin 2t$$

Need to find the tangent to the curve at $(-1,1)$.

Differentiate the parametric equations with respect to t to obtain that

$$\frac{dx}{dt}=-\sin t-2\sin 2t \quad \text{and} \quad \frac{dy}{dt}=\cos t+2\cos 2t$$

Thus

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy/dt}{dx/dt}\\&=\frac{\cos t+2\cos 2t}{-\sin t-2\sin 2t}\end{aligned}$$

Need to find the parameter t when the point $(x, y) = (-1, 1)$.

$$\begin{aligned} x^2 + y^2 &= (\cos t + \cos 2t)^2 + (\sin t + \sin 2t)^2 \\ &= \cos^2 t + \cos^2 2t + 2\cos t \cos 2t + \sin^2 t + \sin^2 2t + 2\sin t \sin 2t \\ &= 1 + 1 + 2(\cos t \cos 2t + \sin t \sin 2t) \\ &= 2 + 2\cos t \end{aligned}$$

So,

$$\begin{aligned} (-1)^2 + (1)^2 &= 2 + 2\cos t \\ 2 &= 2 + 2\cos t \\ \cos t &= 0 \\ t &= \frac{\pi}{2} \end{aligned}$$

The slope of the tangent at $t = \frac{\pi}{2}$ is

$$\begin{aligned} \left(\frac{dy}{dx}\right)_{t=\frac{\pi}{2}} &= \frac{\cos\frac{\pi}{2} + 2\cos\pi}{-\sin\frac{\pi}{2} - 2\sin\pi} \\ &= \frac{0 + 2(-1)}{-1 - 0} \\ &= 2 \end{aligned}$$

Recollect that, the equation of the tangent line at (x_1, y_1) with slope m is

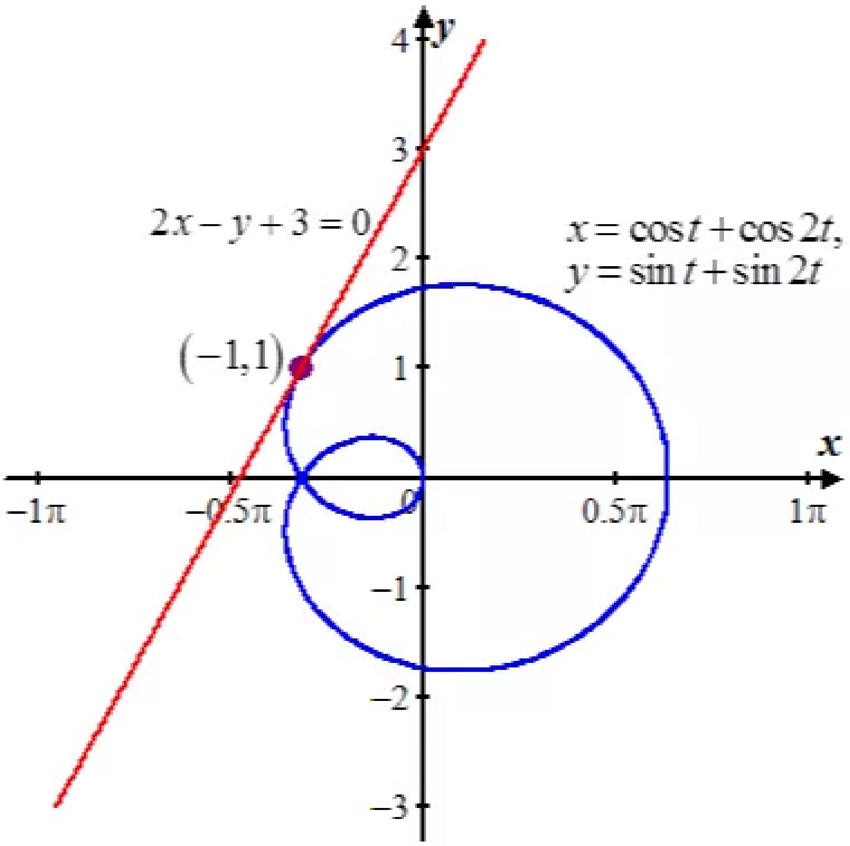
$$y - y_1 = m(x - x_1)$$

The equation of the tangent at $(-1, 1)$ is

$$\begin{aligned} y - 1 &= 2(x + 1) \\ y - 1 &= 2x + 2 \\ 2x - y + 3 &= 0 \end{aligned}$$

Thus the equation of the tangent to the curve at point $(-1, 1)$ is $\boxed{2x - y + 3 = 0}$.

Sketch the graph of the curve and the tangent line is shown below:



Q11E

Given curve is $x = t^2 + 1,\ y = t^2 + t$

$$\frac{dx}{dt} = 2t,\ \frac{dy}{dt} = 2t + 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+1}{2t} = 1 + \frac{1}{2t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(1 + \frac{1}{2t}\right)}{2t} = \frac{\frac{1}{2}\left(-\frac{1}{t^2}\right)}{2t} = \boxed{-\frac{1}{4t^3}}$$

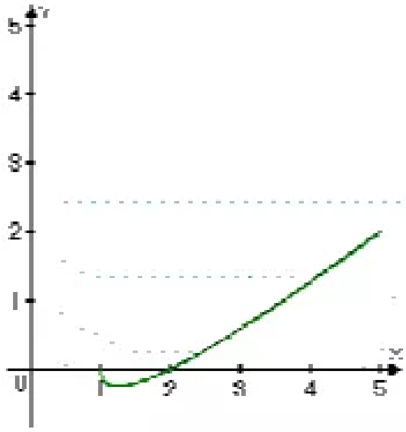
Observe that when $t < 0, t^3 < 0$ and so, $-\frac{1}{4t^3} > 0$

It is clear that when t < 0, $\frac{d^2y}{dx^2} > 0$

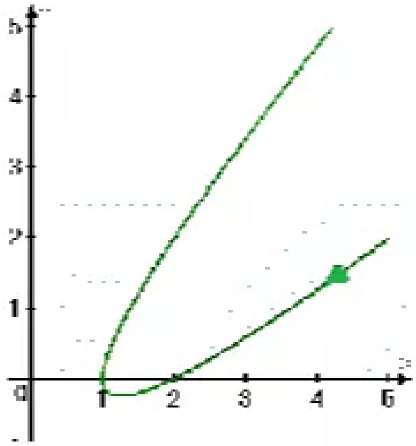
Further, the curve has concave upward ness where $\frac{d^2y}{dx^2} > 0$

Keeping this in view, we confirm that the given curve has concave upward ness at t <0

The following graph verifies this result. In this graph the concave upwardness continued till (1,0)



then started the concave down wa rd ness.



The first graph shows the concave upward ness for t <0

Then second graph shows the remaining part is concave down ward.

Q12E

Consider the curves

$x = t^3 + 1 \quad \cdots\cdots(1)$

$y = t^2 - t \quad \cdots\cdots(2)$

Now differentiate (1) on both sides

$\frac{dx}{dt} = 3t^2$ Since $\frac{d}{dt}t^n = nt^{n-1}$ and $\frac{d}{dt}(\text{constant}) = 0$

Again differentiate (2) on both sides

$\frac{dy}{dt} = 2t - 1$ Since $\frac{d}{dt}t^n = nt^{n-1}$

Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{2t-1}{3t^2}$$

$$\boxed{\frac{dy}{dx} = \frac{2}{3t} - \frac{1}{3t^2}}$$

Now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$= \dfrac{\frac{d}{dt}\left(\frac{2}{3t} - \frac{1}{3t^2}\right)}{3t^2}$ Since $\frac{dy}{dx} = \frac{2}{3t} - \frac{1}{3t^2}$ and $\frac{dx}{dt} = 3t^2$

$$= \frac{\frac{2}{3}\left[\frac{d}{dt}\left(\frac{1}{t}\right)\right] - \frac{1}{3}\left[\frac{d}{dt}\left(\frac{1}{t^2}\right)\right]}{3t^2}$$

$= \dfrac{-\frac{2}{3t^2} + \frac{2}{3t^3}}{3t^2}$ Since $\frac{d}{dt}\left(\frac{1}{t}\right) = -\frac{1}{t^2}$; $\frac{d}{dt}\left(\frac{1}{t^2}\right) = -\frac{2}{t^3}$

$$= \boxed{\frac{2}{9t^4}\left(\frac{1}{t} - 1\right)}$$

Concavity Test:

(a)If $\frac{d^2y}{dx^2} > 0$ for all *x* in *I*, then the graph of *f* is concave upward on *I*.

(b)If $\frac{d^2y}{dx^2} < 0$ for all *x* in *I*, then the graph of *f* is concave downward on *I*.

Now consider

$$\frac{d^2y}{dt^2} = \frac{2}{9t^4}\left(\frac{1}{t} - 1\right)$$

Now see that for every *t*, $\frac{2}{9t^4} > 0$

So, $\frac{d^2y}{dx^2} > 0 \text{ if and only if } \frac{1}{t} - 1 > 0$

$$\frac{d^2y}{dx^2} > 0 \text{ if and only if } \frac{1}{t} > 1$$

$$\frac{d^2y}{dx^2} > 0 \text{ if and only if } t < 1$$

Also, when $t < 0$, $\frac{1}{t} - 1 < 0$

This leads to $\frac{d^2y}{dx^2} < 0, \text{ when } t < 0$

Thus by **Concavity Test** and putting the above results together, the given curve has concavity upward on $(0,1)$.

Q13E

Consider the curve $x = e^t, y = te^{-t}$

Now

$$\frac{dx}{dt} = e^t, \frac{dy}{dt} = te^{-t}(-1) + e^{-t} \cdot 1$$
$$= e^{-t}(1-t)$$

Therefore,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$= \frac{e^{-t}(1-t)}{e^t}$$
$$= \boxed{e^{-2t}(1-t)}$$

Now

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{d}{dt}\left(e^{-2t}(1-t)\right)}{e^t}$$

$$= \frac{e^{-2t}(-1)+(1-t)e^{-2t}(-2)}{e^t}$$

$$= \frac{e^{-2t}(2t-3)}{e^t}$$

$$= \boxed{e^{-3t}(2t-3)}$$

Concavity Test:

(a) If $\frac{d^2y}{dx^2} > 0$ for all *x* in *I*, then the graph of *f* is concave upward on *I*.

(b) If $\frac{d^2y}{dx^2} < 0$ for all *x* in *I*, then the graph of *f* is concave downward on *I*.

Always, $e^{-3t} \geq 0$

So, $e^{-3t}(2t-3) = \frac{d^2y}{dx^2} > 0$ if and only if $2t-3>0$

So, the given curve is concave upward at *t* where $\frac{d^2y}{dx^2} > 0 \Leftrightarrow 2t-3>0$

That is the given curve has concavity upward at $\boxed{t > \frac{3}{2}}$.

Q14

Sol: Given parametric curve is $x = t^2+1,\ y = e^t - 1$

$$\frac{dx}{dt} = 2t,\ \frac{dy}{dt} = e^t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{e^t}{2t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{d}{dt}\left(\frac{e^t}{2t}\right)}{2t} = \frac{\left(\frac{2te^t - 2e^t}{4t^2}\right)}{2t}$$

$$= \frac{e^t(t-1)}{4t^3} = e^t\left(\frac{1}{4t^2} - \frac{1}{4t^3}\right)$$

$$= \frac{e^t}{4t^2} - \frac{e^t}{4t^3}$$

We know that $\frac{e^t}{4t^2} > 0$ for every $t \neq 0$

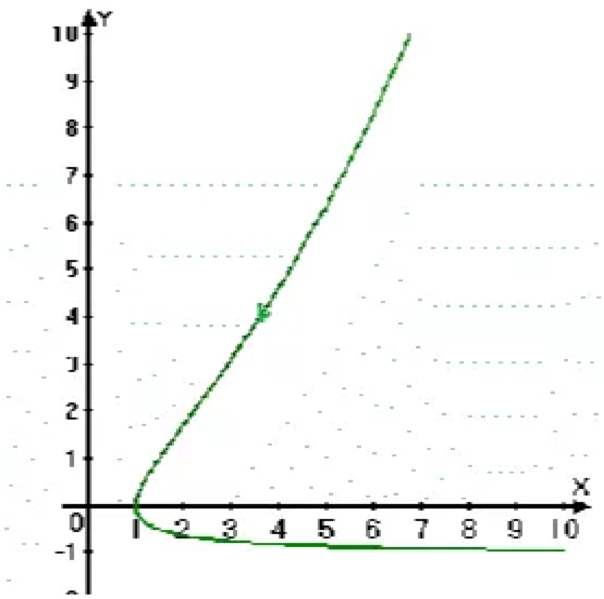
So, $\frac{d^2y}{dx^2} = \frac{e^t}{4t^2} - \frac{e^t}{4t^3} > 0$ if and only if $-\frac{e^t}{4t^3} < 0$

But we know that $e^t > 0$ so, $-\frac{e^t}{4t^3} < 0$ if and only if $t < 0$.

$\therefore \frac{d^2y}{dx^2} > 0$ if and only if $t < 0$

So, the given curve has concavity upward when $t < 0$

We verify this result in the following graph.



Q15E

Consider the parametric curve,

$x = 2\sin t,\ y = 3\cos t,\ 0 < t < 2\pi$.

Need to find the values of the derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

From the given parametric curve, $x = 2\sin t,\ y = 3\cos t,\ 0 < t < 2\pi$,

$$\frac{dx}{dt} = 2\cos t, \quad \frac{dy}{dt} = -3\sin t$$

Since, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, so,

$$\frac{dy}{dx} = \frac{-3\sin t}{2\cos t}$$

$$= -\frac{3}{2}\tan t \quad \text{Since, } \tan t = \frac{\sin t}{\cos t}$$

Thus, the value of $\frac{dy}{dx}$ is $\boxed{\frac{dy}{dx} = \frac{-3}{2}\tan t}$.

Now, consider:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{\frac{dx}{dt}}$$

$$= \frac{\frac{d}{dt}\left(-\frac{3}{2}\tan t\right)}{2\cos t} \quad \text{Since, } \frac{dy}{dx} = -\frac{3}{2}\tan t \text{ and } \frac{dx}{dt} = 2\cos t$$

$$= \frac{-\frac{3}{2}\cdot\frac{d}{dt}(\tan t)}{2\cos t} \quad \text{Since, } \frac{d}{dt}(cf(t)) = c\frac{d}{dt}(f(t))$$

$$= \frac{-\frac{3}{2}\sec^2 t}{2\cos t} \quad \text{Since, } \frac{d}{dt}(\tan t) = \sec^2 t$$

$$= -\frac{3}{4}\sec^3 t \quad \text{Since, } \frac{1}{\cos t} = \sec t$$

Thus, the value of $\frac{d^2y}{dx^2}$ is $\boxed{\frac{d^2y}{dx^2} = -\frac{3}{4}\sec^3 t}$.

The given parametric curve is concave upward when

$$\sec^3 t < 0$$
$$\sec t < 0$$
$$\cos t < 0$$
$$\frac{\pi}{2} < t < \frac{3\pi}{2}$$

Therefore, the parametric curve $x = 2\sin t,\ y = 3\cos t,\ 0 < t < 2\pi$ is concave upward for the values of t: $\boxed{\frac{\pi}{2} < t < \frac{3\pi}{2}}$.

Q16E

We have $x = \cos 2t, \quad y = \cos t \quad 0 < t < \pi$

Then $\frac{dx}{dt} = -2\sin 2t, \quad \frac{dy}{dt} = -\sin t$

so $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{-2\sin 2t}$$
$$= \frac{\sin t}{4\sin t\cos t}$$
$$= \frac{1}{4\cos t}$$
$$\boxed{\frac{dy}{dt} = \frac{1}{4}\sec t}$$

Therefore $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{1}{4}\sec t\tan t}{-2\sin 2t}$$
$$= -\frac{1}{8}\frac{\sin t/\cos^2 t}{\sin 2t}$$
$$= \frac{-1}{8}\frac{\sin t/\cos^2 t}{2\sin t\cos t}$$

Or $$\boxed{\frac{d^2y}{dx^2} = \frac{-1}{16}\sec^3 t}$$

The curve is concave upward when $\sec^3 t < 0 \Rightarrow \sec t < 0$

$$\Rightarrow \cos t < 0 \Rightarrow \boxed{\frac{\pi}{2} < t < \pi}$$

Q17E

Sol: Given parametric curve is $x = t^3 - 3t, y = t^2 - 3$

$$\frac{dx}{dt} = 3t^2 - 3, \frac{dy}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{2t}{3(t^2 - 1)}$$

The tangent to the given is horizontal when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

$$\frac{dy}{dt} = 2t = 0 \Rightarrow t = 0$$

Using this value of t in the given curve , we get $x = 0, y = -3$

$\therefore$ The point on the curve where the tangent is horizontal is (0,-3)

The tangent is vertical at $\frac{dx}{dt} = 0$

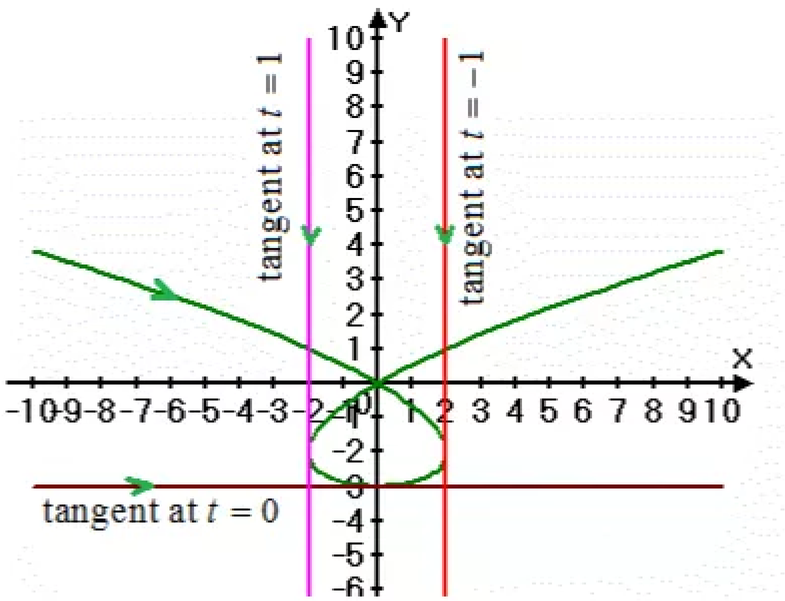
$$\frac{dx}{dt} = 3(t^2 - 1) = 0 \Rightarrow t = \pm 1$$

Using this in the given curve, the points on the curve where the tangent is vertical are

$$\left(x = 1^3 - 3(1), y = (1)^2 - 3\right), \left(x = (-1)^3 - 3(-1), y = (-1)^2 - 3\right)$$

=(-2, -2) and (2, -2)

We verify the above observations in the following graph.



Q18E

Given $x = t^3 - 3t, y = t^3 - 3t^2$

The tangent is horizontal when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

$\frac{dy}{dt} = 3t^2 - 6t = 0 \Rightarrow t = 0, 2$

Using t = 0, 2 in the given parametric equations, we get the points (0, 0) and (2,-4)

Further, we can easily see that when t =0, 2, $\frac{dx}{dt} = 3t^2 - 3 \neq 0$

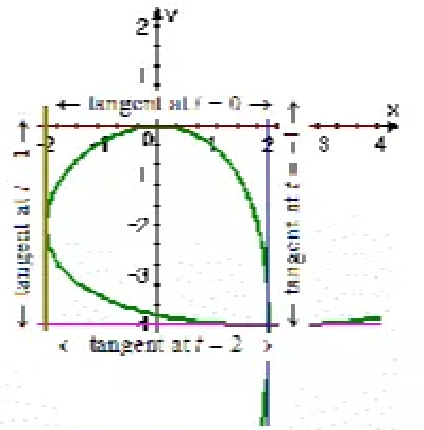
$\therefore$ the points at which the tangents are horizontal = (0, 0), (2, -4)

The tangent is vertical at $\frac{dx}{dt} = 0$

$\frac{dx}{dt} = 3t^2 - 3 = 0 \Rightarrow t = \pm 1$

The corresponding point on the given curve C is $\boxed{(-2,-2) \text{ and } (2,-4)}$.

We verify these observations in the following figure.



Q19E

Sol: Given $x = \cos\theta, y = \cos 3\theta$

The tangent is horizontal when $\frac{dy}{d\theta} = 0$

$\frac{dy}{d\theta} = -3\sin 3\theta$

we know that $\sin x = 0 \Leftrightarrow x = n\pi$

$\therefore -3\sin 3\theta = 0 \Rightarrow 3\theta = \pi$

$\Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}$

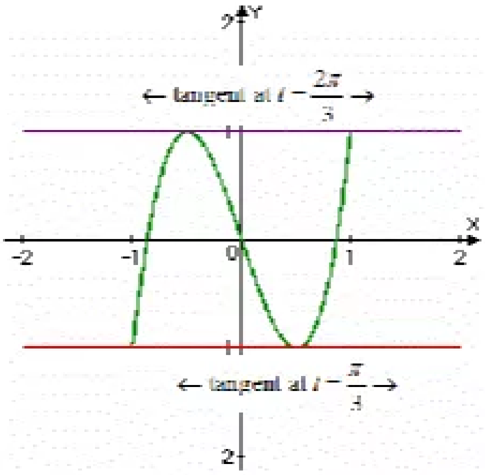
Substituting $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$ in the given curve, we get $\left(\cos\left(\frac{\pi}{3}\right), \cos 3\left(\frac{\pi}{3}\right)\right) = \left(\frac{1}{2}, -1\right)$,

$\left(\cos\left(\frac{2\pi}{3}\right), \cos 3\left(\frac{2\pi}{3}\right)\right) = \left(-\frac{1}{2}, 1\right)$

The points where the tangents are horizontal to the given curve are $\left(\frac{1}{2}, -1\right)$ and $\left(-\frac{1}{2}, 1\right)$

The tangent to the given curve is vertical at $\frac{dx}{d\theta} = 0$

Observe that $\frac{dx}{d\theta} = -\sin\theta = 0$ is not possible while the curve is defined for $\theta \geq 0$



So, the curve has no vertical tangents.

Q20E

Sol: Given $x = e^{\sin\theta}, y = e^{\cos\theta}$

The tangent is horizontal when $\frac{dy}{d\theta} = 0, \frac{dx}{d\theta} \neq 0$

$\frac{dy}{d\theta} = e^{\cos\theta}(-\sin\theta) = 0 \Rightarrow \theta = 0, -\pi,$

So, the given curve has horizontal tangents at $\theta = 0, -\pi$

Substituting $\theta = 0, -\pi$ in the given curve, we get $\left(e^{\sin 0}, e^{\cos 0}\right) = (1, e)$ and

$\left(e^{\sin(-\pi)}, e^{\cos(-\pi)}\right) = \left(e^{-\sin\pi}, e^{\cos\pi}\right) = \left(e^{0}, e^{-1}\right) = \left(1, \frac{1}{e}\right)$ where the tangents are horizontal.

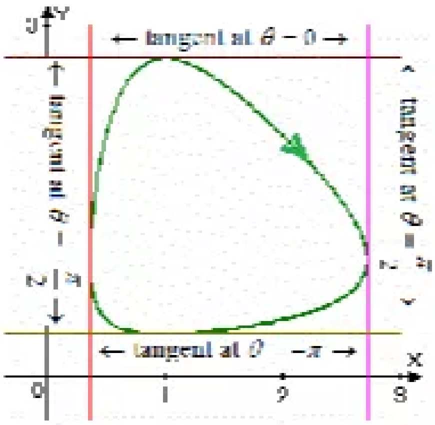
The tangent is vertical at $\frac{dx}{d\theta} = 0, \frac{dy}{d\theta} \neq 0$

$$\frac{dx}{d\theta} = e^{\sin\theta} \cos\theta = 0 \Rightarrow \theta = \pm\frac{\pi}{2}$$

Substituting $\theta = \pm\frac{\pi}{2}$ in the given curve, the points where the tangents are horizontal are

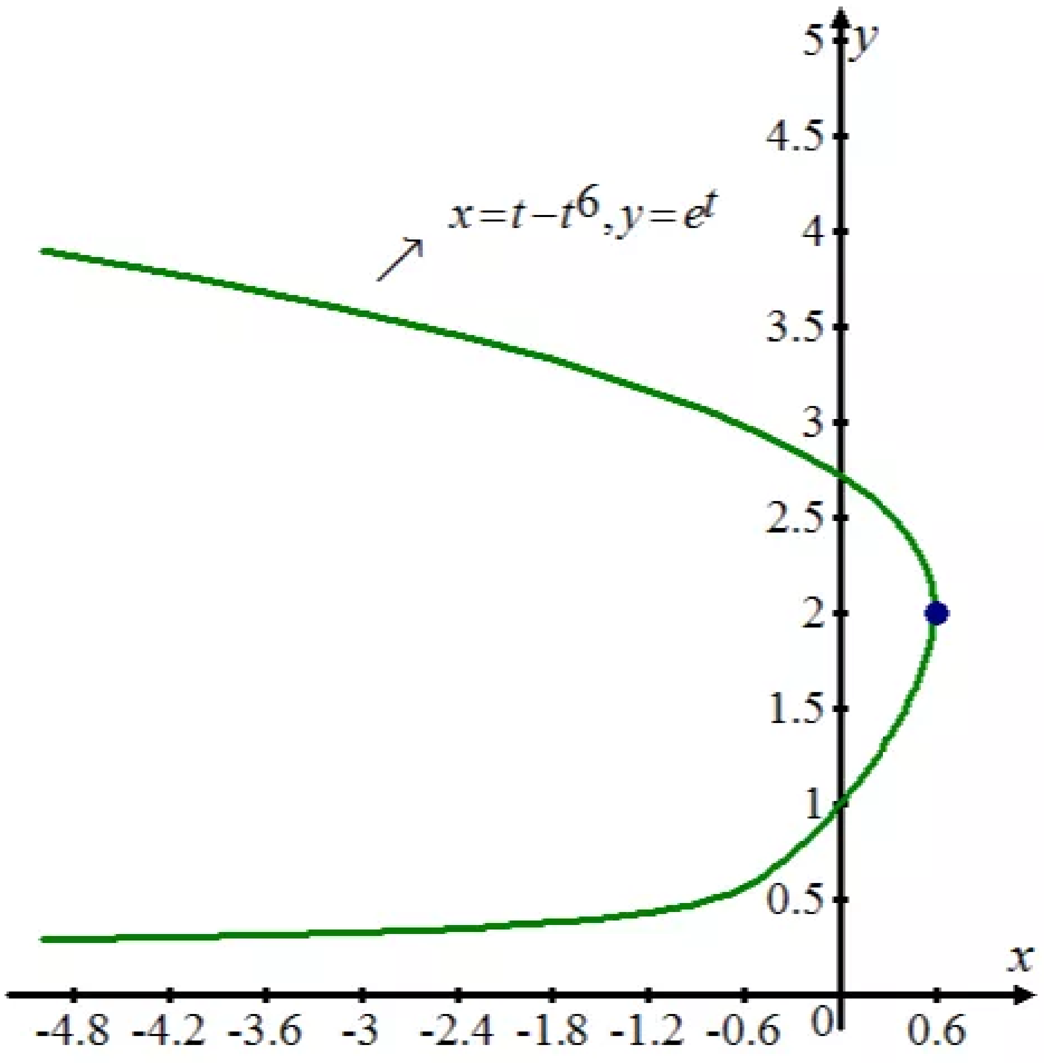
$\boxed{(e,1) \text{ and } \left(\frac{1}{e},1\right)}$.

The following figure confirms these observations.



Q21E

The graph of the curve $x = t - t^6, y = e^t$ is shown below:



From the graph, observe that the right most point on the curve is $\boxed{(0.6, 2)}$

The slope of the tangent to the curve is given by,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{\frac{d}{dt}\left(e^t\right)}{\frac{d}{dt}\left(t - t^6\right)}$$

$$= \frac{e^t}{1 - 6t^5} \quad \text{Use } \frac{d}{dt}\left(t^n\right) = nt^{n-1}$$

At the right most point on the curve, the slope of the tangent is undefined (because the tangent is vertical there).

$$\frac{dy}{dx} = \infty$$

That is,

$$\frac{e^t}{1-6t^5} = \infty$$

So we have,

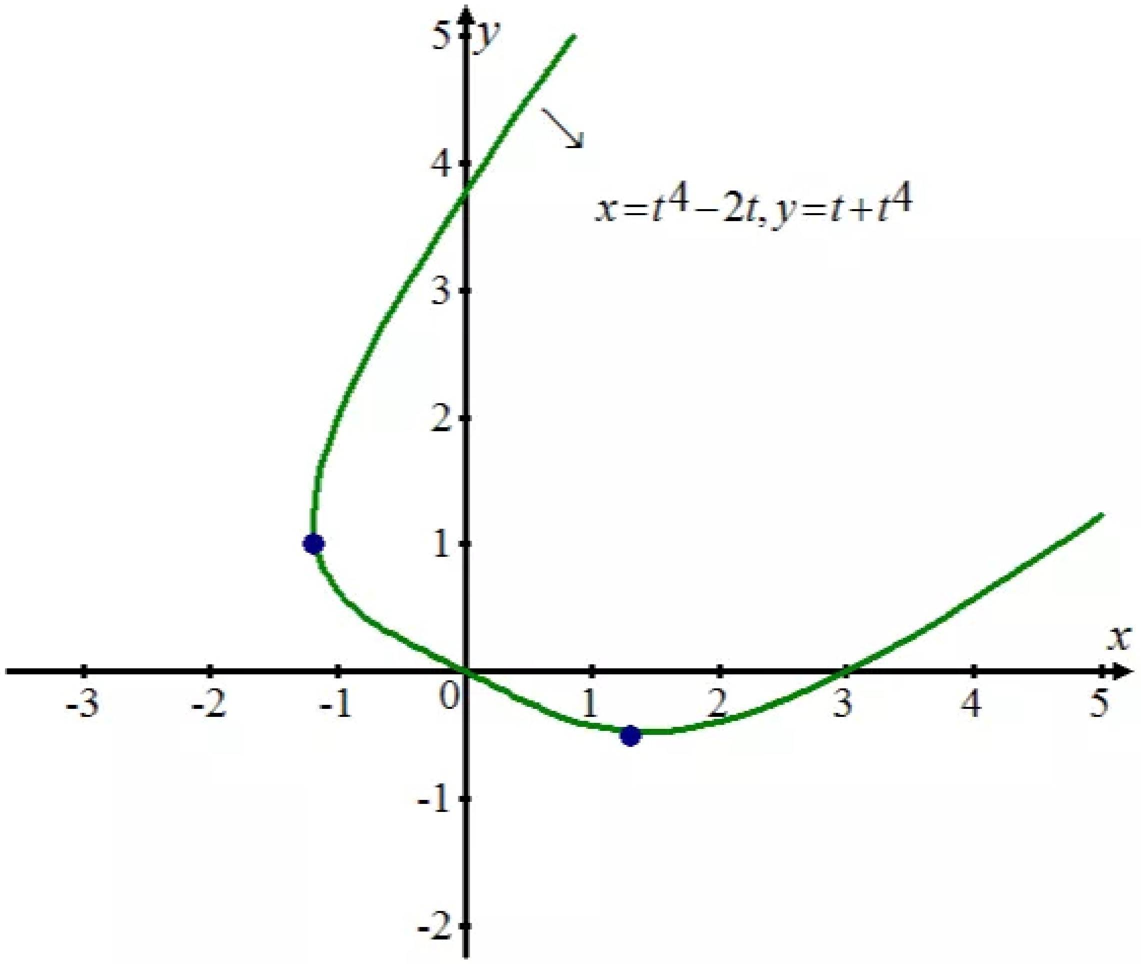
$$\begin{aligned} 1-6t^5 &= 0 \\ 1 &= 6t^5 \\ t^5 &= \frac{1}{6} \\ t &= \frac{1}{6^{\frac{1}{5}}} \end{aligned}$$

So the exact coordinates of the right most point on the curve is,

$$\begin{aligned} (x,y) &= \left(t-t^6, e^t\right) \\ &= \left(\frac{1}{6^{\frac{1}{5}}} - \frac{1}{6^{\frac{6}{5}}}, e^{6^{-\frac{1}{5}}}\right) \\ &= \left(\frac{1}{6^{\frac{1}{5}}} - \frac{1}{6\cdot 6^{\frac{1}{5}}}, e^{6^{-\frac{1}{5}}}\right) \\ &= \left(\frac{6-1}{6\cdot 6^{\frac{1}{5}}}, e^{6^{-\frac{1}{5}}}\right) \\ &= \left(\frac{5}{6^{\frac{6}{5}}}, e^{6^{-\frac{1}{5}}}\right) \\ &= \boxed{\left(5\cdot 6^{-\frac{6}{5}}, e^{6^{-\frac{1}{5}}}\right)} \end{aligned}$$

Q22E

The graph of the curve $x = t^4 - 2t, y = t + t^4$ is shown below:



From the graph, observe that the left most point and the lowest points on the graph approximately are $\boxed{(-1.2,1),(1.3,-0.5)}$

The slope of the tangent to the curve is given by,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{\frac{d}{dt}\left(t + t^4\right)}{\frac{d}{dt}\left(t^4 - 2t\right)}$$

$\frac{dy}{dx} = \frac{1 + 4t^3}{4t^3 - 2}$ Use $\frac{d}{dt}\left(t^n\right) = nt^{n-1}$

At the left most point, the tangent to the curve is vertical, and the slope of this vertical tangent is undefined.

That is,

$$\frac{dy}{dx}=\frac{1+4t^3}{4t^3-2}=\infty$$

So, we have

$$\begin{aligned}4t^3-2&=0\\ t^3&=\frac{1}{2}\\ t&=\frac{1}{2^{\frac{1}{3}}}\end{aligned}$$

Hence the left most point on the curve is given as,

$$\begin{aligned}(x,y)&=\left(t^4-2t,t+t^4\right)\\ &=\left(\frac{1}{2^{\frac{4}{3}}}-\frac{2}{2^{\frac{1}{3}}},\frac{1}{2^{\frac{1}{3}}}+\frac{1}{2^{\frac{4}{3}}}\right)\\ &=\left(\frac{1-2\cdot 2}{2^{\frac{4}{3}}},\frac{2+1}{2^{\frac{4}{3}}}\right)\\ &=\boxed{\left(-3\cdot 2^{-\frac{4}{3}},3\cdot 2^{-\frac{4}{3}}\right)}\end{aligned}$$

Also, at the lowest point on the curve, the tangent is parallel to x-axis, so its slope is 0.

That is,

$$\frac{dy}{dx}=\frac{1+4t^3}{4t^3-2}=0$$

So, we have

$$1+4t^3=0$$
$$4t^3=-1$$
$$t=\left(-\frac{1}{4}\right)^{\frac{1}{3}}$$

Hence the lowest point on the curve is given as,

$$(x,y)=\left(t^4-2t,t+t^4\right)$$
$$=\left(\left(-\frac{1}{4}\right)^{\frac{4}{3}}-2\cdot\left(-\frac{1}{4}\right)^{\frac{1}{3}},\left(-\frac{1}{4}\right)^{\frac{1}{3}}+\left(-\frac{1}{4}\right)^{\frac{4}{3}}\right)$$
$$=\left(\left(-\frac{1}{4}\right)^{\frac{1}{3}}\left(-\frac{1}{4}-2\right),\left(-\frac{1}{4}\right)^{\frac{1}{3}}\left(1-\frac{1}{4}\right)\right)$$
$$=\boxed{\left(-\frac{9}{4}\left(-\frac{1}{4}\right)^{\frac{1}{3}},\frac{3}{4}\left(-\frac{1}{4}\right)^{\frac{1}{3}}\right)}$$

Q23

We have $x=t^4-2t^3-2t^2$ and $y=t^3-t$

Then $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3t^2-1}{4t^3-6t^2-4t}$

For horizontal tangent $\frac{dy}{dt}=0$

$$\Rightarrow 3t^2-1=0$$
$$\Rightarrow t=\pm\,1/\sqrt{3}$$

At $t=\pm 1/\sqrt{3}\Rightarrow y=\frac{1}{3\sqrt{3}}-\frac{1}{\sqrt{3}}$ and $y=-\frac{1}{3\sqrt{3}}+\frac{1}{\sqrt{3}}$

$$\Rightarrow y=\frac{-2}{3\sqrt{3}}\approx -0.38 \text{ and } y=\frac{2}{3\sqrt{3}}\approx +0.38$$

$at\ t = \pm 1/\sqrt{3} \quad \Rightarrow x = -\frac{5}{9} - \frac{2}{3\sqrt{3}} \approx -0.9$ and $x = -\frac{5}{9} + \frac{2}{3\sqrt{3}} \approx -0.170$

For vertical tangent $\frac{dx}{dt} = 0$

$$\Rightarrow 4t^3 - 6t^2 - 4t = 0$$
$$\Rightarrow 2t\left(2t^2 - 3t - 1\right) = 0$$
$$\Rightarrow 2t(t-2)(2t+1) = 0$$
$$\Rightarrow t = 0 \text{ or } t = 2 \text{ or } t = -1/2$$

For t = 0, x = 0, y = 0
For t = 2, x = -8, y = 6
For t =-1/2, x = -0.1875, y = 0.375

We see that maximum x value is 8 and minimum value of x is 0
And minimum value of y = - 0.38
Maximum value of y = 6
So first we graph the curve in viewing rectangle [-8.5, 0.5]by [-0.5, 6.5]
This corresponds to $t \in [-1.2,\ 2.05]$

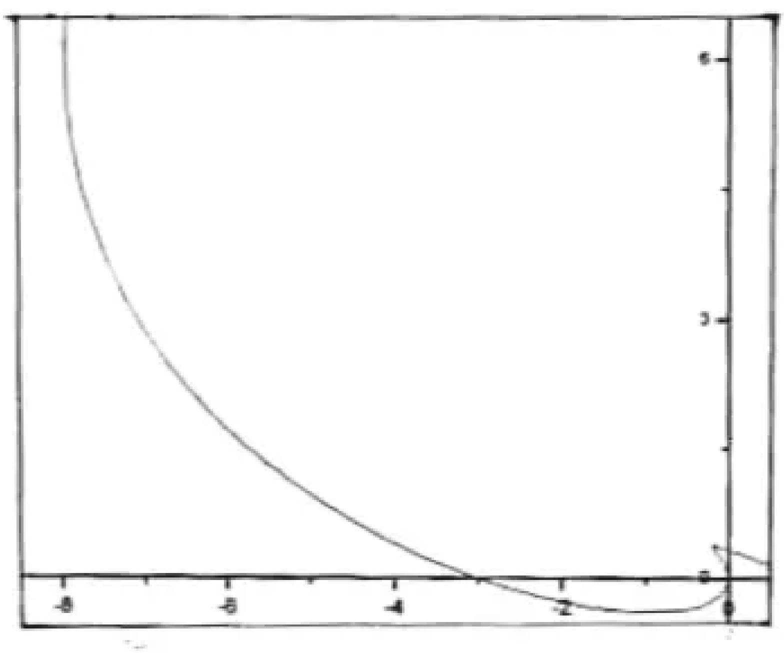


Fig. 1

But we see that on right hand side, we have very small portion of graph so we choose viewing rectangle [-8.5, 3] by [-1, 6.5] this corresponds to t $\in$[-1.2, 2.2]

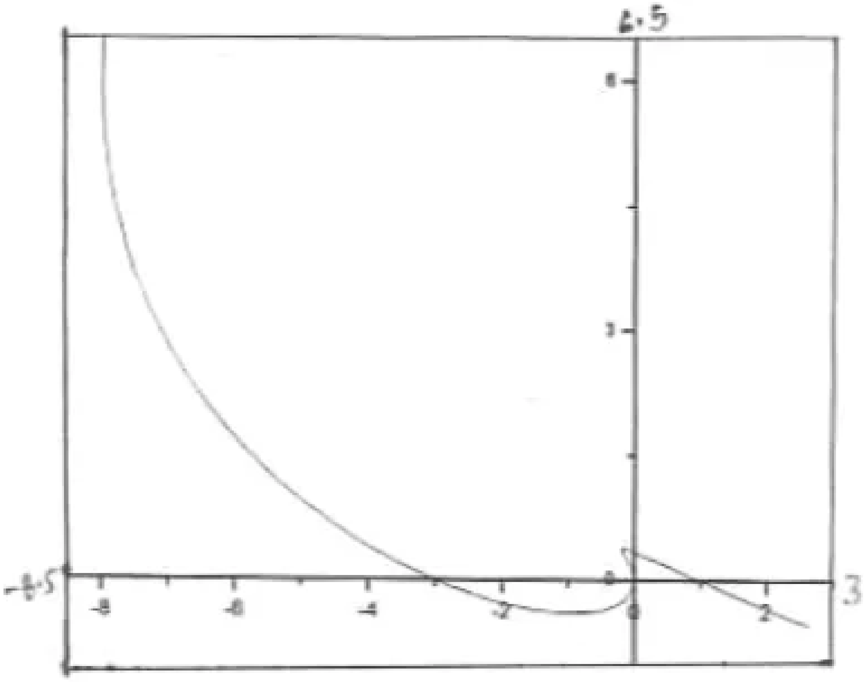


Fig. 2

Q24E

We graph the curve $x = t^4 + 4t^3 - 8t^2$ and $y = 2t^2 - t$ in the viewing rectangle [-5, 5] by [-5, 5], we see that loop of the graph is in the interval (-3, 0) so we choose viewing rectangle [-3.7, 0.2] by [-0.2, 1.4] in figure -2

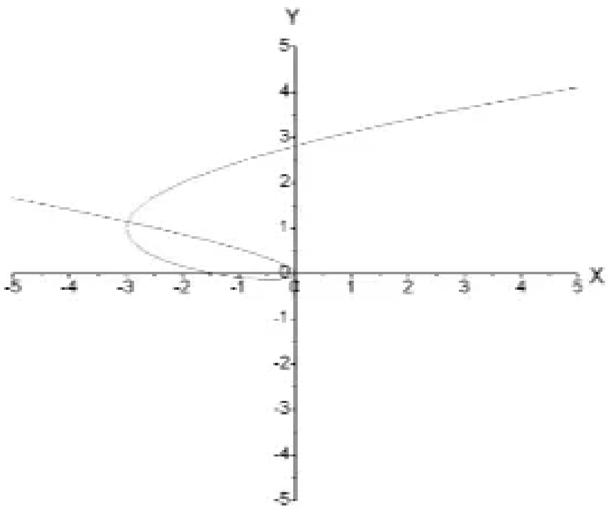


Fig.1

Graph has a horizontal tangent at about (-0.4,-0.1)
And vertical tangent at about (-3, 1) and (0, 0)

Now $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$= \frac{4t-1}{4t^3+12t^2-16t}$$

So for horizontal tangents,

We must have $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dt} = 0$$

$$\Rightarrow t = \frac{1}{4}$$

For $t = 1/4$, $x = (-0.4336)$, $y = -0.125$
This is shown in firgure2

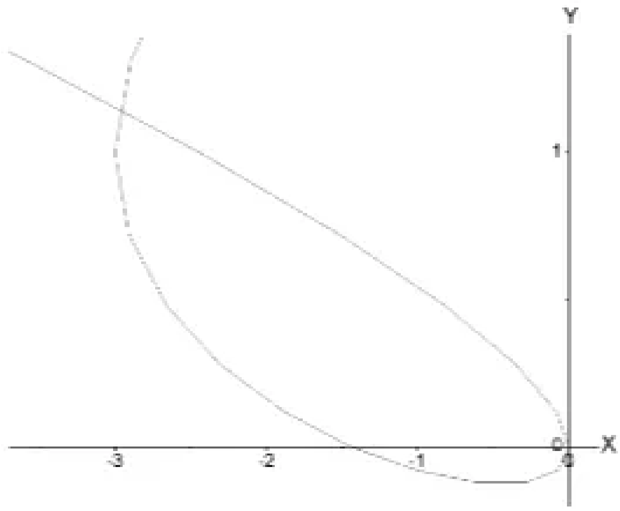


Fig.2

For the vertical tangent we must have $\frac{dx}{dt} = 0$

$$\Rightarrow 4t^3+12t^2-16t = 0$$

$$\Rightarrow 4t\left(t^2+3t-4\right) = 0$$

$$\Rightarrow 4t(t+4)(t-1) = 0$$

$t = 0, t = -4, t = +1$ For which $x = 0$, -3,-128

We do not have the point (-128, 36) in figure 2 so we also choose a rectangle [-130,130] by [0, 55] which shows another ventral tangent.

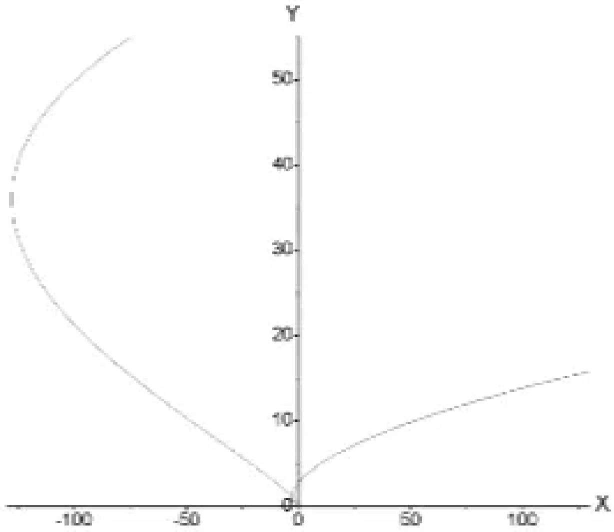


Fig.3

Q25E

We have $x = \cos t$, $y = \sin t \cos t$

then $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = -\sin^2 t + \cos^2 t = \cos 2t$

Let the tangent pass through (0, 0)

So $x = 0$

$\Rightarrow \cos t = 0$

$\Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots\ldots\ldots$

When $t = \pi/2$

$$\frac{dx}{dt} = -\sin(\pi/2) = -1$$

and $$\frac{dy}{dt} = \cos(2\pi/2) = -1$$

Then equation of tangent at (0, 0) is $\boxed{y = x}$

When $t = \frac{3\pi}{2}$

$$\frac{dx}{dt} = -\sin(3\pi/2) = 1$$

and $$\frac{dy}{dt} = \cos(3\pi) = -1$$

so $$\frac{dy}{dx} = -1$$

Then equation of tangent is $\boxed{y = -x}$

Thus the curve has two tangents $\boxed{y = x}$ and $\boxed{y = -x}$ at (0, 0)

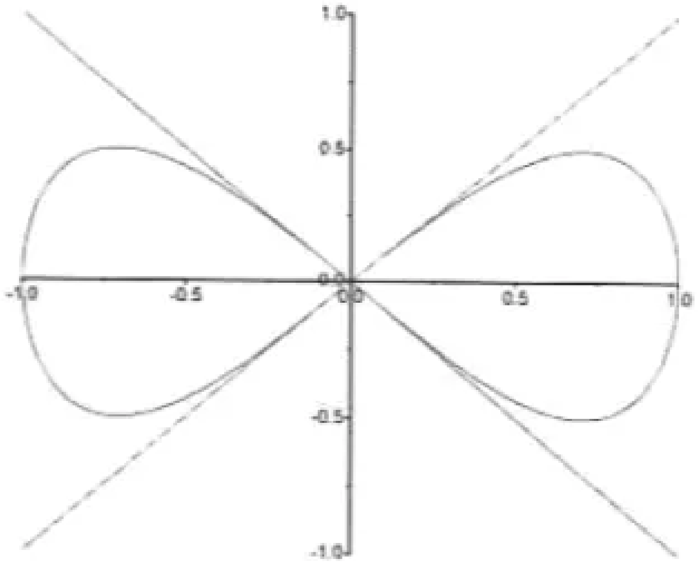
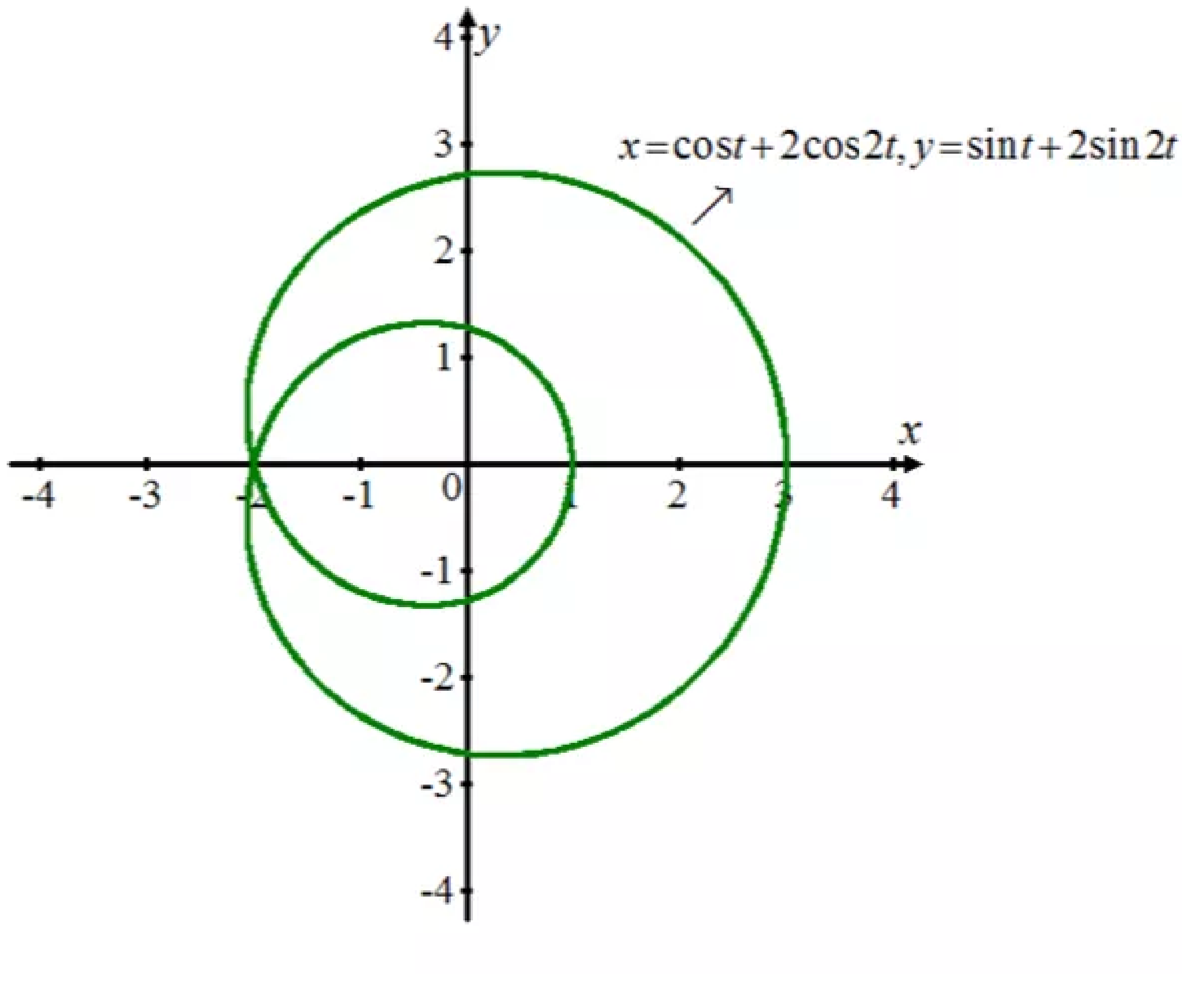


Fig. 1

Q26E

The graph of the curve $x = \cos t + 2\cos 2t, y = \sin t + 2\sin 2t$ is shown below:



From this figure, observe that, this curve intersects itself at the point $\boxed{(-2,0)}$.

The slope of the tangent to the curve at any point is given by,

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$=\frac{\frac{d}{dt}(\sin t+2\sin 2t)}{\frac{d}{dt}(\cos t+2\cos 2t)}$$

$$=\frac{\cos t+2(2\cos 2t)}{-\sin t+2(-2\sin 2t)}$$

$$=\frac{\cos t+4\cos 2t}{-\sin t-4\sin 2t}$$

Therefore, $\frac{dy}{dx}=\frac{\cos t+4\cos 2t}{-\sin t-4\sin 2t}$ ...... (1)

As the point $(-2,0)$ lies on the curve, substitute $x=-2, y=0$ in the equations,

$$-2=\cos t+2\cos 2t, 0=\sin t+2\sin 2t$$

To find the value of t solve these two equations as follows.

First consider,

$$\sin t+2\sin 2t=0$$
$$\sin t+2(2\sin t\cos t)=0$$
$$\sin t(1+4\cos t)=0$$
$$\sin t=0 \text{ (or) } 1+4\cos t=0$$
$$t=0 \text{ (or) } \cos t=-\frac{1}{4}$$

Suppose $t=0$

Put this value in $-2=\cos t+2\cos 2t$ to get,

$$-2=\cos 0+2\cos 0$$
$$-2=1+2(1)$$
$$-2=3$$

This is wrong. So $t\neq 0$

So $\cos t = -\frac{1}{4}$, put this value in $-2 = \cos t + 2\cos 2t$ to get,

$$-2 = \cos t + 2\left(2\cos^2 t - 1\right)$$

$$-2 = -\frac{1}{4} + 2\left(2\left(-\frac{1}{4}\right)^2 - 1\right)$$

$$-2 = -\frac{1}{4} + 2\left(\frac{1}{8} - 1\right)$$

$$-2 = -\frac{1}{4} + \frac{1}{4} - 2$$

$$-2 = -2$$

This is true, so the correct value of *t* is given as $\cos t = -\frac{1}{4}$

To find the value of $\sin t$ use the result,

$$\sin^2 t + \cos^2 t = 1$$

$$\sin t = \pm\sqrt{1 - \cos^2 t}$$

$$\sin t = \pm\sqrt{1 - \left(-\frac{1}{4}\right)^2}$$

$$\sin t = \pm\sqrt{\frac{15}{16}}$$

$$\sin t = \pm\frac{\sqrt{15}}{4}$$

Rewrite the equation (1) as follows.

$$\frac{dy}{dx} = \frac{\cos t + 4\cos 2t}{-\sin t - 4\sin 2t}$$

$$\frac{dy}{dx} = \frac{\cos t + 4\left(2\cos^2 t - 1\right)}{-\sin t - 4\left(2\sin t\cos t\right)}$$

$$\frac{dy}{dx} = \frac{\cos t + 4\left(2\cos^2 t - 1\right)}{-\sin t - 4\left(2\sin t\cos t\right)} \quad \ldots\ldots (2)$$

Now substitute the values of $\sin t = \frac{\sqrt{15}}{4}$, and $\cos t = -\frac{1}{4}$ in the equation (2), we get

$$\frac{dy}{dx} = \frac{\left(-\frac{1}{4}\right) + 4\left(2\left(-\frac{1}{4}\right)^2 - 1\right)}{-\frac{\sqrt{15}}{4} - 4\left(2\left(\frac{\sqrt{15}}{4}\right)\left(-\frac{1}{4}\right)\right)}$$

$$= \frac{-\frac{1}{4} + 4\left(-\frac{7}{8}\right)}{-\frac{\sqrt{15}}{4} + \frac{\sqrt{15}}{2}}$$

$$= \frac{-\frac{1}{4} - \frac{7}{2}}{\frac{\sqrt{15}}{4}}$$

$$= \frac{-\frac{15}{4}}{\frac{\sqrt{15}}{4}}$$

$$= -\sqrt{15}$$

Therefore the equation of the tangent line at the point $(x_1, y_1) = (-2, 0)$, and having the slope $m = -\sqrt{15}$ is,

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -\sqrt{15}(x + 2)$$

$$\boxed{y = -\sqrt{15}x - 2\sqrt{15}}$$

Substitute the values of $\sin t=-\frac{\sqrt{15}}{4}$, and $\cos t=-\frac{1}{4}$ in the equation (2), we get

$$\frac{dy}{dx}=\frac{\left(-\frac{1}{4}\right)+4\left(2\left(-\frac{1}{4}\right)^{2}-1\right)}{\frac{\sqrt{15}}{4}-4\left(2\left(-\frac{\sqrt{15}}{4}\right)\left(-\frac{1}{4}\right)\right)}$$

$$=\frac{-\frac{1}{4}+4\left(-\frac{7}{8}\right)}{\frac{\sqrt{15}}{4}-\frac{\sqrt{15}}{2}}$$

$$=\frac{-\frac{1}{4}-\frac{7}{2}}{-\frac{\sqrt{15}}{4}}$$

$$=\frac{-\frac{15}{4}}{-\frac{\sqrt{15}}{4}}$$

$$=\sqrt{15}$$

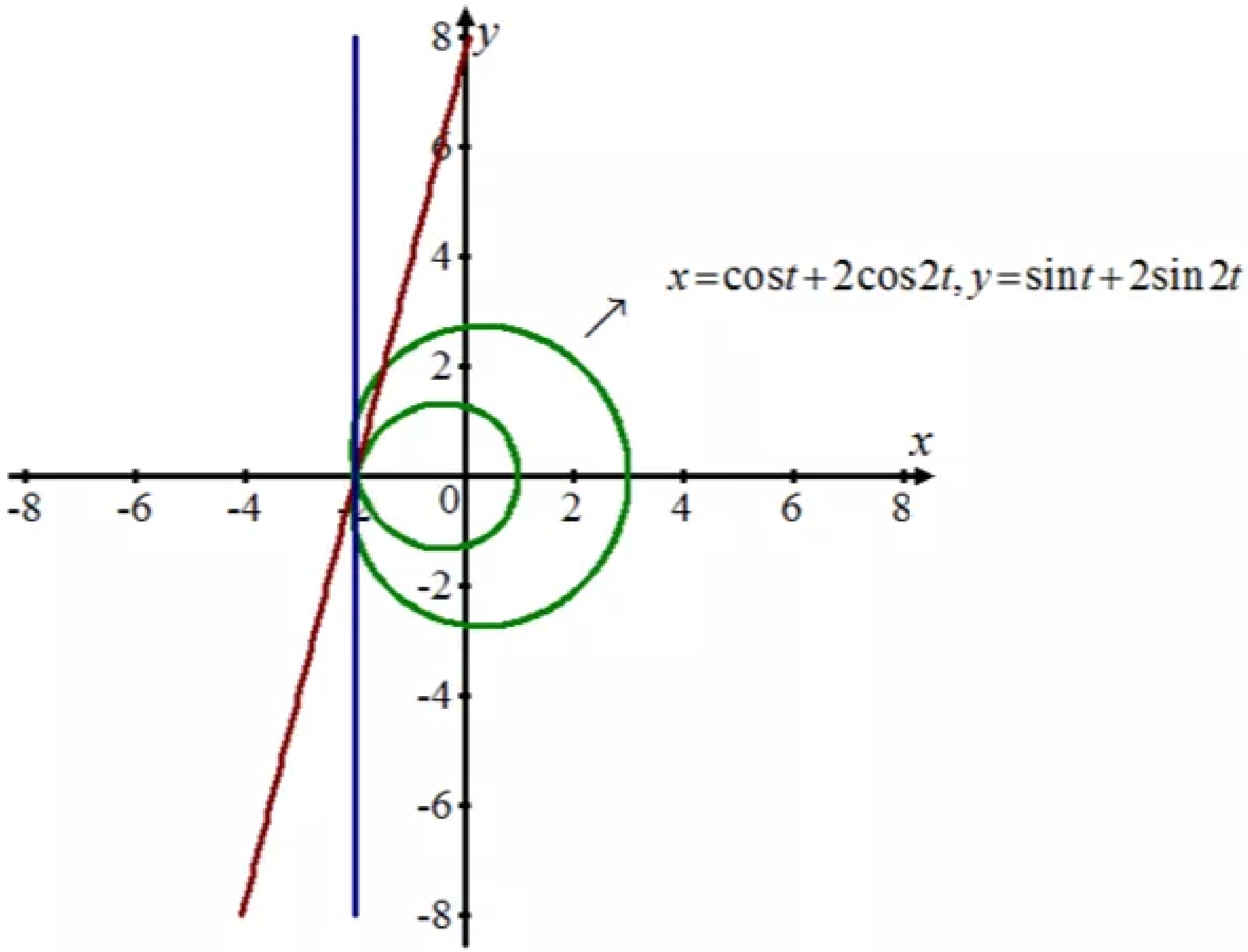
Therefore the equation of the tangent line at the point $(x_1,y_1)=(-2,0)$, and having the slope $m=\sqrt{15}$ is,

$$y-y_1=m(x-x_1)$$

$$y-0=\sqrt{15}(x+2)$$

$$\boxed{y=\sqrt{15}x+2\sqrt{15}}$$

The graph of the given curve with the tangents at $(-2,0)$, is shown below:



Q27E

(a) We have $x = r\theta - d\sin\theta$

$y = r - d\cos\theta$

Then $\frac{dx}{d\theta} = r - d\cos\theta$

And $\frac{dy}{d\theta} = d\sin\theta$

Step 2 of 3

Therefore $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{d\sin\theta}{r - d\cos\theta}$

Slope of the tangent line is $= \boxed{\frac{dy}{dx} = \frac{(d\sin\theta)}{(r - d\cos\theta)}}$

(b) For vertical tangent we must have $\frac{dx}{d\theta}=0$

$$\Rightarrow r-d\cos\theta=0$$
$$\Rightarrow r=d\cos\theta$$
$$\Rightarrow \theta=\cos^{-1}(r/d)$$

This is defined when $-1\le\frac{r}{d}\le 1$ since $|\cos\theta|\le 1$

So we must have r < d for vertical tangent

If d< r, then r/d >1 so θ is not defined

Thus $\boxed{\text{for } d<r, \text{ trochoid does not have vertical tangent}}$.

Q28E

(A) We have $x=a\cos^3\theta$

$y=a\sin^3\theta$

Then $\frac{dx}{d\theta}=3a\cos^2\theta(-\sin\theta)$

$$\Rightarrow \frac{dx}{d\theta}=-3a\sin\theta\cos^2\theta$$

And $\frac{dy}{d\theta}=3a\sin^2\theta\cos\theta$

Then slope of the tangent is $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$

$$\Rightarrow \frac{dy}{dx}=\frac{3a\sin^2\theta\cos\theta}{-3a\sin\theta\cos^2\theta}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\sin\theta}{\cos\theta} \quad \text{or} \quad \boxed{\frac{dy}{dx}=-\tan\theta}$$

(B) For horizontal tangent we must have $\frac{dy}{dx}=0$

$$\Rightarrow -\tan\theta=0$$
$$\Rightarrow \theta=\pm n\pi$$

Corresponding points are $\boxed{(\pm a,0)}$

For vertical tangent we must have $\cos\theta=0$

Or $\theta=2n\pi\pm\pi/2$

Corresponding points are $\boxed{(0, \pm a)}$

(C) We have $\frac{dy}{dx} = \pm 1$

Or $\frac{dy}{dx} = -\tan\theta = \pm 1$

$\Rightarrow \tan\theta = 1$ or -1

Or $\theta = n\pi \pm \pi/4$ where $n = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$

Corresponding point are $\boxed{\left(\pm\frac{\sqrt{2}}{4}a, \ \pm\frac{\sqrt{2}}{4}a\right)}$

Q29E

Consider the following curves:

$x = 2t^3$ And $y = 1 + 4t - t^2$

The objective is to find the points on the curve $x = 2t^3$ and $y = 1 + 4t - t^2$, where the slope of the tangent line equals to 1.

In order to do this, use the tangents formula to find the slope $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{ if } \frac{dx}{dt} \neq 0.$$

Differentiate the equation $x = 2t^3$ with respect to t.

$$\frac{dx}{dt} = 6t^2$$

Also differentiate the equation $y = 1 + 4t - t^2$ with respect to t

$$\frac{dy}{dt} = 4 - 2t$$

Now, using the formula,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{4-2t}{6t^2} \\ &= \frac{(2-t)}{3t^2}\end{aligned}$$

Given that, the slope of the line equals to one.

So, equate $\frac{dy}{dx}=1$

$$\frac{(2-t)}{3t^2}=1$$

$$2-t=3t^2$$

$$3t^2+t-2=0$$

$$(3t-2)(t+1)=0$$

This implies,

$t=\frac{2}{3}$ Or $t=-1$

When $t=\frac{2}{3}$, plug it into the parametric function,

$$\begin{aligned} x &= 2t^3 \\ &= 2\left(\frac{2}{3}\right)^3 \\ &= \frac{16}{27} \end{aligned}$$

And,

$$\begin{aligned} y &= 1+4t-t^2 \\ &= 1+4\left(\frac{2}{3}\right)-\left(\frac{2}{3}\right)^2 \\ &= 1+\frac{8}{3}-\frac{4}{9} \\ &= \frac{29}{9} \end{aligned}$$

So, the point is $\left(\frac{16}{27},\frac{29}{9}\right)$.

When $t=-1$, plug it into the parametric function,

$$x = 2t^3$$
$$= 2(-1)^3$$
$$= -2$$

And,

$$y = 1+4t-t^2$$
$$= 1+4(-1)-(-1)^2$$
$$= 1-5-1$$
$$= -4$$

So, the tangent line have slope 1, at the point $(-2,-4)$.

Hence, the required points are $\boxed{\left(\frac{16}{27},\frac{29}{9}\right) \text{and } (-2,-4)}$.

Q30E

We have $x = 3t^2 + 1$

And $y = 2t^3 + 1$

Then $\frac{dx}{dt} = 6t$ and $\frac{dy}{dt} = 6t^2$

The slope of the tangent at any point

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
$$= \frac{6t^2}{6t}$$
$$\Rightarrow \boxed{\frac{dy}{dx} = t}$$

Then at the point corresponding to the value t, an equation of the tangent line is

$$\left[y-\left(2t^3+1\right)\right] = t\left[x-\left(3t^2+1\right)\right]$$

Since this tangent line passes through the point (4, 3) then

$$\left[3-\left(2t^3+1\right)\right] = t\left[4-\left(3t^2+1\right)\right]$$

Or $3-2t^3-1 = 4t-3t^3-t$

Or $t^3-3t+2=0$

Or $(t-1)^2(t+2)=0$

Or $t=1$ or $t=-2$

When $t = 1$, $x = 4$ and $y = 3$ and slope 1

Then the equation of the tangent at (4, 3) with slopes 1, is

$$(y-3)=1(x-4)$$

$$\Rightarrow y = x-4+3$$

$$\Rightarrow \boxed{y = x-1}$$

When $t = -2$, $x = 13$ and $y = -15$

Then the equation of the tangent line at (13, -15) with the slope (-2) is

$$(y+15)=-2(x-13)$$

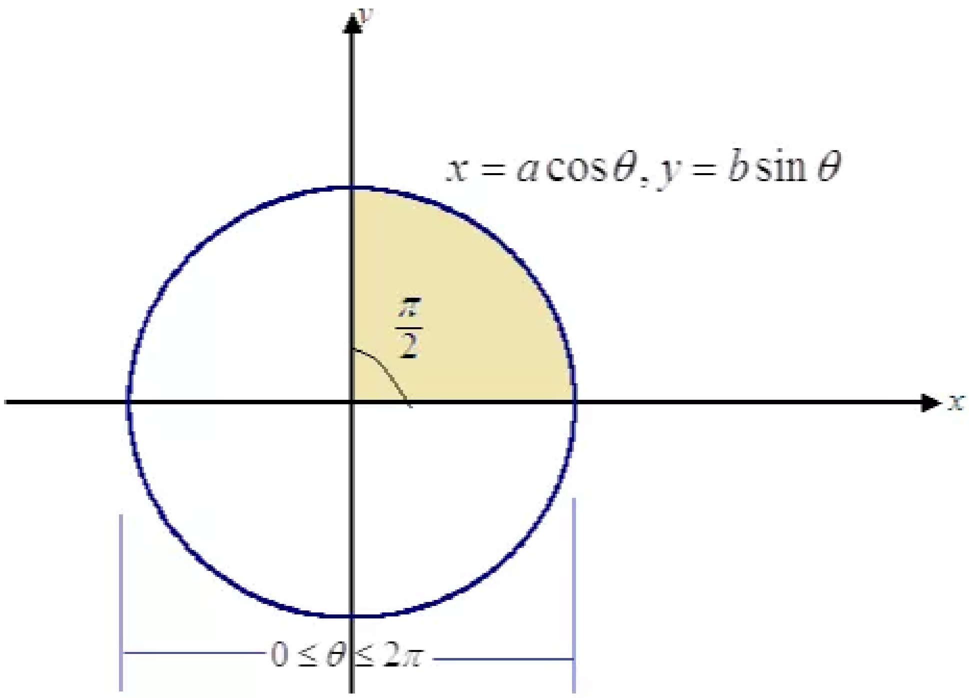
$$\Rightarrow \boxed{y=-2x+11}$$

Q31E

Consider the parametric equations

$$x = a\cos\theta, y = b\sin\theta, \; 0 \le \theta \le 2\pi \quad \text{......} \; (1)$$

Sketch the graph of the curves as follows.



Recall that,

The area of curve traced out by parametric equations $x = f(t), y = g(t)$, $\alpha \le t \le \beta$ is

$$A = \int_{\alpha}^{\beta} g(t) f'(t)\,dt$$

So, to find the area, first find dx.

Differentiate $x = a\cos\theta$ on each side.

Then

$$dx = -a\sin\theta\, d\theta$$

Therefore, the area is

$$A = \int_0^{2\pi} y dx$$

Since the ellipse is symmetric about the axis

Then area,

$$A = 4\int_{\frac{\pi}{2}}^{0} y dx$$

$$= 4\int_{\pi/2}^{0} b\sin\theta(-a\sin\theta)d\theta$$

$$= -4ab\int_{\pi/2}^{0} \sin^2\theta\, d\theta$$

$$= -4ab\left(-\int_0^{\pi/2} \frac{1-\cos 2\theta}{2} d\theta\right)$$

Use $\int_a^b f(x)dx = -\int_b^a f(x)dx$ and $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

Continue the above steps.

$$A = \frac{4ab}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2\theta)d\theta$$

$$= 2ab\left[\int_0^{\frac{\pi}{2}} d\theta - \int_0^{\frac{\pi}{2}} \cos 2\theta d\theta\right]$$

$$= 2ab\left[[\theta]_0^{\frac{\pi}{2}} - \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}\right]$$

$$= 2ab\left[\frac{\pi}{2} - \left[\frac{\sin\pi}{2} - \frac{\sin 0}{2}\right]\right]$$ Apply the limits of integration

$$= 2ab\left[\frac{\pi}{2}\right]$$

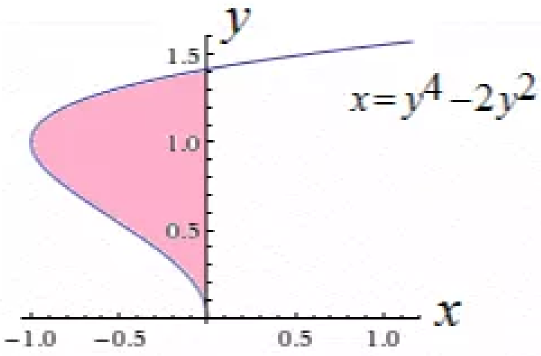
$$= \pi ab$$

Thus, the area of the ellipse enclosed by the parametric equations given by (1) is $\boxed{\pi ab \text{ units}}$.

Q32E

We need to find the area enclosed by the curve $x = t^2 - 2t$, $y = \sqrt{t}$ and the y-axis.

Graph showings the area enclosed by the given curves:



The formula for the area under a curve $x = F(y)$, is $A = \int_a^b F(y)dy$ where $F(y) \geq 0$.

In this case, we will solve for x as a function of y from the given equations.

$y = \sqrt{t}$

$\Rightarrow y^2 = t$

So $x = y^4 - 2y^2$

The intercept of the curve and the y-axis occurs when $x = 0$

That means

$0 = y^4 - 2y^2$

$\Rightarrow 0 = y^2\left(y^2 - 2\right)$

$\Rightarrow y = 0\ ,\ y = \sqrt{2}$

Observe from the graph that $x = F\left(y\right)$ is negative.

So required area is

$$A = -\int_0^{\sqrt{2}} x dy$$

$$A = -\int_0^{\sqrt{2}} \left(y^4 - 2y^2\right) dy$$

$$= -\frac{1}{5}\left[y^5\right]_0^{\sqrt{2}} + \frac{2}{3}\left[y^3\right]_0^{\sqrt{2}}$$

$$= -\frac{1}{5}\left[\left(\sqrt{2}\right)^5 - 0\right] + \frac{2}{3}\left(\left(\sqrt{2}\right)^3 - 0\right)$$

$$= -\frac{1}{5}\left(4\sqrt{2}\right) + \frac{2}{3}\left(2\sqrt{2}\right)$$

$$= \frac{4\sqrt{2}}{3} - \frac{4\sqrt{2}}{5}$$

$$= 4\sqrt{2}\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= 4\sqrt{2}\left(\frac{2}{15}\right)$$

$$= \frac{8\sqrt{2}}{15}$$

Thus area enclosed by the given curves is given by $\boxed{\frac{8\sqrt{2}}{15}}$

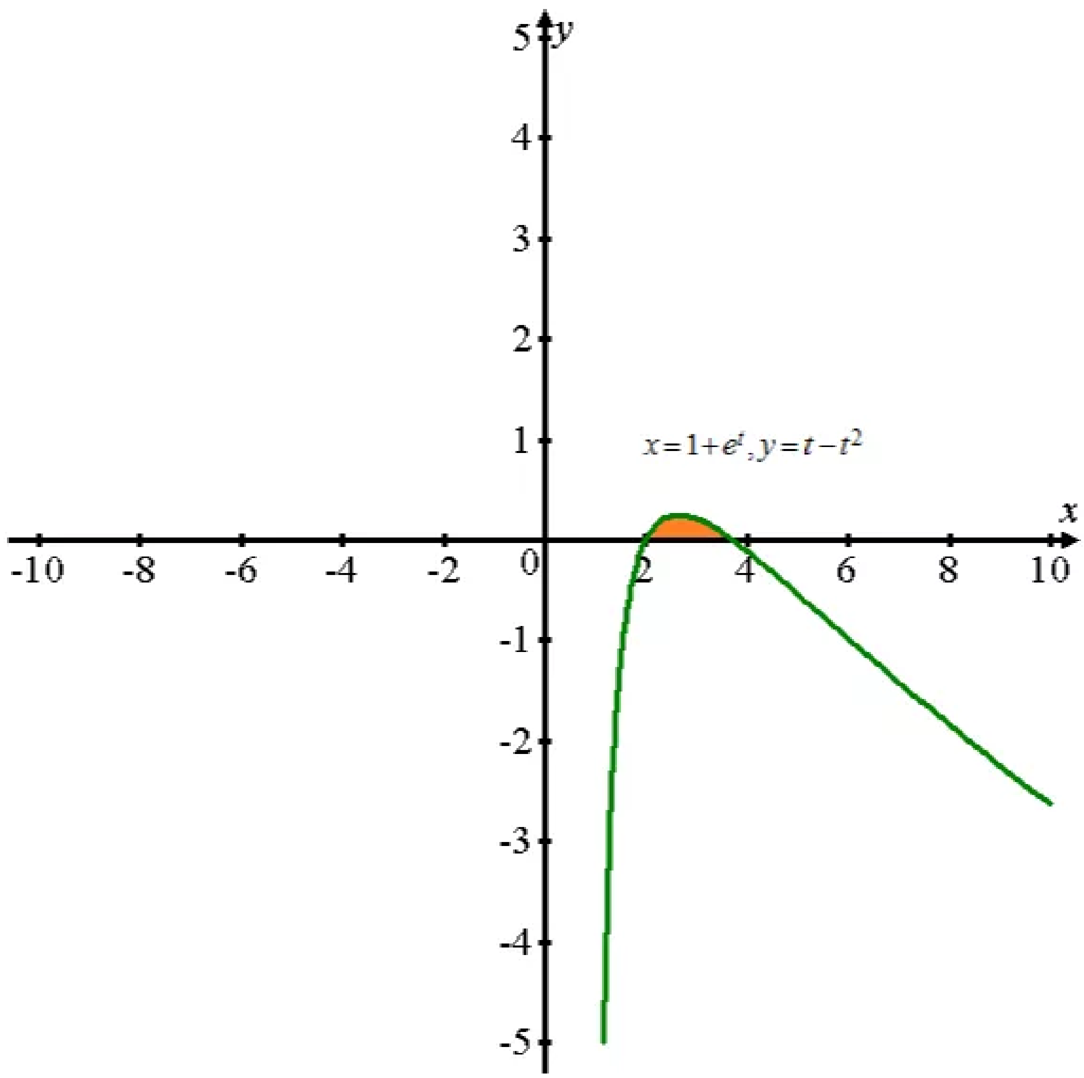
Q33E

Consider the following curves:

$x = 1 + e^t$, $y = t - t^2$

Need to find the area enclosed by the *x*-axis.

Sketch the given parametric curves:



Area bounded by the curves $x = f(t)$ and $y = g(t)$ is as follows:

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t) f'(t)\,dt$$, where $\alpha \le t \le \beta$

$y = g(t) = t - t^2$

$x = f(t) = 1 + e^t$

$f'(t) = e^t$

The intercept of the curve and the x-axis occurs when $y = 0$.

That is, $t - t^2 = 0$

$t(1-t) = 0$

$t = 0$, $t = 1$

Therefore, the area is $A = \int_0^1 (t - t^2) e^t\,dt$

$$= \int_0^1 te^t\,dt - \int_0^1 t^2 e^t\,dt \quad ......(1)$$

Consider the first integral $\int_0^1 te^t\,dt$.

Let $u = t$ and $dv = e^t\,dt$

Then $du = dt$ and $v = e^t$

Use the formula for integration by parts.

$$\int u\,dv = uv - \int v\,du$$

$$\int_0^1 te^t\,dt = \left[te^t\right]_0^1 - \int_0^1 e^t\,dt$$

$$= (e - 0) - \left[e^t\right]_0^1$$

$$= e - (e - 1)$$

$$= 1$$

Now consider the second integral $\int_0^1 t^2 e^t dt$.

Let $u = t^2$ and $dv = e^t dt$

Then $du = 2t dt$ and $v = e^t$

Use the formula for integration by parts.

$$\int u dv = uv - \int v du$$

$$\int_0^1 t^2 e^t dt = \left[t^2 e^t\right]_0^1 - 2\int_0^1 te^t dt$$

$$= (e-0) - 2(1) \qquad \left(\text{from integral } \int_0^1 te^t dt = 1\right)$$

$$= e - 2$$

Substitute the values of first and second integral in (1).

Therefore, the area $A = 1 - (e-2)$

$$= 3 - e$$

Thus, the area enclosed by the given curve is $\boxed{3-e}$.

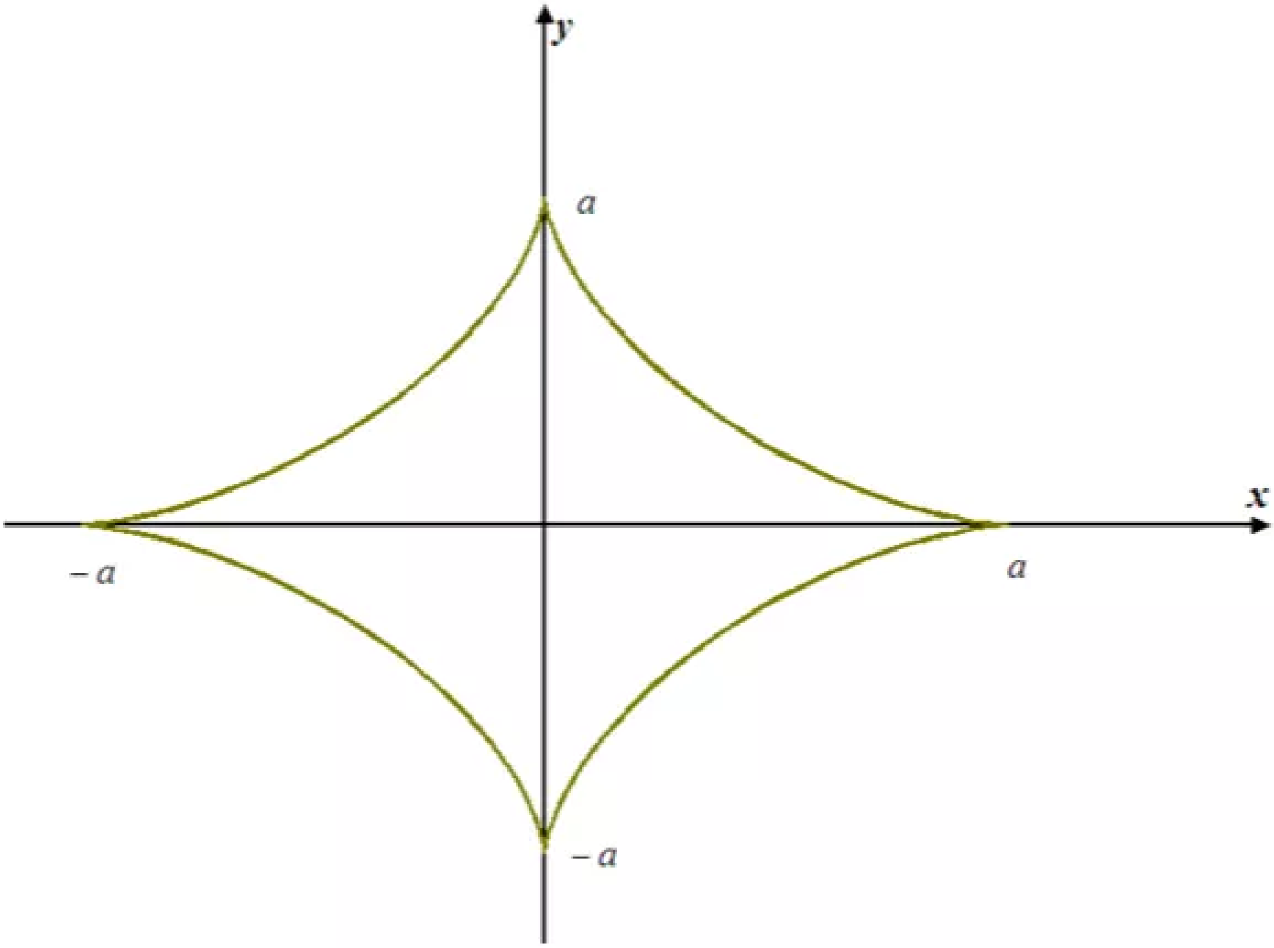
Q34E

Consider the parametric equations

$$x = a\cos^3\theta$$
$$y = a\sin^3\theta$$

Find the area of the region enclosed by the above curves.

The region as shown below:



The derivative of *x* is as follows:

$$x = a\left(\cos^3\theta\right)$$

$$x = a\cdot\frac{d}{d\theta}\left(\cos^3\theta\right)$$

$$x' = -3a\cos^2\theta\sin\theta$$

To find the area of underneath the curve, consider the entire area swept and the region bouned by the curve is $0 \le \theta \le 2\pi$.

Area of the region is,

$$A = \int_{\alpha}^{\beta} y(t)x'(t)dt$$

$$A = \int_{0}^{2\pi}\left(-3a\cos^2\theta\sin\theta\right)\left(a\sin^3\theta\right)d\theta$$

$$A = -3a^2\int_{0}^{2\pi}\left(\cos^2\theta\right)\left(\sin^4\theta\right)d\theta$$

The recurrence relation is,

$$\int \sin^m x\cos^n x dx = -\frac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}\cos^n x dx \text{ If } m \ne n$$

To find the area of the region, first find the integral $\int_0^{2\pi} \sin^4\theta\cos^2\theta d\theta$.

Let $m=4, n=2$

$$\int \sin^4 x\cos^2 x dx = -\frac{\sin^{4-1}x\cos^{2+1}x}{4+2}+\frac{4-1}{4+2}\int \sin^{4-2}\cos^2 x dx$$
$$= -\frac{\sin^3 x\cos^3 x}{6}+\frac{3}{6}\int \sin^2\cos^2 x dx$$
$$= -\frac{\sin^3 x\cos^3 x}{6}+\frac{1}{2}\left(-\frac{\sin x\cos^3 x}{4}+\frac{1}{4}\int \cos^2(x)\,dx\right)$$
$$= -\frac{\sin^3 x\cos^3 x}{6}+\frac{1}{2}\left(-\frac{\sin x\cos^3 x}{4}+\frac{1}{4}\int \frac{1+\cos(2x)}{2}\,dx\right)$$
$$= -\frac{\sin^3 x\cos^3 x}{6}-\frac{\sin x\cos^3 x}{8}+\frac{1}{16}x+\frac{1}{32}\sin 2x$$

Substitute this into the area of the region,

$$A=-3a^2\int_0^{2\pi}\cos^2\theta\sin^4\theta d\theta$$
$$=-3a^2\left(-\frac{\sin^3 x\cos^3 x}{6}-\frac{\sin x\cos^3 x}{8}+\frac{1}{16}x+\frac{1}{32}\sin 2x\right)_0^{2\pi}$$
$$=-3a^2\left(\frac{2\pi}{16}\right) \qquad (\text{Since } \sin(2\pi)=0,\ \sin(0)=0\ )$$
$$=-3a^2\left(\frac{\pi}{8}\right)$$

Hence, the area should be positive and the area of the region is $A=\frac{3a^2\pi}{8}$.

Q35E

We have $x = r\theta - d\sin\theta = f(\theta)$ (Let)
$y = r - d\cos\theta = g(\theta)$ (Let)

For d < r

One arch of the torched lies in the interval $0 \le \theta \le 2\pi$

since $f(\theta) = r\theta - d\sin\theta$

then $f'(\theta) = r - d\cos\theta$

Therefore $A=\left[r^2\theta-rd\sin\theta\right]_0^{2\pi}+d^2\int_0^{2\pi}\frac{(\cos 2\theta+1)}{2}d\theta$ $\qquad (\cos 2\theta=2\cos^2\theta-1)$

$$=\left[r^2\theta-2rd\sin\theta\right]_0^{2\pi}+\frac{d^2}{2}\left[\frac{\sin 2\theta}{2}+\theta\right]_0^{2\pi}$$

$$=\left[2\pi r^2\right]+\frac{d^2}{2}[2\pi]$$

$$\Rightarrow \boxed{A=2\pi r^2+\pi d^2}$$

Q36E

(A) We have parametric equations $x=t^2 \quad =f(t) \qquad (\text{Let})$

And $y=t^3-3t=g(t) \qquad (\text{Let})$

We sketch the curve

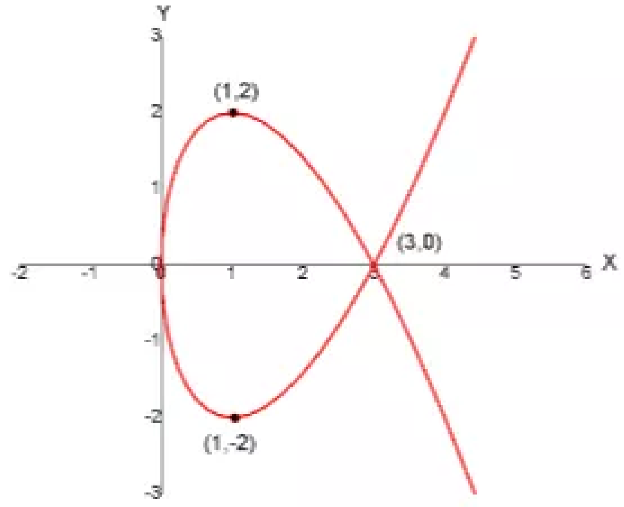


Fig.1

We see that the area is symmetric about x-axis
So we double the area of the region lies above the x-axis

So area of the loop is $A=2\int_0^3(0-y)\,dx$

$$=2\int_0^{\sqrt{3}}-g(t)f'(t)\,dt \qquad [x=3 \text{ then } t=\sqrt{3}$$

$$=-2\int_0^{\sqrt{3}}\left(t^3-3t\right)(2t)\,dt$$

$$=-4\int_0^{\sqrt{3}}\left(t^4-3t^2\right)dt$$

$$\text{Therefore area } = -4\left[\frac{t^5}{5} - t^3\right]_0^{\sqrt{3}}$$

$$= -4\left[\frac{\left(\sqrt{3}\right)^5}{5} - 3\sqrt{3}\right]$$

$$= 4\left[3\sqrt{3} - \frac{9}{5}\sqrt{3}\right] = \frac{4\sqrt{3}}{5}[15-9]$$

$$\boxed{A = 24\sqrt{3}/5}$$

Note [here for $0 \le t \le \sqrt{3}$ the curve lies below x-axis or the line $y = 0$

So area of the region lies under x-axis is $\int_0^3 (0 - y)\,dx$]

(B) We consider only upper or lower half of the curve for finding the volume of solid. Since if we rotate half of the loop about x-axis, we get same volume as the volume gotten by rotating full loop.

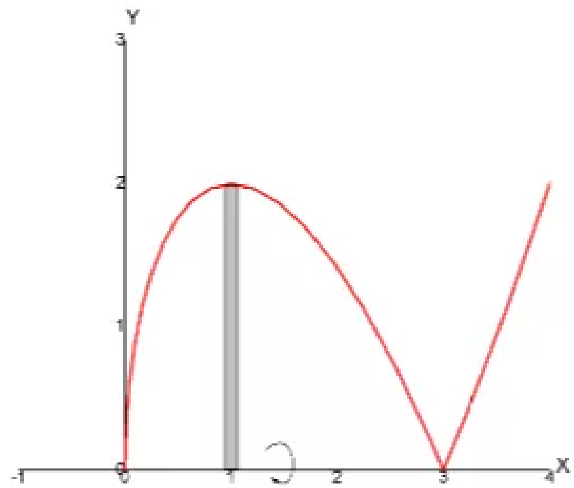


Fig.2

From the equation $x = t^2$ we have $t = \pm\sqrt{x}$

Then $y = t\left(t^2 - 3\right)$

$= \pm\sqrt{x}\,(x-3)$

Equation of upper half of the loop is $y = -\sqrt{x}\,(x-3)$, $\quad 0 \le x \le 3$

We consider a vertical strip in this region, after rotation we get a disk with

Radius $= -\sqrt{x}\,(x-3)$

Then cross sectional area of disk $A(x) = \pi\left(-\sqrt{x}\,(x-3)\right)^2$

$= \pi x (x-3)^2$

So the volume of resulting solid is $V = \int_0^3 A(x)\,dx$

$$= \int_0^3 \pi x(x-3)^2\, dx$$

$$= \pi \int_0^3 \left(x^3 + 9x - 6x^2\right) dx$$

$$= \pi \left[\frac{x^4}{4} + \frac{9x^2}{2} - 2x^3\right]_0^3$$

$$= \pi \left[\frac{81}{4} + \frac{81}{2} - 54\right]$$

$$\Rightarrow \boxed{V = (6.75)\pi}$$

(C) Let $F(x) = -\sqrt{x}(x-3)$ and $G(x) = \sqrt{x}(x-3)$

For $0 \le x \le 3$, $f(x) \ge G(x)$

Then $\bar{x} = \frac{1}{A}\int_0^3 x\left[F(x) - G(x)\right] dx$

$$= \frac{5}{24\sqrt{3}} \int_0^3 x\left[-\sqrt{x}(x-3) - \sqrt{x}(x-3)\right] dx \quad \text{since} \left[A = \frac{24\sqrt{3}}{5}\right]$$

$$= \frac{5}{24\sqrt{3}} \int_0^3 -2x\sqrt{x}(x-3)\, dx$$

$$= -\frac{5}{12\sqrt{3}} \int_0^3 x^{3/2}(x-3)\, dx$$

$$= -\frac{5}{12\sqrt{3}} \int_0^3 \left(x^{5/2} - 3x^{3/2}\right) dx$$

Therefore $\bar{x} = -\frac{5}{12\sqrt{3}}\left[\frac{2x^{7/2}}{2} - \frac{6x^{5/2}}{5}\right]_0^3$

$$= -\frac{5}{12\sqrt{3}}\left[\frac{2}{7}3^{7/2} - \frac{6}{5}3^{5/2}\right]$$

$$= -\frac{5}{12\sqrt{3}}3^{5/2}\left[\frac{2(3)}{7} - \frac{6}{5}\right]$$

$$= -\frac{5\times 9\sqrt{3}}{12\sqrt{3}}\left(\frac{6}{7} - \frac{6}{5}\right)$$

$$= -\frac{15}{4}\left(-\frac{12}{35}\right) = 9/7$$

$\Rightarrow \bar{x} = \frac{9}{7}$ and by symmetry $\bar{y} = 0$

So centroid is $\boxed{(\bar{x}, \bar{y}) = (9/7,\ 0)}$

Q37E

Consider the parametric equation of a curve:

$$x = f(t)$$
$$y = g(t)$$
$$\alpha \le t \le \beta$$

The length of the above curve is determined as shown below:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Consider the equation of a curve:

$$x = t + e^{-t}$$
$$y = t - e^{-t}$$
$$0 \le t \le 2$$

Evaluate the derivative of the above parameters:

$$\frac{dx}{dt} = 1 - e^{-t}$$
$$\frac{dy}{dt} = 1 + e^{-t}$$

Determine the length of the curve:

$$\begin{aligned} L &= \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^2 \sqrt{\left(1-e^{-t}\right)^2 + \left(1+e^{-t}\right)^2}\, dt \\ &= \int_0^2 \sqrt{\left(1+e^{-2t} - 2e^{-2t} + 1 + e^{-2t} + 2e^{-2t}\right)} dt \\ &= \int_0^2 \sqrt{2\left(1+e^{-2t}\right)} dt \\ &= 3.1416 \end{aligned}$$

Solve the above value to obtain the result:

$$L = \left\{ \frac{2\sqrt{e^{-2t}+1}\left[e^t \sinh^{-1}\left(e^t\right) - \sqrt{e^{2t}+1}\right]}{\sqrt{e^{2t}+1}} \right\}\Bigg|_0^2$$
$$= 4.4428$$

Hence, the length of the curve is $\boxed{4.4428}$.

Q38

Sol: Given curve is $x = t^2 - t, y = t^4, 1 \le t \le 4$

The length of the curve is $L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$\frac{dx}{dt} = 2t - 1, \frac{dy}{dt} = 4t^3$$

$$\therefore L = \int_1^4 \sqrt{(2t-1)^2 + (4t^3)^2}\, dt$$

$$= \int_1^4 \sqrt{16t^6 + 4t^2 - 4t + 1}\, dt$$

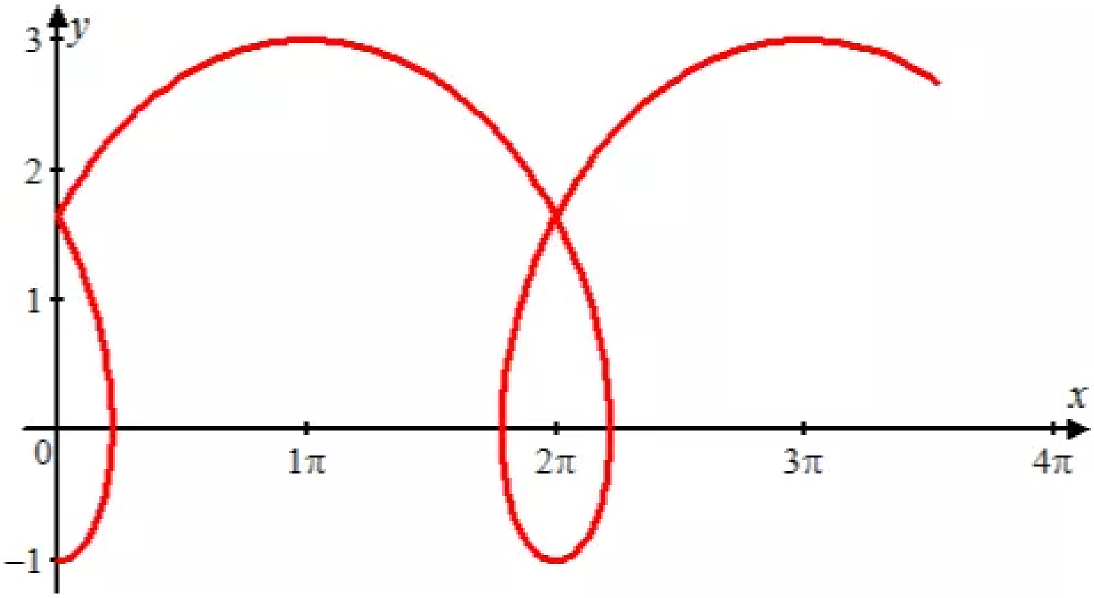
$$= \boxed{255.3756}$$

Q39E

Set up an integral that represents the length of the curve then use the calculator to find the length correct to four decimal places.

Given curve is $x = t - 2\sin t, y = 1 - 2\cos t, 0 \le t \le 4\pi$

At first sketch the given curve (use computer to graph the given function).



The length of the curve is $L=\int_{\alpha}^{\beta}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$$\frac{dx}{dt}=1-2\cos t,\ \frac{dy}{dt}=2\sin t$$

$$\text{Therefore } L=\int_0^{4\pi}\sqrt{(1-2\cos t)^2+(2\sin t)^2}\,dt$$

$$=\int_0^{4\pi}\sqrt{\left(4\cos^2 t+4\sin^2 t\right)-4\cos t+1}\,dt$$

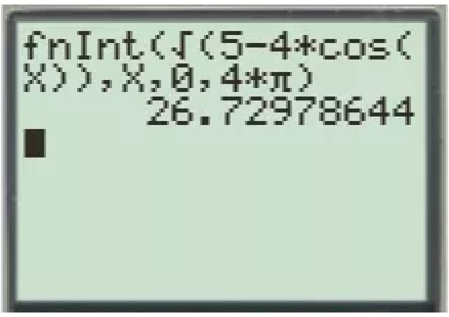
$$=\int_0^{4\pi}\sqrt{5-4\cos t}\cdot dt$$

It is difficult to solve this integral manually, so use the calculator to find the exact value of the integral.

Press the buttons 2nd and 0 respectively to obtain the following window.

In this window choose the option "FnInt" and then press ENTER then the following window will appear.

Now enter the integrand and the limits of integration as shown in the following window and then press the button $\boxed{\text{ENTER}}$ to obtain the final result.



Therfore the length of the give curve in the given interval is $\boxed{26.7298 \text{ units}}$.

Q40E

Sol: Given curve is $x = t + \sqrt{t}, y = t - \sqrt{t}, 0 \le t \le 1$

The length of the curve is $L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$\frac{dx}{dt} = 1 + \frac{1}{2\sqrt{t}}, \frac{dy}{dt} = 1 - \frac{1}{2\sqrt{t}}$$

$$\therefore L = \int_0^1 \sqrt{\left(1 + \frac{1}{2\sqrt{t}}\right)^2 + \left(1 - \frac{1}{2\sqrt{t}}\right)^2}\, dt$$

$$= \int_0^1 \left(\sqrt{2 + \frac{1}{2t}}\right) dt$$

$$= \boxed{2.09154}$$

Q41E

We have	$x = 1 + 3t^2$	$y = 4 + 2t^3$
Then	$\frac{dx}{dt} = 6t$	$\frac{dy}{dt} = 6t^2$

The length of the curve is

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt$$

$$= \int_0^1 \sqrt{36t^2 + 36t^4}\, dt$$

$$= \int_0^1 6t\sqrt{1 + t^2}\, dt$$

$$= 3\int_0^1 \sqrt{1 + t^2}\,(2t)\, dt$$

Now substitute $1+t^2=u \quad \Rightarrow 2t\,dt = du$

And when $\begin{cases} t=0, & u=1 \\ t=1, & u=2 \end{cases}$

Therefore, $L = 3\int_1^2 \sqrt{u}\,du$

$$= 3\left[\frac{u^{3/2}}{3/2}\right]_1^2$$

$$= 2\left[2^{3/2} - 1^{3/2}\right]$$

$$= \boxed{2\left(2\sqrt{2}-1\right)}$$

Q42E

Consider the curves,

$x = e^t + e^{-t}$

$y = 5 - 2t$

The objective is to find the length of the curve on $0 \le t \le 3$.

Recall the formula for arc length,

Arc Length: If a curve C is described by the parametric equations

$x(t) = f(t), y(t) = g(t)$, $\alpha \le t \le \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \quad \text{...... (1)}$$

Consider the equations,

$x = e^t + e^{-t}, y = 5 - 2t$

Differentiate both equations with respect to 't' then

$$\frac{dx}{dt} = e^t - e^{-t}, \frac{dy}{dt} = -2$$

Now find $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(e^t - e^{-t}\right)^2 + (-2)^2$$

$$= e^{2t} + e^{-2t} - 2 + 4$$

$$= e^{2t} + e^{-2t} + 2$$

$$= \left(e^t + e^{-t}\right)^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = e^t + e^{-t}$$

Therefore, the length of the curve over $0 \le t \le 3$ is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_{0}^{3} \left(e^t + e^{-t}\right) dt$$

$$= \left[e^t - e^{-t}\right]_0^3$$

$$= e^3 - e^{-3} - 1 + 1$$

$$= e^3 - e^{-3}$$

The length of the given curve is about $\boxed{e^3 - e^{-3}}$ units.

Q43E

Consider the curve bounded by $x = t\sin t,\ y = t\cos t,\quad 0 \le t \le 1$.

Write the expression to find the length of the curve bounded by

$x = f(t), y = g(t),$ and $\alpha \le t \le \beta$.

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Differentiate the equations with respect to *t*.

$$\frac{dx}{dt} = t\cos t + \sin t,$$

$$\frac{dy}{dt} = -t\sin t + \cos t$$

Replace $\frac{dx}{dt}$ with $t\cos t+\sin t$ and $\frac{dy}{dt}$ with $-t\sin t+\cos t$ to find the length of the curve.

$$L=\int_0^1\sqrt{(t\cos t+\sin t)^2+(-t\sin t+\cos t)^2}\,dt$$

$$=\int_0^1\sqrt{t^2\cos^2 t+2t\cos t\sin t+\sin^2 t+t^2\sin^2 t-2t\sin t\cos t+\cos^2 t}\,dt$$

$$=\int_0^1\sqrt{t^2\left(\cos^2 t+\sin^2 t\right)+\left(\cos^2 t+\sin^2 t\right)}\,dt$$

$$=\int_0^1\sqrt{t^2+1}\,dt$$

$$=\left[\frac{t}{2}\sqrt{t^2+1}+\frac{1}{2}\ln\left(t+\sqrt{t^2+1}\right)\right]_0^1$$

Since $\int\sqrt{u^2+a^2}\,du=\frac{u}{2}\sqrt{u^2+a^2}+\frac{a^2}{2}\ln\left(u+\sqrt{u^2+a^2}\right)+C$

$$L=\boxed{\frac{1}{2}\sqrt{2}+\frac{1}{2}\ln\left(1+\sqrt{2}\right)}.$$

Q44E

$x=3\cos t-\cos 3t,\ y=3\sin t-\sin 3t,\qquad 0\le t\le\pi$

$dx/dt=-3\sin(t)+3\sin(3t)$

$dy/dt=3\cos(t)-3\cos(3t)$

$$L=\int_\alpha^\beta\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$=\int_0^\pi\sqrt{\left[-3\sin(t)+3\sin(3t)\right]^2+\left[3\cos(t)-3\cos(3t)\right]^2}\,dt$$

$$=\int_0^\pi\sqrt{9\sin^2(t)-18\sin(t)\sin(3t)+9\sin^2(3t)+9\cos^2(t)-18\cos(t)\cos(3t)+9\cos^2(3t)}$$

$$=\int_0^\pi\sqrt{18\left[1-\sin(t)\sin(3t)-\cos(t)\cos(3t)\right]}\,dt$$

$$=\sqrt{18}\int_0^\pi\sqrt{1-\sin(t)\sin(3t)-\cos(t)\cos(3t)}\,dt$$

$$=3\sqrt{2}\int_0^\pi\sqrt{1-\frac{\cos(2t)-\cos(4t)}{2}-\frac{\cos(2t)+\cos(4t)}{2}}\,dt$$

$$=3\sqrt{2}\int_0^\pi\sqrt{1-\cos(2t)}\,dt$$

$$=3\sqrt{2}\int_0^\pi\sqrt{2\sin^2 t}\,dt$$

$$=6\int_0^\pi\sin(t)\,dt$$

$$=-6\left[\cos(t)\right]_0^\pi$$

$$=-6(-1-1)$$

$$=12$$

Q45E

Consider the following curves:

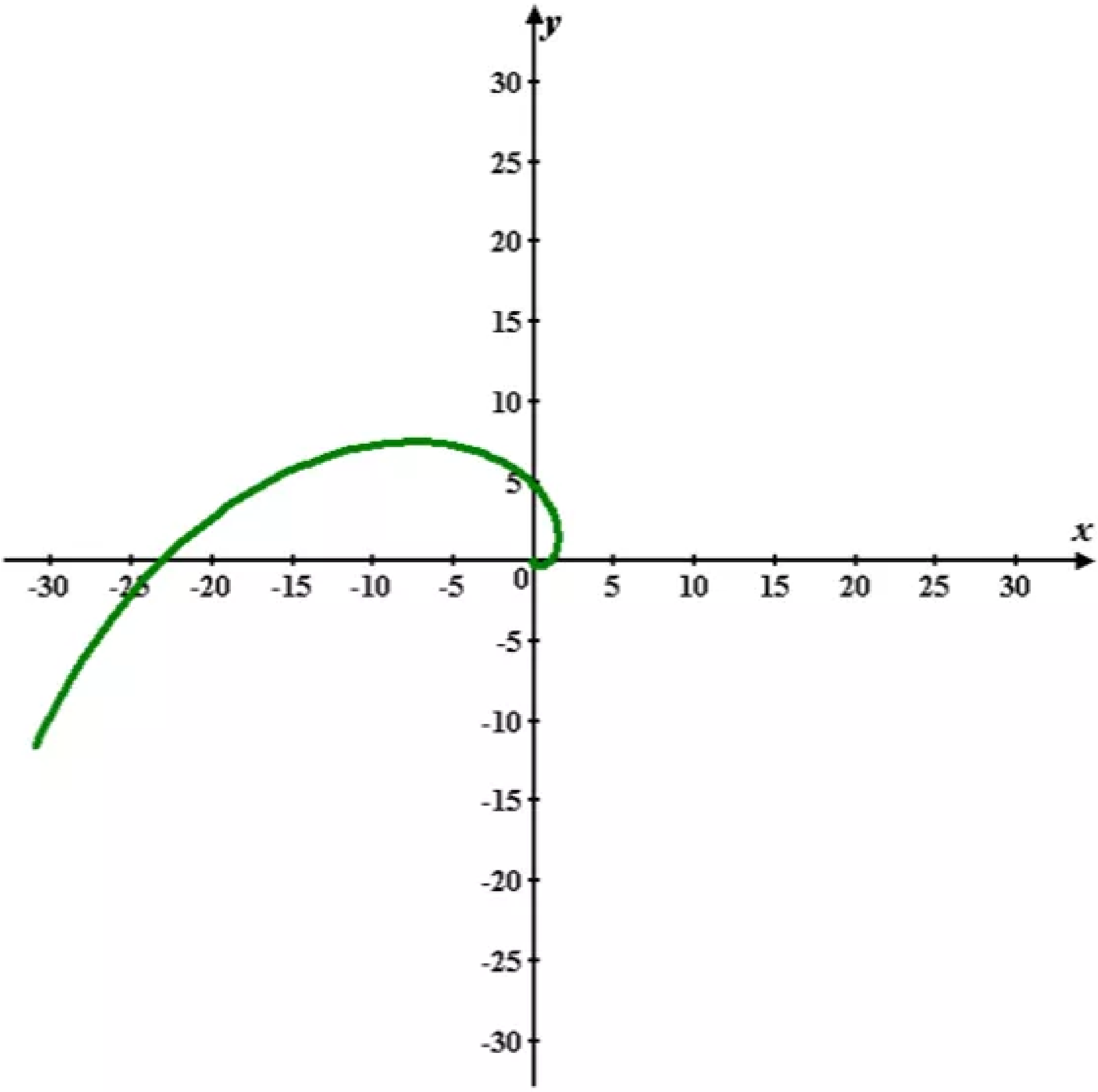
$x = e^t \cos t, y = e^t \sin t, 0 \le t \le \pi$

The objective is to sketch the graph and then find the length of the curve.

The formula for arc length is,

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The graph of $x = e^t \cos t, y = e^t \sin t, 0 \le t \le \pi$ is shown below:



Differentiate the equation $x = e^t \cos t$ with respect to t.

$$\frac{dx}{dt} = e^t \cos t - e^t \sin t$$

Also differentiate the equation $y = e^t \sin t$ with respect to t

$$\frac{dy}{dt} = e^t \sin t + e^t \cos t$$

Now, using the arc length formula, the exact length of the curve is,

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_0^\pi \sqrt{\left(e^t \cos t - e^t \sin t\right)^2 + \left(e^t \sin t + e^t \cos t\right)^2}\, dt$$
$$= \int_0^\pi \sqrt{2e^{2t} \sin^2 t + 2e^{2t} \cos^2 t + 2e^{2t} \sin t \cos t - 2e^{2t} \sin t \cos t}\, dt$$
$$= \int_0^\pi \sqrt{2e^{2t}\left(\sin^2 t + \cos^2 t\right)}\, dt$$
$$= \int_0^\pi \sqrt{2e^{2t}}\, dt$$
$$= \sqrt{2}\int_0^\pi e^t\, dt$$
$$= \sqrt{2}\left(e^\pi - e^0\right)$$
$$= \sqrt{2}\left(e^\pi - 1\right)$$

Hence, the length of the arc is $L = \boxed{\sqrt{2}\left(e^\pi - 1\right)}$.

Q46E

$x = \cos t + \ln\left(\tan\frac{t}{2}\right)$, $y = \sin t$, $\pi/4 \le t \le 3\pi/4$

We sketch the curve

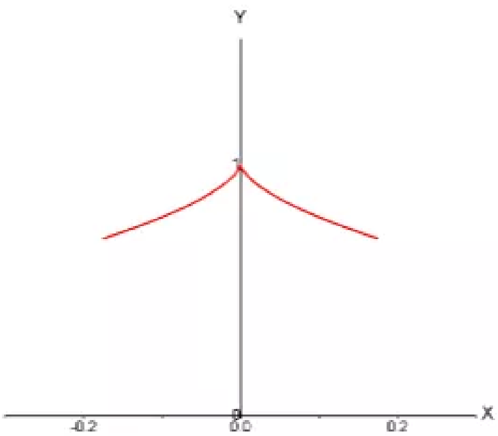


Fig.1

Now $$\frac{dx}{dt} = -\sin t + \frac{\frac{1}{2}\sec^2(t/2)}{\tan(t/2)}$$

$$= -\sin t + \frac{1/\cos^2(t/2)}{2\sin(t/2)/\cos(t/2)}$$

$$= -\sin t + \frac{1}{2\sin\frac{t}{2}\cos t/2}$$

$$= -\sin t + \frac{1}{\sin t}$$

And $$\frac{dy}{dt} = \cos t$$

Then $$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(-\sin t + \frac{1}{\sin t}\right)^2 + \cos^2 t$$

$$= \sin^2 t + \frac{1}{\sin^2 t} - 2 + \cos^2 t$$

$$= \left(\sin^2 t + \cos^2 t\right) - 2 + \frac{1}{\sin^2 t}$$

$$= 1 - 2 + \frac{1}{\sin^2 t}$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \csc^2 t - 1$$

$$= \cot^2 t$$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = |\cot t|$$

Then length of the curve is

$$L = \int_{\pi/4}^{3\pi/4} |\cot t| dt = 2\int_{\pi/4}^{\pi/2} \cot t dt$$ [because graph is symmetric about y axis and at $t = \pi/2, x = 0$]

$$= 2\left[\ln(\sin t)\right]_{\pi/4}^{\pi/2}$$

$$= 2\left[\ln(\sin \pi/2) - \ln(\sin \pi/4)\right]$$

$$= 2\left[\ln(1) - \ln\left(1/\sqrt{2}\right)\right]$$

$$= 2\left[0 - \ln\left(1/\sqrt{2}\right)\right]$$

$$= -2\ln\left(1/\sqrt{2}\right)$$

$$= -2.\ln(2)^{-1/2}$$

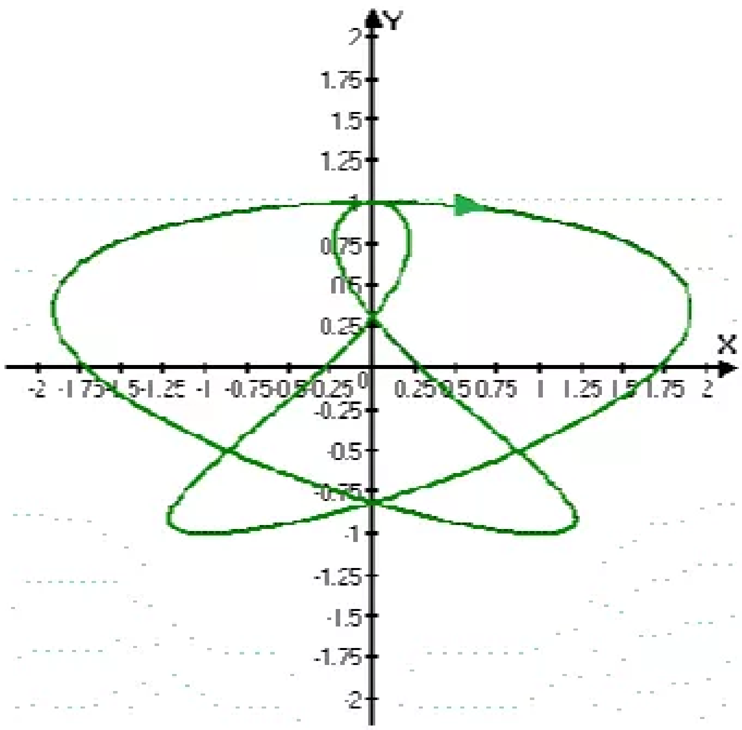
$$= -2\left(-\frac{1}{2}\right)\ln 2$$ [$\ln m^n = n \ln m$

$$= \boxed{\ln 2}$$

Q47E

given curve is $x = \sin t + \sin 1.5t, y = \cos t$

The graph is



The length of this curve is 12.54 units starting from any point which can be observed from any graphing utility.

Length of the curve is $L=\int_{\alpha}^{\beta}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$$\frac{dx}{dt}=\cos t+1.5\cos 1.5t$$
$$\frac{dy}{dt}=\sin t$$

$$L=\int_{0}^{12.54}\sqrt{\left(\cos t+1.5\cos 1.5t\right)^2+\sin^2 t}\,dt \quad =16.6443$$

Q48E

We have $x=3t-t^3$ and $y=3t^2$

First we sketch the curve.

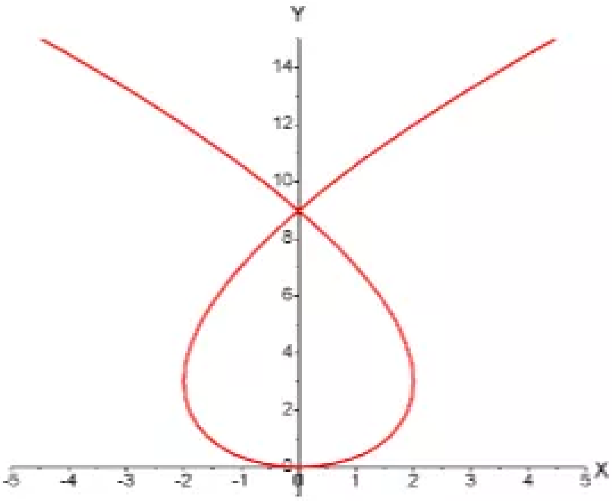


Fig.1

We see that loop starts from $x=0$ and ends at $x=0$

So $\quad 3t-t^3=0$

$$\Rightarrow t\left(3-t^2\right)=0$$

$$\Rightarrow t=0 \qquad or \qquad t=\pm\sqrt{3}$$

So we have interval $-\sqrt{3}\le t\le\sqrt{3}$

Since $\quad x=3t-t^3$ and $\quad y=3t^2$

Then $\quad \frac{dx}{dt}=3-3t^2$ and $\frac{dy}{dt}=6t$

Then length of the curve is

$$L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$\Rightarrow L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{9\left(1-t^2\right)^2 + 36t^2}\ dt$$

$$\Rightarrow L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{\left(9+9t^4-18t^2\right)+36t^2}\ dt$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{9\left(1+t^2\right)^2}\ dt$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} 3\left(1+t^2\right)dt$$

Therefore $L = 3\left[t+\frac{t^3}{3}\right]_{-\sqrt{3}}^{\sqrt{3}}$

$$= 3\left[\sqrt{3}+\frac{3\sqrt{3}}{3}+\sqrt{3}+\frac{3\sqrt{3}}{3}\right]$$

$$= 3\left[4\sqrt{3}\right]$$

$$\Rightarrow \boxed{L = 12\sqrt{3}}$$

Q49E

We have $x = t - e^t$

And $y = t + e^t$ $\quad -6 \le t \le 6$

Then $\frac{dx}{dt} = \left(1-e^t\right)$ *and* $\frac{dy}{dt} = \left(1+e^t\right)$

Then length of the curve is

$$L = \int_{-6}^{6} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$\Rightarrow L = \int_{-6}^{6} \sqrt{\left(1-e^t\right)^2 + \left(1+e^t\right)^2}\ dt$$

$$= \int_{-6}^{6} \sqrt{1+e^{2t}-2e^t+1+e^{2t}+2e^t}\ dt$$

$$= \int_{-6}^{6} \sqrt{2+2e^{2t}}\ dt$$

We use Simpson's rule for approximating the length

Let $f(t)=\sqrt{2+2e^{2t}}$

For n = 6 $\quad \Delta t=\frac{6+6}{6}=2$

Subintervals are $[-6,-4][-4,-2],[-2,0],[0,2],[2,4],[4,6]$

By Simpson's rule

$$L\approx\frac{\Delta t}{3}\left[f(-6)+4f(-4)+2f(-2)+4f(0)+2f(2)+4f(4)+f(6)\right]$$

$$=\frac{2}{3}\left[\sqrt{2+2e^{-12}}+4\sqrt{2+2e^{-8}}+2\sqrt{2+2e^{-4}}+4\sqrt{4}+2\sqrt{2+2e^{4}}+4\sqrt{2+2e^{4}}+\sqrt{2+2e^{12}}\right]$$

$$\approx 612.3053 \Rightarrow \boxed{L\approx 612.3053}$$

Q50E

We have $x=2a\cot\theta$ and $y=2a\sin^2\theta$. $\quad \pi/4\le\theta\le\pi/2$

Then $\frac{dx}{d\theta}=-2a\csc^2\theta$ and $\frac{dy}{d\theta}=4a\sin\theta\cos\theta$

$\Rightarrow \quad \frac{dy}{d\theta}=2a\sin 2\theta$

Then length of the curve is

$$L=\int_{\pi/4}^{\pi/2}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$\Rightarrow \quad L=\int_{\pi/4}^{\pi/2}\sqrt{4a^2\csc^4\theta+4a^2\sin^2 2\theta}\,d\theta$$

$$\Rightarrow \quad L=2a\int_{\pi/4}^{\pi/2}\sqrt{\csc^4\theta+\sin^2 2\theta}\,d\theta \qquad \text{----------- (1)}$$

Now we use Simpson's rule to approximate the length of the curve.

Let, $\quad f(\theta)=\sqrt{\csc^4\pi+\sin^2 2\theta}$

For, $\quad n=4 \qquad \Delta\theta=\frac{\pi/2-\pi/4}{4}=\frac{\pi}{16}$

Then subintervals are $\left[\frac{\pi}{4},\frac{5\pi}{16}\right],\left[\frac{5\pi}{16},\frac{6\pi}{16}\right],\left[\frac{6\pi}{16},\frac{7\pi}{16}\right],\left[\frac{7\pi}{16},\frac{\pi}{2}\right]$

By Simpson's rule, the length of the curve is

$$L\approx 2a.\frac{\Delta\theta}{3}\left[f(\pi/4)+4f(5\pi/16)+2f(6\pi/16)+4f(7\pi/16)+f(\pi/2)\right]$$

$$=\frac{a\pi}{24}\times\left[f(\pi/4)+4f(5\pi/16)+2f(6\pi/16)+4f(7\pi/16)+f(\pi/2)\right]$$

$$\approx\frac{a\pi}{24}\times(17.269319)$$

$$\Rightarrow \boxed{L\approx 2.260549a}$$

Q51E

We have $x = \sin^2 t$ and $y = \cos^2 t$ $\quad 0 \le t \le 3\pi$
We sketch the curve.

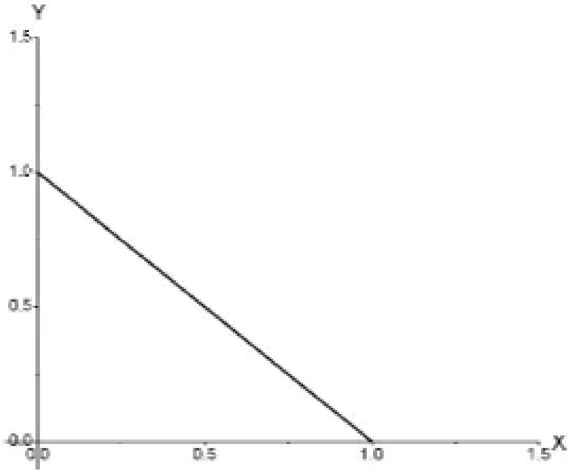


Fig.1

This is a straight line in the interval $[0, 3\pi]$
Since $x = \sin^2 t = 0$, at $t = 0, \pi, 2\pi, 3\pi$.
This means particle starts from $x = 0$ and comes back three times at 0 and so particle moves six times along the line.
[In other wards for $x = 0$, $t = 0$ and $x = 1$, $t = \pi/2$.
So subintervals are $[0,\pi/2],[\pi/2,\pi],[\pi,3\pi/2],[3\pi/2,2\pi],[2\pi,5\pi/2],[5\pi/2,3\pi]$
So particle covers the distance **six times the length of the curve for $0 \le t \le \pi/2$**].

Now we find length of the curve,

$$\frac{dx}{dt} = 2\sin t\cos t \quad \text{and} \quad \frac{dy}{dt} = -2\cos t\sin t$$

$$\Rightarrow \quad \frac{dx}{dt} = \sin 2t \quad \text{and} \quad \frac{dy}{dt} = -\sin 2t$$

Then length of the curve is $L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$\Rightarrow \quad L = \int_0^{\pi/2} \sqrt{\sin^2 2t + \sin^2 2t}\, dt$$

Or $\quad L = \int_0^{\pi/2} \sqrt{2\sin^2 2t}\, dt$

Or $\quad L = \sqrt{2}\int_0^{\pi/2} (\sin 2t)\, dt$

Or $\quad L = \sqrt{2}\left[-\frac{\cos 2t}{2}\right]_0^{\pi/2}$

Or $\quad L = \sqrt{2}\left[-\frac{\cos \pi}{2} + \frac{\cos 0}{2}\right]$

Or $\quad L = \sqrt{2}\left[\frac{1}{2} + \frac{1}{2}\right]$

$$\Rightarrow \quad \boxed{L = \sqrt{2}}$$

Therefore total distance, traveled by the particle is $= 6 \times L$

$$\boxed{= 6\sqrt{2}}$$

Q52E

We have $x = \cos^2 t$, $y = \cos t$, $0 \le t \le 4\pi$

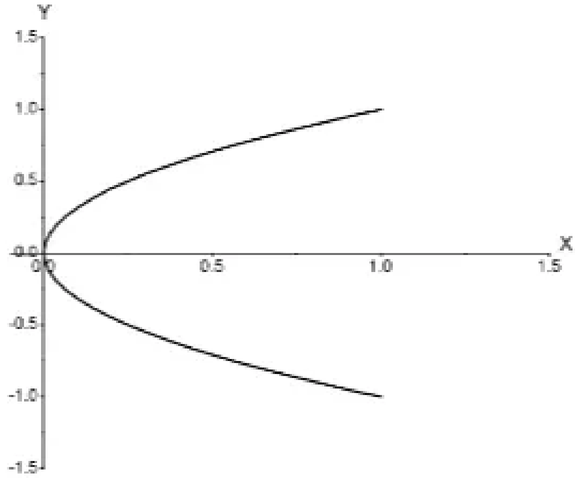


Fig.1

From these equation we have $x = y^2$

Curve starts from (1, 1) and comes back at (1, 1) two times because,

At $t = 0$, $x = 1$, $y = 1$.

At $t = 2\pi$ $x = 1$, $y = 1$.

At $t = 4\pi$ $x = 1$, $y = 1$.

So the particle moves four times along the parabola and then distance traveled by the particle $= 4 \times$ length of the curve from $t = 0$ to π.

We have, $x = \cos^2 t$ and $y = \cos t$.

Then $\dfrac{dx}{dt} = -2\cos t \sin t$ and $\dfrac{dy}{dt} = -\sin t$

Then length of the curve $L = \int_0^{\pi} \sqrt{\left[\dfrac{dx}{dt}\right]^2 + \left[\dfrac{dy}{dt}\right]^2}\, dt$

$$\Rightarrow \quad L = \int_0^{\pi} \sqrt{4\cos^2 t \sin^2 t + \sin^2 t}\, dt$$

$$\Rightarrow \quad L = \int_0^{\pi} \sin t \sqrt{4\cos^2 t + 1}\, dt$$

Let, $\cos t = z \Rightarrow -\sin t\, dt = dz$

When $t = 0, z = 1$ and when $t = \pi, z = -1$

So, $L = -\int_1^{-1} \sqrt{1 + 4z^2}\, dz = \int_{-1}^{1} \sqrt{1 + 4z^2}\, dz$

$$\Rightarrow \quad L = 2\int_{-1}^{1} \sqrt{\frac{1}{4} + z^2}\, dz$$

Using $\int \sqrt{a^2+u^2}du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right)+C$

We have the length of the curve

$$L = 2\left[\frac{z}{2}\sqrt{\frac{1}{4}+z^2} + \frac{1}{8}\ln\left(z+\sqrt{\frac{1}{4}+z^2}\right)\right]_{-1}^{1}$$

$$= 2\left[\frac{1}{2}\sqrt{5/4} + \frac{1}{8}\ln\left(1+\sqrt{5/4}\right) + \frac{1}{2}\sqrt{5/4} - \frac{1}{8}\ln\left(\sqrt{5/4}-1\right)\right]$$

$$= 2\left[\sqrt{5/4} + \frac{1}{8}\ln\frac{\left(1+\sqrt{5/4}\right)}{\left(\sqrt{5/4}-1\right)}\right]$$

$$\Rightarrow \quad L = \sqrt{5} + \frac{1}{4}\ln\frac{\left(\sqrt{5/4}+1\right)}{\left(\sqrt{5/4}-1\right)}$$

$$\Rightarrow L = \sqrt{5} + \frac{1}{4}\ln\frac{\left(\sqrt{5}+2\right)}{\left(\sqrt{5}-2\right)}$$

Therefore total distance traveled by the particle is $d = 4 \times L$.

$$\text{Or,} \quad d = \boxed{4\sqrt{5} + \ln\frac{\left(\sqrt{5}+2\right)}{\left(\sqrt{5}-2\right)}}$$

$$\text{Since} \quad \ln\frac{\left(\sqrt{5}+2\right)}{\left(\sqrt{5}-2\right)} = \ln\frac{\left(\sqrt{5}+2\right)}{\left(\sqrt{5}-2\right)} \times \frac{\left(\sqrt{5}+2\right)}{\left(\sqrt{5}+2\right)}$$

$$= \ln\frac{\left(\sqrt{5}+2\right)^2}{(5-4)}$$

$$= 2\ln\left(\sqrt{5}+2\right)$$

Then we can write the length of the curve as $d = \boxed{4\sqrt{5} + 2\ln\left(\sqrt{5}+2\right)}$

Q53E

We have $x = a\sin\theta$ and $y = b\cos\theta$, a > b > 0.

We divide the ellipse in to four equal parts. First part lies in the first quadrant, for this θ varies from 0 to π/2.

So, Total length of the ellipse = 4x length of the first part of the ellipse.

Since, $x = a\sin\theta$ and $y = b\cos\theta$

Then $\frac{dx}{d\theta} = a\cos\theta$ and $\frac{dy}{d\theta} = -b\sin\theta$

Then, length of the ellipse $= 4\times\int_0^{\pi/2}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$

$$= 4\times\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta$$

$$= 4\times\int_0^{\pi/2}\sqrt{a^2\left(1-\sin^2\theta\right)+b^2\sin^2\theta}\,d\theta$$

$$= 4\times\int_0^{\pi/2}a\sqrt{1-\sin^2\theta+\frac{b^2}{a^2}\sin^2\theta}\,d\theta$$

$$= 4\times\int_0^{\pi/2}\sqrt{1-\left(1-\frac{b^2}{a^2}\right)\sin^2\theta}\,d\theta$$

$$= 4\times\int_0^{\pi/2}\sqrt{1-\left(\frac{a^2-b^2}{a^2}\right)\sin^2\theta}\,d\theta$$

Since $c=\sqrt{a^2-b^2}$ and e = c/a so $e^2=\frac{\left(a^2-b^2\right)}{a^2}$

Then, length of the ellipse = $\boxed{4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}\;d\theta}$

Q54E

We have $x=a\cos^3\theta$ and $y=a\sin^3\theta$ $a>0$.

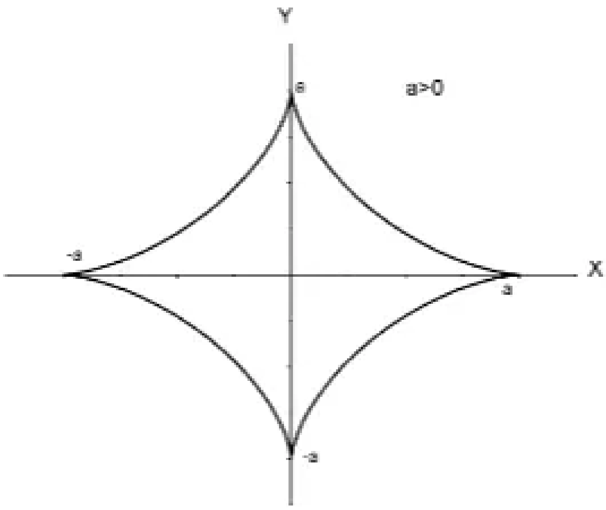


Fig.1

We see that curve is symmetric about origin and lies in the all four quadrants so,
Total length of the curve = 4 × length of the curve lying in first quadrant.
Since $x=a\cos^3\theta$ and $y=a\sin^3\theta$,

Then $\frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta$ and $\frac{dy}{d\theta}=3a\sin^2\theta\cos\theta$

In the first quadrant θ varies 0 to $\pi/2$
Then length of the curve.

$$L = 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 4\int_0^{\pi/2} \sqrt{9a^2 \cos^4\theta \sin^2\theta + 9a^2 \sin^4\theta \cos^2\theta}\, d\theta$$

$$= 4\int_0^{\pi/2} 3a\sqrt{\sin^2\theta\cos^2\theta\left(\cos^2\theta + \sin^2\theta\right)}\, d\theta$$

Since $\cos^2\theta + \sin^2\theta = 1$, then

$$L = 4\int_0^{\pi/2} 3a\sqrt{\sin^2\theta\cos^2\theta}\, d\theta$$

$$= 12a\int_0^{\pi/2} \sin\theta\cos\theta d\theta$$

$$= 12a\int_0^{\pi/2} \frac{\sin 2\theta}{2} d\theta \qquad [\sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{12a}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= 3a[-\cos\pi + \cos 0]$$

$$= 3a[1+1]$$

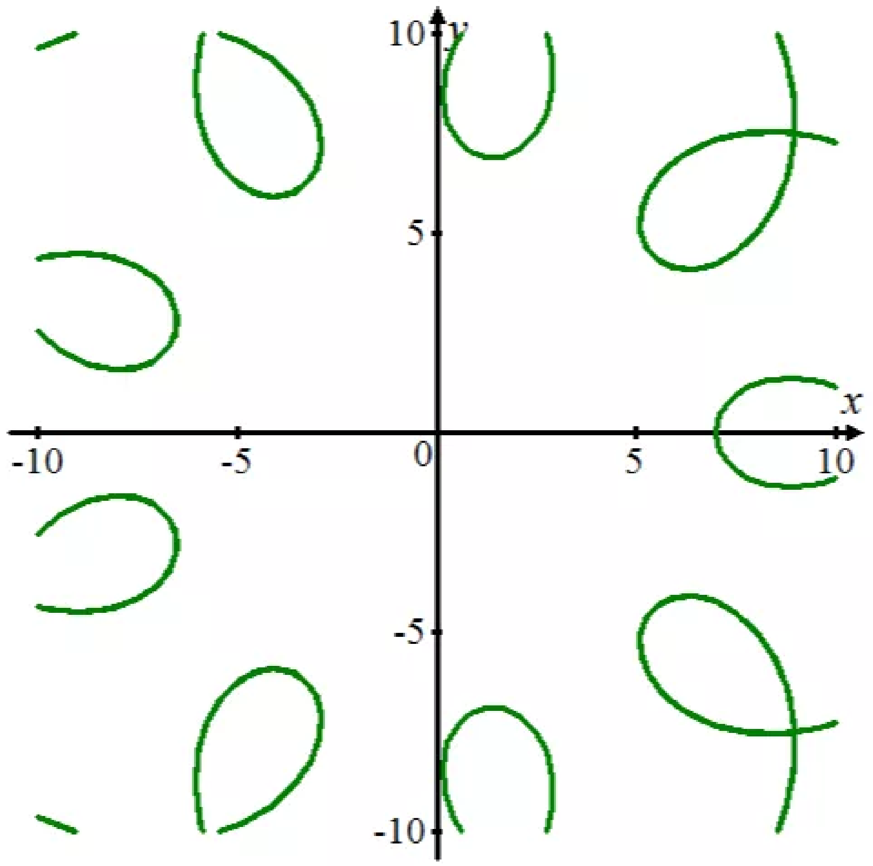
$$\Rightarrow \boxed{L = 6a}$$

Q55E

Consider the "epitrochoid" with parametric equations,

$$x = 11\cos t - 4\cos\left(\frac{11t}{2}\right), y = 11\sin t - 4\sin\left(\frac{11t}{2}\right)$$

To graph this curve, use "advanced grapher", the graph of the curve is shown below:



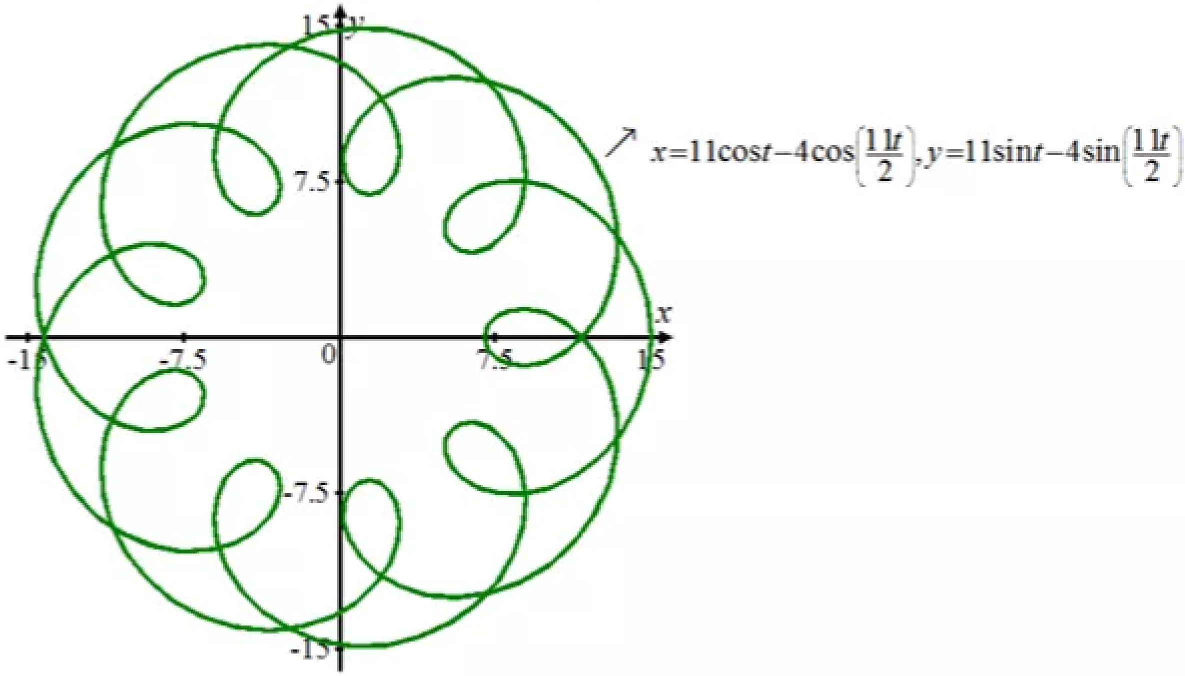
The period of $\cos t$ is 2π , and the period of $\cos\left(\frac{11t}{2}\right)$ is $\frac{2\pi}{\frac{11}{2}} = \frac{4\pi}{11}$

And, the period of $\sin t$ is 2π , and the period of $\sin\left(\frac{11t}{2}\right)$ is $\frac{2\pi}{\frac{11}{2}} = \frac{4\pi}{11}$

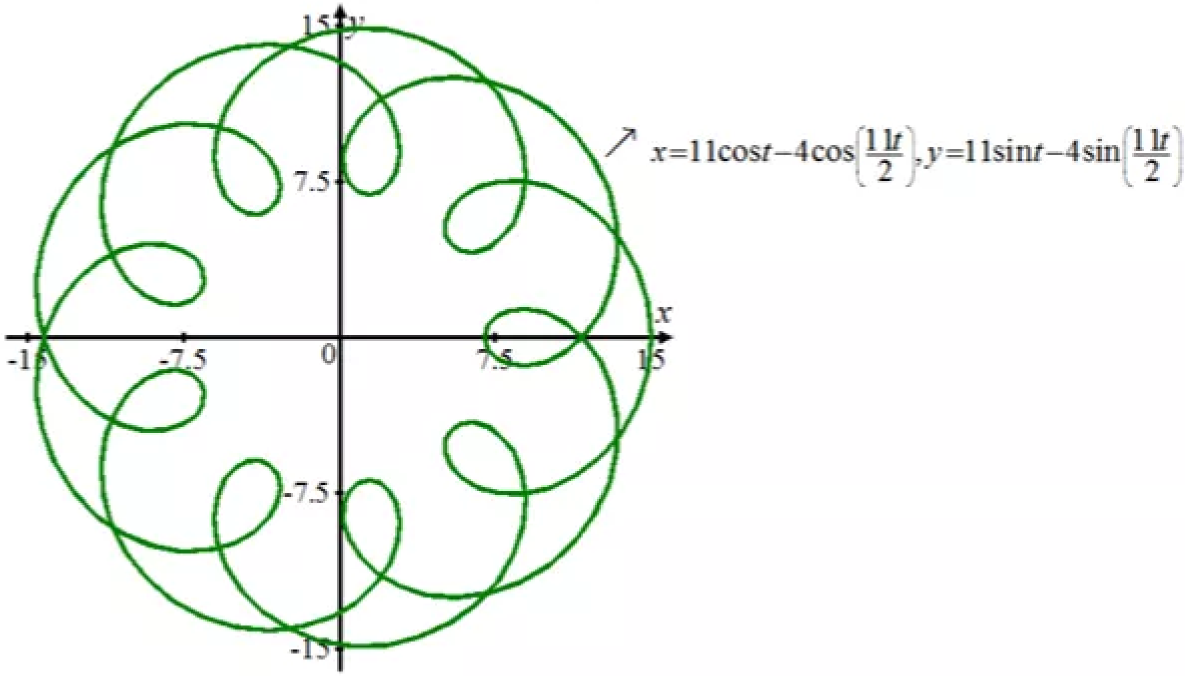
So the period of $x = 11\cos t - 4\cos\left(\frac{11t}{2}\right), y = 11\sin t - 4\sin\left(\frac{11t}{2}\right)$ is the L.C.M of $2\pi, \frac{4\pi}{11} = 4\pi$

Therefore the parameter interval is, $[0, 4\pi]$

So the parameter in this interval gives the complete graph of the given curve.



So the parameter in this interval gives the complete graph of the given curve.



(b) If the curve C is given by the parametric equations, $x=f(t), y=g(t)$, $\alpha \le t \le \beta$ then the length of C is,

$$L=\int_{\alpha}^{\beta}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

So the length of this curve is given by,

$$L=\int_0^{4\pi}\sqrt{\left(\frac{d}{dt}\left(11\cos t-4\cos\left(\frac{11t}{2}\right)\right)\right)^2+\left(\frac{d}{dt}\left(11\sin t-4\sin\left(\frac{11t}{2}\right)\right)\right)^2}\,dt$$

$$=\int_0^{4\pi}\sqrt{\left(-11\sin t+22\sin\left(\frac{11t}{2}\right)\right)^2+\left(11\cos t-22\cos\left(\frac{11t}{2}\right)\right)^2}\,dt$$

$$=\int_0^{4\pi}\sqrt{\begin{array}{l}121\left(\sin^2 t+\cos^2 t\right)+(22)^2\left(\sin^2\left(\frac{11t}{2}\right)+\cos^2\left(\frac{11t}{2}\right)\right)\\-(22\times 22)\sin t\sin\left(\frac{11t}{2}\right)-(22\times 22)\cos t\cos\left(\frac{11t}{2}\right)\end{array}}\,dt$$

$$=\int_0^{4\pi}\sqrt{121+484-484\left(\sin t\sin\left(\frac{11t}{2}\right)+\cos t\cos\left(\frac{11t}{2}\right)\right)}\,dt$$

$$=\int_0^{4\pi}\sqrt{605-484\cos\left(\frac{9t}{2}\right)}\,dt \qquad \text{Use } \cos A\cos B+\sin A\sin B=\cos(A-B)$$

By Maple software, we get

$$\int_0^{4\cdot\pi}\sqrt{605-484\cos\left(\frac{9x}{2}\right)}\,dx$$

$$264\,\mathrm{EllipticE}\left(\frac{2}{3}\sqrt{2}\right)$$

$$\xrightarrow{\text{at 5 digits}}$$

$$294.02$$

Therefore the length of the epitrochoid is approximately $\boxed{294}$

Q56E

(a)

It is given that

$$x=C(t)=\int_0^t\cos\left(\frac{\pi u^2}{2}\right)du$$

$$y=S(t)=\int_0^t\sin\left(\frac{\pi u^2}{2}\right)du$$

By using maple software to the graph can be drawn as shown below:

Maple Inputs:

```
> with(plots):
> a1:=plot(int(cos(Pi*u^2/2), u = 0..t),t = -5..5, thickness = 3, color = black):
> a2:=plot(int(sin(Pi*u^2/2), u = 0..t),t = -5..5, thickness = 3, color = blue):
> display(a1,a2);
```

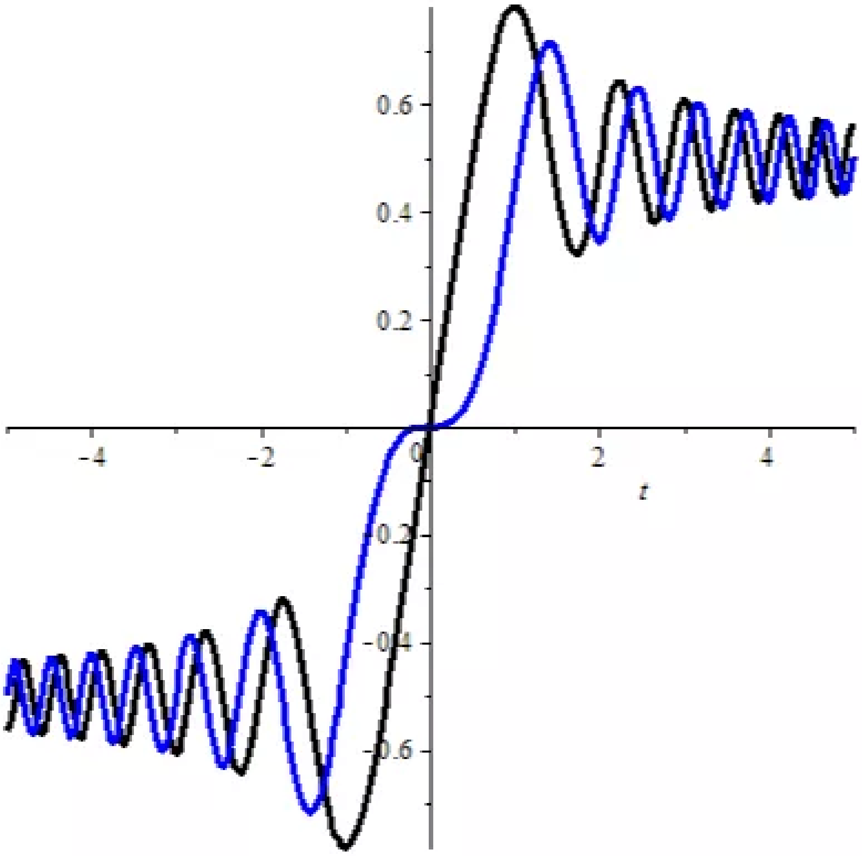
Maple Outputs

> $with(plots):$

> $a1 := plot\left(int\left(\cos\left(\frac{\pi \cdot u^2}{2}\right), u = 0..t\right), t = -5..5, thickness = 3, color = black\right):$

> $a2 := plot\left(int\left(\sin\left(\frac{\pi \cdot u^2}{2}\right), u = 0..t\right), t = -5..5, thickness = 3, color = blue\right):$

> $display(a1, a2);$



From the graph it can be assume that when $t \to \infty$ the value of the function approaches 0.5 and when $t \to -\infty$ the value of the function approaches -0.5.

Hence

> int(sqrt((sin(Pi*t^2/2)*(-Pi*t))^2+(cos(Pi*t^2/2)*Pi*t)^2),t = 0..t);

$$\frac{1}{2} t \pi \sqrt{t^2 \left(\sin\left(\frac{\pi t^2}{2}\right)^2 + \cos\left(\frac{\pi t^2}{2}\right)^2\right)}$$

> simplify(%);

$$\frac{1}{2} t^2 \pi \operatorname{csgn}(t)$$

Q57E

Consider the curve $x = t\sin t, y = t\cos t, \quad 0 \le t \le \frac{\pi}{2}$

Always, the curve given by the parametric equations $x = f(t), y = g(t)$, $\alpha \le t \le \beta$ is rotated about the x – axis, where f', g' are continuous functions and $g(t) \ge 0$, then the area of the resulting surface is given by

$$S = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Since $x = t\sin t, y = t\cos t$

Therefore, $\frac{dx}{dt} = t\cos t + \sin t, \frac{dy}{dt} = -t\sin t + \cos t$

Hence, the required area of the surface is

$$S = \int_0^{\pi/2} 2\pi \cdot t\cos t \cdot \sqrt{(t\cos t + \sin t)^2 + (-t\sin t + \cos t)^2}\, dt$$

$$= \int_0^{\pi/2} 2\pi t\cos t\sqrt{t^2\cos^2 t + 2t\cos t\sin t + \sin^2 t + t^2\sin^2 t - 2t\sin t\cos t \_ \cos^2 t}\, dt$$

$$= \int_0^{\pi/2} 2\pi t\cos t\sqrt{t^2(\cos^2 t + \sin^2 t) + (\cos^2 t + \sin^2 t)}$$

$$= \int_0^{\pi/2} \left(2\pi t\cos t\sqrt{t^2 + 1}\right) dt$$

$\approx \boxed{4.7394}$ (Using a calculator)

Q58E

Sol: Given curve is $x = \sin t, y = \sin 2t, 0 \le t \le \frac{\pi}{2}$

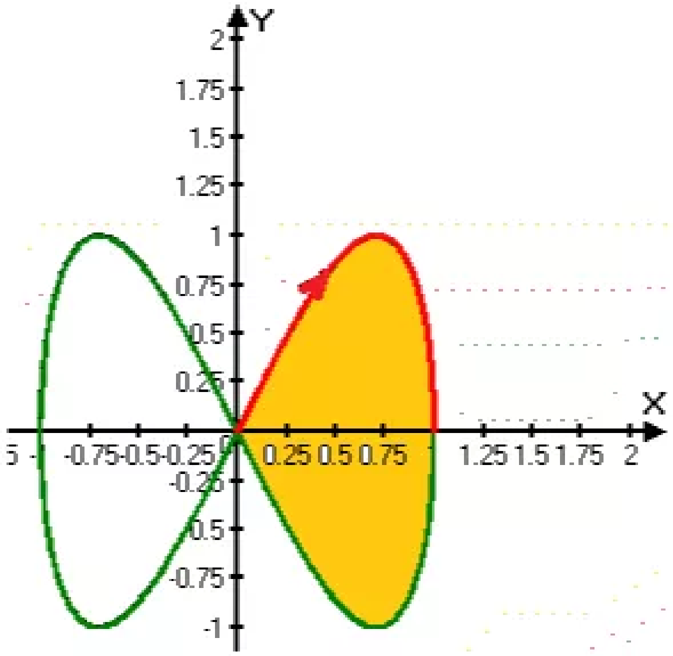
The surface area of the region enclosed by the curve and by rotating about X axis is

$$S = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = \cos t, \frac{dy}{dt} = 2\cos 2t$$

$$\therefore S = \int_0^{\pi/2} 2\pi \cdot \sin(2t) \cdot \sqrt{(\cos t)^2 + (2\cos 2t)^2}\, dt$$

$$= \boxed{8.17192}$$



The red color piece of the curve is the given curve between 0 and $\frac{\pi}{2}$.

When it is rotated about X axis, the area of the surface enclosed is shown in yellow color.

Q59E

Given curve is $x = 1 + te^t, y = (t^2+1)e^t, 0 \le t \le 1$

The surface area when the given parametric curve is rotated about X axis is given by

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

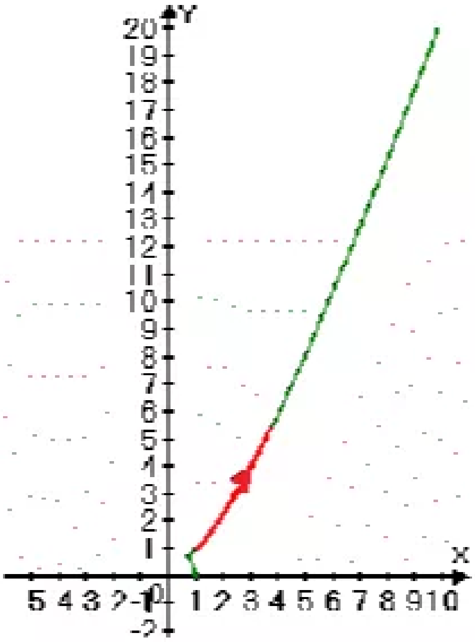
$$\frac{dx}{dt} = e^t(t+1), \frac{dy}{dt} = e^t(t+1)^2$$

$$\therefore S = \int_0^1 2\pi \cdot (t^2+1)e^t \cdot \sqrt{\left(e^t(t+1)\right)^2 + \left(e^t(t+1)^2\right)^2}\, dt$$

$$= \int_0^1 2\pi \cdot (t^2+1)e^{2t} \cdot \sqrt{(t+1)^2 + (t+1)^4}\, dt$$

$$= \int_0^1 2\pi \cdot e^{2t}(t+1)(t^2+1) \cdot \sqrt{t^2+2t+2}\, dt$$

$$= \boxed{103.5999}$$



The red color portion denotes the given curve between t = 0 and t = 1.

Q60E

Sol: Given curve is $x = t^2 - t^3, y = t + t^4, 0 \le t \le 1$

The surface area when the given parametric curve is rotated about X axis is given by

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

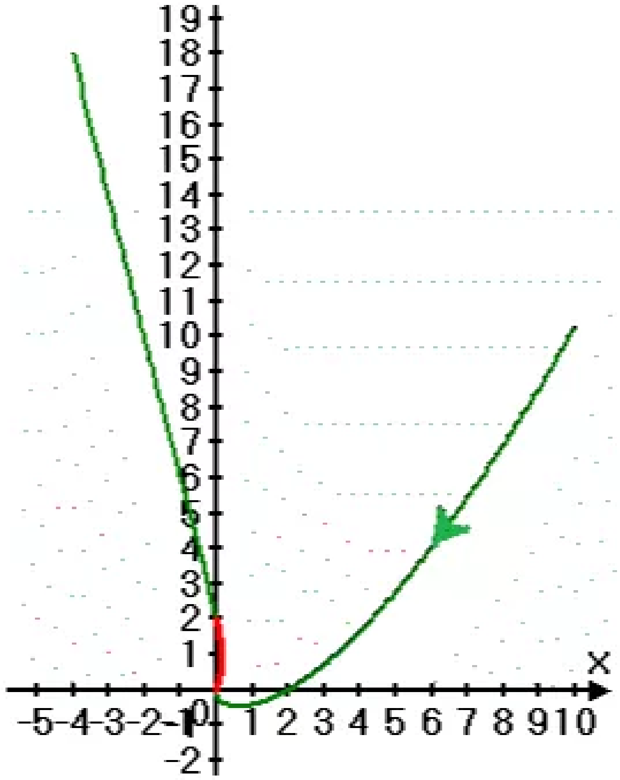
$$\frac{dx}{dt} = 2t - 3t^2, \frac{dy}{dt} = 1 + 4t^3$$

$$\therefore S = \int_0^1 2\pi(t + t^4) \cdot \sqrt{\left(2t - 3t^2\right)^2 + \left(1 + 4t^3\right)^2} \cdot dt$$

$$= \int_0^1 2\pi(t + t^4) \cdot \sqrt{\left(2t - 3t^2\right)^2 + \left(1 + 4t^3\right)^2} \cdot dt$$

$$= \int_0^1 2\pi(t + t^4) \cdot \sqrt{16t^6 + 9t^4 - 4t^3 + 4t^2 + 1} \cdot dt$$

$$= \boxed{12.7176}$$



The part of the curve in the red color is $0 \le t \le 1$

Q61E

We have $x = t^3$ $y = t^2$

Then $\frac{dx}{dt} = 3t^2$ $\frac{dy}{dt} = 2t$

The surface area of the surface obtained by rotating the curve about the x-axis is

$$S = \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 2\pi t^2 \sqrt{9t^4 + 4t^2}\, dt$$

$$= \int_0^1 2\pi t^2 \times t\sqrt{9t^2 + 4}\, dt$$

$$= 2\pi \int_0^1 t^2 \sqrt{9t^2 + 4}\ t\, dt$$

Now substitute $9t^2+4=u \Rightarrow \quad 18t\,dt = du$

When $\begin{cases} t=0, u=4 \\ t=1, u=13 \end{cases}$

Then, $S = 2\pi \int_4^{13} \frac{u-4}{9}\sqrt{u}\,\frac{du}{18}$

$$= \frac{\pi}{81}\int_4^{13}\left(u^{3/2}-4u^{1/2}\right)du$$

$$= \frac{\pi}{81}\left[\frac{2}{5}u^{5/2}-4\times\frac{2}{3}u^{3/2}\right]_4^{13}$$

$$= \frac{2\pi}{81}\left[\left(\frac{13^{5/2}}{5}-\frac{4}{3}13^{3/2}\right)-\left(\frac{4^{5/2}}{5}-\frac{4.4^{3/2}}{3}\right)\right]$$

$$= \frac{2\pi}{81}\left[13^{3/2}\left(\frac{13}{5}-\frac{4}{3}\right)-\left(\frac{32}{5}-\frac{32}{3}\right)\right]$$

$$= \frac{2}{81}\pi\left[13\sqrt{13}\left(\frac{19}{15}\right)+\frac{64}{15}\right] = \boxed{\frac{2\pi}{1215}\left[247\sqrt{13}+64\right]}$$

Q62E

We have $\quad x = 3t - t^3, \quad y = 3t^2, \quad 0 \le t \le 1$

Then $\quad \frac{dx}{dt} = 3-3t^2, \quad \frac{dy}{dt} = 6t$

The surface area of the surface obtained by rotating curve about the x-axis is

$$S = \int_0^1 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$= 2\pi\int_0^1 3t^2\sqrt{\left(3-3t^2\right)^2+\left(6t\right)^2}\,dt$$

$$= 6\pi\int_0^1 t^2\sqrt{9+9t^4-18t^2+36t^2}\,dt$$

$$= 6\pi\int_0^1 t^2\sqrt{9t^4+18t^2+9}\,dt$$

$$= 6\pi\int_0^1 t^2\sqrt{\left(3t^2+3\right)^2}\,dt$$

$$= 6\pi\int_0^1 t^2\left(3t^2+3\right)dt$$

$$= 18\pi\int_0^1\left(t^4+t^2\right)dt$$

$$= 18\pi\left(\frac{t^5}{5}+\frac{t^3}{3}\right)_0^1$$

$$= 18\pi\left(\frac{1}{5}+\frac{1}{3}\right)$$

$$= 18\pi\left(\frac{8}{15}\right)$$

$$= \boxed{\frac{48}{5}\pi}$$

Q63E

We have $x = a\cos^3\theta$ $\quad y = a\sin^3\theta$

Then $\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$, $\quad \frac{dy}{d\theta} = 3a\sin^2\theta.\cos\theta$

$= -3a\sin\theta\cos^2\theta$, $\quad = 3a\cos\theta\sin^2\theta$

The surface area of the curve about the x-axis is

$$S = \int_0^{\pi/2} 2\pi y\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi/2} 2\pi a\sin^2\theta\sqrt{9a^2\sin^2\theta\cos^4\theta + 9a^2\cos^2\theta\sin^4\theta}\ d\theta$$

$$= \int_0^{\pi/2} 2a\pi\sin^3\theta\sqrt{9a^2\sin^2\theta\cos^2\theta\left(\cos^2\theta + \sin^2\theta\right)}\ d\theta$$

$$= \int_0^{\pi/2} 2a\pi\sin^3\theta.3a\sin\theta\cos\theta\, d\theta$$

$$= 6a^2\pi\int_0^{\pi/2} \sin^4\theta\cos\theta\ d\theta$$

Substitute $\sin\theta = u \Rightarrow \cos\theta\ d\theta = du$

When, $\begin{cases}\theta = 0, u = 0\\ \theta = \frac{\pi}{2}, u = 1\end{cases}$

Then, $S = 6a^2\pi\int_0^1 u^4 du$

$$= 6a^2\pi\left[\frac{u^5}{5}\right]_0^1$$

$$= \boxed{\frac{6}{5}\pi a^2}$$

Q64E

202-11.2-62E

We have $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$

We sketch the curve.

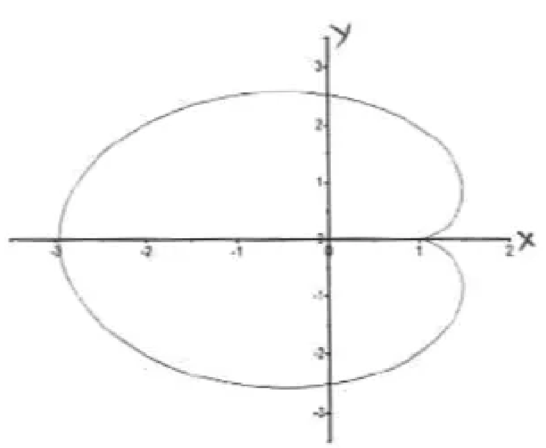


Fig. 1

We see that graph makes this loop with in the interval $0 \le \theta \le 2\pi$. We have to rotate this loop about x-axis so if we rotate half of this loop which is lying above x -axis, we get same solid as obtained by rotating full loop about x-axis.
For half of the graph θ varies from 0 to π.

We have $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$

Then $\frac{dx}{d\theta} = -2\sin\theta + 2\sin 2\theta$ and $\frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta$

Surface area

$$S = \int_0^{\pi} 2\pi y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi} 2\pi(2\sin\theta - \sin 2\theta)\sqrt{(2\sin 2\theta - 2\sin\theta)^2 + (2\cos\theta - 2\cos 2\theta)^2}\, d\theta$$

$$= 4\pi\int_0^{\pi} (2\sin\theta - \sin 2\theta)\sqrt{(\sin 2\theta - \sin\theta)^2 + (\cos\theta - \cos 2\theta)^2}\, d\theta$$

$$= 4\pi\int_0^{\pi} (2\sin\theta - \sin 2\theta)\sqrt{\sin^2 2\theta + \sin^2\theta - 2\sin\theta\sin 2\theta + \cos^2\theta + \cos^2 2\theta - 2\cos\theta\cos 2\theta}$$

Since, $\sin^2 2\theta + \cos^2 2\theta = 1$ and $\sin^2\theta + \cos^2\theta = 1$

Then $S = 4\pi\int_0^{\pi} (2\sin\theta - \sin 2\theta)\sqrt{1 + 1 - 2\sin\theta\sin 2\theta - 2\cos\theta\cos 2\theta}\, d\theta$

$$= 4\pi\int (2\sin\theta - 2\sin\theta\cos\theta)\sqrt{2 - 2\sin\theta(2\sin\theta\cos\theta) - 2\cos\theta(1 - 2\sin^2\theta)}\, d\theta$$

$$= 4\pi\int_0^{\pi} 2\sin\theta(1 - \cos\theta)\sqrt{2 - 4\sin^2\theta\cos\theta - 2\cos\theta + 4\cos\theta\sin^2\theta}\, d\theta$$

$$= 4\pi\int_0^{\pi} 2\sin\theta(1 - \cos\theta)\sqrt{2 - 2\cos\theta}\, d\theta$$

$$= 8\pi\sqrt{2}\int_0^{\pi} \sin\theta(1 - \cos\theta)^{3/2}\, d\theta$$

Let, $1 - \cos\theta = t \Rightarrow \sin\theta\, d\theta = dt$, when $\begin{cases} \theta = 0, t = 0 \\ \theta = \pi, t = 2 \end{cases}$

Then, $S = 8\pi\sqrt{2}\int_0^2 t^{3/2} dt$

$$\Rightarrow \quad S = 8\pi\sqrt{2}\left[\frac{2t^{5/2}}{5}\right]_0^2$$

$$= \frac{16\pi\sqrt{2}}{5}\left[\left(\sqrt{2}\right)^5 - 0\right]$$

$$\Rightarrow S = \frac{16\pi \times 8}{5}$$

$$= 128\,\pi/5$$

Thus $\boxed{S = 128\pi/5}$

Q65E

Consider the curves

$x = 3t2$, $y = 2t3$, $0 \le t \le 5$

Then $\frac{dx}{dt} = 6t, \frac{dy}{dt} = 6t^2$

If the curve is given by the parametric equations $x = f(t), y = g(t), \alpha \le t \le \beta,$ is rotated about the y-axis, where f' and g' are continuous $g(t) \ge 0$, then the area of the resulting surface is $S = \int_{\alpha}^{\beta} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$

The surface area of the curve about the y-axis.

$$S = \int_0^5 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$= \int_0^5 2\pi \left(3t^2\right) \sqrt{36t^2 + 36t^4} \, dt$$

$$= 18\pi \int_0^5 t^2 \sqrt{1+t^2} (2t) \, dt$$

Substitute $1 + t^2 = u, \ t^2 = u - 1$

And $2tdt = du$

When $\begin{cases} t = 0, u = 1 \\ t = 5, u = 26 \end{cases}$

Then $S = 18\pi \int_1^{26} (u-1)\sqrt{u} \, du$

$$= 18\pi \int_1^{26} \left(u^{3/2} - u^{1/2}\right) du$$

$$= 18\pi \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^{26}$$

$$= 36\pi \left[\left(\frac{26^{5/2}}{5} - \frac{26^{3/2}}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)\right]$$

Continue the above

$$= 36\pi\left[26^{3/2}\left(\frac{26}{5}-\frac{1}{3}\right)-\left(\frac{-2}{15}\right)\right]$$

$$= 36\pi\left[26\sqrt{26}\left(\frac{73}{15}\right)+\frac{2}{15}\right]$$

$$= 36\times\frac{2}{15}\pi\left[949\sqrt{26}+1\right]$$

Therefore $\boxed{S = \frac{24}{5}\pi\left[949\sqrt{26}+1\right]}$

Q66E

We have $x = e^t - t$ $y = 4e^{t/2}$, $0 \le t \le 1$

Then $\frac{dx}{dt} = e^t - 1$ $\frac{dy}{dt} = 4e^{t/2}\left(\frac{1}{2}\right) = 2e^{t/2}$

The surface area of the surface obtained by rotating the curve about the y-axis is

$$S = \int_0^1 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^1 2\pi\left(e^t - t\right)\sqrt{\left(e^t-1\right)^2+\left(2e^{t/2}\right)^2}\,dt$$

$$= \int_0^1 2\pi\left(e^t - t\right)\sqrt{e^{2t}+1-2e^t+4e^t}\,dt$$

$$= \int_0^1 2\pi\left(e^t - t\right)\sqrt{e^{2t}+2e^t+1}\,dt$$

$$= \int_0^1 2\pi\left(e^t - t\right)\sqrt{\left(e^t+1\right)^2}\,dt$$

$$= 2\pi\int_0^1 \left(e^t - t\right)\left(e^t+1\right)dt$$

$$= 2\pi\int_0^1 \left[e^{2t}+(1-t)e^t - t\right]dt$$

Therefore $S = 2\pi\left[\frac{e^{2t}}{2} - \frac{t^2}{2} + \left\{(1-t)e^t - \int(-1)e^t dt\right\}\right]_0^1$

$$= 2\pi\left[\frac{e^{2t}}{2} - \frac{t^2}{2} + (1-t)e^t + e^t\right]_0^1$$

$$= \pi\left[e^{2t} - t^2 + 2(1-t)e^t + 2e^t\right]_0^1$$

$$= \pi\left[\left(e^2 - 1 + 0 + 2e\right) - \left(e^0 - 0 + 2e^0 + 2e^0\right)\right]_0^1$$

$$= \pi\left[e^2 - 1 + 2e - 1 - 2 - 2\right]_0^1$$

$$= \boxed{\pi\left(e^2 + 2e - 6\right)}$$

Q67E

Consider the function f is invertible.

It is need to show that the parametric curve $x = f(t), y = g(t), a \le t \le b$ can be written in the form $y = F(x)$.

Now consider,

$y = g(t)$

$= g\left(f^{-1}(x)\right)$ Since f is invertible, $x = f(t) \Leftrightarrow f^{-1}(x) = t$

$= \left(g \circ f^{-1}\right)(x)$ By the definition of composition of functions

Suppose that $F = g \circ f^{-1}$

Then we have,

$y = F(x)$

So the parametric curve $x = f(t), y = g(t), a \le t \le b$ can be put in the form $y = F(x)$.

Q68E

Let $x = f(t)$ and $y = g(t)$ where $a \le x \le b$ for $\alpha \le t \le \beta$, $\{f(\alpha) = a, \quad f(\beta) = b\}$

The surface area of the surface obtained by rotating the curve $y = f(x), a \le x \le b$ about x - axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Since, $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$ if $\frac{dx}{dt}\neq 0$

Then $S=\int_\alpha^\beta 2\pi y\sqrt{1+\frac{(dy/dt)^2}{(dx/dt)^2}}\,\frac{dx}{dt}dt$

$$\Rightarrow \quad S=\int_\alpha^\beta 2\pi y\sqrt{\frac{\left[(dx/dt)^2+(dy/dt)^2\right]}{(dx/dt)^2}}\,\frac{dx}{dt}dt$$

$$\Rightarrow \quad S=\int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,\frac{1}{(dx/dt)}\left(\frac{dx}{dt}\right)dt$$

$$\Rightarrow \quad \boxed{S=\int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt}$$

This is the area of the resulting surface obtained by rotating the curve about x- Axis, where parametric equations are $y=g(t)$, $x=f(t)$, $\alpha\le t\le\beta$

Q69E

(A) We have $\phi=\tan^{-1}(dy/dx)$

Then $\frac{d\phi}{dt}=\frac{1}{1+(dy/dx)^2}\frac{d}{dt}\left(\frac{dy}{dx}\right)$ (by chain rule.)

Since, $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\dot y}{\dot x}$

Then $\frac{d\phi}{dt}=\frac{1}{1+(\dot y/\dot x)^2}\frac{d}{dt}(\dot y/\dot x)$

$$\Rightarrow \quad \frac{d\phi}{dt}=\frac{\dot x^2}{(\dot x^2+\dot y^2)}\times\frac{\dot x\ddot y-\dot y\ddot x}{\dot x^2}$$ [By Quotient rule] $\left[\text{Where } \ddot y=\frac{d^2y}{dt^2},\ \ddot x=d^2x/dt^2\right.$

$$\Rightarrow \quad \frac{d\phi}{dt}=\frac{\dot x\ddot y-\dot y\ddot x}{(\dot x^2+\dot y^2)} \qquad \text{------------ (1)}$$

Now we have, $ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$$\Rightarrow \quad \frac{ds}{dt}=\sqrt{\dot x^2+\dot y^2}$$

Then curvature $k=\left|\frac{d\phi}{ds}\right|=\left|\frac{d\phi/dt}{ds/dt}\right|=\left|\frac{(\dot x\ddot y-\dot y\ddot x)}{(\dot x^2+\dot y^2)\sqrt{\dot x^2+\dot y^2}}\right|$

$$\Rightarrow \quad \boxed{k=\frac{|\dot x\ddot y-\ddot x\dot y|}{\left[\dot x^2+\dot y^2\right]^{3/2}}}$$ Because $\dot x^2+\dot y^2\ge 0$ for all x.

(B) Now we have been given x = x and y = f(x)

Then $\frac{dx}{dx} = 1$ *and* $\frac{dy}{dx} = f'(x)$

$\Rightarrow \quad \dot{x} = 1$ and $\dot{y} = f'(x)$

And $\quad \ddot{x} = 0 \; , \; \ddot{y} = f''(x)$

Putting these values in formula $k = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{\left[\dot{x}^2 + \dot{y}^2\right]^{3/2}}$

The curvature $$k = \frac{\left|1.\frac{d^2y}{dx^2} - 0\right|}{\left[1^2 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

$$\Rightarrow \boxed{k = \frac{\left|d^2y/dx^2\right|}{\left[1 + (dy/dx)^2\right]^{3/2}}}$$

Q70E

(A) We have $y = x^2$

Then $\frac{dy}{dx} = 2x$ and $\frac{d^2y}{dx^2} = 2$

Putting these values in formula of curvature $k = \frac{\left|d^2y/dx^2\right|}{\left[1 + (dy/dx)^2\right]^{3/2}}$

$$\Rightarrow \quad k = \frac{2}{\left[1 + (2x)^2\right]^{3/2}}$$

$$\Rightarrow \quad \boxed{k = \frac{2}{\left(1 + 4x^2\right)^{3/2}}}$$

Then curvature at the point (1, 1) is $k = \frac{2}{\left(1 + 4.1^2\right)^{3/2}}$

$$\Rightarrow \quad \boxed{k = 2/\left(5\sqrt{5}\right)}$$

(B) We have to maximize $k = \frac{2}{(1+4x^2)^{3/2}}$

We rewrite $k = 2(1+4x^2)^{-3/2}$

Differentiating with respect to x.

$$\frac{dk}{dx} = 2\left(-\frac{3}{2}\right)(1+4x^2)^{-5/2}(8x) \qquad \text{(By chain rule.)}$$

$$= -3(1+4x^2)^{-5/2}(8x)$$

$$= -24x(1+4x^2)^{-5/2}$$

$$\Rightarrow \quad \frac{dk}{dx} = \frac{-24x}{(1+4x^2)^{5/2}}, \quad \frac{dk}{dx} = 0 \text{ when } x = 0.$$

Since $\frac{dk}{dx} > 0$ for $x < 0$ and $\frac{dk}{dx} < 0$ for $x > 0$

So, k has absolute maximum at $x = 0$ by first derivative test.

When $x = 0, y = 0$.

So, the curve $y = x^2$ has maximum curvature at the point $\boxed{(0, 0)}$

Q71E

Parametric equation are $x = \theta - \sin\theta$, $y = 1 - \cos\theta$

Then $\dot{x} = \frac{dx}{d\theta} = 1 - \cos\theta$ and $\dot{y} = \frac{dy}{d\theta} = \sin\theta$

And $\ddot{x} = \frac{d^2x}{d\theta^2} = \sin\theta$ and $\ddot{y} = \frac{d^2y}{d\theta^2} = \cos\theta$

Putting these values in the formula of curvature $k = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{\left[\dot{x}^2 + \dot{y}^2\right]^{3/2}}$

$$\Rightarrow \quad k = \frac{|(1-\cos\theta)\cos\theta - \sin\theta.(\sin\theta)|}{\left[(1-\cos\theta)^2 + (\sin\theta)^2\right]^{3/2}}$$

$$\Rightarrow \quad k = \frac{|\cos\theta - \cos^2\theta - \sin^2\theta|}{\left[1 + \cos^2\theta - 2\cos\theta + \sin^2\theta\right]^{3/2}}$$

$$\Rightarrow \quad k = \frac{|\cos\theta - 1|}{[2 - 2\cos\theta]^{3/2}}$$

For finding highest point of the cycloid we have to find the maximum of y.

Since $y = 1 - \cos\theta$ and we have $-1 \le \cos\theta \le 1$. for all θ.

When $\cos\theta = 1$, $y = 0$.

And when $\cos\theta = -1$, $\boxed{y = 2}$ which is maximum.

So highest point of the cycloid corresponds to $\theta = \pi, 3\pi, 5\pi \ldots$

Taking $\theta = \pi$.
We find the curvature of the cycloid.

$$\Rightarrow \quad k = \frac{|\cos\pi - 1|}{[2 - 2\cos\pi]^{3/2}}$$

$$\Rightarrow \quad k = \frac{|-1-1|}{[2+2]^{3/2}}$$

$$\Rightarrow \quad k = \frac{2}{(4)^{3/2}}$$

$$\Rightarrow \quad \boxed{k = \frac{1}{4}}$$

Q72E

(A) Equation of the straight line is $y = mx + c$

Regarding this line, the parametric equations are

$$x = x, \qquad y = f(x) = mx + c$$

Then $\dot{x} = 1$, $\dot{y} = \frac{dy}{dx}$
$= m$

And $\ddot{x} = 0$, $\ddot{y} = \frac{d^2y}{dx^2}$
$= 0$

Now the formula of the curvature of the curve is $k = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{\left[\dot{x}^2 + \dot{y}^2\right]^{3/2}}$

Putting the values, we have $k = \frac{|1.0 - 0.m|}{|1^2 + 0^2|^{3/2}}$

$$\Rightarrow \quad k = \frac{0-0}{1} \Rightarrow \boxed{k = 0}$$

Thus the curvature at each point of a straight line is $k = 0$.

(B) Equation of a circle of radius r is $x^2 + y^2 = r^2$

Then $y = \pm\sqrt{r^2 - x^2}$

Since curvature of the circle at any point will be same so, we consider only upper half of the circle and equation is $y = \sqrt{r^2 - x^2}$

Differentiating with respect to x.

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{r^2 - x^2}}$$

$$= \frac{-x}{\sqrt{r^2 - x^2}}$$

Then $$\frac{d^2y}{dx^2} = \frac{-\sqrt{r^2 - x^2} + x.\dfrac{1(-2x)}{2\sqrt{r^2 - x^2}}}{\left(r^2 - x^2\right)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-\sqrt{r^2 - x^2} - \dfrac{x^2}{\sqrt{r^2 - x^2}}}{\left(r^2 - x^2\right)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-\left(r^2 - x^2\right) - x^2}{\left(r^2 - x^2\right)\sqrt{r^2 - x^2}}$$

$$= \frac{-r^2}{\left[r^2 - x^2\right]^{3/2}}$$

Curvature of the circle at any point (x, y) is

$$k = \frac{\left|\dfrac{d^2y}{dx^2}\right|}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

$$= \frac{\left|\dfrac{-r^2}{\left[r^2 - x^2\right]^{3/2}}\right|}{\left[1 + \dfrac{x^2}{r^2 - x^2}\right]^{3/2}}$$

$$\Rightarrow \quad k = \frac{r^2\left(r^2 - x^2\right)^{3/2}}{\left(r^2\right)^{3/2}\left(r^2 - x^2\right)^{3/2}}$$

$$= \frac{r^2}{r^3}$$

$$= \frac{1}{r}$$

$$\Rightarrow \quad \boxed{k = \frac{1}{r}}$$

Thus the curvature at each point of a circle of radius r is $k = 1/r$.

Q73E

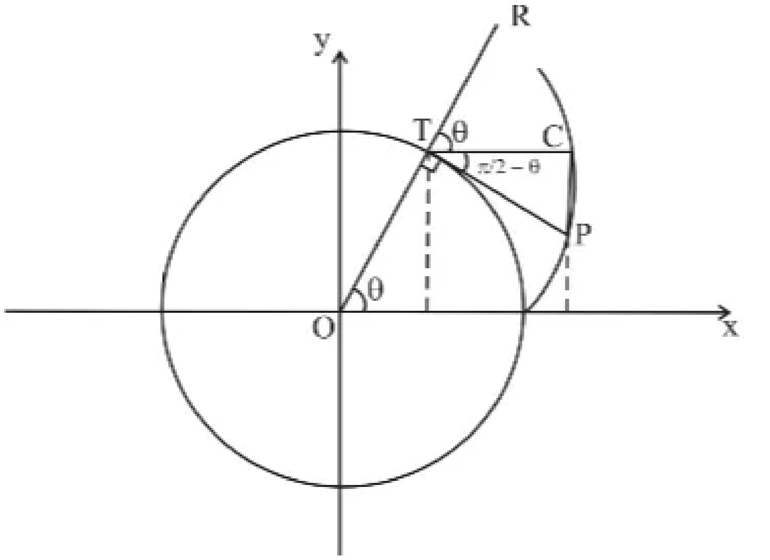


Fig. 1

We increase $|OT|$ up to the point R and draw a line segment from T parallel to the x-axis and a line segment from P parallel to y-axis. Both the lines intersect each other at C then $\angle PCT = \frac{\pi}{2}$

And $\angle RTC = \theta = \angle TOX$

Since, $\angle OTP = \frac{\pi}{2}$ then $\angle RTP = \frac{\pi}{2}$

And $\angle CTP = \angle RTP - \angle RTC = \frac{\pi}{2} - \theta$

Now parametric equations of a circle of radius r are

$x = r\cos\theta$ and $y = r\sin\theta$

Then co-ordinates of the point T are ($r\cos\theta$, $r\sin\theta$)

Given that initial position of P is (r, 0)

Then $|TP|$ = radius × angle at the center of the circle.

$= r \times \theta$

$= r\theta.$

Then by the triangle CTP,

$$|CT| = r\theta.\cos\left(\frac{\pi}{2} - \theta\right) = r\theta\sin\theta \qquad \ldots\ldots\ldots (1)$$

And $$|CP| = r\theta.\sin\left(\frac{\pi}{2} - \theta\right) = r\theta\cos\theta \qquad \ldots\ldots\ldots (2)$$

Then co-ordinates of the point P are

$$x = r\cos\theta + |CT|$$

$$\Rightarrow \quad x = r\cos\theta + r\theta\sin\theta = r(\cos\theta + \theta\sin\theta)$$

And $\quad y = r\sin\theta - |CP|$

$$\Rightarrow \quad y = r\sin\theta - r\theta\cos\theta$$

$$= r(\sin\theta - \theta\cos\theta)$$

Thus parametric equations are,

$$\Rightarrow \quad \boxed{\begin{aligned} x &= r(\cos\theta + \theta\sin\theta) \\ y &= r(\sin\theta - \theta\cos\theta) \end{aligned}}$$

Q74E

Consider the data:

If the cow walks with the rope taut, it traces out the portion of the involute, corresponding to the range $0 \le \theta \le \pi$, arriving at the point $(-r, \pi r)$ when $\theta = \pi$.

Now fully extended with the rope, the cow walks in a semicircle of radius πr, arriving at $(-r, -\pi r)$.

Finally, the cow traces out another portion of the involute, namely the reflection about the x-axis of the initial involute path (this corresponds to the range $-\pi \le \theta \le 0$).

Refer to the figure: Observe that the total grazing area is $2(A_1 + A_3)$.

Here, the area A_3 is one quarter of the area of a circle of radius, πr.

So

$$A_3 = \frac{1}{4}\pi(\pi r)^2$$

$$= \frac{1}{4}\pi(\pi^2 r^2)$$

$$= \frac{1}{4}\pi^3 r^2$$

Now, compute $A_1 + A_2$ and then subtract $A_2 = \frac{1}{2}\pi r^2$ to obtain A_1.

To find $A_1 + A_2$, first note that the right most point of the involute is $\left(\frac{\pi r}{2}, r\right)$.

The left most point of the involute is $(-r, \pi r)$.

Thus, write as follows:

$$\begin{aligned} A_1 + A_2 &= \int_{\theta=\pi}^{\frac{\pi}{2}} y\, dx - \int_{\theta=0}^{\frac{\pi}{2}} y\, dx \\ &= \int_{\theta=\pi}^{\frac{\pi}{2}} y\, dx + \int_{\frac{\pi}{2}}^{\theta=0} y\, dx \\ &= \int_{\theta=\pi}^{0} y\, dx \end{aligned}$$

Write the following:

$$\begin{aligned} ydx &= r(\sin\theta - \theta\cos\theta) r\theta\cos\theta d\theta \\ &= r^2\left(\theta\sin\theta\cos\theta - \theta^2\cos^2\theta\right)d\theta \end{aligned}$$

Integrate on both sides:

$$\begin{aligned} \frac{1}{r^2}\int ydx &= \int\left(\theta\sin\theta\cos\theta - \theta^2\cos^2\theta\right)d\theta \\ &= \int\left(\theta\sin\theta\cos\theta - \theta^2\left(\frac{1+\cos 2\theta}{2}\right)\right)d\theta \\ &= \frac{1}{2}\left[\frac{\theta^2}{2}\sin\theta\cos\theta + \theta\cos^2\theta + \theta\sin^2\theta - \frac{\theta^3}{6} - \frac{1}{8}\left(2\theta^2 - 1\right)\sin 2\theta - \frac{1}{4}\theta\cos 2\theta\right] \\ &= -\theta\cos^2\theta - \frac{1}{2}\left(\theta^2 - 1\right)\sin\theta\cos\theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta + c \end{aligned}$$

This enables us, to compute $A_1 + A_2$ as follows:

$$A_1 + A_2 = \int_{\theta=\pi}^{0} y\,dx$$

$$= r^2\left[-\theta\cos^2\theta - \frac{1}{2}(\theta^2 - 1)\sin\theta\cos\theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta\right]_{\pi}^{0}$$

$$= r^2\left[0 - \left(-\pi - \frac{\pi^3}{6} + \frac{\pi}{2}\right)\right]$$

$$= r^2\left(\frac{\pi}{2} + \frac{\pi^3}{6}\right)$$

Now, the area, A_1 can be calculated as follows:

$$A_1 = (A_1 + A_2) - A_2$$

$$= r^2\left(\frac{\pi}{2} + \frac{\pi^3}{6}\right) - \frac{1}{2}\pi r^2$$

$$= \frac{1}{6}\pi^3 r^2$$

So, the grazing area can be calculated as follows:

$$2(A_1 + A_3) = 2\left(\frac{1}{6}\pi^3 r^2 + \frac{1}{4}\pi^3 r^2\right)$$

$$= \frac{5}{6}\pi^3 r^2$$

Hence, the required area is $\boxed{\frac{5}{6}\pi^3 r^2}$.