CHAPTER XXVII.

RECURRING CONTINUED FRACTIONS.

355. We have seen in Chap. XXV. that a *terminating* continued fraction with rational quotients can be reduced to an ordinary fraction with integral numerator and denominator, and therefore cannot be equal to a surd; but we shall prove that a quadratic surd can be expressed as an *infinite* continued fraction whose quotients recur. We shall first consider a numerical example.

Example. Express $\sqrt{19}$ as a continued fraction, and find a series of fractions approximating to its value.

$$\sqrt{19}=4+(\sqrt{19}-4)=4+\frac{3}{\sqrt{19}+4};$$

$$\frac{\sqrt{19}+4}{3}=2+\frac{\sqrt{19}-2}{3}=2+\frac{5}{\sqrt{19}+2};$$

$$\frac{\sqrt{19}+2}{5}=1+\frac{\sqrt{19}-3}{5}=1+\frac{2}{\sqrt{19}+3};$$

$$\frac{\sqrt{19}+3}{2}=3+\frac{\sqrt{19}-3}{2}=3+\frac{5}{\sqrt{19}+3};$$

$$\frac{\sqrt{19}+3}{5}=1+\frac{\sqrt{19}-2}{5}=1+\frac{3}{\sqrt{19}+2};$$

$$\frac{\sqrt{19}+2}{3}=2+\frac{\sqrt{19}-4}{3}=2+\frac{1}{\sqrt{19}+4};$$

$$\sqrt{19}+4=8+(\sqrt{19}-4)=8+\ldots\ldots\ldots\ldots$$

after this the quotients 2, 1, 3, 1, 2, 8 recur; hence

$$\sqrt{19}=4+\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{8+}\cdots$$

It will be noticed that the quotients recur as soon as we come to a quotient which is double of the first. In Art. 361 we shall prove that this is always the case.

[*Explanation.* In each of the lines above we perform the same series of operations. For example, consider the second line: we first find the greatest integer in $\frac{\sqrt{19}+4}{3}$; this is 2, and the remainder is $\frac{\sqrt{19}+4}{3}-2$, that is $\frac{\sqrt{19}-2}{3}$. We then multiply numerator and denominator by the surd conjugate to $\sqrt{19}-2$, so that after inverting the result $\frac{5}{\sqrt{19}+2}$, we begin a new line with a rational denominator.]

The first seven convergents formed as explained in Art. 336 are

$$\frac{4}{1},\ \frac{9}{2},\ \frac{13}{3},\ \frac{48}{11},\ \frac{61}{14},\ \frac{170}{39},\ \frac{1421}{326}.$$

The error in taking the last of these is less than $\frac{1}{(326)^2}$, and is therefore less than $\frac{1}{(320)^2}$ or $\frac{1}{102400}$, and *a fortiori* less than $\cdot 00001$. Thus the seventh convergent gives the value to at least four places of decimals.

356. *Every periodic continued fraction is equal to one of the roots of a quadratic equation of which the coefficients are rational.*

Let x denote the continued fraction, and y the periodic part, and suppose that

$$x = a + \frac{1}{b+}\,\frac{1}{c+}\,\cdots\cdots\,\frac{1}{h+}\,\frac{1}{k+}\,\frac{1}{y},$$

and

$$y = m + \frac{1}{n+}\,\cdots\cdots\,\frac{1}{u+}\,\frac{1}{v+}\,\frac{1}{y},$$

where $a, b, c, \ldots h, k, m, n, \ldots u, v$ are positive integers.

Let $\frac{p}{q}$, $\frac{p'}{q'}$ be the convergents to x corresponding to the quotients h, k respectively; then since y is the complete quotient, we have $x = \frac{p'y+p}{q'y+q}$; whence $y = \frac{p-qx}{q'x-p'}$.

Let $\frac{r}{s}$, $\frac{r'}{s'}$ be the convergents to y corresponding to the quotients u, v respectively; then $y = \frac{r'y+r}{s'y+s}$.

Substituting for y in terms of x and simplifying we obtain a quadratic of which the coefficients are rational.

The equation $s'y^2 + (s - r')y - r = 0$, which gives the value of y, has its roots real and of opposite signs; if the positive value of y be substituted in $x = \dfrac{p'y + p}{q'y + q}$, on rationalising the denominator the value of x is of the form $\dfrac{A + \sqrt{B}}{C}$, where A, B, C are integers, B being positive since the value of y is real.

Example. Express $1 + \dfrac{1}{2+}\,\dfrac{1}{3+}\,\dfrac{1}{2+}\,\dfrac{1}{3+}\,\dots$ as a surd.

Let x be the value of the continued fraction; then $x - 1 = \dfrac{1}{2+}\,\dfrac{1}{3 + (x-1)}$; whence $2x^2 + 2x - 7 = 0$.

The continued fraction is equal to the positive root of this equation, and is therefore equal to $\dfrac{\sqrt{15} - 1}{2}$.

EXAMPLES. XXVII. a.

Express the following surds as continued fractions, and find the sixth convergent to each:

1. $\sqrt{3}$. 2. $\sqrt{5}$. 3. $\sqrt{6}$. 4. $\sqrt{8}$.

5. $\sqrt{11}$. 6. $\sqrt{13}$. 7. $\sqrt{14}$. 8. $\sqrt{22}$.

9. $2\sqrt{3}$. 10. $4\sqrt{2}$. 11. $3\sqrt{5}$. 12. $4\sqrt{10}$.

13. $\dfrac{1}{\sqrt{21}}$. 14. $\dfrac{1}{\sqrt{33}}$. 15. $\sqrt{\dfrac{6}{5}}$. 16. $\sqrt{\dfrac{7}{11}}$.

17. Find limits of the error when $\dfrac{268}{65}$ is taken for $\sqrt{17}$.

18. Find limits of the error when $\dfrac{916}{191}$ is taken for $\sqrt{23}$.

19. Find the first convergent to $\sqrt{101}$ that is correct to five places of decimals.

20. Find the first convergent to $\sqrt{15}$ that is correct to five places of decimals.

Express as a continued fraction the positive root of each of the following equations:

21. $x^2 + 2x - 1 = 0$. 22. $x^2 - 4x - 3 = 0$. 23. $7x^2 - 8x - 3 = 0$.

24. Express each root of $x^2 - 5x + 3 = 0$ as a continued fraction.

25. Find the value of $3 + \dfrac{1}{6+}\,\dfrac{1}{6+}\,\dfrac{1}{6+}\,\dots\dots$

26. Find the value of $\dfrac{1}{1+}\,\dfrac{1}{3+}\,\dfrac{1}{1+}\,\dfrac{1}{3+}\,\dots\dots$

27. Find the value of $3+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\cdots\cdots$

28. Find the value of $5+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{10+}\cdots\cdots$

29. Shew that

$$3+\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{6+}\cdots=3\left(1+\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\cdots\cdots\right).$$

30. Find the difference between the infinite continued fractions

$$\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{5+}\cdots,\quad \frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{5+}\cdots.$$

*357. *To convert a quadratic surd into a continued fraction.*

Let N be a positive integer which is not an exact square, and let a_1 be the greatest integer contained in $\sqrt{N}$; then

$$\sqrt{N}=a_1+(\sqrt{N}-a_1)=a_1+\frac{r_1}{\sqrt{N}+a_1},\ \text{if } r_1=N-a_1^2.$$

Let b_1 be the greatest integer contained in $\frac{\sqrt{N}+a_1}{r_1}$; then

$$\frac{\sqrt{N}+a_1}{r_1}=b_1+\frac{\sqrt{N}-b_1r_1+a_1}{r_1}=b_1+\frac{\sqrt{N}-a_2}{r_1}=b_1+\frac{r_2}{\sqrt{N}+a_2};$$

where $\qquad a_2=b_1r_1-a_1$ and $r_1r_2=N-a_2^2$.

Similarly

$$\frac{\sqrt{N}+a_2}{r_2}=b_2+\frac{\sqrt{N}-a_3}{r_2}=b_2+\frac{r_3}{\sqrt{N}+a_3};$$

where $\qquad a_3=b_2r_2-a_2$ and $r_2r_3=N-a_3^2$;

and so on; and generally

$$\frac{\sqrt{N}+a_{n-1}}{r_{n-1}}=b_{n-1}+\frac{\sqrt{N}-a_n}{r_{n-1}}=b_{n-1}+\frac{r_n}{\sqrt{N}+a_n};$$

where $\qquad a_n=b_{n-1}r_{n-1}-a_{n-1}$ and $r_{n-1}r_n=N-a_n^2$.

Hence $\qquad \sqrt{N}=a_1+\frac{1}{b_1+}\,\frac{1}{b_2+}\,\frac{1}{b_3+}\,\frac{1}{b_4+}\cdots\cdots;$

and thus $\sqrt{N}$ can be expressed as an infinite continued fraction.

We shall presently prove that this fraction consists of recurring periods; it is evident that the period will begin whenever any complete quotient is first repeated.

We shall call the series of quotients

$$\sqrt{N}, \quad \frac{\sqrt{N}+a_1}{r_1}, \quad \frac{\sqrt{N}+a_2}{r_2}, \quad \frac{\sqrt{N}+a_3}{r_3}, \ldots\ldots$$

the first, second, third, fourth......complete quotients.

*358. From the preceding article it appears that the quantities $a_1, r_1, b_1, b_2, b_3 \ldots\ldots$ are positive integers; we shall now prove that the quantities $a_2, a_3, a_4, \ldots\ldots, r_2, r_3, r_4, \ldots$are also positive integers.

Let $\frac{p}{q}, \frac{p'}{q'}, \frac{p''}{q''}$ be three consecutive convergents to $\sqrt{N}$, and let $\frac{p''}{q''}$ be the convergent corresponding to the partial quotient b_n. The complete quotient at this stage is $\frac{\sqrt{N}+a_n}{r_n}$; hence

$$\sqrt{N} = \frac{\dfrac{\sqrt{N}+a_n}{r_n}p'+p}{\dfrac{\sqrt{N}+a_n}{r_n}q'+q} = \frac{p'\sqrt{N}+a_n p'+r_n p}{q'\sqrt{N}+a_n q'+r_n q}.$$

Clearing of fractions and equating rational and irrational parts, we have

$$a_n p' + r_n p = Nq', \quad a_n q' + r_n q = p';$$

whence $a_n(pq'-p'q) = pp'-qq'N$, $r_n(pq'-p'q) = Nq'^2 - p'^2$.

But $pq'-p'q=\pm 1$, and $pq'-p'q$, $pp'-qq'N$, $Nq'^2-p'^2$ have the same sign [Art. 344]; hence a_n and r_n are positive integers. Since two convergents precede the complete quotient $\frac{\sqrt{N}+a_2}{r_2}$, this investigation holds for all values of n greater than 1.

*359. *To prove that the complete and partial quotients recur.*

In Art. 357 we have proved that $r_n r_{n-1} = N - a_n^2$. Also r_n and r_{n-1} are positive integers; hence a_n must be less than $\sqrt{N}$, thus a_n cannot be greater than a_1, and therefore it cannot have any values except $1, 2, 3, \ldots a_1$; that is, *the number of different values of* a_n *cannot exceed* a_1.

Again, $a_{n+1} = r_n b_n - a_n$, that is $r_n b_n = a_n + a_{n+1}$, and therefore $r_n b_n$ cannot be greater than $2a_1$; also b_n is a positive integer; hence r_n cannot be greater than $2a_1$. Thus r_n cannot have any values except $1, 2, 3, \ldots 2a_1$; that is, *the number of different values of* r_n *cannot exceed* $2a_1$.

Thus the complete quotient $\frac{\sqrt{N}+a_n}{r_n}$ cannot have more than $2a_1^2$ different values; that is, *some one complete quotient, and therefore all subsequent ones, must recur.*

Also b_n is the greatest integer in $\frac{\sqrt{N}+a_n}{r_n}$; hence *the partial quotients must also recur, and the number of partial quotients in each cycle cannot be greater than* $2a_1^2$.

*360. *To prove that* $a_1 < a_n + r_n$.

We have $$a_{n-1} + a_n = b_{n-1} r_{n-1};$$
$$\therefore\ a_{n-1} + a_n = \text{ or } > r_{n-1},$$
since b_{n-1} is a positive integer;
$$\therefore\ \sqrt{N} + a_n > r_{n-1}.$$
But $$N - a_n^2 = r_n r_{n-1};$$
$$\therefore\ \sqrt{N} - a_n < r_n;$$
$$\therefore\ a_1 - a_n < r_n,$$
which proves the proposition.

*361. *To shew that the period begins with the second partial quotient and terminates with a partial quotient double of the first.*

Since, as we have seen in Art. 359, a recurrence must take place, let us suppose that the $(n+1)^{\text{th}}$ complete quotient recurs at the $(s+1)^{\text{th}}$; then
$$a_s = a_n,\ \ r_s = r_n,\ \text{ and }\ b_s = b_n;$$
we shall prove that
$$a_{s-1} = a_{n-1},\ \ r_{s-1} = r_{n-1},\ \ b_{s-1} = b_{n-1}.$$
We have
$$r_{s-1} r_s = N - a_s^2 = N - a_n^2 = r_{n-1} r_n = r_{n-1} r_s;$$
$$\therefore\ \ r_{s-1} = r_{n-1}.$$
Again,
$$a_{n-1} + a_n = b_{n-1} r_{n-1},\ \ a_{s-1} + a_s = b_{s-1} r_{s-1} = b_{s-1} r_{n-1};$$
$$\therefore\ \ a_{n-1} - a_{s-1} = r_{n-1}(b_{n-1} - b_{s-1});$$
$$\therefore\ \ \frac{a_{n-1} - a_{s-1}}{r_{n-1}} = b_{n-1} - b_{s-1} = \text{zero, or an integer.}$$

But, by Art. 360, $a_1 - a_{n-1} < r_{n-1}$, and $a_1 - a_{s-1} < r_{s-1}$; that is $a_1 - a_{s-1} < r_{n-1}$; therefore $a_{n-1} - a_{s-1} < r_{n-1}$; hence $\dfrac{a_{n-1} - a_{s-1}}{r_{n-1}}$ is less than unity, and therefore must be zero.

Thus $a_{s-1} = a_{n-1}$, and also $b_{s-1} = b_{n-1}$.

Hence if the $(n+1)^{\text{th}}$ complete quotient recurs, the n^{th} complete quotient must also recur; therefore the $(n-1)^{\text{th}}$ complete quotient must also recur; and so on.

This proof holds as long as n is not less than 2 [Art. 358], hence the complete quotients recur, beginning with the second quotient $\dfrac{\sqrt{N}+a_1}{r_1}$. It follows therefore that the recurrence begins with the second partial quotient b_1; we shall now shew that it terminates with a partial quotient $2a_1$.

Let $\dfrac{\sqrt{N}+a_n}{r_n}$ be the complete quotient which just precedes the second complete quotient $\dfrac{\sqrt{N}+a_1}{r_1}$ when it recurs; then $\dfrac{\sqrt{N}+a_n}{r_n}$ and $\dfrac{\sqrt{N}+a_1}{r_1}$ are two consecutive complete quotients; therefore

$$a_n + a_1 = r_n b_n, \quad r_n r_1 = N - a_1^2;$$

but $N - a_1^2 = r_1$; hence $r_n = 1$.

Again, $a_1 - a_n < r_n$, that is < 1; hence $a_1 - a_n = 0$, that is

$$a_n = a_1.$$

Also $a_n + a_1 = r_n b_n = b_n$; hence $b_n = 2a_1$; which establishes the proposition.

*362. *To shew that in any period the partial quotients equidistant from the beginning and end are equal, the last partial quotient being excluded.*

Let the last complete quotient be denoted by $\dfrac{\sqrt{N}+a_n}{r_n}$; then

$$r_n = 1, \quad a_n = a_1, \quad b_n = 2a_1.$$

We shall prove that

$$r_{n-1} = r_1, \quad a_{n-1} = a_2, \quad b_{n-1} = b_1;$$
$$r_{n-2} = r_2, \quad a_{n-2} = a_3, \quad b_{n-2} = b_2;$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

We have

$$r_{n-1} = r_n r_{n-1} = N - a_n^2 = N - a_1^2 = r_1.$$

Also

$$a_{n-1} + a_1 = a_{n-1} + a_n = r_{n-1} b_{n-1} = r_1 b_{n-1};$$

and

$$a_1 + a_2 = r_1 b_1;$$

$$\therefore \quad a_2 - a_{n-1} = r_1 (b_1 - b_{n-1});$$

$$\therefore \quad \frac{a_2 - a_{n-1}}{r_1} = b_1 - b_{n-1} = \text{zero, or an integer.}$$

But $\dfrac{a_2 - a_{n-1}}{r_1} < \dfrac{a_1 - a_{n-1}}{r_1}$, that is $< \dfrac{a_1 - a_{n-1}}{r_{n-1}}$, which is less than unity; thus $a_2 - a_{n-1} = 0$; hence $a_{n-1} = a_2$, and $b_{n-1} = b_1$.

Similarly $\quad r_{n-2} = r_2, \quad a_{n-2} = a_3, \quad b_{n-2} = b_2$; and so on.

*363. From the results of Arts. 361, 362, it appears that when a quadratic surd $\sqrt{N}$ is converted into a continued fraction, it must take the following form

$$a_1 + \frac{1}{b_1 +} \frac{1}{b_2 +} \frac{1}{b_3 +} \cdots\cdots \frac{1}{b_3 +} \frac{1}{b_2 +} \frac{1}{b_1 +} \frac{1}{2a_1 +} \cdots\cdots$$

*364. *To obtain the penultimate convergents of the recurring periods.*

Let n be the number of partial quotients in the recurring period; then the penultimate convergents of the recurring periods are the n^{th}, $2n^{\text{th}}$, $3n^{\text{th}}$,...... convergents; let these be denoted by

$$\frac{p_n}{q_n}, \quad \frac{p_{2n}}{q_{2n}}, \quad \frac{p_{3n}}{q_{3n}}, \ldots\ldots \text{ respectively.}$$

Now $\displaystyle \sqrt{N} = a_1 + \frac{1}{b_1 +} \frac{1}{b_2 +} \frac{1}{b_3 +} \cdots\cdots \frac{1}{b_{n-1} +} \frac{1}{2a_1 +} \cdots\cdots$

so that the partial quotient corresponding to $\dfrac{p_{n+1}}{q_{n+1}}$ is $2a_1$; hence

$$\frac{p_{n+1}}{q_{n+1}} = \frac{2a_1 p_n + p_{n-1}}{2a_1 q_n + q_{n-1}}.$$

The complete quotient at the same stage consists of the period

$$2a_1 + \frac{1}{b_1 +} \frac{1}{b_2 +} \cdots\cdots \frac{1}{b_{n-1} +} \cdots\cdots,$$

and is therefore equal to $a_1 + \sqrt{N}$; hence

$$\sqrt{N} = \frac{(a_1 + \sqrt{N})\, p_n + p_{n-1}}{(a_1 + \sqrt{N})\, q_n + q_{n-1}}.$$

Clearing of fractions and equating rational and irrational parts, we obtain

$$a_1 p_n + p_{n-1} = Nq_n, \quad a_1 q_n + q_{n-1} = p_n \quad \ldots\ldots\ldots\ldots (1).$$

Again $\dfrac{p_{2n}}{q_{2n}}$ can be obtained from $\dfrac{p_n}{q_n}$ and $\dfrac{p_{n-1}}{q_{n-1}}$ by taking for the quotient

$$2a_1 + \frac{1}{b_1 +}\,\frac{1}{b_2 +}\,\cdots\cdots\,\frac{1}{b_{n-1}},$$

which is equal to $a_1 + \dfrac{p_n}{q_n}$. Thus

$$\frac{p_{2n}}{q_{2n}} = \frac{\left(a_1 + \dfrac{p_n}{q_n}\right) p_n + p_{n-1}}{\left(a_1 + \dfrac{p_n}{q_n}\right) q_n + q_{n-1}} = \frac{Nq_n + \dfrac{p_n}{q_n} \cdot p_n}{p_n + \dfrac{p_n}{q_n} \cdot q_n}, \text{ from (1);}$$

$$\therefore \quad \frac{p_{2n}}{q_{2n}} = \frac{1}{2}\left(\frac{p_n}{q_n} + \frac{Nq_n}{p_n}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In like manner we may prove that if $\dfrac{p_{cn}}{q_{cn}}$ is the penultimate convergent in the c^{th} recurring period,

$$a_1 p_{cn} + p_{cn-1} = Nq_{cn}, \quad a_1 q_{cn} + q_{cn-1} = p_{cn},$$

and by using these equations, we may obtain $\dfrac{p_{3n}}{q_{3n}}, \dfrac{p_{4n}}{q_{4n}}, \ldots\ldots$ successively.

It should be noticed that equation (2) holds for all multiples of n; thus

$$\frac{p_{2cn}}{q_{2cn}} = \frac{1}{2}\left(\frac{p_{cn}}{q_{cn}} + \frac{Nq_{cn}}{p_{cn}}\right);$$

the proof being similar to that already given.

*365. In Art. 356, we have seen that a periodic continued fraction can be expressed as the root of a quadratic equation with rational coefficients.

Conversely, we might prove by the method of Art. 357 that an expression of the form $\frac{A+\sqrt{B}}{C}$, where A, B, C are positive integers, and B not a perfect square, can be converted into a recurring continued fraction. In this case the periodic part will not usually begin with the second partial quotient, nor will the last partial quotient be double the first.

For further information on the subject of recurring continued fractions we refer the student to Serret's *Cours d'Algèbre Supérieure*, and to a pamphlet on *The Expression of a Quadratic Surd as a Continued Fraction*, by Thomas Muir, M.A., F.R.S.E.

*EXAMPLES. XXVII. b.

Express the following surds as continued fractions, and find the fourth convergent to each:

1. $\sqrt{a^2+1}$. 2. $\sqrt{a^2-a}$. 3. $\sqrt{a^2-1}$.

4. $\sqrt{1+\frac{1}{a}}$. 5. $\sqrt{a^2+\frac{2a}{b}}$. 6. $\sqrt{a^2-\frac{a}{n}}$.

7. Prove that

$$\sqrt{9a^2+3}=3a+\frac{1}{2a+}\,\frac{1}{6a+}\,\frac{1}{2a+}\,\frac{1}{6a+}\,\dots\dots,$$

and find the fifth convergent.

8. Shew that

$$p+\frac{2}{1+}\,\frac{1}{p+}\,\frac{1}{1+}\,\frac{1}{p+}\,\frac{1}{1+}\,\dots\dots=\sqrt{p^2+4p}.$$

9. Shew that

$$p\left(a_1+\frac{1}{pqa_2+}\,\frac{1}{a_3+}\,\frac{1}{pqa_4+}\,\dots\dots\right)=pa_1+\frac{1}{qa_2+}\,\frac{1}{pa_3+}\,\frac{1}{qa_4+}\,\dots\dots$$

10. If $\sqrt{a^2+1}$ be expressed as a continued fraction, shew that

$$2(a^2+1)\,q_n=p_{n-1}+p_{n+1}, \quad 2p_n=q_{n-1}+q_{n+1}.$$

11. If

$$x=\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_1+}\,\frac{1}{a_2+}\,\dots,$$

$$y=\frac{1}{2a_1+}\,\frac{1}{2a_2+}\,\frac{1}{2a_1+}\,\frac{1}{2a_2+}\,\dots,$$

$$z=\frac{1}{3a_1+}\,\frac{1}{3a_2+}\,\frac{1}{3a_1+}\,\frac{1}{3a_2+}\,\dots,$$

shew that $\quad x(y^2-z^2)+2y(z^2-x^2)+3z(x^2-y^2)=0.$

12. Prove that

$$\left(a+\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\cdots\right)\left(\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\cdots\right)=\frac{a}{b}.$$

13. If
$$x=a+\frac{1}{b+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{a+}\cdots,$$

$$y=b+\frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{b+}\cdots,$$

shew that
$$(ab^2+a+b)\,x-(a^2b+a+b)\,y=a^2-b^2.$$

14. If $\dfrac{p_n}{q_n}$ be the n^{th} convergent to $\sqrt{a^2+1}$, shew that

$$\frac{p_2{}^2+p_3{}^2+\ldots+p^2{}_{n+1}}{q_2{}^2+q_3{}^2+\ldots+q^2{}_{n+1}}=\frac{p_{n+1}p_{n+2}-p_1p_2}{q_{n+1}q_{n+2}-q_1q_2}.$$

15. Shew that

$$\left(\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\cdots\right)\left(c+\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{c+}\cdots\right)=\frac{1+bc}{1+ab}.$$

16. If $\dfrac{p_r}{q_r}$ denote the r^{th} convergent to $\dfrac{\sqrt{5}+1}{2}$, shew that

$$p_3+p_5+\ldots+p_{2n-1}=p_{2n}-p_2,\quad q_3+q_5+\ldots+q_{2n-1}=q_{2n}-q_2.$$

17. Prove that the difference of the infinite continued fractions

$$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\cdots,\qquad \frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{c+}\cdots,$$

is equal to $\dfrac{a-b}{1+ab}$.

18. If $\sqrt{N}$ is converted into a continued fraction, and if n is the number of quotients in the period, shew that

$$q_{2n}=2p_nq_n,\qquad p_{2n}=2p_n{}^2+(-1)^{n+1}.$$

19. If $\sqrt{N}$ be converted into a continued fraction, and if the penultimate convergents in the first, second, ...k^{th} recurring periods be denoted by n_1, n_2,...n_k respectively, shew that

$$\frac{n_k+\sqrt{N}}{n_k-\sqrt{N}}=\left(\frac{n_1+\sqrt{N}}{n_1-\sqrt{N}}\right)^k.$$