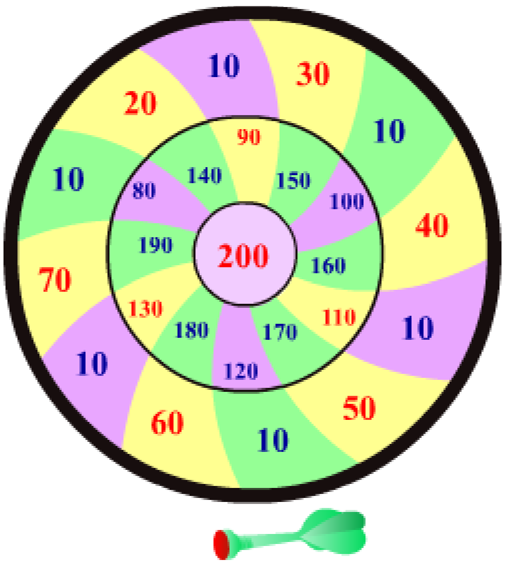
Probability

Observing an Experiment

It is not always possible to tell the exact outcome of a particular action. Take, for example, a dart board.



A dart is repeatedly thrown toward the dartboard, targeting a random number in each throw. We do not know which number is targeted in a particular throw. What we do know is that there is a fixed group of numbers and each time the targeted number is one of them.

We know that the likelihood of occurrence of an unpredictable event is studied under the theory of probability. So, we can say that there is a certain probability for each number to be targeted in the above experiment.

Let us learn more about probability and the meanings of terms associated with it, for example, ‘experiment’ and ‘outcome’.

Did You Know?

The word ‘probability’ has evolved from the Latin word ‘probabilitas’, which can be considered to have the same meaning as the word ‘probity’. In olden days in Europe, ‘probity’ was a measure of authority of a witness in a legal case, and it often correlated with the nobility of the witness.

The modern meaning of probability, however, focuses on the statistical observation of the likelihood of occurrence of an event.

Know More

Probability is widely applicable in daily life and in researches pertaining to different fields. It is an important factor in the diverse worlds of share market, philosophy, artificial intelligence or machine learning, statistics, etc. All gambling is based on probability. In gambling, one considers all possibilities and then tries to predict a result that is most likely to happen. The concept of probability is perhaps the most interesting topic to discuss in mathematics.

Terms Related to Probability

Experiment: When an operation is planned and done under controlled conditions, it is known as an experiment. For example, tossing a coin, throwing a die, drawing a card from a pack of playing cards without seeing, etc., are all experiments. A chance experiment is one in which the result is unknown or not predetermined.

Outcomes: Different results obtained in an experiment are known as outcomes. For example, on tossing a coin, if the result is a head, then the outcome is a head; if the result is a tail, then the outcome is a tail.

Random: An experiment is random if it is done without any conscious decision. For example, drawing a card from a well-shuffled pack of playing cards is a random experiment if it is done without seeing the card or figuring it out by touching.

Trial: A trial is an action or an experiment that results in one or several outcomes. For example, if a coin is tossed five times, then each toss of the coin is called a trial.

Sample space: The set of all possible outcomes of an experiment is called the sample space. It is denoted by the English letter 'S' or Greek letter 'Ω' (omega). In the experiment of tossing a coin, there are only two possible outcomes—a head (H) and a tail (T).

∴ Sample space (S) = {H, T}

Event: The event of an experiment is one or more outcomes of the experiment. For example, tossing a coin and getting a head or a tail is an event. Throwing a die and getting a face marked with an odd number (i.e., 1, 3 or 5) or an even number (2, 4 or 6) is also an event.

Know More

Initially, the word 'probable' meant the same as the word 'approvable' and was used in the same sense to support or approve of opinions and actions. Any action described as 'probable' was considered the most likely and sensible action to be taken by a rational and sensible person.

Whiz Kid

Equally Likely: If each outcome of an experiment has the same probability of occurring, then the outcomes are said to be equally likely outcomes.

Know Your Scientist

Girolamo Cardano (1501−1576) was a great Italian mathematician, physicist, astrologer and gambler. His interest in gambling led him to do more research on the concept of probability and formulate its rules.

He was often short of money and kept himself solvent through his gambling skills. He was also a very good chess player. He wrote a book named *Liber de Ludo Aleae*. In this book about games of chance, he propounded the basic concepts of probability.

Solved Examples

Easy

Example 1:

A fair die is thrown. What is the sample space of this experiment?

Solution:

When a die is thrown, we can have six outcomes, namely, 1, 2, 3, 4, 5 and 6.

We know that sample space is the collection of all possible outcomes of an experiment.

∴ Sample space (S) = {1, 2, 3, 4, 5, 6}

Example 2:

Which of the following are experiments?

i)Tossing a coin

ii)Rolling a six-sided die

iii)Getting a head on a tossed coin

Solution:

Tossing a coin and rolling a six-sided die are experiments, while getting a head on a tossed coin is the outcome of an experiment.

Medium

Example 1:

What is the sample space when two coins are tossed together?

Solution:

When two coins are tossed together, we can get four possible outcomes. These are as follows:

i)A head (H) on one coin and a tail (T) on the other

ii)A head (H) on one coin and a head (H) on the other

iii)A tail (T) on one coin and a head (H) on the other

iv)A tail (T) on one coin and a tail (T) on the other

∴ Sample space (S) = {HT, HH, TH, TT}

Equally Likely Outcomes

There are many situations where on a particular day, you take a chance and the things do not go the way you want. However on the other days, they do.

For example, suppose Prachi takes her umbrella everyday to her office. However, on one day she forgot to take the umbrella and it rained that day.

Sometimes it happens that you leave home just 10 minutes before the school timings and still manage to reach at time. Whereas the other day, when you left home 30 minutes earlier, still you could not reach at time because of a heavy traffic jam.

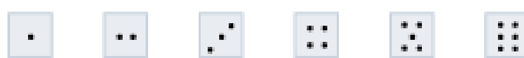
In these kinds of examples, chances of a certain thing occurring and not occurring are not equal. But there are also some cases where there are equal chances of an event to occur or not to occur.

For example, suppose you play a game with your friend where you toss a coin to decide who will play first. When you toss a coin, you can either get a tail or a head. There is no other possibility.

Also, when you toss a coin, you cannot always get what you want out of Head or Tail. **There are equal chances of getting a head or a tail.** Such an experiment is called a **random experiment.**

The results of an experiment are called **outcomes** of the experiment. Here, when you toss a coin, head or tail are the only two outcomes of this experiment.

Consider another example. Suppose you throw a dice while playing a game. There are 6 possible outcomes (1, 2, 3, 4, 5, or 6). There is no other possibility. Moreover, the chance of getting any of these outcomes is the same.



Let us note the results we obtain, when we throw a dice, once, twice, thrice, and so on. We will observe that as the number of throws increases, the chances of getting each of 1, 2 … 6 come closer and closer to one another.

That is the numbers of each of the six outcomes become almost equal to each other. In this case, we may say that the different outcomes of the experiment are **equally likely,** i.e. each of the outcomes has the same chance of occurring.

Let us now look at an example.

Example 1:

Which of the following experiments results in equally likely outcomes?

1. **The school bus of Archit comes daily on time but the day he reaches early, the bus comes late.**

2. **Tossing of a coin 10 times**

Solution:

1. The chances of the bus to come on time or not on time are not equal. Thus, this experiment does not result in equally likely outcomes.

2. The experiment of tossing a coin 10 times will result in equally likely outcomes, since there are equal chances of getting a head or a tail.

Probability Of Events

Suppose Shashank throws a dice. There are six different outcomes. The outcomes of an experiment or a collection of outcomes makes an **event.**

In this example, getting the number 1 on the top face of the dice is an event. Similarly, getting the other numbers (2, 3, 4, 5, or 6) are also known as events.

Can we tell what will be the probability of getting 2 on the top face of the dice?

There are six possible outcomes and all are equally likely to occur. **The probability is the ratio of getting an outcome to the total number of outcomes.**

Probability of an event

$$= \frac{\textit{Number of outcomes that make an event}}{\textit{Total number of outcomes of the experiment}}$$

Mathematically,

$$P(A) = \frac{n(A)}{n(S)}$$

Here,

P(*A*) represents the probability of an event *A*.

n(*S*) represents the total number of outcomes or number of elements in sample space.

n(*A*) represents the number of outcomes that make event A or the number of elements in set *A*.

It should be noted that sample space S is the universal set here, so all elements of set A belong to set S i.e, $A \subseteq S$.

Now, let us find the solution of above discussed case.

In the above example, the total number of outcomes are 6.

∴ Probability of getting a number 2 on the top face = 1/6

What is the probability of getting 6 on the top face of the dice?

Since all the outcomes are equally likely to occur,

∴ Probability of getting a number 6 on the top face = 1/6

Similarly, for other numbers (1, 3, 4, and 5) as well, the probability of showing up on the top face is 1/6.

This is how we can find out the probability of the occurrence of an outcome in an experiment.

Can we calculate the probability of the occurrence of a multiple of 3 on the top face of the dice?

Yes, we can.

Consider the multiples of 3 out of six possible outcomes. The multiples of 3 are 3 and 6 out of six possible outcomes.

Thus, probability of getting a multiple of 3

$$= \frac{\text{Number of multiples of 3}}{\text{Total number of possible outcomes}} = \frac{2}{6} = \frac{1}{3} \quad \frac{\text{Multiples of 3}}{\text{Total number of possible outcomes}} = \frac{2}{6} = \frac{1}{3}$$

Properties of probability:

Property 1: Probability of a certain event is 1 and probability of an impossible event is 0.

Probability of an impossible event is denoted by P(ϕ) and probability of a certain event is denoted by P(S).

Therefore, $P(\phi) = 0$ and $P(S) = 1$.

Proof:

We have

$$P(A) = \frac{n(A)}{n(S)}$$

Impossible event means that there is no way in which the event can occur. So, number of outcomes making event A will be 0 or set A will be empty set i.e., ϕ.

$$\therefore P(\phi) = \frac{n(\phi)}{n(S)} = \frac{0}{n(S)} = 0$$

Certain event means that there is only one possibility. So, the number of outcomes in sample space S as well as in set A will be equal i.e., 1.

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{n(S)}{n(S)} = 1$$

Hence proved.

Let us consider few examples based on this property.

Consider the events **"son being older than his father", "Saturday comes before Friday in a week"** and **"taking water from an empty mug".** All these are impossible events and thus, the probability of occurrence of each of these events is 0.

Now, consider the events **"Sun is larger than earth**" and **"Sunday comes before Monday in a week".** Both of these are certain events and thus, the probability of occurrence each of these events is 1.

Property 2: If the sample space S is a finite set and A is an event of S then probability of event A lies between 0 and 1 both inclusive.

Therefore, $0 \le P(A) \le 1$.

Proof:

Since $A \subseteq S$, we have

$\phi \subseteq A \subseteq S$

$\Rightarrow n(\phi) \le n(A) \le n(S)$

$\Rightarrow 0 \le n(A) \le n(S)$ $[n(\phi) = 0]$

On dividing the inequality by $n(S)$, we get

$$\frac{0}{n(S)} \le \frac{n(A)}{n(S)} \le \frac{n(S)}{n(S)}$$

$\Rightarrow 0 \le P(A) \le 1$

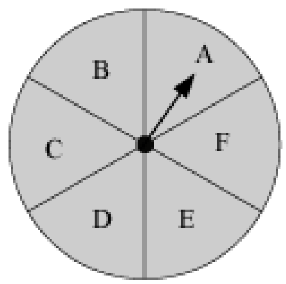
Hence proved.

These are very important properties related to probability which prove to be very helpful at times.

Now, let us have a look at some examples.

Example 1:

The given figure shows a wheel in which six English alphabets are written in six equal sectors of the wheel. Suppose we spin the wheel. What is the possibility of the pointer stopping in the sector containing alphabet A?



Solution:

The total number of possible outcomes is 6. The pointer can stop at six different sectors (A, B, C, D, E, F).

Thus, probability of pointer stopping in the sector containing A = 1/6

Example 2:

A bag has 6 blue and 4 red balls. A ball is drawn from the bag without looking into the bag.

1. **What is the probability of getting a blue ball?**
2. **What is the probability of getting a red ball?**

Solution:

In a bag, there are 6 blue and 4 red balls.

$\therefore$ Total number of outcomes = 6 + 4 = 10

1. Getting a blue ball consists of 6 outcomes, since there are 6 blue balls.

Probability of getting a blue ball $= \frac{6}{10} = \frac{3}{5}$

2. Getting a red ball consists of 4 outcomes, since there are 4 red balls.

   Probability of getting a red ball $= \frac{4}{10} = \frac{2}{5}$

Example 3:

When a dice is thrown, what is the probability of getting

(a) A prime number

(b) An even number

(c) An odd number

(d) A number less than or equal to 2

(e) A number more than or equal to 4

Solution:

When a dice is thrown, the total number of outcomes is 6.

(a) The prime numbers out of six possible outcomes are 2, 3, and 5. Thus, getting a prime number consists of 3 outcomes.

∴ Probability of getting a prime number $= \frac{3}{6} = \frac{1}{2}$

(b) Out of the possible outcomes, the even numbers are 2, 4, and 6. Thus, the number of outcomes of getting an even numbers is 3.

∴ Probability of getting an even number $= \frac{3}{6} = \frac{1}{2}$

(c) The odd numbers are 1, 3, and 5. Thus, the number of outcomes of getting an odd number is 3.

∴ Probability of getting an odd number $= \frac{3}{6} = \frac{1}{2}$

(d) The numbers less than or equal to 2 are 1 and 2. Thus, there are 2 possible outcomes.

∴ Required probability $= \frac{2}{6} = \frac{1}{3}$

(e) The numbers more than or equal to 4 are 4, 5, and 6. Thus, there are 3 possible outcomes.

∴ Required probability $= \frac{3}{6} = \frac{1}{2}$

Theoretical and Experimental Probability

Consider an experiment of tossing a coin. Before tossing a coin, we are not sure whether head or tail will come up. To measure this uncertainty, we will find the probability of getting a head and the probability of getting a tail.

A student tosses a coin 1000 times out of which 520 times head comes up and 480 times tail comes up.

The probability of getting a head is the ratio of the number of times head comes up to the total number of times he tosses the coin.

Probability of getting a head $= \frac{520}{1000}$

= 0.52

Similarly, probability of getting a tail $= \frac{480}{1000}$

= 0.48

These are the probabilities obtained from the result of an experiment when we actually perform the experiment. The probabilities that we found above are called **experimental (or empirical) probabilities.**

On the other hand, the probability we find through the theoretical approach without actually performing the experiment is called theoretical probability.

The **theoretical probability (or classical probability)** of an event E, is denoted by P(E) and is defined as

$$P(E) = \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes of the experiment}}$$

Here, we assume that the outcomes of the experiment are equally likely.

When a coin is tossed, there are two possible outcomes. We can either get a head or a tail and these two outcomes are equally likely. The chance of getting a head or a tail is 1.

Thus, probability of getting a head P(E)

$$= \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes of the experiment}} = \frac{1}{2}$$

Similarly, probability of getting a tail = 1/2

Here, 1/2 (or 0.5) is the theoretical probability.

Relation between Experimental and Theoretical Probabilities:

There is a fact that the experimental probability may or may not be equal to the theoretical probability.

For example, if we take a coin and toss it by a particular number of times then the theoretical probability of getting a head or a tail will be $\frac{1}{2} = 0.5$ in each trial, but if we observe the outcomes of all the trials and calculate the experimental probability for head or tail then it will not be exactly equal to theoretical probability.

For approval, we can consider the theoretical and experimental probabilities of getting tail in the above experiment.

We have:
Theoretical probability of getting head = 0.52
Also, in our experiment we obtained:

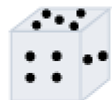
Experimental probability of getting tail = 0.5

It can be observed that experimental probability is not exactly equal to theoretical probability, but very close to it.

Also, experimental probability of the same event can vary according to the number of trials.

Finding Probability Using Complement of a Known Event

Consider the experiment of throwing a dice. Any of the numbers 1, 2, 3, 4, 5, or 6 can come up on the upper face of the dice. We can easily find the probability of getting a number 5 on the upper face of the dice?



Mathematically, probability of any event E can be defined as follows.

$$P(E) = \frac{\text{Number of outcomes favourable to event } E}{\text{Number of all possible outcomes}} = \frac{n(E)}{n(S)}$$

Here, S represents the sample space and $n(S)$ represents the number of outcomes in the sample space.

For this experiment, we have

Sample space (S) = {1, 2, 3, 4, 5, 6}. Thus, S is a finite set.

So, we can say that the possible outcomes of this experiment are 1, 2, 3, 4, 5, and 6.

$\therefore$ Number of all possible outcomes = 6

Number of favourable outcomes of getting the number 5 = 1

$\therefore$ Probability (getting 5) = 1/6

Similarly, we can find the probability of getting other numbers also.

P (getting 1) = 1/6, P (getting 2) = 1/6, P (getting 3) = 1/6, P (getting 4) = 1/6 and

P (getting 6) = 1/6

Let us add the probability of each separate observation.

This will give us the sum of the probabilities of all possible outcomes.

P (getting 1) + *P* (getting 2) + *P* (getting 3) + *P* (getting 4) + *P* (getting 5) + *P* (getting 6)

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$

∴ "**Sum of the probabilities of all elementary events is 1".**

Now, let us find the probability of **not** getting 5 on the upper face.

The outcomes favourable to this event are 1, 2, 3, 4, and 6.

∴ Number of favourable outcomes = 5

∴ P (not getting 5) = 5/6

We can also see that *P* (getting 5) + *P* (not getting 5) $= \frac{1}{6} + \frac{5}{6} = 1$

∴ "**Sum of probabilities of occurrence and non occurrence of an event is 1".**

i.e. **If *E* is the event, then *P* (*E*) + *P* (not *E*) = 1** … (1)

or we can write ***P(E) = 1 − P* (not *E*)**

Here, the events of getting a number 5 and not getting 5 are complements of each other as we cannot find an observation which is common to the two observations.

Thus, event **not *E*** is the complement of event *E.* Complement of event *E* is denoted by $\bar{E}$ or *E***'**.

Using equation (1), we can write

$$P(E) + P(\bar{E}) = 1$$

or

$$P(\bar{E}) = 1 - P(E)$$

This is a very important property about the probability of complement of an event and it is stated as follows:

If *E* is an event of finite sample space S, then $P(\overline{E}) = 1 - P(E)$ where $\overline{E}$ is the complement of event *E*.

Now, let us prove this property algebraically.

Proof:

We have,

$E \cup \overline{E} = S$ and $E \cap \overline{E} = \phi$

$\Rightarrow n(E \cup \overline{E}) = n(S)$ and $n(E \cap \overline{E}) = n(\phi)$

$\Rightarrow n(E \cup \overline{E}) = n(S)$ and $n(E \cap \overline{E}) = 0$ ...(1)

Now,

$n(E \cup \overline{E}) = n(S)$

$\Rightarrow n(E) + n(\overline{E}) - n(E \cap \overline{E}) = n(S)$

$\Rightarrow n(E) + n(\overline{E}) - 0 = n(S)$ [Using (1)]

$\Rightarrow n(\overline{E}) = n(S) - n(E)$

On dividing both sides by $n(S)$, we get

$$\frac{n(\overline{E})}{n(S)} = \frac{n(S)}{n(S)} - \frac{n(E)}{n(S)}$$

$\Rightarrow P(\overline{E}) = 1 - P(E)$

Hence proved.

Let us solve some examples based on this concept.

Example 1:

One card is drawn from a well shuffled deck. What is the probability that the card will be

(i) a king?

(ii) not a king?

Solution:

Let *E* be the event 'the card is a king' and *F* be the event 'the card is not a king'.

(i) Since there are 4 kings in a deck.

∴ Number of outcomes favourable to *E* = 4

Number of possible outcomes = 52

∴ $P(E) = \frac{4}{52} = \frac{1}{13}$

2. Here, the events E and F are complements of each other.

∴ $P(E) + P(F) = 1$

$P(F) = 1 - \frac{1}{13}$

$= \frac{12}{13}$

Example 2:

If the probability of an event A is 0.12 and B is 0.88 and they belong to the same set of observations, then show that A and B are complementary events.

Solution:

It is given that $P(A) = 0.12$ and $P(B) = 0.88$

Now, P(A) + P(B) = 0.12 + 0.88 = 1

∴ The events A and B are complementary events.

Example 3:

Savita and Babita are playing badminton. The probability of Savita winning the match is 0.52. What is the probability of Babita winning the match?

Solution:

Let E be the event 'Savita winning the match' and F be the event 'Babita wining the match'.

It is given that $P(E) = 0.52$

Here, E and F are complementary events because if Babita wins the match, Savita will surely lose the match and vice versa.

$\therefore P(E) + P(F) = 1$

$0.52 + P(F) = 1$

$P(F) = 1 - 0.52 = 0.48$

Thus, the probability of Babita winning the match is 0.48.

Example 4:

In a box, there are 2 red, 5 blue, and 7 black marbles. One marble is drawn from the box at random. What is the probability that the marble drawn will be (i) red (ii) blue (iii) black (iv) not blue?

Solution:

Since the marble is drawn at random, all the marbles are equally likely to be drawn.

Total number of marbles = 2 + 5 + 7 = 14

Let A be the event 'the marble is red', B be the event 'the marble is blue' and C be the event 'the marble is black.

(i) Number of outcomes favourable to event A = 2

$\therefore\ P(A) = \frac{2}{14} = \frac{1}{7}$

(ii) Number of outcomes favourable to event B = 5

$\therefore P(B) = \frac{5}{14}$

(iii) Number of outcomes favourable to event C = 7

$\therefore\ P(C) = \frac{7}{14} = \frac{1}{2}$

(iv) We have, $P(B) = \frac{5}{14}$

The event of drawing a marble which is not blue is the complement of event B.

$\therefore\ P(\overline{B}) = 1 - P(B) = 1 - \frac{5}{14} = \frac{9}{14}$

Thus, the probability of drawing a marble which is not blue is $\frac{9}{14}$.