Chapter 6

Vector Algebra

Contents

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The word vector is derived from Latin word meaning "to carry". The subject vector analysis was developed in the later part of the 19th century by the American Physicist and mathematician Josiah Willard Gibbs (1839-1903 A.D.) and the English enginner Oliver Heaviside (1850-1925 A.D.) workign independently. Many of the ideas came earlier, however, especially from Irish mathematician William Rowen Hamilton (1805-1865 A.D.) Scottish physicist James Clerk Maxwell (1831-1879 A.D) and H.G. Grassmann (1809-1877 A.D.). It was Hamiliton who introduced scalar and vector terms In 1844, Grassman published his work in lineale ausdehnungslehre and in 1883 appeared Hamilton's Lectures on Quatermions. Hamilton's method of quaternions was a solution to the problem of multiplying vectors in three dimensional space. Maxwell used some of Hamilton's ideas in his study of electro-magnetic theory.

Vector Algebra

6.1 Introduction

Vectors represent one of the most important mathematical systems, which is used to handle certain types of problems in Geometry, Mechanics and other branches of Applied Mathematics, Physics and Engineering.

Scalar and vector quantities : Physical quantities are divided into two categories – scalar quantities and vector quantities. Those quantities which have only magnitude and which are not related to any fixed direction in space are called *scalar quantities*, or briefly scalars. Examples of scalars are mass, volume, density, work, temperature etc.

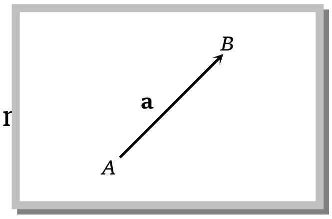
A scalar quantity is represented by a real number along with a suitable unit.

Second kind of quantities are those which have both magnitude and direction. Such quantities are called vectors. Displacement, velocity, acceleration, momentum, weight, force etc. are examples of vector quantities.

6.2 Representation of Vectors

Geometrically a vector is represented by a line segment. For example, $\mathbf{a} = \overrightarrow{AB}$. Here A is called the initial point and B, the terminal point or tip.

Magnitude or modulus of $\mathbf{a}$ is expressed as $|\mathbf{a}| = |\overrightarrow{AB}| = AB$.



Note : ❒ The magnitude of a vector is always a non-negative real r

❒ Every vector $\overrightarrow{AB}$ has the following three characteristics:

Length : The length of $\overrightarrow{AB}$ will be denoted by $|\overrightarrow{AB}|$ or AB.

Support : The line of unlimited length of which AB is a segment is called the support of the vector $\overrightarrow{AB}$.

Sense : The sense of $\overrightarrow{AB}$ is from A to B and that of $\overrightarrow{BA}$ is from B to A. Thus, the sense of a directed line segment is from its initial point to the terminal point.

6.3 Types of Vector

(1) **Zero or null vector :** A vector whose magnitude is zero is called zero or null vector and it is represented by $\vec{O}$.

The initial and terminal points of the directed line segment representing zero vector are coincident and its direction is arbitrary.

(2) **Unit vector :** A vector whose modulus is unity, is called a unit vector. The unit vector in the direction of a vector $\mathbf{a}$ is denoted by $\hat{\mathbf{a}}$, read as "*a cap*". Thus, $|\hat{\mathbf{a}}| = 1$.

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\text{Vector } a}{\text{Magnitude of } a}$$

Note : ❒ Unit vectors parallel to x-axis, y-axis and z-axis are denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively.

❒ Two unit vectors may not be equal unless they have the same direction.

(3) **Like and unlike vectors :** Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions.

(4) **Collinear or parallel vectors :** Vectors having the same or parallel supports are called collinear vectors.

(5) **Co-initial vectors :** Vectors having the same initial point are called *co-initial vectors*.

(6) **Co-planar vectors :** A system of vectors is said to be coplanar, if their supports are parallel to the same plane.

Note : ❒ Two vectors having the same initial point are always coplanar but such three or more vectors may or may not be coplanar.

(7) **Coterminous vectors :** Vectors having the same terminal point are called *coterminous vectors*.

(8) **Negative of a vector :** The vector which has the same magnitude as the vector $\mathbf{a}$ but opposite direction, is called the negative of $\mathbf{a}$ and is denoted by $-\mathbf{a}$. Thus, if $\overrightarrow{PQ}=\mathbf{a}$, then $\overrightarrow{QP}=-\mathbf{a}$.

(9) **Reciprocal of a vector :** A vector having the same direction as that of a given vector $\mathbf{a}$ but magnitude equal to the reciprocal of the given vector is known as the reciprocal of $\mathbf{a}$ and is denoted by $\mathbf{a}^{-1}$. Thus, if $|\mathbf{a}|=a, |\mathbf{a}^{-1}|=\frac{1}{a}$

Note : ❒ A unit vector is self reciprocal.

(10) **Localized and free vectors :** A vector which is drawn parallel to a given vector through a specified point in space is called a localized vector. For example, a force acting on a rigid body is a localized vector as its effect depends on the line of action of the force. If the value of a vector depends only on its length and direction and is independent of its position in the space, it is called a free vector.

(11) **Position vectors :** The vector $\overrightarrow{OA}$ which represents the position of the point A with respect to a fixed point O (called origin) is called position vector of the point A. If (x, y, z) are co-ordinates of the point A, then $\overrightarrow{OA}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

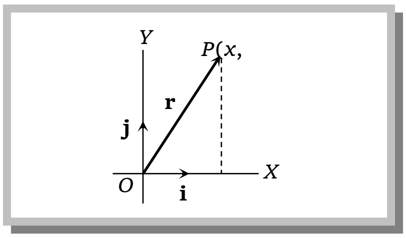
(12) **Equality of vectors :** Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be equal, if

(i) $|\mathbf{a}|=|\mathbf{b}|$ (ii) They have the same or parallel support and (iii) The same sense.

6.4 Rectangular resolution of a Vector in Two and Three dimensional systems

(1) Any vector $\mathbf{r}$ can be expressed as a linear combination of two unit vectors $\mathbf{i}$ and $\mathbf{j}$ at right angle *i.e.*, $\mathbf{r}=x\mathbf{i}+y\mathbf{j}$

The vector $x\mathbf{i}$ and $y\mathbf{j}$ are called the perpendicular component vectors of $\mathbf{r}$. The scalars x and y are called the components or resolved parts of $\mathbf{r}$ in the directions of x-axis and y-axis



respectively and the ordered pair (*x*, *y*) is known as co-ordinates of point whose position vector is $\mathbf{r}$.

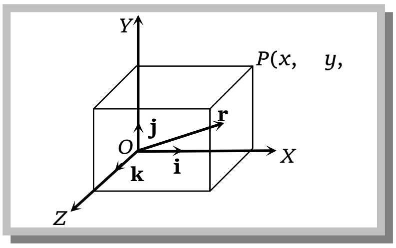
Also the magnitude of $\mathbf{r} = \sqrt{x^2 + y^2}$ and if θ be the inclination of $\mathbf{r}$ with the *x*-axis, then $\theta = \tan^{-1}(y/x)$

(2) If the coordinates of *P* are (*x, y, z*) then the position vector of $\mathbf{r}$ can be written as $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

The vectors $x\mathbf{i}, y\mathbf{j}$ and $z\mathbf{k}$ are called the right angled components of $\mathbf{r}$.

The scalars x, y, z are called the components or resolved parts of $\mathbf{r}$ in the directions of *x*-axis, *y*-axis and *z*-axis respectively and ordered triplet (*x, y, z*) is known as coordinates of *P* whose position vector is $\mathbf{r}$.

Also the magnitude or modulus of $\mathbf{r} = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$



Direction cosines of $\mathbf{r}$ are the cosines of angles that the vector $\mathbf{r}$ makes with the positive direction of *x, y* and *z*-axes.

$\cos\alpha = l = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{x}{|\mathbf{r}|}$, $\cos\beta = m = \dfrac{y}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{y}{|\mathbf{r}|}$ and

$\cos\gamma = n = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{z}{|\mathbf{r}|}$

Clearly, $l^2 + m^2 + n^2 = 1$. Here $\alpha = \angle POX$, $\beta = \angle POY$ $\gamma = \angle POZ$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors along OX, OY, OZ respectively.

Example: 1 If $\mathbf{a}$ is a non-zero vector of modulus *a* and *m* is a non-zero scalar, then $m\mathbf{a}$ is a unit vector if

(a) $m = \pm 1$ (b) $m = |\mathbf{a}|$ (c) $m = \dfrac{1}{|\mathbf{a}|}$ (d) $m = \pm 2$

Solution: (c) As $m\mathbf{a}$ is a unit vector, $|m\mathbf{a}| = 1 \Rightarrow |m||\mathbf{a}| = 1 \Rightarrow |m| = \dfrac{1}{|\mathbf{a}|} \Rightarrow m = \pm\dfrac{1}{|\mathbf{a}|}$

Example: 2 For a non-zero vector $\mathbf{a}$, the set of real numbers, satisfying $|(5 - x)\mathbf{a}| < |2\mathbf{a}|$ consists of all *x* such that

(a) $0 < x < 3$ (b) $3 < x < 7$ (c) $-7 < x < -3$ (d) $-7 < x < 3$

Solution: (b) We have, $|(5 - x)\mathbf{a}| < |2\mathbf{a}|$

$\Rightarrow |5 - x||\mathbf{a}| < 2|\mathbf{a}| \Rightarrow |5 - x| < 2 \Rightarrow -2 < 5 - x < 2 \Rightarrow 3 < x < 7$.

Example: 3 The direction cosines of the vector $3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ are

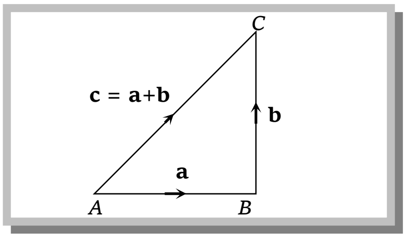
(a) $\dfrac{3}{5}, \dfrac{-4}{5}, \dfrac{1}{5}$ (b) $\dfrac{3}{5\sqrt{2}}, \dfrac{-4}{5\sqrt{2}}, \dfrac{1}{\sqrt{2}}$ (c) $\dfrac{3}{\sqrt{2}}, \dfrac{-4}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}$ (d) $\dfrac{3}{5\sqrt{2}}, \dfrac{4}{5\sqrt{2}}, \dfrac{1}{\sqrt{2}}$

Solution: (b) $\mathbf{r} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$; $|\mathbf{r}| = \sqrt{3^2 + (-4)^2 + 5^2} = 5\sqrt{2}$

Hence, direction cosines are $\dfrac{3}{5\sqrt{2}}, \dfrac{-4}{5\sqrt{2}}, \dfrac{5}{5\sqrt{2}}$ *i.e.,* $\dfrac{3}{5\sqrt{2}}, \dfrac{-4}{5\sqrt{2}}, \dfrac{1}{\sqrt{2}}$.

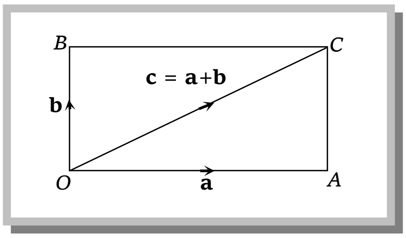
6.5 Properties of Vectors

(1) **Addition of vectors**



(i) **Triangle law of addition :** If two vectors are represented by two consecutive sides of a triangle then their sum is represented by the third side of the triangle, but in opposite direction. This is known as the triangle law of addition of vectors. Thus, if $\overrightarrow{AB} = \mathbf{a}, \overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$ then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ *i.e.,* $\mathbf{a} + \mathbf{b} = \mathbf{c}$.

(ii) **Parallelogram law of addition :** If two vectors are represented by two adjacent sides of a parallelogram, then their sum is represented by the diagonal of the parallelogram whose initial point is the same as the initial point of the given vectors. This is known as parallelogram law of addition of vectors.



Thus, if $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$

Then $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ *i.e.,* $\mathbf{a} + \mathbf{b} = \mathbf{c}$, where *OC* is a diagonal of the parallelogram *OABC*.

(iii) **Addition in component form :** If the vectors are defined in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, *i.e.,* if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then their sum is defined as $\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$.

Properties of vector addition : Vector addition has the following properties.

(a) **Binary operation :** The sum of two vectors is always a vector.

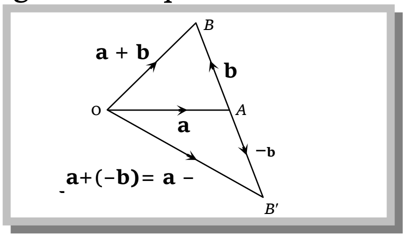
(b) **Commutativity :** For any two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

(c) **Associativity :** For any three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

(d) **Identity :** Zero vector is the identity for addition. For any vector $\mathbf{a}$, $\mathbf{0} + \mathbf{a} = \mathbf{a} = \mathbf{a} + \mathbf{0}$

(e) **Additive inverse :** For every vector $\mathbf{a}$ its negative vector $-\mathbf{a}$ exists such that $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$ *i.e.,* $(-\mathbf{a})$ is the additive inverse of the vector **a**.

(2) **Subtraction of vectors :** If $\mathbf{a}$ and $\mathbf{b}$ are two vectors, then their subtraction $\mathbf{a} - \mathbf{b}$ is defined as $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$ where $-\mathbf{b}$ is the negative of $\mathbf{b}$ having magnitude equal to that of $\mathbf{b}$ and direction opposite to $\mathbf{b}$.



If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Then $\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} + (a_3 - b_3)\mathbf{k}$.

Properties of vector subtraction

(i) $\mathbf{a} - \mathbf{b} \neq \mathbf{b} - \mathbf{a}$

(ii) $(\mathbf{a} - \mathbf{b}) - \mathbf{c} \neq \mathbf{a} - (\mathbf{b} - \mathbf{c})$

(iii) Since any one side of a triangle is less than the sum and greater than the difference of the other two sides, so for any two vectors *a* and *b*, we have

(a) $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$

(b) $|\mathbf{a} + \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$

(c) $|\mathbf{a} - \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$

(d) $|\mathbf{a} - \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$

(3) **Multiplication of a vector by a scalar :** If $\mathbf{a}$ is a vector and *m* is a scalar (*i.e.,* a real number) then $m\mathbf{a}$ is a vector whose magnitude is *m* times that of $\mathbf{a}$ and whose direction is the same as that of $\mathbf{a}$, if *m* is positive and opposite to that of $\mathbf{a}$, if *m* is negative.

$\therefore$ Magnitude of $m\mathbf{a} = |m\mathbf{a}| \Rightarrow m$ (magnitude of $\mathbf{a}$) = $m|\mathbf{a}|$

Again if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ then $m\mathbf{a} = (ma_1)\mathbf{i} + (ma_2)\mathbf{j} + (ma_3)\mathbf{k}$

Properties of Multiplication of vectors by a scalar : The following are properties of multiplication of vectors by scalars, for vectors $\mathbf{a}, \mathbf{b}$ and scalars m, n

(i) $m(-\mathbf{a}) = (-m)\mathbf{a} = -(m\mathbf{a})$ (ii) $(-m)(-\mathbf{a}) = m\mathbf{a}$

(iii) $m(n\mathbf{a}) = (mn)\mathbf{a} = n(m\mathbf{a})$ (iv) $(m+n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$

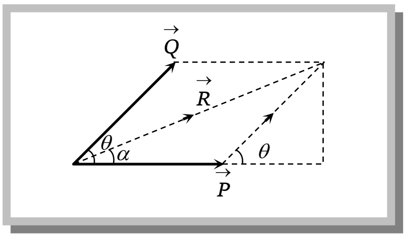
(v) $m(\mathbf{a}+\mathbf{b}) = m\mathbf{a} + m\mathbf{b}$

(4) **Resultant of two forces**

$\vec{R} = \vec{P} + \vec{Q}$

$|\vec{R}| = R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

where $|\vec{P}| = P$, $|\vec{Q}| = Q$, $\tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$

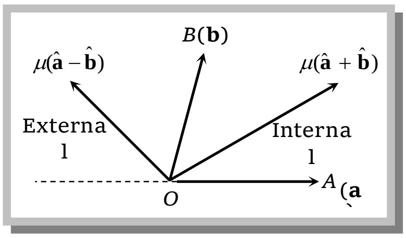


Deduction : When $|\vec{P}| = |\vec{Q}|$, *i.e.*, $P = Q$, $\tan\alpha = \dfrac{P\sin\theta}{P + P\cos\theta} = \dfrac{\sin\theta}{1+\cos\theta} = \tan\dfrac{\theta}{2}$; $\therefore$ $\alpha = \dfrac{\theta}{2}$

Hence, the angular bisector of two unit vectors $\mathbf{a}$ and $\mathbf{b}$ is along the vector sum $\mathbf{a} + \mathbf{b}$.

Important Tips

- *The internal bisector of the angle between any two vectors is along the vector sum of the corresponding unit vectors.*
- *The external bisector of the angle between two vectors is along the vector difference of the corresponding unit vectors.*



Example: 4 If $ABCDEF$ is a regular hexagon, then $\overrightarrow{AD} + \overrightarrow{EB} + \overrightarrow{FC} =$ **[Karnataka CET 2002]**

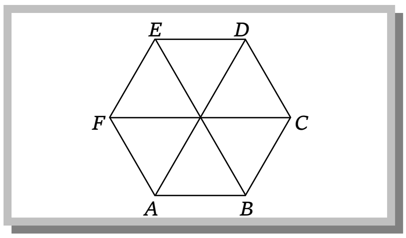
(a) $\vec{O}$ (b) $2\overrightarrow{AB}$ (c) $3\overrightarrow{AB}$ (d) $4\overrightarrow{AB}$

Solution: (d) We have $\overrightarrow{AD} + \overrightarrow{EB} + \overrightarrow{FC}$

$= (\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}) + (\overrightarrow{ED} + \overrightarrow{DC} + \overrightarrow{CB}) + \overrightarrow{FC}$

$= \overrightarrow{AB} + (\overrightarrow{BC} + \overrightarrow{CB}) + (\overrightarrow{CD} + \overrightarrow{DC}) + \overrightarrow{ED} + \overrightarrow{FC}$

$= \overrightarrow{AB} + \vec{O} + \vec{O} + \overrightarrow{AB} + 2\overrightarrow{AB} = 4\overrightarrow{AB}$ $[\overrightarrow{ED} = \overrightarrow{AB}, \overrightarrow{FC} = 2\overrightarrow{AB}]$



Example: 5 The unit vector parallel to the resultant vector of $2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ is

(a) $\frac{1}{7}(3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k})$ (b) $\dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$ (c) $\dfrac{\mathbf{i}+\mathbf{j}+2\mathbf{k}}{\sqrt{6}}$ (d) $\dfrac{1}{\sqrt{69}}(-\mathbf{i} - \mathbf{j} + 8\mathbf{k})$

Solution: (a) Resultant vector $\mathbf{r} = (2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) + (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

Unit vector parallel to $\mathbf{r}$ $= \dfrac{1}{|\mathbf{r}|}\mathbf{r} = \dfrac{1}{\sqrt{3^2 + 6^2 + (-2)^2}}(3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}) = \dfrac{1}{7}(3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k})$

Example: 6 If the sum of two vectors is a unit vector, then the magnitude of their difference is

[Kurukshetra CEE 1996; Rajasthan PET 1996]

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) 1

Solution: (b) Let $|\mathbf{a}|=1$, $|\mathbf{b}|=1$ and $|\mathbf{a}+\mathbf{b}|=1 \Rightarrow |\mathbf{a}+\mathbf{b}|^2=1 \Rightarrow 1+1+2\cos\theta=1 \Rightarrow \cos\theta=-\frac{1}{2} \Rightarrow \theta=120^o$

$\therefore \; |\mathbf{a}-\mathbf{b}|^2=1+1-2\cos\theta=3 \Rightarrow |\mathbf{a}-\mathbf{b}|=\sqrt{3}$.

Example: 7 The length of longer diagonal of the parallelogram constructed on 5**a** + 2**b** and **a** – 3**b**, it is given that $|\mathbf{a}|=2\sqrt{2}, |\mathbf{b}|=3$ and angle between **a** and **b** is $\frac{\pi}{4}$, is

(a) 15 (b) $\sqrt{113}$ (c) $\sqrt{593}$ (d) $\sqrt{369}$

Solution: (c) Length of the two diagonals will be $d_1=|(5\mathbf{a}+2\mathbf{b})+(\mathbf{a}-3\mathbf{b})|$ and $d_2=|(5\mathbf{a}+2\mathbf{b})-(\mathbf{a}-3\mathbf{b})|$

$\Rightarrow d_1=|6\mathbf{a}-\mathbf{b}|$, $d_2=|4\mathbf{a}+5\mathbf{b}|$

Thus, $d_1=\sqrt{|6\mathbf{a}|^2+|-\mathbf{b}|^2+2|6\mathbf{a}||-\mathbf{b}|\cos(\pi-\pi/4)} = \sqrt{36(2\sqrt{2})^2+9+12.2\sqrt{2}.3.\left(-\frac{1}{\sqrt{2}}\right)}$ = 15.

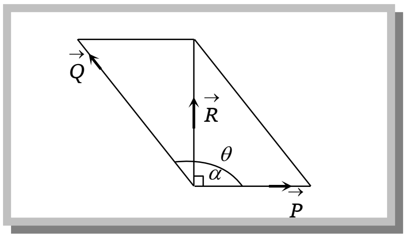
$$d_2=\sqrt{|4\mathbf{a}|^2+|5\mathbf{b}|^2+2|4\mathbf{a}||5\mathbf{b}|\cos\frac{\pi}{4}} = \sqrt{16\times 8+25\times 9+40\times 2\sqrt{2}\times 3\times\frac{1}{\sqrt{2}}} = \sqrt{593}.$$

$\therefore$ Length of the longer diagonal = $\sqrt{593}$

Example: 8 The sum of two forces is $18\ N$ and resultant whose direction is at right angles to the smaller force is $12N$. The magnitude of the two forces are

(a) 13, 5 (b) 12, 6 (c) 14, 4 (d) 11, 7

Solution: (a) We have, $|\vec{P}|+|\vec{Q}|=18N$; $|\vec{R}|=|\vec{P}+\vec{Q}|=12N$



$\alpha=90^o \Rightarrow P+Q\cos\theta=0 \Rightarrow Q\cos\theta=-P$

Now, $R^2=P^2+Q^2+2PQ\cos\theta \Rightarrow R^2=P^2+Q^2+2P(-P)=Q^2-P^2$

$\Rightarrow 12^2=(P+Q)(Q-P)=18(Q-P)$

$\Rightarrow Q-P=8$ and $Q+P=18 \Rightarrow Q=13$, $P=5$

$\therefore$ Magnitude of two forces are 5*N*, 13*N*.

Example: 9 The vector **c**, directed along the internal bisector of the angle between the vectors $\mathbf{a}=7\mathbf{i}-4\mathbf{j}-4\mathbf{k}$ and $\mathbf{b}=-2\mathbf{i}-\mathbf{j}+2\mathbf{k}$ with $|\mathbf{c}|=5\sqrt{6}$, is

(a) $\frac{5}{3}(\mathbf{i}-7\mathbf{j}+2\mathbf{k})$ (b) $\frac{5}{3}(5\mathbf{i}+5\mathbf{j}+2\mathbf{k})$ (c) $\frac{5}{3}(\mathbf{i}+7\mathbf{j}+2\mathbf{k})$ (d) $\frac{5}{3}(-5\mathbf{i}+5\mathbf{j}+2\mathbf{k})$

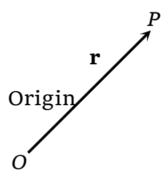
Solution: (a) Let $\mathbf{a}=7\mathbf{i}-4\mathbf{j}-4\mathbf{k}$ and $\mathbf{b}=-2\mathbf{i}-\mathbf{j}+2\mathbf{k}$

Now required vector $\mathbf{c}=\lambda\left(\frac{\mathbf{a}}{|\mathbf{a}|}+\frac{\mathbf{b}}{|\mathbf{b}|}\right) = \lambda\left(\frac{7\mathbf{i}-4\mathbf{j}-4\mathbf{k}}{9}+\frac{-2\mathbf{i}-\mathbf{j}+2\mathbf{k}}{3}\right) = \frac{\lambda}{9}(\mathbf{i}-7\mathbf{j}+2\mathbf{k})$

$|\mathbf{c}|^2=\frac{\lambda^2}{81}\times 54=150 \Rightarrow \lambda=\pm 15 \Rightarrow \mathbf{c} = \pm\frac{5}{3}(\mathbf{i}-7\mathbf{j}+2\mathbf{k})$

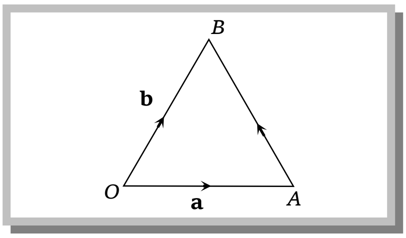
6.6 Position Vector

If a point *O* is fixed as the origin in space (or plane) and *P* is any point, then $\overrightarrow{OP}$ is called the position vector of *P* with respect to *O*.



If we say that *P* is the point **r**, then we mean that the position vector of *P* is **r** with respect to some origin *O*.

(1) **$\overrightarrow{AB}$ in terms of the position vectors of points A and B :** If $\mathbf{a}$ and $\mathbf{b}$ are position vectors of points A and B respectively. Then, $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$



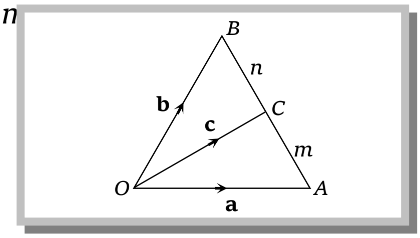
In ΔOAB, we have $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$

$\Rightarrow$ $\overrightarrow{AB}$ = (Position vector of B) – (Position vector of A)

$\Rightarrow$ $\overrightarrow{AB}$ = (Position vector of head) – (Position vector of tail)

(2) **Position vector of a dividing point**

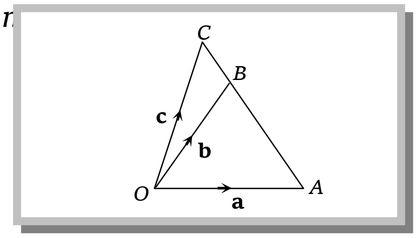
(i) **Internal division :** Let A and B be two points with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, and let C be a point dividing AB internally in the ratio n



Then the position vector of C is given by

$$\overrightarrow{OC} = \frac{m\mathbf{b} + n\mathbf{a}}{m+n}$$

(ii) **External division :** Let A and B be two points with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively and let C be a point dividing AB externally in the ratio



Then the position vector of C is given by

$$\overrightarrow{OC} = \frac{m\mathbf{b} - n\mathbf{a}}{m-n}$$

Important Tips

☞ *Position vector of the mid point of AB is* $\frac{\mathbf{a}+\mathbf{b}}{2}$

☞ *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are position vectors of vertices of a triangle, then position vector of its centroid is* $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$

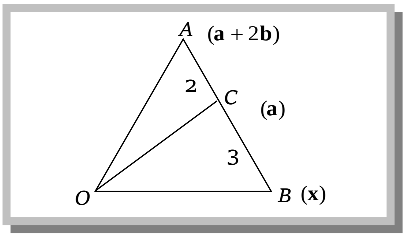
☞ *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *are position vectors of vertices of a tetrahedron, then position vector of its centroid is* $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}$.

Example: 10 If position vector of a point A is $\mathbf{a}+2\mathbf{b}$ and $\mathbf{a}$ divides AB in the ratio 2 : 3, then the position vector of B is **[MP PET 2002]**

(a) $2\mathbf{a}-\mathbf{b}$ (b) $\mathbf{b}-2\mathbf{a}$ (c) $\mathbf{a}-3\mathbf{b}$ (d) $\mathbf{b}$

Solution: (c) Let position vector of B is $\mathbf{x}$.

The point $C(\mathbf{a})$ divides AB in 2 : 3.



$\therefore$ $\mathbf{a} = \frac{2\mathbf{x} + 3(\mathbf{a}+2\mathbf{b})}{2+3}$

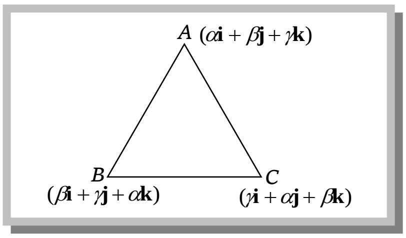
$\Rightarrow 5\mathbf{a} = 2\mathbf{x} + 3\mathbf{a} + 6\mathbf{b}$

$\therefore$ $\mathbf{x} = \mathbf{a} - 3\mathbf{b}$

Example: 11 Let α, β, γ be distinct real numbers. The points with position vectors $\alpha\mathbf{i}+\beta\mathbf{j}+\gamma\mathbf{k}$, $\beta\mathbf{i}+\gamma\mathbf{j}+\alpha\mathbf{k}$, $\gamma\mathbf{i}+\alpha\mathbf{j}+\beta\mathbf{k}$

[IIT Screening 1994]

(a) Are collinear
(b) Form an equilateral triangle
(c) Form a scalene triangle
(d) Form a right angled triangle



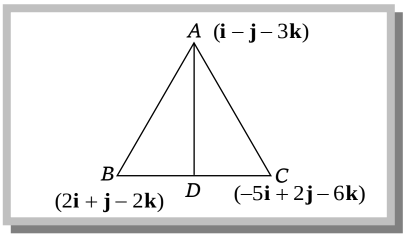
Solution: (b) $AB = \sqrt{(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2} = BC = CA$

$\therefore$ ABC is an equilateral triangle.

Example: 12 The position vectors of the vertices *A, B, C* of a triangle are $\mathbf{i}-\mathbf{j}-3\mathbf{k}$, $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $-5\mathbf{i}+2\mathbf{j}-6\mathbf{k}$ respectively. The length of the bisector *AD* of the angle *BAC* where *D* is on the segment *BC*, is

(a) $\frac{3}{4}\sqrt{10}$ (b) $\frac{1}{4}$ (c) $\frac{11}{2}$ (d) None of these

Solution: (a) $|\overrightarrow{AB}| = |(2\mathbf{i}+\mathbf{j}-2\mathbf{k})-(\mathbf{i}-\mathbf{j}-3\mathbf{k})| = |\mathbf{i}+2\mathbf{j}+\mathbf{k}| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$



$|\overrightarrow{AC}| = |(-5\mathbf{i}+2\mathbf{j}-6\mathbf{k})-(\mathbf{i}-\mathbf{j}-3\mathbf{k})| = |-6\hat{\mathbf{i}}+3\hat{\mathbf{j}}-3\hat{\mathbf{k}}| = \sqrt{(-6)^2+3^2+(-3)^2}$

$= \sqrt{54} = 3\sqrt{6}$.

$BD : DC = AB : AC = \frac{\sqrt{6}}{3\sqrt{6}} = \frac{1}{3}$.

$\therefore$ Position vector of *D* $= \frac{1.(-5\mathbf{i}+2\mathbf{j}-6\mathbf{k})+3(2\mathbf{i}+\mathbf{j}-2\mathbf{k})}{1+3} = \frac{1}{4}(\mathbf{i}+5\mathbf{j}-12\mathbf{k})$

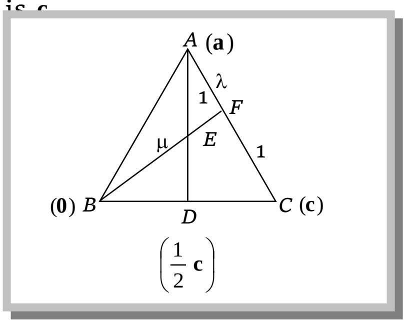
$\therefore$ $\overrightarrow{AD} =$ position vector of *D* – Position vector of *A* $= \frac{1}{4}(\mathbf{i}+5\mathbf{j}-12\mathbf{k})-(\mathbf{i}-\mathbf{j}-3\mathbf{k}) = \frac{1}{4}(-3\mathbf{i}+9\mathbf{j}) = \frac{3}{4}(-\mathbf{i}+3\mathbf{j})$

$|\overrightarrow{AD}| = \frac{3}{4}\sqrt{(-1)^2+3^2} = \frac{3}{4}\sqrt{10}$.

Example: 13 The median *AD* of the triangle *ABC* is bisected at *E*, *BE* meets *AC* in *F*. Then *AF* : *AC* =

(a) 3/4 (b) 1/3 (c) 1/2 (d) 1/4

Solution: (b) Let position vector of *A* with respect to *B* is $\mathbf{a}$ and that of *C* *w.r.t. B* is $\mathbf{c}$.



Position vector of *D* *w.r.t. B* $= \frac{\mathbf{0}+\mathbf{c}}{2} = \frac{\mathbf{c}}{2}$

Position vector of *E* $= \frac{\mathbf{a}+\frac{\mathbf{c}}{2}}{2} = \frac{\mathbf{a}}{2}+\frac{\mathbf{c}}{4}$ .....(i)

Let $AF : FC = \lambda : 1$ and $BE : EF = \mu : 1$

Position vector of *F* $= \frac{\lambda\mathbf{c}+\mathbf{a}}{1+\lambda}$

Now, position vector of *E* $= \frac{\mu\left(\frac{\lambda\mathbf{c}+\mathbf{a}}{1+\lambda}\right)+1.\mathbf{0}}{\mu+1}$ ....(ii). From (i) and (ii) ,

$\frac{\mathbf{a}}{2}+\frac{\mathbf{c}}{4} = \frac{\mu}{(1+\lambda)(1+\mu)}\mathbf{a}+\frac{\lambda\mu}{(1+\lambda)(1+\mu)}\mathbf{c}$

$\Rightarrow \frac{1}{2} = \frac{\mu}{(1+\lambda)(1+\mu)}$ and $\frac{1}{4} = \frac{\lambda\mu}{(1+\lambda)(1+\mu)} \Rightarrow \lambda = \frac{1}{2}$, $\therefore \frac{AF}{AC} = \frac{AF}{AF+FC} = \frac{\lambda}{1+\lambda} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$.

6.7 Linear Combination of Vectors

A vector $\mathbf{r}$ is said to be a linear combination of vectors $\mathbf{a},\mathbf{b},\mathbf{c}$..... etc, if there exist scalars *x*, *y*, *z* etc., such that $\mathbf{r} = x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+....$

Examples : Vectors $\mathbf{r}_1 = 2\mathbf{a}+\mathbf{b}+3\mathbf{c}, \mathbf{r}_2 = \mathbf{a}+3\mathbf{b}+\sqrt{2}\mathbf{c}$ are linear combinations of the vectors $\mathbf{a},\mathbf{b},\mathbf{c}$.

(1) **Collinear and Non-collinear vectors :** Let $\mathbf{a}$ and $\mathbf{b}$ be two collinear vectors and let $\mathbf{x}$ be the unit vector in the direction of $\mathbf{a}$. Then the unit vector in the direction of $\mathbf{b}$ is $\mathbf{x}$ or $-\mathbf{x}$ according as $\mathbf{a}$ and $\mathbf{b}$ are like or unlike parallel vectors. Now, $\mathbf{a} = |\mathbf{a}|\hat{\mathbf{x}}$ and $\mathbf{b} = \pm|\mathbf{b}|\hat{\mathbf{x}}$.

$\therefore \quad \mathbf{a} = \left(\frac{|\mathbf{a}|}{|\mathbf{b}|}\right)|\mathbf{b}|\,\hat{\mathbf{x}} \Rightarrow \mathbf{a} = \left(\pm\frac{|\mathbf{a}|}{|\mathbf{b}|}\right)\mathbf{b} \Rightarrow \mathbf{a} = \lambda\mathbf{b}$, where $\lambda = \pm\frac{|\mathbf{a}|}{|\mathbf{b}|}$.

Thus, if $\mathbf{a}, \mathbf{b}$ are collinear vectors, then $\mathbf{a} = \lambda\mathbf{b}$ or $\mathbf{b} = \lambda\mathbf{a}$ for some scalar λ.

(2) **Relation between two parallel vectors**

(i) If $\mathbf{a}$ and $\mathbf{b}$ be two parallel vectors, then there exists a scalar k such that $\mathbf{a} = k\mathbf{b}$.

i.e., there exist two non-zero scalar quantities x and y so that $x\mathbf{a} + y\mathbf{b} = \mathbf{0}$.

If $\mathbf{a}$ and $\mathbf{b}$ be two non-zero, non-parallel vectors then $x\mathbf{a} + y\mathbf{b} = \mathbf{0} \Rightarrow x = 0$ and $y = 0$.

Obviously $x\mathbf{a} + y\mathbf{b} = \mathbf{0} \Rightarrow \begin{cases} \mathbf{a} = \mathbf{0}, \mathbf{b} = \mathbf{0} \\ \text{or} \\ x = 0, y = 0 \\ \text{or} \\ \mathbf{a} \parallel \mathbf{b} \end{cases}$

(ii) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then from the property of parallel vectors, we have

$$\mathbf{a} \parallel \mathbf{b} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

(3) **Test of collinearity of three points :** Three points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are collinear iff there exist scalars x, y, z not all zero such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}$, where $x + y + z = 0$. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j}$, then the points with position vector $\mathbf{a}, \mathbf{b}, \mathbf{c}$ will be collinear iff $\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0$.

(4) **Test of coplanarity of three vectors :** Let $\mathbf{a}$ and $\mathbf{b}$ two given non-zero non-collinear vectors. Then any vectors $\mathbf{r}$ coplanar with $\mathbf{a}$ and $\mathbf{b}$ can be uniquely expressed as $\mathbf{r} = x\mathbf{a} + y\mathbf{b}$ for some scalars x and y.

(5) **Test of coplanarity of Four points :** Four points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are coplanar iff there exist scalars x, y, z, u not all zero such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + u\mathbf{d} = \mathbf{0}$, where $x + y + z + u = 0$.

Four points with position vectors

$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$

will be coplanar, *iff* $\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} = 0$

6.8 Linear Independence and Dependence of Vectors

(1) **Linearly independent vectors :** A set of non-zero vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots\ldots \mathbf{a}_n$ is said to be linearly independent, if $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots\ldots + x_n\mathbf{a}_n = \mathbf{0} \Rightarrow x_1 = x_2 = \ldots\ldots = x_n = 0$.

(2) **Linearly dependent vectors :** A set of vectors $\mathbf{a}_1, \mathbf{a}_2, ...\mathbf{a}_n$ is said to be linearly dependent if there exist scalars $x_1, x_2, ..., x_n$ not all zero such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_n\mathbf{a}_n = \mathbf{0}$

Three vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ will be linearly dependent vectors *iff* $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$.

Properties of linearly independent and dependent vectors

(i) Two non-zero, non-collinear vectors are linearly independent.

(ii) Any two collinear vectors are linearly dependent.

(iii) Any three non-coplanar vectors are linearly independent.

(iv) Any three coplanar vectors are linearly dependent.

(v) Any four vectors in 3-dimensional space are linearly dependent.

Example: 14 The points with position vectors $60\mathbf{i} + 3\mathbf{j}$, $40\mathbf{i} - 8\mathbf{j}, a\mathbf{i} - 52\mathbf{j}$ are collinear, if a =

[Rajasthan PET 1991; IIT 1983; MP PET 2002]

(a) – 40 (b) 40 (c) 20 (d) None of these

Solution: (a) As the three points are collinear, $x(60\mathbf{i} + 3\mathbf{j}) + y(40\mathbf{i} - 8\mathbf{j}) + z(a\mathbf{i} - 52\mathbf{j}) = 0$

such that x, y, z are not all zero and $x + y + z = 0$.

$\Rightarrow (60x + 40y + az)\mathbf{i} + (3x - 8y - 52z)\mathbf{j} = 0$ and $x + y + z = 0$

$\Rightarrow 60x + 40y + az = 0$, $3x - 8y - 52z = 0$ and $x + y + z = 0$

For non-trivial solution, $\begin{vmatrix} 60 & 40 & a \\ 3 & -8 & -52 \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow a = -40$

Trick : If *A*, *B*, *C* are given points, then $\overrightarrow{AB} = k(\overrightarrow{BC}) \Rightarrow -20\mathbf{i} - 11\mathbf{j} = k[(a-40)\mathbf{i} - 44\mathbf{j}]$

On comparing, $-11 = -44k \Rightarrow k = \frac{1}{4}$ and $-20 = \frac{1}{4}(a - 40) \Rightarrow a = -40$.

Example: 15 If the position vectors of *A, B, C, D* are $2\mathbf{i} + \mathbf{j}, \mathbf{i} - 3\mathbf{j}, 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{i} + \lambda\mathbf{j}$ respectively and $\overrightarrow{AB} \parallel \overrightarrow{CD}$, then λ will be

[Rajasthan PET 1988]

(a) – 8 (b) – 6 (c) 8 (d) 6

Solution: (b) $\overrightarrow{AB} = (\mathbf{i} - 3\mathbf{j}) - (2\mathbf{i} + \mathbf{j}) = -\mathbf{i} - 4\mathbf{j}$; $\overrightarrow{CD} = (\mathbf{i} + \lambda\mathbf{j}) - (3\mathbf{i} + 2\mathbf{j}) = -2\mathbf{i} + (\lambda - 2)\mathbf{j}$; $\overrightarrow{AB} \parallel \overrightarrow{CD} \Rightarrow \overrightarrow{AB} = x\overrightarrow{CD}$

$-\mathbf{i} - 4\mathbf{j} = x\{-2\mathbf{i} + (\lambda - 2)\mathbf{j}\} \Rightarrow -1 = -2x, -4 = (\lambda - 2)x \Rightarrow x = \frac{1}{2}, \lambda = -6$.

Example: 16 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors such that no two of these are collinear. If the vector $\mathbf{a} + 2\mathbf{b}$ is collinear with $\mathbf{c}$ and $\mathbf{b} + 3\mathbf{c}$ is collinear with $\mathbf{a}$ (λ being some non-zero scalar) then $\mathbf{a} + 2\mathbf{b} + 6\mathbf{c}$ equals **[AIEEE 2004]**

(a) **0** (b) $\lambda\mathbf{b}$ (c) $\lambda\mathbf{c}$ (d) $\lambda\mathbf{a}$

Solution: (a) As $\mathbf{a} + 2\mathbf{b}$ and $\mathbf{c}$ are collinear $\mathbf{a} + 2\mathbf{b} = \lambda\mathbf{c}$ ......(i)

Again $\mathbf{b} + 3\mathbf{c}$ is collinear with $\mathbf{a}$

$\therefore \mathbf{b} + 3\mathbf{c} = \mu\mathbf{a}$ .....(ii)

Now, $\mathbf{a} + 2\mathbf{b} + 6\mathbf{c} = (\mathbf{a} + 2\mathbf{b}) + 6\mathbf{c} = \lambda\mathbf{c} + 6\mathbf{c} = (\lambda + 6)\mathbf{c}$ .....(iii)

Also, $\mathbf{a} + 2\mathbf{b} + 6\mathbf{c} = \mathbf{a} + 2(\mathbf{b} + 3\mathbf{c}) = \mathbf{a} + 2\mu\mathbf{a} = (2\mu + 1)\mathbf{a}$ ......(iv)

From (iii) and (iv), $(\lambda+6)\mathbf{c}=(2\mu+1)\mathbf{a}$

But $\mathbf{a}$ and $\mathbf{c}$ are non-zero , non-collinear vectors,

$\therefore \quad \lambda+6=0=2\mu+1$. Hence, $\mathbf{a}+2\mathbf{b}+6\mathbf{c}=\mathbf{0}$.

Example: 17 If the vectors $4\mathbf{i}+11\mathbf{j}+m\mathbf{k}, 7\mathbf{i}+2\mathbf{j}+6\mathbf{k}$ and $\mathbf{i}+5\mathbf{j}+4\mathbf{k}$ are coplanar, then m is **[Karnataka CET 2003]**

(a) 38 (b) 0 (c) 10 (d) – 10

Solution: (c) As the three vectors are coplanar, one will be a linear combination of the other two.

$\therefore \quad 4\mathbf{i}+11\mathbf{j}+m\mathbf{k}=x(7\mathbf{i}+2\mathbf{j}+6\mathbf{k})+y(\mathbf{i}+5\mathbf{j}+4\mathbf{k}) \Rightarrow 4=7x+y$ .....(i)

$11=2x+5y$ .....(ii)

$m=6x+4y$ .....(iii)

From (i) and (ii), $x=\frac{3}{11}, y=\frac{23}{11}$; From (iii), $m=6\times\frac{3}{11}+4\times\frac{23}{11}=10$.

Trick : $\because$ Vectors $4\mathbf{i}+11\mathbf{j}+m\mathbf{k}$, $7\mathbf{i}+2\mathbf{j}+6\mathbf{k}$ and $\mathbf{i}+5\mathbf{j}+4\mathbf{k}$ are coplanar.

$$\therefore \begin{vmatrix} 4 & 11 & m \\ 7 & 2 & 6 \\ 1 & 5 & 4 \end{vmatrix}=0$$

$\Rightarrow \quad 4(8-30)-11(28-6)+m(35-2)=0 \Rightarrow -88-11\times 22+33m=0 \Rightarrow -8-22+3m=0 \Rightarrow 3m=30 \Rightarrow m=10$.

Example: 18 The value of λ for which the four points $2\mathbf{i}+3\mathbf{j}-\mathbf{k}, \mathbf{i}+2\mathbf{j}+3\mathbf{k}, 3\mathbf{i}+4\mathbf{j}-2\mathbf{k}, \mathbf{i}-\lambda\mathbf{j}+6\mathbf{k}$ are coplanar **[MP PET 2004]**

(a) 8 (b) 0 (c) – 2 (d) 6

Solution: (c) The given four points are coplanar

$\therefore \quad x(2\mathbf{i}+3\mathbf{j}-\mathbf{k})+y(\mathbf{i}+2\mathbf{j}+3\mathbf{k})+z(3\mathbf{i}+4\mathbf{j}-2\mathbf{k})+w(\mathbf{i}-\lambda\mathbf{j}+6\mathbf{k})=\mathbf{0}$ and $x+y+z+w=0$,

where x, y, z, w are not all zero.

$\Rightarrow \quad (2x+y+3z+w)\mathbf{i}+(3x+2y+4z-\lambda w)\mathbf{j} + (-x+3y-2z+6w)\mathbf{k}=0$ and $x+y+z+w=0$

$\Rightarrow \quad 2x+y+3z+w=0$, $3x+2y+4z-\lambda w=0$, $-x+3y-2z+6w=0$ and $x+y+z+w=0$

For non-trivial solution, $\begin{vmatrix} 2 & 1 & 3 & 1 \\ 3 & 2 & 4 & -\lambda \\ -1 & 3 & -2 & 6 \\ 1 & 1 & 1 & 1 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & 0 & 0 & -(\lambda+2) \\ -1 & 3 & -2 & 6 \\ 1 & 1 & 1 & 1 \end{vmatrix}=0$, Operating $[R_2 \to R_2-R_1-R_4]$

$$\Rightarrow -(\lambda+2)\begin{vmatrix} 2 & 1 & 3 \\ -1 & 3 & -2 \\ 1 & 1 & 1 \end{vmatrix}=0 \Rightarrow \lambda=-2.$$

Example: 19 If $\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{b}=4\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and $\mathbf{c}=\mathbf{i}+\alpha\mathbf{j}+\beta\mathbf{k}$ are linearly dependent vectors and $|\mathbf{c}|=\sqrt{3}$, then **[IIT 1998]**

(a) $\alpha=1, \beta=-1$ (b) $\alpha=1, \beta=\pm 1$ (c) $\alpha=-1, \beta=\pm 1$ (d) $\alpha=\pm 1, \beta=1$

Solution: (d) The given vectors are linearly dependent hence, there exist scalars x, y, z not all zero, such that $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=\mathbf{0}$

i.e., $x(\mathbf{i}+\mathbf{j}+\mathbf{k})+y(4\mathbf{i}+3\mathbf{j}+4\mathbf{k})+z(\mathbf{i}+\alpha\mathbf{j}+\beta\mathbf{k})=\mathbf{0}$,

i.e., $(x+4y+z)\mathbf{i}+(x+3y+\alpha z)\mathbf{j}+(x+4y+\beta z)\mathbf{k}=\mathbf{0}$

$\Rightarrow \quad x+4y+z=0$, $x+3y+\alpha z=0$, $x+4y+\beta z=0$

For non-trivial solution, $\begin{vmatrix} 1 & 4 & 1 \\ 1 & 3 & \alpha \\ 1 & 4 & \beta \end{vmatrix}=0 \Rightarrow \beta=1$

$|c|^2=3 \Rightarrow 1+\alpha^2+\beta^2=3 \Rightarrow \alpha^2=2-\beta^2=2-1=1$; $\therefore \alpha=\pm 1$

Trick : $|\mathbf{c}|=\sqrt{1+\alpha^2+\beta^2}=\sqrt{3} \Rightarrow \alpha^2+\beta^2=2$

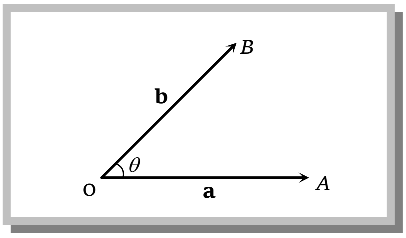
$\because \quad \mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly dependent, hence $\begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 1 & \alpha & \beta \end{vmatrix}=0 \Rightarrow \beta=1$.

$\therefore \quad \alpha^2=1 \Rightarrow \alpha=\pm 1$.

6.9 Product of Two Vectors

Product of two vectors is processed by two methods. When the product of two vectors results is a scalar quantity, then it is called scalar product. It is also known as dot product because we are putting a dot (.) between two vectors.

When the product of two vectors results is a vector quantity then this product is called vector product. It is also known as cross product because we are putting a cross (×) between two vectors.

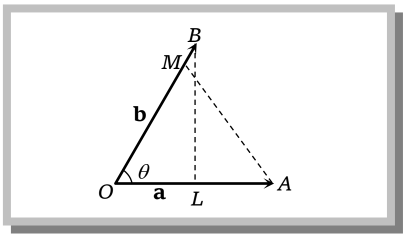


(1) **Scalar or Dot product of two vectors :** If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero vectors and θ be the angle between them, then their scalar product (or dot product) is denoted by $\mathbf{a}\,.\,\mathbf{b}$ and is defined as the scalar $|\,\mathbf{a}\,||\,\mathbf{b}\,|\cos\theta$, where $|\,\mathbf{a}\,|$ and $|\,\mathbf{b}\,|$ are modulii of $\mathbf{a}$ and $\mathbf{b}$ respectively and $0 \le \theta \le \pi$.

Important Tips

- $\mathbf{a}.\mathbf{b} \in R$
- $\mathbf{a}.\mathbf{b} \le |\,\mathbf{a}\,||\,\mathbf{b}\,|$
- $\mathbf{a}.\mathbf{b} > 0 \Rightarrow$ *angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is acute.*
- $\mathbf{a}.\mathbf{b} < 0 \Rightarrow$ *angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is obtuse.*
- *The dot product of a zero and non-zero vector is a scalar zero.*

(i) **Geometrical Interpretation of scalar product :** Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Draw $BL \perp OA$ and $AM \perp OB$.



From Δs OBL and OAM, we have $OL = OB\cos\theta$ and $OM = OA\cos\theta$. Here OL and OM are known as projection of $\mathbf{b}$ on $\mathbf{a}$ and $\mathbf{a}$ on $\mathbf{b}$ respectively.

Now $\mathbf{a}\,.\,\mathbf{b} = |\,\mathbf{a}\,||\,\mathbf{b}\,|\cos\theta = |\,\mathbf{a}\,|(OB\cos\theta) = |\,\mathbf{a}\,|(OL)$

$= (\text{Magnitude of } \mathbf{a})(\text{Projection of } \mathbf{b} \text{ on } \mathbf{a})$ .....(i)

Again, $\mathbf{a}\,.\,\mathbf{b} = |\,\mathbf{a}\,||\,\mathbf{b}\,|\cos\theta = |\,\mathbf{b}\,|(|\,\mathbf{a}\,|\cos\theta) = |\,\mathbf{b}\,|(OA\cos\theta) = |\,\mathbf{b}\,|(OM)$

a.b = (Magnitude of **b**) (Projection of **a** on **b**) .....(ii)

Thus geometrically interpreted, the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.

(ii) **Angle between two vectors :** If $\mathbf{a}, \mathbf{b}$ be two vectors inclined at an angle θ, then, $\mathbf{a}\,.\,\mathbf{b} = |\,\mathbf{a}\,||\,\mathbf{b}\,|\cos\theta$

$$\Rightarrow \cos\theta = \frac{\mathbf{a}\,.\,\mathbf{b}}{|\,\mathbf{a}\,||\,\mathbf{b}\,|} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{a}\,.\,\mathbf{b}}{|\,\mathbf{a}\,||\,\mathbf{b}\,|}\right)$$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$; $\theta = \cos^{-1}\left(\dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$

(2) **Properties of scalar product**

(i) **Commutativity :** The scalar product of two vector is commutative *i.e.,* $\mathbf{a}\,.\,\mathbf{b} = \mathbf{b}\,.\,\mathbf{a}$.

(ii) **Distributivity of scalar product over vector addition:** The scalar product of vectors is distributive over vector addition *i.e.,*

(a) $\mathbf{a}.(\mathbf{b}+\mathbf{c})=\mathbf{a}.\mathbf{b}+\mathbf{a}.\mathbf{c}$ (Left distributivity) (b) $(\mathbf{b}+\mathbf{c}).\mathbf{a}=\mathbf{b}.\mathbf{a}+\mathbf{c}.\mathbf{a}$ (Right distributivity)

(iii) Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors $\mathbf{a}.\mathbf{b}=0 \Leftrightarrow \mathbf{a}\perp\mathbf{b}$.

As $\mathbf{i},\mathbf{j},\mathbf{k}$ are mutually perpendicular unit vectors along the co-ordinate axes, therefore $\mathbf{i}.\mathbf{j}=\mathbf{j}.\mathbf{i}=0$; $\mathbf{j}.\mathbf{k}=\mathbf{k}.\mathbf{j}=0$; $\mathbf{k}.\mathbf{i}=\mathbf{i}.\mathbf{k}=0$.

(iv) For any vector $\mathbf{a}, \mathbf{a}.\mathbf{a}=|\mathbf{a}|^2$.

As $\mathbf{i},\mathbf{j},\mathbf{k}$ are unit vectors along the co-ordinate axes, therefore $\mathbf{i}.\mathbf{i}=|\mathbf{i}|^2=1$, $\mathbf{j}.\mathbf{j}=|\mathbf{j}|^2=1$ and $\mathbf{k}.\mathbf{k}=|\mathbf{k}|^2=1$

(v) If m is a scalar and $\mathbf{a},\mathbf{b}$ be any two vectors, then $(m\mathbf{a}).\mathbf{b}=m(\mathbf{a}.\mathbf{b})=\mathbf{a}.(m\mathbf{b})$

(vi) If m, n are scalars and $\mathbf{a},\mathbf{b}$ be two vectors, then $m\mathbf{a}.n\mathbf{b}=mn(\mathbf{a}.\mathbf{b})=(mn\,\mathbf{a}).\mathbf{b}=\mathbf{a}.(mn\,\mathbf{b})$

(vii) For any vectors $\mathbf{a}$ and $\mathbf{b}$, we have (a) $\mathbf{a}.(-\mathbf{b})=-(\mathbf{a}.\mathbf{b})=(-\mathbf{a}).\mathbf{b}$ (b) $(-\mathbf{a}).(-\mathbf{b})=\mathbf{a}.\mathbf{b}$

(viii) For any two vectors $\mathbf{a}$ and $\mathbf{b}$, we have

(a) $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2+2\mathbf{a}.\mathbf{b}$ (b) $|\mathbf{a}-\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2-2\mathbf{a}.\mathbf{b}$

(c) $(\mathbf{a}+\mathbf{b}).(\mathbf{a}-\mathbf{b})=|\mathbf{a}|^2-|\mathbf{b}|^2$ (d) $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}|+|\mathbf{b}| \Rightarrow \mathbf{a}\parallel\mathbf{b}$

(e) $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2 \Rightarrow \mathbf{a}\perp\mathbf{b}$ (f) $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}| \Rightarrow \mathbf{a}\perp\mathbf{b}$

(3) **Scalar product in terms of components :** If $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, then, $\mathbf{a}.\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$. Thus, scalar product of two vectors is equal to the sum of the products of their corresponding components. In particular, $\mathbf{a}.\mathbf{a}=|\mathbf{a}|^2=a_1^2+a_2^2+a_3^2$.

Example: 20 $(\mathbf{a}.\mathbf{i})\mathbf{i}+(\mathbf{a}.\mathbf{j})\mathbf{j}+(\mathbf{a}.\mathbf{k})\mathbf{k}=$ **[Karnataka CET 2004]**

(a) $\mathbf{a}$ (b) $2\mathbf{a}$ (c) $3\mathbf{a}$ (d) $\mathbf{0}$

Solution: (a) Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$

$\therefore$ $\mathbf{a}.\mathbf{i}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}).\mathbf{i}=a_1$, $\mathbf{a}.\mathbf{j}=a_2, \mathbf{a}.\mathbf{k}=a_3$

$\therefore$ $(\mathbf{a}.\mathbf{i})\mathbf{i}+(\mathbf{a}.\mathbf{j})\mathbf{j}+(\mathbf{a}.\mathbf{k})\mathbf{k}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}=\mathbf{a}$.

Example: 21 If $|\mathbf{a}|=3, |\mathbf{b}|=4$ then a value of λ for which $\mathbf{a}+\lambda\mathbf{b}$ is perpendicular to $\mathbf{a}-\lambda\mathbf{b}$ is

(a) 9/16 (b) 3/4 (c) 3/2 (d) 4/3

Solution: (b) $\mathbf{a}+\lambda\mathbf{b}$ is perpendicular to $\mathbf{a}-\lambda\mathbf{b}$

$\therefore$ $(\mathbf{a}+\lambda\mathbf{b}).(\mathbf{a}-\lambda\mathbf{b})=0 \Rightarrow |\mathbf{a}|^2-\lambda(\mathbf{a}.\mathbf{b})+\lambda(\mathbf{b}.\mathbf{a})-\lambda^2|\mathbf{b}|^2=0 \Rightarrow |\mathbf{a}|^2-\lambda^2|\mathbf{b}|^2=0 \Rightarrow \lambda=\pm\frac{|\mathbf{a}|}{|\mathbf{b}|}=\pm\frac{3}{4}$

Example: 22 A unit vector in the plane of the vectors $2\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{i}-\mathbf{j}+\mathbf{k}$ and orthogonal to $5\mathbf{i}+2\mathbf{j}+6\mathbf{k}$ is

(a) $\frac{6\mathbf{i}-5\mathbf{k}}{\sqrt{61}}$ (b) $\frac{3\mathbf{j}-\mathbf{k}}{\sqrt{10}}$ (c) $\frac{2\mathbf{i}-5\mathbf{j}}{\sqrt{29}}$ (d) $\frac{2\mathbf{i}+\mathbf{j}-2\mathbf{k}}{3}$

Solution: (b) Let a unit vector in the plane of $2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{i}-\mathbf{j}+\mathbf{k}$ be

$\hat{\mathbf{a}}=\alpha(2\mathbf{i}+\mathbf{j}+\mathbf{k})+\beta(\mathbf{i}-\mathbf{j}+\mathbf{k})=(2\alpha+\beta)\mathbf{i}+(\alpha-\beta)\mathbf{j}+(\alpha+\beta)\mathbf{k}$

As $\hat{\mathbf{a}}$ is unit vector, we have

$=(2\alpha+\beta)^2+(\alpha-\beta)^2+(\alpha+\beta)^2=1$

$\Rightarrow$ $6\alpha^2+4\alpha\beta+3\beta^2=1$ .....(i)

As $\hat{\mathbf{a}}$ is orthogonal to $5\mathbf{i}+2\mathbf{j}+6\mathbf{k}$, we get $5(2\alpha+\beta)+2(\alpha-\beta)+6(\alpha+\beta)=0 \Rightarrow 18\alpha+9\beta=0 \Rightarrow \beta=-2\alpha$

From (i), we get $6\alpha^2 - 8\alpha^2 + 12\alpha^2 = 1 \Rightarrow \alpha = \pm\frac{1}{\sqrt{10}} \Rightarrow \beta = \mp\frac{2}{\sqrt{10}}$. Thus $\hat{\mathbf{a}} = \pm\left(\frac{3}{\sqrt{10}}\mathbf{j} - \frac{1}{\sqrt{10}}\mathbf{k}\right)$

Example: 23 If θ be the angle between the vectors $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, then

(a) $\cos\theta = \frac{4}{21}$ (b) $\cos\theta = \frac{3}{19}$ (c) $\cos\theta = \frac{2}{19}$ (d) $\cos\theta = \frac{5}{21}$

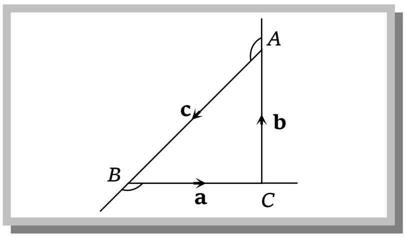
Solution: (a) Angle between $\mathbf{a}$ and $\mathbf{b}$ is given by, $\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{(2\mathbf{i}+2\mathbf{j}-\mathbf{k}).(6\mathbf{i}-3\mathbf{j}+2\mathbf{k})}{\sqrt{2^2+2^2+(-1)^2}.\sqrt{6^2+(-3)^2+2^2}} = \frac{12-6-2}{3.7} = \frac{4}{21}$

Example: 24 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be vectors with magnitudes 3, 4 and 5 respectively and $\mathbf{a}+\mathbf{b}+\mathbf{c} = 0$, then the values of $\mathbf{a}.\mathbf{b}+\mathbf{b}.\mathbf{c}+\mathbf{c}.\mathbf{a}$ is **[DCE 2001; AIEEE 2002; UPSEAT 2002]**

(a) 47 (b) 25 (c) 50 (d) – 25

Solution: (d) We observe, $|\mathbf{a}|^2 + |\mathbf{b}|^2 = 3^2 + 4^2 = 5^2 = |\mathbf{c}|^2$

$\therefore \quad \mathbf{a}.\mathbf{b} = 0$



$\mathbf{b}.\mathbf{c} = |\mathbf{b}||\mathbf{c}|.\cos\left(\pi - \cos^{-1}\frac{4}{5}\right) = 4\times 5\left\{-\cos\left(\cos^{-1}\frac{4}{5}\right)\right\} = 4\times 5\times\left(-\frac{4}{5}\right) = -16$

$\mathbf{c}.\mathbf{a} = |\mathbf{c}||\mathbf{a}|.\cos\left(\pi - \cos^{-1}\frac{3}{5}\right) = 5.3.\left\{-\cos\left(\cos^{-1}\frac{3}{5}\right)\right\} = 5.3.\left(-\frac{3}{5}\right) = -9$

$\therefore \quad \mathbf{a}.\mathbf{b}+\mathbf{b}.\mathbf{c}+\mathbf{c}.\mathbf{a} = 0 - 16 - 9 = -25$

Trick: $\because \quad \mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$

Squaring both the sides $|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2 = 0$

$\Rightarrow \quad |\mathbf{a}^2| + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}.\mathbf{b}+\mathbf{b}.\mathbf{c}+\mathbf{c}.\mathbf{a}) = 0 \Rightarrow 2(\mathbf{a}.\mathbf{b}+\mathbf{b}.\mathbf{c}+\mathbf{c}.\mathbf{a}) = -(9+16+25) \Rightarrow \mathbf{a}.\mathbf{b}+\mathbf{b}.\mathbf{c}+\mathbf{c}.\mathbf{a} = -25$.

Example: 25 The vectors $\mathbf{a} = 2\lambda^2\mathbf{i} + 4\lambda\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} - 2\mathbf{j} + \lambda\mathbf{k}$ make an obtuse angle whereas the angle between $\mathbf{b}$ and $\mathbf{k}$ is acute and less than $\pi/6$, then domain of λ is

(a) $0 < \lambda < \frac{1}{2}$ (b) $\lambda > \sqrt{159}$ (c) $-\frac{1}{2} < \lambda < 0$ (d) Null set

Solution: (d) As angle between $\mathbf{a}$ and $\mathbf{b}$ is obtuse, $\mathbf{a}.\mathbf{b} < 0$

$\Rightarrow \quad (2\lambda^2\mathbf{i} + 4\lambda\mathbf{j} + \mathbf{k}).(7\mathbf{i} - 2\mathbf{j} + \lambda\mathbf{k}) < 0 \Rightarrow 14\lambda^2 - 8\lambda + \lambda < 0 \Rightarrow \lambda(2\lambda - 1) < 0 \Rightarrow 0 < \lambda < \frac{1}{2}$ ......(i)

Angle between $\mathbf{b}$ and $\mathbf{k}$ is acute and less than $\frac{\pi}{6}$.

$\mathbf{b}.\mathbf{k} = |\mathbf{b}|.|\mathbf{k}|\cos\theta \Rightarrow \lambda = \sqrt{53+\lambda^2}.1.\cos\theta \Rightarrow \cos\theta = \frac{\lambda}{\sqrt{53+\lambda^2}}$

$\theta < \frac{\pi}{6} \Rightarrow \cos\theta > \cos\frac{\pi}{6} \Rightarrow \cos\theta > \frac{\sqrt{3}}{2} \Rightarrow \frac{\lambda}{\sqrt{53+\lambda^2}} > \frac{\sqrt{3}}{2} \Rightarrow 4\lambda^2 - 3(53+\lambda^2) > 0 \Rightarrow \lambda^2 > 159 \Rightarrow \lambda < -\sqrt{159}$

or $\quad \lambda > \sqrt{159}$ ......(ii)

From (i) and (ii), $\lambda = \phi$. $\therefore$ Domain of λ is null set.

Example: 26 In cartesian co-ordinates the point *A* is (x_1, y_1) where $x_1 = 1$ on the curve $y = x^2 + x + 10$. The tangent at *A* cuts the *x*-axis at *B*. The value of the dot product $\overrightarrow{OA}.\overrightarrow{AB}$ is

(a) $-\frac{520}{3}$ (b) -148 (c) 140 (d) 12

Solution: (b) Given curve is $y = x^2 + x + 10$ ......(i)

when $x = 1$, $y = 1^2 + 1 + 10 = 12$

$\therefore \quad A \equiv (1, 12)$; $\therefore \overrightarrow{OA} = \mathbf{i} + 12\mathbf{j}$

From (i), $\frac{dy}{dx} = 2x+1$

Equation of tangent at A is $y-12 = \left(\frac{dy}{dx}\right)_{(1,12)}(x-1) \Rightarrow \quad y-12 = (2\times1+1)(x-1) \Rightarrow y-12 = 3x-3$

$\therefore \quad y = 3(x+3)$

This tangent cuts x-axis (*i.e.,* $y=0$) at $(-3,0)$

$\therefore \quad B \equiv (-3,0)$

$\overrightarrow{OB} = -3\mathbf{i} + 0.\,\mathbf{j} = -3\mathbf{i}$; $\overrightarrow{OA}.\,\overrightarrow{AB} = \overrightarrow{OA}\,.(\overrightarrow{OB} - \overrightarrow{OA}) = (\mathbf{i}+12\mathbf{j}).(-3\mathbf{i}-\mathbf{i}-12\mathbf{j}) = (\mathbf{i}+12\mathbf{j}).(-4\mathbf{i}-12\mathbf{j}) = -4-144 = -148$.

Example: 27 If three non-zero vectors are $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$. If $\mathbf{c}$ is the unit vector perpendicular to the vectors $\mathbf{a}$ and $\mathbf{b}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{6}$, then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to **[IIT 1986]**

(a) 0 (b) $\frac{3(\Sigma a_1^2)(\Sigma b_1^2)(\Sigma c_1^2)}{4}$ (c) 1 (d) $\frac{(\Sigma a_1^2)(\Sigma b_1^2)}{4}$

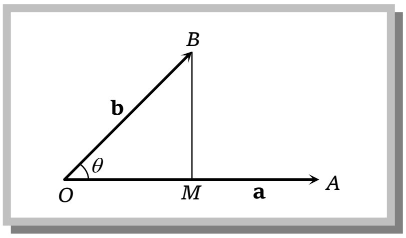
Solution: (d) As $\mathbf{c}$ is the unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$, we have $|\mathbf{c}| = 1, \mathbf{a}.\mathbf{c} = 0 = \mathbf{b}.\mathbf{c}$

Now, $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1^2+a_2^2+a_3^2 & a_1b_1+a_2b_2+a_3b_3 & a_1c_1+a_2c_2+a_3c_3 \\ a_1b_1+a_2b_2+a_3b_3 & b_1^2+b_2^2+b_3^2 & b_1c_1+b_2c_2+b_3c_3 \\ a_1c_1+a_2c_2+a_3c_3 & b_1c_1+b_2c_2+b_3c_3 & c_1^2+c_2^2+c_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} |\mathbf{a}|^2 & \mathbf{a}.\mathbf{b} & \mathbf{a}.\mathbf{c} \\ \mathbf{a}.\mathbf{b} & |\mathbf{b}|^2 & \mathbf{b}.\mathbf{c} \\ \mathbf{a}.\mathbf{c} & \mathbf{b}.\mathbf{c} & |\mathbf{c}|^2 \end{vmatrix} = \begin{vmatrix} |\mathbf{a}|^2 & \mathbf{a}.\mathbf{b} & 0 \\ \mathbf{a}.\mathbf{b} & |\mathbf{b}|^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}.\mathbf{b})^2$$

$$= |\mathbf{a}|^2|\mathbf{b}|^2 - \left(|\mathbf{a}||\mathbf{b}|\cos\frac{\pi}{6}\right)^2 = |\mathbf{a}|^2|\mathbf{b}|^2\left(1-\frac{3}{4}\right) = \frac{1}{4}|\mathbf{a}|^2|\mathbf{b}|^2 = \frac{1}{4}(\Sigma a_1^2)(\Sigma b_1^2)$$

(4) **Components of a vector along and perpendicular to another vector :** If $\mathbf{a}$ and $\mathbf{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Draw $BM \perp OA$. In ΔOBM, we have $\overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB} \Rightarrow \mathbf{b} = \overrightarrow{OM} + \overrightarrow{MB}$



Thus, $\overrightarrow{OM}$ and $\overrightarrow{MB}$ are components of $\mathbf{b}$ along $\mathbf{a}$ and perpendicular to $\mathbf{a}$ respectively.

Now, $\overrightarrow{OM} = (OM)\,\hat{\mathbf{a}} = (OB\cos\theta)\,\hat{\mathbf{a}} = (|\mathbf{b}|\cos\theta)\hat{\mathbf{a}} = \left(|\mathbf{b}|\frac{(\mathbf{a}.\mathbf{b})}{|\mathbf{a}||\mathbf{b}|}\right)\hat{\mathbf{a}} = \left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|}\right)\hat{\mathbf{a}} = \left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$

$\therefore \quad \mathbf{b} = \overrightarrow{OM} + \overrightarrow{MB} \Rightarrow \overrightarrow{MB} = \mathbf{b} - \overrightarrow{OM} = \mathbf{b} - \left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$

Thus, the components of $\mathbf{b}$ along and perpendicular to $\mathbf{a}$ are $\left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$ and $\mathbf{b} - \left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$ respectively.

Example: 28 The projection of $\mathbf{a} = 2\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ on $\mathbf{b} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$ is

(a) $\frac{1}{\sqrt{14}}$ (b) $\frac{2}{\sqrt{14}}$ (c) $\sqrt{14}$ (d) $\frac{-2}{\sqrt{14}}$

Solution: (b) Projection of $\mathbf{a}$ on $\mathbf{b}$ $= \mathbf{a}.\hat{\mathbf{b}} = \mathbf{a}.\dfrac{\mathbf{b}}{|\mathbf{b}|} = \dfrac{(2\mathbf{i}+3\mathbf{j}-2\mathbf{k}).(\mathbf{i}+2\mathbf{j}+3\mathbf{k})}{|\mathbf{i}+2\mathbf{j}+3\mathbf{k}|} = \dfrac{2+6-6}{\sqrt{14}} = \dfrac{2}{\sqrt{14}}$

Example: 29 Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be such that $|\mathbf{u}| = 1, |\mathbf{v}| = 2, |\mathbf{w}| = 3$. If the projection $\mathbf{v}$ along $\mathbf{u}$ is equal to that of $\mathbf{w}$ along $\mathbf{u}$ and $\mathbf{v}, \mathbf{w}$ are perpendicular to each other then $|\mathbf{u}-\mathbf{v}+\mathbf{w}|$ equals

(a) 14 (b) $\sqrt{7}$ (c) $\sqrt{14}$ (d) 2

Solution: (c) Without loss of generality, we can assume $\mathbf{v} = 2\mathbf{i}$ and $\mathbf{w} = 3\mathbf{j}$. Let $\mathbf{u} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $|\mathbf{u}| = 1 \Rightarrow$ $x^2+y^2+z^2 = 1$ .....(i)

Projection of $\mathbf{v}$ along $\mathbf{u}$ = Projection of $\mathbf{w}$ along $\mathbf{u}$

$\Rightarrow \mathbf{v}.\mathbf{u} = \mathbf{w}.\mathbf{u} \Rightarrow 2\mathbf{i}.(x\mathbf{i}+y\mathbf{j}+2\mathbf{k}) = 3\mathbf{j}.(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) \Rightarrow 2x = 3y \Rightarrow 3y-2x = 0$

Now, $|\mathbf{u}-\mathbf{v}-\mathbf{w}| = |x\mathbf{i}+y\mathbf{j}+z\mathbf{k}-2\mathbf{i}+3\mathbf{j}| = |(x-2)\mathbf{i}+(y+3)\mathbf{j}+z\mathbf{k}| = \sqrt{(x-2)^2+(y-3)^2+z^2}$

$$= \sqrt{(x^2+y^2+z^2)+2(3y-2x)+13} = \sqrt{1+2\times 0+13} = \sqrt{14}\,.$$

Example: 30 Let $\mathbf{b} = 3\mathbf{j}+4\mathbf{k}$, $\mathbf{a} = \mathbf{i}+\mathbf{j}$ and let $\mathbf{b}_1$ and $\mathbf{b}_2$ be component vectors of $\mathbf{b}$ parallel and perpendicular to $\mathbf{a}$. If $\mathbf{b}_1 = \dfrac{3}{2}\mathbf{i}+\dfrac{3}{2}\mathbf{j}$, then $\mathbf{b}_2 =$

(a) $\dfrac{3}{2}\mathbf{i}+\dfrac{3}{2}\mathbf{j}+4\mathbf{k}$ (b) $-\dfrac{3}{2}\mathbf{i}+\dfrac{3}{2}\mathbf{j}+4\mathbf{k}$ (c) $-\dfrac{3}{2}\mathbf{i}+\dfrac{3}{2}\mathbf{j}$ (d) None of these

Solution: (b) $\mathbf{b} = \mathbf{b}_1+\mathbf{b}_2$

$\therefore \mathbf{b}_2 = \mathbf{b}-\mathbf{b}_1 = (3\mathbf{j}+4\mathbf{k}) - \left(\dfrac{3}{2}\mathbf{i}+\dfrac{3}{2}\mathbf{j}\right) = -\dfrac{3}{2}\mathbf{i}+\dfrac{3}{2}\mathbf{j}+4\mathbf{k}$

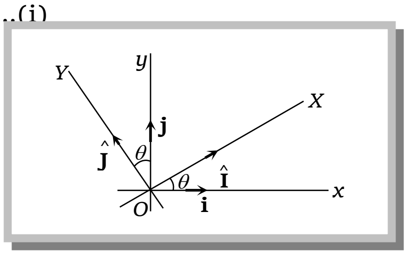
Clearly, $\mathbf{b}_1 = \dfrac{3}{2}(\mathbf{i}+\mathbf{j}) = \dfrac{3}{2}\mathbf{a}$ *i.e.,* $\mathbf{b}_1$ is parallel to $\mathbf{a}$

$\mathbf{b}_2.\mathbf{a} = \left(-\dfrac{3}{2}\mathbf{i}+\dfrac{3}{2}\mathbf{j}+4\mathbf{k}\right).(\mathbf{i}+\mathbf{j}) = 0$; $\therefore \mathbf{b}_2$ is $\perp r$ to $\mathbf{a}$.

Example: 31 A vector $\mathbf{a}$ has components $2p$ and 1 with respect to a rectangular cartesian system. The system is rotated through a certain angle about the origin in the anti-clockwise sense. If $\mathbf{a}$ has components $p+1$ and 1 with respect to the new system, then

(a) $p = 0$ (b) $p = 1$ or $-\dfrac{1}{3}$ (c) $p = -1$ or $\dfrac{1}{3}$ (d) $p = 1$ or -1

Solution: (b) Without loss of generality, we can write $\mathbf{a} = 2p\mathbf{i}+\mathbf{j} = (p+1)\hat{\mathbf{I}}+\hat{\mathbf{J}}$ .....(i)



Now, $\hat{\mathbf{I}} = \cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}$

$\hat{\mathbf{J}} = -\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j}$

$\therefore$ From (i), $2p\mathbf{i}+\mathbf{j} = (p+1)(\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j})+(-\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j})$

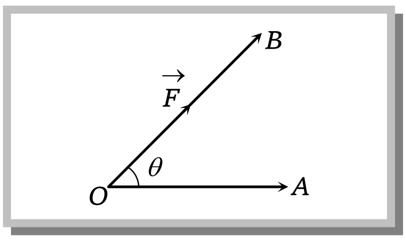
$\Rightarrow 2p\mathbf{i}+\mathbf{j} = \{(p+1)\cos\theta-\sin\theta\}\,\mathbf{i}+\{(p+1)\sin\theta+\cos\theta\}\,\mathbf{j}$

$\Rightarrow 2p = (p+1)\cos\theta-\sin\theta$ ....(ii) and $1 = (p+1)\sin\theta+\cos\theta$ ....(iii)

Squaring and adding, $4p^2+1 = (p+1)^2+1$

$\Rightarrow (p+1)^2 = 4p^2 \Rightarrow p+1 = \pm 2p \Rightarrow p = 1, -\dfrac{1}{3}$.

(5) **Work done by a force :** A force acting on a particle is said to do work if the particle is displaced in a direction which is not perpendicular to the force.



The work done by a force is a scalar quantity and its measure is equal to the product of the magnitude of the force and the resolved part of the displacement in the direction of the force.

If a particle be placed at *O* and a force $\vec{F}$ represented by $\overrightarrow{OB}$ be acting on the particle at *O*. Due to the application of force $\vec{F}$ the particle is displaced in the direction of $\overrightarrow{OA}$. Let $\overrightarrow{OA}$ be the displacement. Then the component of $\overrightarrow{OA}$ in the direction of the force $\vec{F}$ is $|\overrightarrow{OA}|\cos\theta$.

$\therefore$ Work done = $|\vec{F}||\overrightarrow{OA}|\cos\theta = \vec{F}.\overrightarrow{OA} = \vec{F}.\mathbf{d}$, where $\mathbf{d} = \overrightarrow{OA}$ Or Work done = (Force) . (Displacement)

If a number of forces are acting on a particle, then the sum of the works done by the separate forces is equal to the work done by the resultant force.

Example: 32 A particle is acted upon by constant forces $4\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $3\mathbf{i}+\mathbf{j}-\mathbf{k}$ which displace it from a point $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ to the point $5\mathbf{i}+4\mathbf{j}+\mathbf{k}$. The work done in standard units by the force is given by

(a) 15 (b) 30 (c) 25 (d) 40

Solution: (d) Total force $\vec{F} = (4\mathbf{i}+\mathbf{j}-3\mathbf{k})+(3\mathbf{i}+\mathbf{j}-\mathbf{k}) = 7\mathbf{i}+2\mathbf{j}-4\mathbf{k}$

Displacement $\mathbf{d} = (5\mathbf{i}+4\mathbf{j}+\mathbf{k})-(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = 4\mathbf{i}+2\mathbf{j}-2\mathbf{k}$

Work done = $\vec{F}.\mathbf{d} = (7\mathbf{i}+2\mathbf{j}-4\mathbf{k}).(4\mathbf{i}+2\mathbf{j}-2\mathbf{k}) = 28+4+8 = 40$.

Example: 33 A groove is in the form of a broken line *ABC* and the position vectors of the three points are respectively $2\mathbf{i}-3\mathbf{j}+2\mathbf{k}$, $3\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $\mathbf{i}+\mathbf{j}+\mathbf{k}$. A force of magnitude $24\sqrt{3}$ acts on a particle of unit mass kept at the point *A* and moves it along the groove to the point *C*. If the line of action of the force is parallel to the vector $\mathbf{i}+2\mathbf{j}+\mathbf{k}$ all along, the number of units of work done by the force is

(a) $144\sqrt{2}$ (b) $144\sqrt{3}$ (c) $72\sqrt{2}$ (d) $72\sqrt{3}$

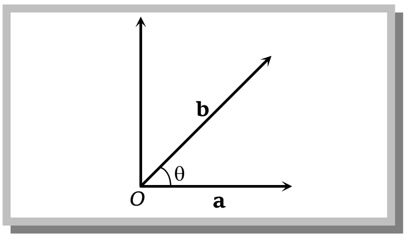
Solution: (c) $\vec{F} = (24\sqrt{3})\dfrac{\mathbf{i}+2\mathbf{j}+\mathbf{k}}{|\mathbf{i}+2\mathbf{j}+\mathbf{k}|} = \dfrac{24\sqrt{3}}{\sqrt{6}}(\mathbf{i}+2\mathbf{j}+\mathbf{k}) = 12\sqrt{2}(\mathbf{i}+2\mathbf{j}+\mathbf{k})$

Displacement $\mathbf{r}$ = position vector of *C* – Position vector of *A* = $(\mathbf{i}+\mathbf{j}+\mathbf{k})-(2\mathbf{i}-3\mathbf{j}+2\mathbf{k}) = (-\mathbf{i}+4\mathbf{j}-\mathbf{k})$

Work done by the force $W = \mathbf{r}.\vec{F} = (-\mathbf{i}+4\mathbf{j}-\mathbf{k}).12\sqrt{2}(\mathbf{i}+2\mathbf{j}+\mathbf{k}) = 12\sqrt{2}(-1+8-1) = 72\sqrt{2}$.

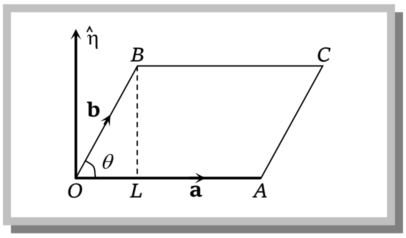
6.10 Vector or Cross product of Two Vectors

Let $\mathbf{a},\mathbf{b}$ be two non-zero, non-parallel vectors. Then the vector product $\mathbf{a}\times\mathbf{b}$, in that order, is defined as a vector whose magnitude is $|\mathbf{a}||\mathbf{b}|\sin\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ whose direction is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ in such a way that $\mathbf{a},\mathbf{b}$ and this direction constitute a right handed system.



In other words, $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\boldsymbol{\eta}}$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $\hat{\boldsymbol{\eta}}$ is a unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a},\mathbf{b},\hat{\boldsymbol{\eta}}$ form a right handed system.

(1) **Geometrical interpretation of vector product :** If $\mathbf{a},\mathbf{b}$ be two non-zero, non-parallel vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively and let θ be the angle between them. Complete the parallelogram *OACB*. Draw $BL \perp OA$.



In ΔOBL, $\sin\theta = \frac{BL}{OB} \Rightarrow BL = OB\sin\theta = |\mathbf{b}|\sin\theta$ ......(i)

Now, $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\boldsymbol{\eta}} = (OA)(BL)\hat{\boldsymbol{\eta}}$

$= (\text{Base} \times \text{Height})\ \hat{\boldsymbol{\eta}} = (\text{area of parallelogram } OACB)\,\hat{\boldsymbol{\eta}}$

= Vector area of the parallelogram *OACB*

Thus, $\mathbf{a}\times\mathbf{b}$ is a vector whose magnitude is equal to the area of the parallelogram having $\mathbf{a}$ and $\mathbf{b}$ as its adjacent sides and whose direction $\hat{\boldsymbol{\eta}}$ is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a},\mathbf{b},\hat{\boldsymbol{\eta}}$ form a right handed system. Hence $\mathbf{a}\times\mathbf{b}$ represents the vector area of the parallelogram having adjacent sides along $\mathbf{a}$ and $\mathbf{b}$.

Thus, area of parallelogram *OACB* = $|\mathbf{a}\times\mathbf{b}|$.

Also, area of $\Delta OAB = \frac{1}{2}$ area of parallelogram *OACB* = $\frac{1}{2}|\mathbf{a}\times\mathbf{b}| = \frac{1}{2}|\overrightarrow{OA}\times\overrightarrow{OB}|$

(2) **Properties of vector product**

(i) Vector product is not commutative *i.e.,* if $\mathbf{a}$ and $\mathbf{b}$ are any two vectors, then $\mathbf{a}\times\mathbf{b} \neq \mathbf{b}\times\mathbf{a}$, however, $\mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a})$

(ii) If $\mathbf{a},\mathbf{b}$ are two vectors and *m* is a scalar, then $m\mathbf{a}\times\mathbf{b} = m(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times m\mathbf{b}$

(iii) If $\mathbf{a},\mathbf{b}$ are two vectors and *m, n* are scalars, then $m\mathbf{a}\times n\mathbf{b} = mn(\mathbf{a}\times\mathbf{b}) = m(\mathbf{a}\times n\mathbf{b}) = n(m\mathbf{a}\times\mathbf{b})$

(iv) Distributivity of vector product over vector addition.

Let $\mathbf{a},\mathbf{b},\mathbf{c}$ be any three vectors. Then

(a) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ (Left distributivity)

(b) $(\mathbf{b}+\mathbf{c})\times\mathbf{a} = \mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}$ (Right distributivity)

(v) For any three vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ we have $\mathbf{a}\times(\mathbf{b}-\mathbf{c}) = \mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}$

(vi) The vector product of two non-zero vectors is zero vector *iff* they are parallel (Collinear) *i.e.,* $\mathbf{a}\times\mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a}\,||\,\mathbf{b}, \mathbf{a},\mathbf{b}$ are non-zero vectors.

It follows from the above property that $\mathbf{a}\times\mathbf{a} = \mathbf{0}$ for every non-zero vector $\mathbf{a}$, which in turn implies that $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$

(vii) Vector product of orthonormal triad of unit vectors **i, j, k** using the definition of the vector product, we obtain $\mathbf{i}\times\mathbf{j} = \mathbf{k}, \mathbf{j}\times\mathbf{k} = \mathbf{i}, \mathbf{k}\times\mathbf{i} = \mathbf{j}$, $\mathbf{j}\times\mathbf{i} = -\mathbf{k}, \mathbf{k}\times\mathbf{j} = -\mathbf{i}, \mathbf{i}\times\mathbf{k} = -\mathbf{j}$

(viii) Lagrange's identity: If $\mathbf{a}$, $\mathbf{b}$ are any two vector then $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}.\mathbf{b})^2$ or $|\mathbf{a}\times\mathbf{b}|^2 + (\mathbf{a}.\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2$

(3) **Vector product in terms of components :** If $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$.

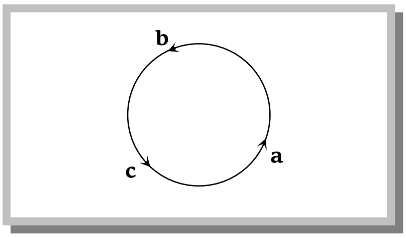
Then, $\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

(4) **Angle between two vectors :** If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then $\sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$

Expression for $\sin\theta$ **:** If $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and θ be angle between $\mathbf{a}$ and $\mathbf{b}$, then

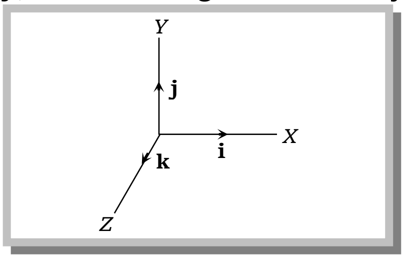
$$\sin^2\theta = \frac{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$$

(5) (i) **Right handed system of vectors :** Three mutually perpendicular vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a right handed system of vector *iff* $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, $\mathbf{b} \times \mathbf{c} = \mathbf{a}$, $\mathbf{c} \times \mathbf{a} = \mathbf{b}$



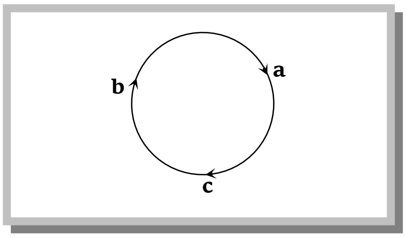
Example: The unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a right-handed system,

$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}$



(ii) **Left handed system of vectors :** The vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, mutually perpendicular to one another form a left handed system of vector *iff*

$\mathbf{c} \times \mathbf{b} = \mathbf{a}, \mathbf{a} \times \mathbf{c} = \mathbf{b}, \mathbf{b} \times \mathbf{a} = \mathbf{c}$



(6) **Vector normal to the plane of two given vectors :** If $\mathbf{a}, \mathbf{b}$ be two non-zero, nonparallel vectors and let θ be the angle between them. $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\, \hat{\boldsymbol{\eta}}$ where $\hat{\boldsymbol{\eta}}$ is a unit vector $\perp$ to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}, \mathbf{b}, \boldsymbol{\eta}$ from a right-handed system.

$$\Rightarrow \quad (\mathbf{a} \times \mathbf{b}) = |\mathbf{a} \times \mathbf{b}|\, \hat{\boldsymbol{\eta}} \Rightarrow \hat{\boldsymbol{\eta}} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

Thus, $\dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$ is a unit vector $\perp$ to the plane of $\mathbf{a}$ and $\mathbf{b}$. Note that $-\dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$ is also a unit vector $\perp$ to the plane of $\mathbf{a}$ and $\mathbf{b}$. Vectors of magnitude 'λ' normal to the plane of $\mathbf{a}$ and $\mathbf{b}$ are given by $\pm \dfrac{\lambda(\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times \mathbf{b}|}$.

Example: 34 If $\mathbf{a}$ is any vector, then $(\mathbf{a} \times \mathbf{i})^2 + (\mathbf{a} \times \mathbf{j})^2 + (\mathbf{a} \times \mathbf{k})^2$ is equal to

[EAMCET 1988; Rajasthan PET 2000; Orissa JEE 2003; MP PET 1984, 2004]

(a) $|\mathbf{a}|^2$ (b) 0 (c) $3|\mathbf{a}|^2$ (d) $2|\mathbf{a}|^2$

Solution: (d) Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$\therefore \quad \mathbf{a} \times \mathbf{i} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times \mathbf{i} = -a_2\mathbf{k} + a_3\mathbf{j}$

$(\mathbf{a} \times \mathbf{i})^2 = (\mathbf{a} \times \mathbf{i}).(\mathbf{a} \times \mathbf{i}) = (-a_2\mathbf{k} + a_3\mathbf{j}).(-a_2\mathbf{k} + a_3\mathbf{j}) = a_2^2 + a_3^2$

Similarly $(\mathbf{a} \times \mathbf{j})^2 = a_3^2 + a_1^2$ and $(\mathbf{a} \times \mathbf{k})^2 = a_1^2 + a_2^2$

$\therefore \quad (\mathbf{a} \times \mathbf{i})^2 + (\mathbf{a} \times \mathbf{j})^2 + (\mathbf{a} \times \mathbf{k})^2 = 2(a_1^2 + a_2^2 + a_3^2) = 2|\mathbf{a}|^2$.

Example: 35 $(\mathbf{a} \times \mathbf{b})^2 + (\mathbf{a}.\mathbf{b})^2$ is equal to

(a) $\mathbf{a}^2 + \mathbf{b}^2$ (b) $\mathbf{a}^2 \mathbf{b}^2$ (c) $2\mathbf{a}.\mathbf{b}$ (d) 1

Solution: (b) $(\mathbf{a}\times\mathbf{b})^2 + (\mathbf{a}.\mathbf{b})^2 = (|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\boldsymbol{\eta}})^2 + (|\mathbf{a}||\mathbf{b}|\cos\theta)^2$

$= |\mathbf{a}|^2|\mathbf{b}|^2(\sin^2\theta(\hat{\boldsymbol{\eta}}.\hat{\boldsymbol{\eta}}) + \cos^2\theta) = (\mathbf{a}.\mathbf{a})(\mathbf{b}.\mathbf{b})(\sin^2\theta + \cos^2\theta) = \mathbf{a}^2\mathbf{b}^2.1 = \mathbf{a}^2\mathbf{b}^2$.

Example: 36 The unit vector perpendicular to the vectors $6\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $3\mathbf{i}-6\mathbf{j}-2\mathbf{k}$, is **[IIT 1989; Rajasthan PET 1996]**

(a) $\frac{2\mathbf{i}-3\mathbf{j}+6\mathbf{k}}{7}$ (b) $\frac{2\mathbf{i}-3\mathbf{j}-6\mathbf{k}}{7}$ (c) $\frac{2\mathbf{i}+3\mathbf{j}-6\mathbf{k}}{7}$ (d) $\frac{2\mathbf{i}+3\mathbf{j}+6\mathbf{k}}{7}$

Solution: (c) Let $\mathbf{a} = 6\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i}-6\mathbf{j}-2\mathbf{k}$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & 3 \\ 3 & -6 & -2 \end{vmatrix} = 14\mathbf{i}+21\mathbf{j}-42\mathbf{k} = 7(2\mathbf{i}+3\mathbf{j}-6\mathbf{k}); \quad |\mathbf{a}\times\mathbf{b}| = 7|2\mathbf{i}+3\mathbf{j}-6\mathbf{k}| = 7.7$$

$\therefore \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|} = \frac{1}{7}(2\mathbf{i}+3\mathbf{j}-6\mathbf{k})$, which is a unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

Example: 37 The sine of the angle between the vectors $\mathbf{a} = 3\mathbf{i}+\mathbf{j}+\mathbf{k}, \mathbf{b} = 2\mathbf{i}-2\mathbf{j}+\mathbf{k}$ is **[Pb. CET 1988]**

(a) $\sqrt{\frac{74}{99}}$ (b) $\sqrt{\frac{25}{99}}$ (c) $\sqrt{\frac{37}{99}}$ (d) $\frac{5}{\sqrt{41}}$

Solution: (a) $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 1 \\ 2 & -2 & 1 \end{vmatrix} = 3\mathbf{i}-\mathbf{j}-8\mathbf{k}$; $\sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} = \frac{\sqrt{74}}{\sqrt{11}.\sqrt{9}} = \sqrt{\frac{74}{99}}$

Example: 38 The vectors $\mathbf{c}$, $\mathbf{a} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $\mathbf{b} = \mathbf{j}$ are such that **a, c, b** form a right handed system, then **c** is **[DCE 1999]**

(a) $z\mathbf{i}-x\mathbf{k}$ (b) 0 (c) $y\mathbf{j}$ (d) $-z\mathbf{i}+x\mathbf{k}$

Solution: (a) $\mathbf{a}, \mathbf{c}, \mathbf{b}$ form a right handed system. Hence, $\mathbf{b}\times\mathbf{a} = \mathbf{c} \Rightarrow \mathbf{c} = \mathbf{j}\times(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) = -x\mathbf{k}+z\mathbf{i} = z\mathbf{i}-x\mathbf{k}$

(7) Area of parallelogram and Triangle

(i) The area of a parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a}\times\mathbf{b}|$.

(ii) The area of a parallelogram with diagonals $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.

(iii) The area of a plane quadrilateral $ABCD$ is $\frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}|$, where AC and BD are its diagonals.

(iv) The area of a triangle with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

(v) The area of a triangle ABC is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$ or $\frac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BA}|$ or $\frac{1}{2}|\overrightarrow{CB}\times\overrightarrow{CA}|$

(vi) If $\mathbf{a},\mathbf{b},\mathbf{c}$ are position vectors of vertices of a ΔABC, then its area $= \frac{1}{2}|(\mathbf{a}\times\mathbf{b})+(\mathbf{b}\times\mathbf{c})+(\mathbf{c}\times\mathbf{a})|$

Note : ❑ Three points with position vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ are collinear if $(\mathbf{a}\times\mathbf{b})+(\mathbf{b}\times\mathbf{c})+(\mathbf{c}\times\mathbf{a}) = \mathbf{0}$

Example: 39 The area of a triangle whose vertices are $A(1,-1,2)$, $B(2,1,-1)$ and $C(3,-1,2)$ is

(a) 13 (b) $\sqrt{13}$ (c) 6 (d) $\sqrt{6}$

Solution: (b) $\overrightarrow{AB}=(2\mathbf{i}+\mathbf{j}-\mathbf{k})-(\mathbf{i}-\mathbf{j}+2\mathbf{k})=\mathbf{i}+2\mathbf{j}-3\mathbf{k}$, $\overrightarrow{AC}=(3\mathbf{i}-\mathbf{j}+2\mathbf{k})-(\mathbf{i}-\mathbf{j}+2\mathbf{k})=2\mathbf{i}$

Area of triangle ABC = $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$ = $\frac{1}{2}|(\mathbf{i}+2\mathbf{j}-3\mathbf{k})\times 2\mathbf{i}|$ = $\frac{1}{2}|-4\mathbf{k}-6\mathbf{j}|=|-3\mathbf{j}-2\mathbf{k}|=\sqrt{13}$

Example: 40 If $\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k}, \mathbf{b}=\mathbf{i}+3\mathbf{j}+5\mathbf{k}$ and $\mathbf{c}=7\mathbf{i}+9\mathbf{j}+11\mathbf{k}$, then the area of the parallelogram having diagonals $\mathbf{a}+\mathbf{b}$ and $\mathbf{b}+\mathbf{c}$ is **[Haryana CET 2002]**

(a) $4\sqrt{6}$ (b) $\frac{1}{2}\sqrt{21}$ (c) $\frac{\sqrt{6}}{2}$ (d) $\sqrt{6}$

Solution: (a) Area of the parallelogram with diagonals $\mathbf{a}+\mathbf{b}$ and $\mathbf{b}+\mathbf{c}$ = $\frac{1}{2}|(\mathbf{a}+\mathbf{b})\times(\mathbf{b}+\mathbf{c})|$

$= \frac{1}{2}|\{(\mathbf{i}+\mathbf{j}+\mathbf{k})+(\mathbf{i}+3\mathbf{j}+5\mathbf{k})\}\times\{(\mathbf{i}+3\mathbf{j}+5\mathbf{k})+(7\mathbf{i}+9\mathbf{j}+11\mathbf{k})\}|$ = $\frac{1}{2}|\{(2\mathbf{i}+4\mathbf{j}+6\mathbf{k})\times(8\mathbf{i}+12\mathbf{j}+16\mathbf{k})\}|$

$= 4|(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\times(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})|$ = $4\left|\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 3\\ 2 & 3 & 4\end{vmatrix}\right| = 4|-\mathbf{i}+2\mathbf{j}-\mathbf{k}|$ = $4\sqrt{6}$

Example: 41 The position vectors of the vertices of a quadrilateral $ABCD$ are **a, b, c** and **d** respectively. Area of the quadrilateral formed by joining the middle points of its sides is

(a) $\frac{1}{4}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{d}+\mathbf{d}\times\mathbf{a}|$

(b) $\frac{1}{4}|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{d}+\mathbf{a}\times\mathbf{d}+\mathbf{b}\times\mathbf{a}|$

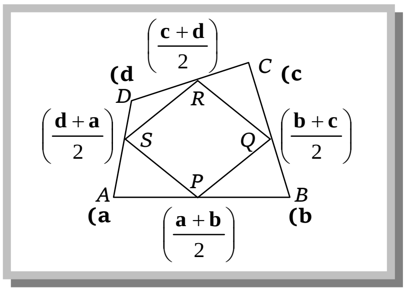
(c) $\frac{1}{4}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{d}+\mathbf{d}\times\mathbf{a}|$

(d) $\frac{1}{4}|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{d}+\mathbf{d}\times\mathbf{b}|$

Solution: (c) Let P, Q, R, S be the middle points of the sides of the quadrilateral $ABCD$.

Position vector of P = $\frac{\mathbf{a}+\mathbf{b}}{2}$, that of $Q=\frac{\mathbf{b}+\mathbf{c}}{2}$, that of R = $\frac{\mathbf{c}+\mathbf{d}}{2}$ and that of S = $\frac{\mathbf{d}+\mathbf{a}}{2}$

Mid point of diagonal $SQ \equiv \left(\frac{\mathbf{d}+\mathbf{a}}{2}+\frac{\mathbf{b}+\mathbf{c}}{2}\right)\frac{1}{2}=\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})$



Similarly mid point of $PR \equiv \frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})$

As the diagonals bisect each other, $PQRS$ is a parallelogram.

$\overrightarrow{SP}=\frac{\mathbf{a}+\mathbf{b}}{2}-\frac{\mathbf{d}+\mathbf{a}}{2}=\frac{\mathbf{b}-\mathbf{d}}{2}$; $\overrightarrow{SR}=\frac{\mathbf{c}+\mathbf{d}}{2}-\frac{\mathbf{d}+\mathbf{a}}{2}=\frac{\mathbf{c}-\mathbf{a}}{2}$

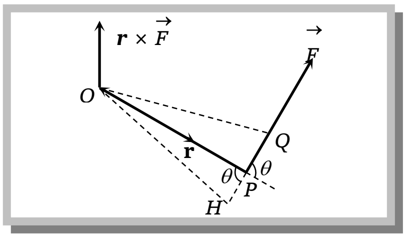
Area of parallelogram $PQRS$ = $|\overrightarrow{SP}\times\overrightarrow{SR}|=\left|\left(\frac{\mathbf{b}-\mathbf{d}}{2}\right)\times\left(\frac{\mathbf{c}-\mathbf{a}}{2}\right)\right|$

$= \frac{1}{4}|\mathbf{b}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}-\mathbf{d}\times\mathbf{c}+\mathbf{d}\times\mathbf{a}| = \frac{1}{4}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{d}+\mathbf{d}\times\mathbf{a}|$.

6.11 Moment of a Force and Couple

(1) **Moment of a force**

(i) **About a point :** Let a force $\vec{F}$ be applied at a point P of a rigid body. Then the moment of $\vec{F}$ about a point O measures the tendency of $\vec{F}$ to turn the body about point O. If this tendency of rotation about O is in anticlockwise direction, the moment is positive, otherwise it is negative.



Let $\mathbf{r}$ be the position vector of P relative to O. Then the moment or torque of $\vec{F}$ about the point O is defined as the vector $\vec{M} = \mathbf{r} \times \vec{F}$.

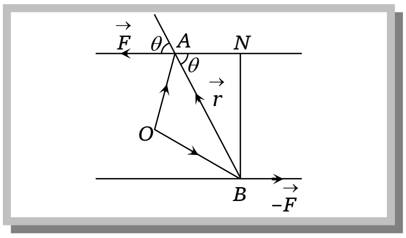
If several forces are acting through the same point P, then the vector sum of the moment of the separate forces about O is equal to the moment of their resultant force about O.

(ii) **About a line:** The moment of a force $\vec{F}$ acting at a point P about a line L is a scalar given by $(\mathbf{r} \times \vec{F}).\hat{\mathbf{a}}$ where $\hat{\mathbf{a}}$ is a unit vector in the direction of the line, and $\overrightarrow{OP} = \mathbf{r}$, where O is any point on the line.

Thus, the moment of a force $\vec{F}$ about a line is the resultant part (component) along this line, of the moment of $\vec{F}$ about any point on the line.

Note : ❑ The moment of a force about a point is a vector while the moment about a straight line is a scalar quantity.

(2) **Moment of a couple :** A system consisting of a pair of equal unlike parallel forces is called a couple. The vector sum of two forces of a couple is always zero vector.



The moment of a couple is a vector perpendicular to the plane of couple and its magnitude is the product of the magnitude of either force with the perpendicular distance between the lines of the forces.

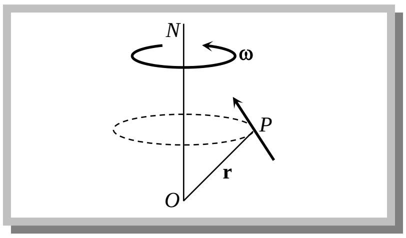
$\vec{M} = \mathbf{r} \times \vec{F}$, where $\mathbf{r} = \overrightarrow{BA}$

$|\vec{M}| = |\overrightarrow{BA} \times \vec{F}| = |\vec{F}||\overrightarrow{BA}|\sin\theta$, where θ is the angle between $\overrightarrow{BA}$ and $\vec{F}$

$= |\vec{F}|(BN) = |\vec{F}|a$

where $a = BN$ is the arm of the couple and $+ve$ or $-ve$ sign is to be taken according as the forces indicate a counter-clockwise rotation or clockwise rotation.

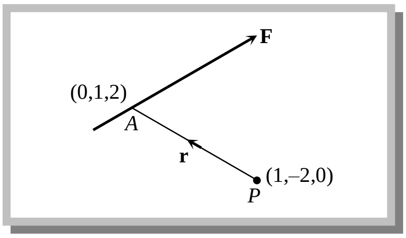
(3) **Rotation about an axis :** When a rigid body rotates about a fixed axis ON with an angular velocity ω, then the velocity $\mathbf{v}$ of a particle P is given by $\mathbf{v} = \omega \times \mathbf{r}$, where $\mathbf{r} = \overrightarrow{OP}$ and $\omega = |\omega|$ (unit vector along ON)



Example: 42 Three forces $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$ are acting on a particle at the point (0, 1, 2). The magnitude of the moment of the forces about the point (1, – 2, 0) is **[MNR 1983]**

(a) $2\sqrt{35}$ (b) $6\sqrt{10}$ (c) $4\sqrt{17}$ (d) None of these

Solution: (b) Total force $\vec{F} = (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) + (\mathbf{i} - \mathbf{j} + \mathbf{k}) = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$



Moment of the forces about $P = \mathbf{r} \times \vec{F} = \overrightarrow{PA} \times \vec{F}$

$\overrightarrow{PA} = (0-1)\mathbf{i} + (1+2)\mathbf{j} + (2-0)\mathbf{k} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

$\therefore$ Moment about $P = (-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \times (4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 2 \\ 4 & 4 & 2 \end{vmatrix} = -2\mathbf{i} + 10\mathbf{j} - 16\mathbf{k}$

Magnitude of the moment $= |-2\mathbf{i} + 10\mathbf{j} - 16\mathbf{k}| = 2\sqrt{1^2 + 5^2 + 8^2} = 2\sqrt{90} = 6\sqrt{10}$

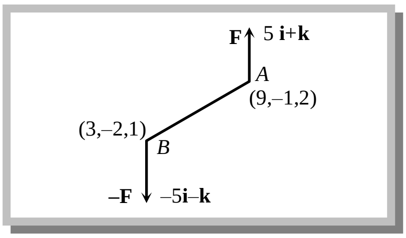
Example: 43 The moment of the couple formed by the forces $5\mathbf{i} + \mathbf{k}$ and $-5\mathbf{i} - \mathbf{k}$ acting at the points $(9, -1, 2)$ and $(3, -2, 1)$ respectively is **[AMU 1998]**

(a) $-\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ (b) $\mathbf{i} - \mathbf{j} - 5\mathbf{k}$ (c) $2\mathbf{i} - 2\mathbf{j} - 10\mathbf{k}$ (d) $-2\mathbf{i} + 2\mathbf{j} + 10\mathbf{k}$

Solution: (b) Moment of the couple,

$= \overrightarrow{BA} \times \vec{F} = \{(9-3)\mathbf{i} + (-1+2)\mathbf{j} + (2-1)\mathbf{k}\} \times (5\mathbf{i} + \mathbf{k})$

$= (6\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (5\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 1 & 1 \\ 5 & 0 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 5\mathbf{k}$



6.12 Scalar Triple Product

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three vectors, then their scalar triple product is defined as the dot product of two vectors $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$. It is generally denoted by $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ or $[\mathbf{abc}]$. It is read as box product of $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Similarly other scalar triple products can be defined as $(\mathbf{b} \times \mathbf{c}).\mathbf{a}, (\mathbf{c} \times \mathbf{a}).\mathbf{b}$. By the property of scalar product of two vectors we can say, $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}).\mathbf{c}$

(1) **Geometrical interpretation of scalar triple product :** The scalar triple product of three vectors is equal to the volume of the parallelopiped whose three coterminous edges are represented by the given vectors. $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a right handed system of vectors. Therefore $(\mathbf{a} \times \mathbf{b}).\mathbf{c} = [\mathbf{abc}] =$ volume of the parallelopiped, whose coterminous edges are **a**, **b** and **c**.

(2) **Properties of scalar triple product**

(i) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are cyclically permuted, the value of scalar triple product remains the same. *i.e.,* $(\mathbf{a} \times \mathbf{b}).\mathbf{c} = (\mathbf{b} \times \mathbf{c}).\mathbf{a} = (\mathbf{c} \times \mathbf{a}).\mathbf{b}$ or $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = [\mathbf{b}\,\mathbf{c}\,\mathbf{a}] = [\mathbf{c}\,\mathbf{a}\,\mathbf{b}]$

(ii) The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude *i.e.,* $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = -[\mathbf{b}\,\mathbf{a}\,\mathbf{c}] = -[\mathbf{c}\,\mathbf{b}\,\mathbf{a}] = -[\mathbf{a}\,\mathbf{c}\,\mathbf{b}]$

(iii) In scalar triple product the positions of dot and cross can be interchanged provided that the cyclic order of the vectors remains same *i.e.,* $(\mathbf{a} \times \mathbf{b}).\mathbf{c} = \mathbf{a}.(\mathbf{b} \times \mathbf{c})$

(iv) The scalar triple product of three vectors is zero if any two of them are equal.

(v) For any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and scalar λ, $[\lambda \mathbf{a}\, \mathbf{b}\, \mathbf{c}] = \lambda[\mathbf{a}\, \mathbf{b}\, \mathbf{c}]$

(vi) The scalar triple product of three vectors is zero if any two of them are parallel or collinear.

(vii) If $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are four vectors, then $[(\mathbf{a}+\mathbf{b})\ \mathbf{c}\ \mathbf{d}] = [\mathbf{a}\, \mathbf{c}\, \mathbf{d}] + [\mathbf{b} \mathbf{c} \mathbf{d}]$

(viii) The necessary and sufficient condition for three non-zero non-collinear vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to be coplanar is that $[\mathbf{a}\, \mathbf{b}\, \mathbf{c}] = 0$ *i.e.*, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar $\Leftrightarrow [\mathbf{a}\, \mathbf{b} \mathbf{c}] = 0$.

(ix) Four points with position vectors **a**, **b**, **c** and **d** will be coplanar, if $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{d}\ \mathbf{c}\ \mathbf{a}] + [\mathbf{d}\ \mathbf{a}\ \mathbf{b}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

(3) **Scalar triple product in terms of components**

(i) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ be three vectors.

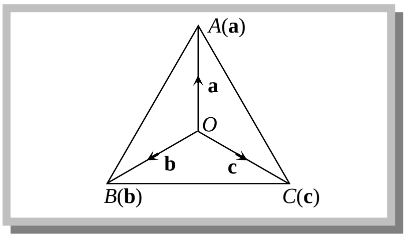
Then, $[\mathbf{a}\, \mathbf{b}\, \mathbf{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(ii) If $\mathbf{a} = a_1\mathbf{l} + a_2\mathbf{m} + a_3\mathbf{n}, \mathbf{b} = b_1\mathbf{l} + b_2\mathbf{m} + b_3\mathbf{n}$ and $\mathbf{c} = c_1\mathbf{l} + c_2\mathbf{m} + c_3\mathbf{n}$, then $[\mathbf{a}\, \mathbf{b}\, \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\mathbf{l}\, \mathbf{m}\, \mathbf{n}]$

(iii) For any three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$

(a) $[\mathbf{a}+\mathbf{b}\ \ \mathbf{b}+\mathbf{c}\ \ \mathbf{c}+\mathbf{a}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ (b) $[\mathbf{a}-\mathbf{b}\ \ \mathbf{b}-\mathbf{c}\ \ \mathbf{c}-\mathbf{a}] = 0$ (c) $[\mathbf{a}\times\mathbf{b}\ \ \mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

(4) **Tetrahedron :** A tetrahedron is a three-dimensional figure formed by four triangle *OABC* is a tetrahedron with ΔABC as the base. OA, OB, OC, AB, BC and CA are known as edges of the tetrahedron. $OA, BC; OB, CA$ and OC, AB are known as the pairs of opposite edges. A tetrahedron in which all edges are equal, is called a regular tetrahedron.



Properties of tetrahedron

(i) If two pairs of opposite edges of a tetrahedron are perpendicular, then the opposite edges of the third pair are also perpendicular to each other.

(ii) In a tetrahedron, the sum of the squares of two opposite edges is the same for each pair.

(iii) Any two opposite edges in a regular tetrahedron are perpendicular.

Volume of a tetrahedron

(i) The volume of a tetrahedron = $\frac{1}{3}$(area of the base) (corresponding altitude)

$= \frac{1}{3}.\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}||\overrightarrow{ED}| = \frac{1}{6}|\overrightarrow{AB} \times \overrightarrow{AC}||\overrightarrow{ED}|\cos 0^o$ for $\overrightarrow{AB} \times \overrightarrow{AC} \| \overrightarrow{ED}$

$= \frac{1}{6}(\overrightarrow{AB} \times \overrightarrow{AC}).\overrightarrow{ED} = \frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{EA} + \overrightarrow{AD}] = \frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]$.

Because $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{EA}$ are coplanar, so $[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{EA}] = 0$

(ii) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are position vectors of vertices *A*, *B* and *C* with respect to *O*, then volume of tetrahedron

$OABC = \frac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

(iii) If $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are position vectors of vertices *A*, *B*, *C*, *D* of a tetrahedron *ABCD*, then

its volume = $\frac{1}{6}[\mathbf{b}-\mathbf{a}\ \mathbf{c}-\mathbf{a}\ \mathbf{d}-\mathbf{a}]$

(5) **Reciprocal system of vectors :** Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three non-coplanar vectors, and let $\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]}$, $\mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{abc}]}$, $\mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{abc}]}$.

$\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are said to form a reciprocal system of vectors for the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ form a reciprocal system of vectors, then

(i) $\mathbf{a}.\mathbf{a}' = \mathbf{b}.\mathbf{b}' = \mathbf{c}.\mathbf{c}' = 1$ (ii) $\mathbf{a}.\mathbf{b}' = \mathbf{a}.\mathbf{c}' = 0; \mathbf{b}.\mathbf{c}' = \mathbf{b}.\mathbf{a}' = 0; \mathbf{c}.\mathbf{a}' = \mathbf{c}.\mathbf{b}' = 0$

(iii) $[\mathbf{a}'\mathbf{b}'\mathbf{c}'] = \frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$ (v) $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar *iff* so are $\mathbf{a}', \mathbf{b}', \mathbf{c}'$.

Example: 44 If **u, v** and **w** are three non-coplanar vectors, then $(\mathbf{u} + \mathbf{v} - \mathbf{w}).[(\mathbf{u} - \mathbf{v}) \times (\mathbf{v} - \mathbf{w})]$ equals **[AIEEE 2003]**

(a) 0 (b) $\mathbf{u}.(\mathbf{v} \times \mathbf{w})$ (c) $\mathbf{u}.(\mathbf{w} \times \mathbf{v})$ (d) $3\mathbf{u}.(\mathbf{v} \times \mathbf{w})$

Solution: (b) $(\mathbf{u} + \mathbf{v} - \mathbf{w}).[\mathbf{u} - \mathbf{v} \times (\mathbf{v} - \mathbf{w})] = (\mathbf{u} + \mathbf{v} - \mathbf{w}).[(\mathbf{u} \times \mathbf{v}) - (\mathbf{u} \times \mathbf{w}) - 0 + (\mathbf{v} \times \mathbf{w})]$

$= [\mathbf{u}\ \mathbf{u}\ \mathbf{v}] + [\mathbf{v}\ \mathbf{u}\ \mathbf{v}] - [\mathbf{w}\ \mathbf{u}\ \mathbf{v}] - [\mathbf{u}\ \mathbf{u}\ \mathbf{w}] - [\mathbf{v}\ \mathbf{u}\ \mathbf{w}] + [\mathbf{w}\ \mathbf{u}\ \mathbf{w}] + [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] + [\mathbf{v}\ \mathbf{v}\ \mathbf{w}] - [\mathbf{w}\ \mathbf{v}\ \mathbf{w}]$

$= 0 + 0 - [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] - 0 + [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] + 0 + [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] + 0 - 0 = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \mathbf{u}.(\mathbf{v} \times \mathbf{w})$.

Example: 45 The value of 'a' so that the volume of parallelopiped formed by $\mathbf{i} + a\mathbf{j} + \mathbf{k}$; $\mathbf{j} + a\mathbf{k}$ and $a\mathbf{i} + \mathbf{k}$ becomes minimum is **[IIT Screening 2003]**

(a) -3 (b) 3 (c) $1/\sqrt{3}$ (d) $\sqrt{3}$

Solution: (c) Volume of the parallelepiped

V = $[\mathbf{i} + a\mathbf{j} + \mathbf{k}\ \ \mathbf{j} + a\mathbf{k}\ \ a\mathbf{i} + \mathbf{k}] = (\mathbf{i} + a\mathbf{j} + \mathbf{k}).\{(\mathbf{j} + a\mathbf{k}) \times (a\mathbf{i} + \mathbf{k})\} = (\mathbf{i} + a\mathbf{j} + \mathbf{k}).\{\mathbf{i} + a^2\mathbf{j} - a\mathbf{k}\} = 1 + a^3 - a$

$\frac{dV}{da} = 3a^2 - 1$; $\frac{d^2V}{da^2} = 6a$; $\frac{dV}{da} = 0 \Rightarrow 3a^2 - 1 = 0 \Rightarrow a = \pm\frac{1}{\sqrt{3}}$

At $a = \frac{1}{\sqrt{3}}, \frac{d^2V}{da^2} = \frac{6}{\sqrt{3}} > 0$

$\therefore$ V is minimum at $a = \frac{1}{\sqrt{3}}$

Example: 46 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be any three non-zero non-coplanar vectors, then any vector $\mathbf{r}$ is equal to

(a) $z\mathbf{a} + x\mathbf{b} + y\mathbf{c}$ (b) $x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$ (c) $y\mathbf{a} + z\mathbf{b} + x\mathbf{c}$ (d) None of these

Where $x = \frac{[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$, $y = \frac{[\mathbf{r}\ \mathbf{c}\ \mathbf{a}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$, $z = \frac{[\mathbf{r}\ \mathbf{a}\ \mathbf{b}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

Solution: (b) As **a, b, c** are three non-coplanar vectors, we may assume $\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$

$[\mathbf{r}\ \mathbf{b}\ \mathbf{c}] = (\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}).(\mathbf{b} \times \mathbf{c}) = \alpha\{\mathbf{a}.(\mathbf{b} \times \mathbf{c})\} = \alpha[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \Rightarrow \alpha = \frac{[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

But $x = \frac{[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{abc}]}$; $\therefore \alpha = x$

Similarly $\beta = y, \gamma = z$; $\therefore$ $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$.

Example: 47 If **a, b, c** are non-coplanar vectors and λ is a real number, then the vectors $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}, \lambda\mathbf{b} + 4\mathbf{c}$ and $(2\lambda - 1)\mathbf{c}$ are non-coplanar for **[AIEEE 2004]**

(a) No value of λ (b) All except one value of λ

(c) All except two values of λ (d) All values of λ

Solution: (c) As $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar vectors. $\therefore$ $[\mathbf{abc}] \neq 0$

Now, $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}, \lambda\mathbf{b} + 4\mathbf{c}$ and $(2\lambda - 1)\mathbf{c}$ will be non-coplanar *iff* $(\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}).\{(\lambda\mathbf{b} + 4\mathbf{c}) \times (2\lambda - 1)\mathbf{c}\} \neq 0$

i.e., $(\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}).\{\lambda(2\lambda - 1)(\mathbf{b} \times \mathbf{c})\} \neq 0$ *i.e.,* $\lambda(2\lambda - 1)[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$

$\therefore$ $\lambda \neq 0, \frac{1}{2}$

Thus, given vectors will be non-coplanar for all values of λ except two values: $\lambda = 0$ and $\frac{1}{2}$.

Example: 48 x, y, z are distinct scalars such that $[x\mathbf{a}+y\mathbf{b}+z\mathbf{c}, x\mathbf{b}+y\mathbf{c}+z\mathbf{a}, x\mathbf{c}+y\mathbf{a}+z\mathbf{b}]=0$ where **a**, **b**, **c** are non-coplanar vectors then

(a) $x+y+z=0$ (b) $xy+yz+zx=0$ (c) $x^3+y^3+z^3=0$ (d) $x^2+y^2+z^2=0$

Solution: (a) **a, b, c** are non-coplanar

$\therefore \; [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]\neq 0$

Now, $[x\mathbf{a}+y\mathbf{b}+z\mathbf{c}, x\mathbf{b}+y\mathbf{c}+z\mathbf{a}, x\mathbf{c}+y\mathbf{a}+z\mathbf{b}]=0$

$\Rightarrow (x\mathbf{a}+y\mathbf{b}+z\mathbf{c}).\{(x\mathbf{b}+y\mathbf{c}+z\mathbf{a})\times(x\mathbf{c}+y\mathbf{a}+z\mathbf{b})\}=0 \Rightarrow (x\mathbf{a}+y\mathbf{b}+z\mathbf{c}).\{(x^2-yz)(\mathbf{b}\times\mathbf{c})+(z^2-xy)(\mathbf{a}\times\mathbf{b})+(y^2-zx)(\mathbf{c}\times\mathbf{a})\}=0$

$\Rightarrow x(x^2-yz)[\mathbf{abc}]+y(y^2-zx)[\mathbf{b}\,\mathbf{c}\,\mathbf{a}]+z(z^2-xy)[\mathbf{c}\,\mathbf{a}\,\mathbf{b}]=0 \Rightarrow \; (x^3-xyz)[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]+(y^3-xyz)[\mathbf{abc}]+(z^3-xyz)[\mathbf{abc}]=0$

As $[\mathbf{abc}]\neq 0$, $x^3+y^3+z^3-3xyz=0 \Rightarrow (x+y+z)(x^2+y^2+z^2-xy-yz-zx)=0$

$\Rightarrow \; \frac{1}{2}(x+y+z)\{(x-y)^2+(y-z)^2+(z-x)^2\}=0 \Rightarrow x+y+z=0$ or $x=y=z$

But x, y, z are distinct. $\therefore \; x+y+z=0$.

6.13 Vector Triple Product

Let $\mathbf{a},\mathbf{b},\mathbf{c}$ be any three vectors, then the vectors $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ are called vector triple product of **a, b, c** .

Thus, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\,.\,\mathbf{c})\mathbf{b}-(\mathbf{a}\,.\,\mathbf{b})\mathbf{c}$

(1) Properties of vector triple product

(i) The vector triple product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is a linear combination of those two vectors which are within brackets.

(ii) The vector $\mathbf{r}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is perpendicular to $\mathbf{a}$ and lies in the plane of $\mathbf{b}$ and $\mathbf{c}$.

(iii) The formula $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\,.\,\mathbf{c})\mathbf{b}-(\mathbf{a}\,.\,\mathbf{b})\mathbf{c}$ is true only when the vector outside the bracket is on the left most side. If it is not, we first shift on left by using the properties of cross product and then apply the same formula.

Thus, $(\mathbf{b}\times\mathbf{c})\times\mathbf{a}=-\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\}=-\{(\mathbf{a}\,.\,\mathbf{c})\mathbf{b}-(\mathbf{a}\,.\,\mathbf{b})\mathbf{c}\}=(\mathbf{a}\,.\,\mathbf{b})\mathbf{c}-(\mathbf{a}\,.\,\mathbf{c})\mathbf{b}$

(iv) If $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and $\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$

then $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_2c_3-b_3c_2 & b_3c_1-b_1c_3 & b_1c_2-b_2c_1\end{vmatrix}$

Note : ❑ Vector triple product is a vector quantity.

❑ $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\neq(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$

Example: 49 Let $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ be non-zero vectors such that $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$. If θ is the acute angle between the vectors $\mathbf{b}$ and $\mathbf{c}$, then $\sin\theta$ equals **[AIEEE 2004]**

(a) $\frac{2\sqrt{2}}{3}$ (b) $\frac{\sqrt{2}}{3}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

Solution: (a) $\because \; (\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a} \Rightarrow (\mathbf{a}\,.\,\mathbf{c})\mathbf{b}-(\mathbf{b}\,.\,\mathbf{c})\mathbf{a}=\frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$

$\Rightarrow \; (\mathbf{a}\,.\,\mathbf{c})\mathbf{b}=\{(\mathbf{b}\,.\,\mathbf{c})+\frac{1}{3}|\mathbf{b}||\mathbf{c}|\}\mathbf{a} \Rightarrow (\mathbf{a}\,.\,\mathbf{c})\mathbf{b}=|\mathbf{b}||\mathbf{c}|\left\{\cos\theta+\frac{1}{3}\right\}\mathbf{a}$

As $\mathbf{a}$ and $\mathbf{b}$ are not parallel, $\mathbf{a}\,.\,\mathbf{c}=0$ and $\cos\theta+\frac{1}{3}=0$

$\Rightarrow \; \cos\theta=-\frac{1}{3} \Rightarrow \sin\theta=\frac{2\sqrt{2}}{3}$

Example: 50 If $\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k}, \mathbf{b}=\mathbf{i}+\mathbf{j}, \mathbf{c}=\mathbf{i}$ and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\lambda\mathbf{a}+\mu\mathbf{b}$, then $\lambda+\mu=$ **[EAMCT 2003]**

(a) 0 (b) 1 (c) 2 (d) 3

Solution: (a) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \lambda\mathbf{a}+\mu\mathbf{b} \Rightarrow (\mathbf{a}.\mathbf{c})\mathbf{b}-(\mathbf{b}.\mathbf{c})\mathbf{a} = \lambda\mathbf{a}+\mu\mathbf{b} \Rightarrow \lambda = -\mathbf{b}.\mathbf{c}$, $\mu = \mathbf{a}.\mathbf{c}$

$\therefore\ \lambda+\mu = \mathbf{a}.\mathbf{c}-\mathbf{b}.\mathbf{c} = (\mathbf{a}-\mathbf{b}).\mathbf{c} = \{(\mathbf{i}+\mathbf{j}+\mathbf{k})-(\mathbf{i}+\mathbf{j})\}.\mathbf{i} = \mathbf{k}.\mathbf{i} = 0$.

Example: 51 If $\mathbf{a},\mathbf{b},\mathbf{c}$ and $\mathbf{p},\mathbf{q},\mathbf{r}$ are reciprocal system of vectors, then $\mathbf{a}\times\mathbf{p}+\mathbf{b}\times\mathbf{q}+\mathbf{c}\times\mathbf{r}$ equals

(a) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ (b) $(\mathbf{p}+\mathbf{q}+\mathbf{r})$ (c) $\mathbf{0}$ (d) $\mathbf{a}+\mathbf{b}+\mathbf{c}$

Solution: (c) $\mathbf{p} = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$, $\mathbf{q} = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$, $\mathbf{r} = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

$\mathbf{a}\times\mathbf{p} = \mathbf{a}\times\frac{(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{(\mathbf{a}.\mathbf{c})\mathbf{b}-(\mathbf{a}.\mathbf{b})\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

Similarly $\mathbf{b}\times\mathbf{q} = \frac{(\mathbf{a}.\mathbf{b})\mathbf{c}-(\mathbf{b}.\mathbf{c})\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$ and $\mathbf{c}\times\mathbf{r} = \frac{(\mathbf{b}.\mathbf{c})\mathbf{a}-(\mathbf{a}.\mathbf{c})\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

$\therefore\ \mathbf{a}\times\mathbf{p}+\mathbf{b}\times\mathbf{q}+\mathbf{c}\times\mathbf{r} = \frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\{(\mathbf{a}.\mathbf{c})\mathbf{b}-(\mathbf{a}.\mathbf{b})\mathbf{c}+(\mathbf{a}.\mathbf{b})\mathbf{c}-(\mathbf{b}.\mathbf{c})\mathbf{a}+(\mathbf{b}.\mathbf{c})\mathbf{a}-(\mathbf{a}.\mathbf{c})\mathbf{b}\} = \frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\times\mathbf{0} = \mathbf{0}$

6.14 Scalar product of Four Vectors

$(\mathbf{a}\times\mathbf{b}).(\mathbf{c}\times\mathbf{d})$ is a scalar product of four vectors. It is the dot product of the vectors $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$.

It is a scalar triple product of the vectors $\mathbf{a},\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$ as well as scalar triple product of the vectors $\mathbf{a}\times\mathbf{b},\mathbf{c}$ and $\mathbf{d}$.

$$(\mathbf{a}\times\mathbf{b}).(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \mathbf{a}.\mathbf{c} & \mathbf{a}.\mathbf{d} \\ \mathbf{b}.\mathbf{c} & \mathbf{b}.\mathbf{d} \end{vmatrix}$$

6.15 Vector product of Four Vectors

(1) $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ is a vector product of four vectors.

It is the cross product of the vectors $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$.

(2) $\mathbf{a}\times\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}, \{(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\}\times\mathbf{d}$ are also different vector products of four vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ and $\mathbf{d}$.

Example: 52 $\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})]$ is equal to **[AMU 2001]**

(a) $(\mathbf{a}\times\mathbf{a}).(\mathbf{b}\times\mathbf{a})$ (b) $\mathbf{a}.(\mathbf{b}\times\mathbf{a})-\mathbf{b}.(\mathbf{a}\times\mathbf{b})$ (c) $[\mathbf{a}.(\mathbf{a}\times\mathbf{b})]\mathbf{a}$ (d) $(\mathbf{a}.\mathbf{a})(\mathbf{b}\times\mathbf{a})$

Solution: (d) $\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})] = \mathbf{a}\times[(\mathbf{a}.\mathbf{b})\mathbf{a}-(\mathbf{a}.\mathbf{a})\mathbf{b}] = (\mathbf{a}.\mathbf{b})(\mathbf{a}\times\mathbf{a})-(\mathbf{a}.\mathbf{a})(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}.\mathbf{b})\mathbf{0}+(\mathbf{a}.\mathbf{a})(\mathbf{b}\times\mathbf{a}) = (\mathbf{a}.\mathbf{a})(\mathbf{b}\times\mathbf{a})$

Example: 53 $[\mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}\ \ \mathbf{a}\times\mathbf{b}]$ is equal to **[MP PET 2004]**

(a) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ (b) $2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ (c) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$ (d) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

Solution: (c) $[(\mathbf{b}\times\mathbf{c})(\mathbf{c}\times\mathbf{a}),(\mathbf{a}\times\mathbf{b})] = (\mathbf{b}\times\mathbf{c}).\{[\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})\} = (\mathbf{b}\times\mathbf{c}).\{[\mathbf{c}\ \mathbf{a}\ \mathbf{b}]\mathbf{a}-[\mathbf{a}\ \mathbf{a}\ \mathbf{b}]\mathbf{c}\}$

$= (\mathbf{b}\times\mathbf{c}).\{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{a}-0\} = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}][\mathbf{abc}] = [\mathbf{abc}]^2$

Example: 54 Let the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be such that $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \mathbf{0}$. Let P_1 and P_2 be planes determined by pair of vectors $\mathbf{a},\mathbf{b}$ and $\mathbf{c},\mathbf{d}$ respectively. Then the angle between P_1 and P_2 is **[IIT Screening 2000]**

(a) 0^o (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Solution: (a) $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \mathbf{0} \Rightarrow (\mathbf{a}\times\mathbf{b})$ is parallel to $(\mathbf{c}\times\mathbf{d})$

Hence plane P_1, determined by vectors $\mathbf{a},\mathbf{b}$ is parallel to the plane P_2 determined by $\mathbf{c},\mathbf{d}$

$\therefore$ Angle between P_1 and $P_2 = 0$ (As the planes P_1 and P_2 are parallel).

6.16 Vector Equations

Solving a vector equation means determining an unknown vector or a number of vectors satisfying the given conditions. Generally, to solve a vector equation, we express the unknown vector as a linear combination of three known non-coplanar vectors and then we determine the coefficients from the given conditions.

If $\mathbf{a},\mathbf{b}$ are two known non-collinear vectors, then $\mathbf{a},\mathbf{b},\mathbf{a}\times\mathbf{b}$ are three non-coplanar vectors.

Thus, any vector $\mathbf{r} = x\mathbf{a}+y\mathbf{b}+z(\mathbf{a}\times\mathbf{b})$ where x,y,z are unknown scalars.

Example: 55 If $\mathbf{a} = \mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{a}.\mathbf{b} = 1$ and $\mathbf{a}\times\mathbf{b} = \mathbf{j}-\mathbf{k}$, then $\mathbf{b} =$ **[IIT Screening 2004]**

(a) $\mathbf{i}$ (b) $\mathbf{i}-\mathbf{j}+\mathbf{k}$ (c) $2\mathbf{j}-\mathbf{k}$ (d) $2\mathbf{i}$

Solution: (a) Let $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$

Now, $\mathbf{j}-\mathbf{k} = \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ b_1 & b_2 & b_3 \end{vmatrix} \Rightarrow b_3 - b_2 = 0, b_1 - b_3 = 1, b_2 - b_1 = -1 \Rightarrow b_3 = b_2, b_1 = b_2 + 1$

Now, $\mathbf{a}.\mathbf{b} = 1 \Rightarrow b_1 + b_2 + b_3 = 1 \Rightarrow 3b_2 + 1 = 1 \Rightarrow b_2 = 0 \Rightarrow b_1 = 1, b_3 = 0$. Thus $\mathbf{b} = \mathbf{i}$

Example: 56 The point of intersection of $\mathbf{r}\times\mathbf{a} = \mathbf{b}\times\mathbf{a}$ and $\mathbf{r}\times\mathbf{b} = \mathbf{a}\times\mathbf{b}$ where $\mathbf{a} = \mathbf{i}+\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i}-\mathbf{k}$ is **[Orissa JEE 2004]**

(a) $3\mathbf{i}+\mathbf{j}-\mathbf{k}$ (b) $3\mathbf{i}-\mathbf{k}$ (c) $3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ (d) None of these

Solution: (a) We have $\mathbf{r}\times\mathbf{b} = \mathbf{a}\times\mathbf{b}$ and $\mathbf{r}\times\mathbf{a} = \mathbf{b}\times\mathbf{a}$

Adding, $\mathbf{r}\times(\mathbf{a}+\mathbf{b}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{a})$

$\Rightarrow$ $\mathbf{r}\times(\mathbf{a}+\mathbf{b}) = \mathbf{0}$ $\Rightarrow$ $\mathbf{r}$ is parallel to $\mathbf{a}+\mathbf{b}$

$\therefore$ $\mathbf{r} = \lambda(\mathbf{a}+\mathbf{b}) = \lambda\{(\mathbf{i}+\mathbf{j}) + (2\mathbf{i}-\mathbf{k})\} = \lambda\{3\mathbf{i}+\mathbf{j}-\mathbf{k}\}$

For $\lambda = 1$, $\mathbf{r} = 3\mathbf{i}+\mathbf{j}-\mathbf{k}$

Example: 57 Let $\mathbf{a} = \mathbf{i}-\mathbf{j}, \mathbf{b} = \mathbf{j}-\mathbf{k}, \mathbf{c} = \mathbf{k}-\mathbf{i}$. If $\hat{\mathbf{d}}$ is a unit vector such that $\mathbf{a}.\mathbf{d} = 0 = [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$, then $\hat{\mathbf{d}}$ is equal to **[IIT 1995]**

(a) $\pm\frac{\mathbf{i}+\mathbf{j}-\mathbf{k}}{\sqrt{3}}$ (b) $\pm\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$ (c) $\pm\frac{\mathbf{i}+\mathbf{j}-2\mathbf{k}}{\sqrt{6}}$ (d) $\pm\mathbf{k}$

Solution: (c) Let $\hat{\mathbf{d}} = \alpha\,\mathbf{i}+\beta\,\mathbf{j}+\gamma\,\mathbf{k}$

$\mathbf{a}.\hat{\mathbf{d}} = 0 \Rightarrow (\mathbf{i}-\mathbf{j}).(\alpha\mathbf{i}+\beta\mathbf{j}+\gamma\mathbf{k}) = 0 \Rightarrow \alpha-\beta = 0 \Rightarrow \alpha = \beta$

$[\mathbf{b}\,\mathbf{c}\,\mathbf{d}] = 0 \Rightarrow (\mathbf{b}\times\mathbf{c}).\mathbf{d} = 0 \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix}.(\alpha\,\mathbf{i}+\beta\,\mathbf{j}+\gamma\,\mathbf{k}) = 0 \Rightarrow (\mathbf{i}+\mathbf{j}+\mathbf{k}).(\alpha\,\mathbf{i}+\beta\,\mathbf{j}+\gamma\,\mathbf{k}) = 0 \Rightarrow \alpha+\beta+\gamma = 0$

$\Rightarrow \gamma = -(\alpha+\beta) = -2\alpha$; $(\beta = \alpha)$

$|\hat{\mathbf{d}}| = 1 \Rightarrow \alpha^2+\beta^2+\gamma^2 = 1 \Rightarrow \alpha^2+\alpha^2+4\alpha^2 = 1 \Rightarrow \alpha = \pm\frac{1}{\sqrt{6}} = \beta$ and $\gamma = \mp\frac{2}{\sqrt{6}}$

$\therefore$ $\hat{\mathbf{d}} = \pm\frac{1}{\sqrt{6}}(\mathbf{i}+\mathbf{j}-2\mathbf{k})$.

Example: 58 Let $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ be three mutually perpendicular vectors of the same magnitude. If a vector $\mathbf{x}$ satisfies equation $\mathbf{p}\times|(\mathbf{x}-\mathbf{q})\times\mathbf{p}| + \mathbf{q}\times|(\mathbf{x}-\mathbf{r})\times\mathbf{q}| + \mathbf{r}\times|(\mathbf{x}-\mathbf{p})\times\mathbf{r}| = 0$, then $\mathbf{x}$ is given by **[IIT 1997]**

(a) $\frac{1}{2}(\mathbf{p}+\mathbf{q}-2\mathbf{r})$ (b) $\frac{1}{2}(\mathbf{p}+\mathbf{q}+\mathbf{r})$ (c) $\frac{1}{3}(\mathbf{p}+\mathbf{q}+\mathbf{r})$ (d) $\frac{1}{3}(2\mathbf{p}+\mathbf{q}-\mathbf{r})$

Solution: (b) Let $|\mathbf{p}| = |\mathbf{q}| = |\mathbf{r}| = \mathbf{k}$

$\therefore$ $\mathbf{p} = k\,\hat{\mathbf{p}}, \mathbf{q} = k\,\hat{\mathbf{q}}, \mathbf{r} = k\,\hat{\mathbf{r}}$

Let $\mathbf{x} = \alpha\,\hat{\mathbf{p}}+\beta\,\hat{\mathbf{q}}+\gamma\,\hat{\mathbf{r}}$

Now, $\mathbf{p}\times\{\mathbf{x}-\mathbf{q}\}\times\mathbf{p}\} = (\mathbf{p}.\mathbf{p})(\mathbf{x}-\mathbf{q}) - \{\mathbf{p}.(\mathbf{x}-\mathbf{q})\}\mathbf{p} = |\mathbf{p}|^2(\mathbf{x}-\mathbf{q}) - \{(\mathbf{p}.\mathbf{x}) - \mathbf{p}.\mathbf{q}\}\mathbf{p}$

$= k^2(\mathbf{x}-\mathbf{q}) - \{|\mathbf{p}|(\hat{\mathbf{p}}.\hat{\mathbf{x}}) - 0\}|\mathbf{p}|\hat{\mathbf{p}} = k^2(\mathbf{x}-\mathbf{q}) - |\mathbf{p}|^2(\hat{\mathbf{p}}.\hat{\mathbf{x}})\hat{\mathbf{p}} = k^2\{\mathbf{x}-\mathbf{q}-\alpha\,\hat{\mathbf{p}})$

$\therefore$ $\mathbf{p}\times\{(\mathbf{x}-\mathbf{q})\times\mathbf{p}\} + \mathbf{q}\times(\mathbf{x}-\mathbf{r})\times\mathbf{q}\} + \mathbf{r}\times\{(\mathbf{x}-\mathbf{p})\times\mathbf{r}\} = \mathbf{0}$

$\Rightarrow$ $k^2\{\mathbf{x}-\mathbf{q}-\alpha\mathbf{p}+\mathbf{x}-\mathbf{r}-\beta\mathbf{q}+\mathbf{x}-\mathbf{p}-\gamma\mathbf{r}\} = 0 \Rightarrow 3\mathbf{x}-(\mathbf{p}+\mathbf{q}+\mathbf{r}) - (\alpha\mathbf{p}+\beta\mathbf{q}+\gamma\mathbf{r}) = 0$

$\Rightarrow$ $3\mathbf{x}-(\mathbf{p}+\mathbf{q}+\mathbf{r}) - \mathbf{x} = 0 \Rightarrow 2\mathbf{x}-(\mathbf{p}+\mathbf{q}+\mathbf{r}) = 0$

$\therefore$ $\mathbf{x} = \frac{1}{2}(\mathbf{p}+\mathbf{q}+\mathbf{r})$

Example: 59 Let the unit vectors $\mathbf{a}$ and $\mathbf{b}$ be perpendicular and the unit vector $\mathbf{c}$ be inclined at an angle θ to both $\mathbf{a}$ and $\mathbf{b}$. If $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a}\times\mathbf{b})$, then **[Orissa JEE 2003]**

(a) $\alpha = \beta = \cos\theta\,,\gamma^2 = \cos 2\theta$ (b) $\alpha = \beta = \cos\theta, \gamma^2 = -\cos 2\theta$

(c) $\alpha = \cos\theta, \beta = \sin\theta, \gamma^2 = \cos 2\theta$ (d) None of these

Solution: (b) We have, $|\mathbf{a}| = |\mathbf{b}| = 1$

$\mathbf{a}\,.\,\mathbf{b} = 0$; (as $\mathbf{a}\perp\mathbf{b}$)

$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a}\times\mathbf{b})$ ......(i)

Taking dot product by $\mathbf{a}$, $\mathbf{a}\,.\,\mathbf{c} = \alpha|\,\mathbf{a}|^2 + \beta(\mathbf{a}\,.\,\mathbf{b}) + \gamma[\mathbf{a}\;\mathbf{a}\;\mathbf{b}]$

$\Rightarrow\; |\mathbf{a}|.|\mathbf{c}|\cos\theta = \alpha.1 + 0 + 0 \;\Rightarrow\; 1.|\mathbf{c}|.\cos\theta = \alpha$

As $|\mathbf{c}| = 1$; $\therefore\;\; \alpha = \cos\theta$

Taking dot product of (i) by $\mathbf{b}$

$\mathbf{b}\,.\,\mathbf{c} = \mathbf{b}\,.\,\mathbf{a} + \beta|\,\mathbf{b}|^2 + \gamma\,[\mathbf{b}\;\mathbf{a}\;\mathbf{b}] \Rightarrow |\,\mathbf{b}||\,\mathbf{c}|\cos\theta = 0 + \beta.1 + 0$

$\therefore\;\; \beta = 1.1.\cos\theta = \cos\theta$

$|\,\mathbf{c}|^2 = 1 \;\Rightarrow\; \alpha^2 + \beta^2 + \gamma^2 = 1 \;\Rightarrow\; \cos^2\theta + \cos^2\theta + \gamma^2 = 1$

$\therefore\;\; \gamma^2 = 1 - 2\cos^2\theta = -\cos 2\theta$

Hence, $\alpha = \beta = \cos\theta, \gamma^2 = -\cos 2\theta$

Example: 60 The locus of a point equidistant from two given points whose position vectors are $\mathbf{a}$ and $\mathbf{b}$ is equal to

(a) $\left[\mathbf{r} - \frac{1}{2}(\mathbf{a}+\mathbf{b})\right].(\mathbf{a}+\mathbf{b}) = 0$ (b) $\left[\mathbf{r} - \frac{1}{2}(\mathbf{a}+\mathbf{b})\right].(\mathbf{a}-\mathbf{b}) = 0$

(c) $\left[\mathbf{r} - \frac{1}{2}(\mathbf{a}+\mathbf{b})\right].\mathbf{a} = 0$ (d) $[\mathbf{r} - (\mathbf{a}+\mathbf{b})].\,\mathbf{b} = 0$

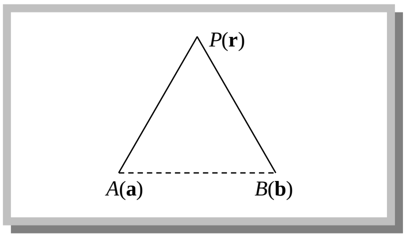
Solution: (b) Let $P\,(\mathbf{r})$ be a point on the locus.

$\therefore\;\; AP = BP$

$\Rightarrow\;\; |\,\mathbf{r}-\mathbf{a}| = |\,\mathbf{r}-\mathbf{b}| \;\Rightarrow\; |\,\mathbf{r}-\mathbf{a}|^2 = |\,\mathbf{r}-\mathbf{b}|^2 \;\Rightarrow\; (\mathbf{r}-\mathbf{a}).(\mathbf{r}-\mathbf{a}) = (\mathbf{r}-\mathbf{b}).(\mathbf{r}-\mathbf{b})$

$\Rightarrow\;\; 2\mathbf{r}.(\mathbf{a}-\mathbf{b}) = \mathbf{a}\,.\,\mathbf{a} - \mathbf{b}.\mathbf{b} \;\Rightarrow\; \mathbf{r}.(\mathbf{a}-\mathbf{b}) = \frac{1}{2}(\mathbf{a}+\mathbf{b}).(\mathbf{a}-\mathbf{b})$

$\therefore\;\; [\mathbf{r} - \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{a}+\mathbf{b})].(\mathbf{a}-\mathbf{b}) = 0$. This is the locus of *P*.



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