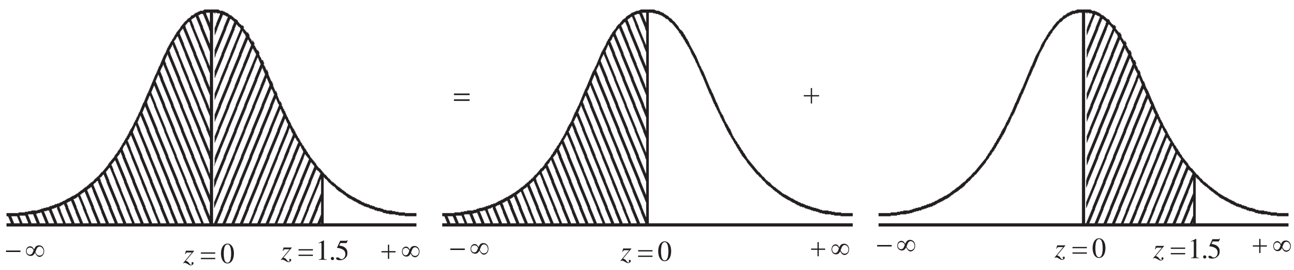


$= P(-\infty < Z \le 0) + P(0 \le Z \le 1.5)$

= 0.5 + 0.4332 (from the tables of standard normal variable)

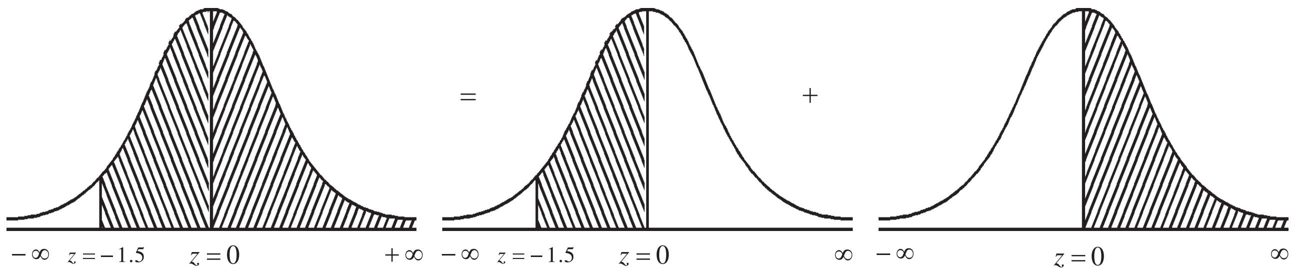
= 0.9332

(2) The probability that a normal variable X will take value more than 14

$$= P(X \ge 14) = P\left(\frac{X-\mu}{\sigma} \ge \frac{14-\mu}{\sigma}\right)$$

$$= P\left(Z \ge \frac{14-20}{4}\right)$$

$$= P(Z \ge -1.5)$$



$= P(-1.5 \le Z \le 0) + P(0 \le Z < \infty)$

$= P(0 \le Z \le 1.5) + 0.5$ $(\because$ Symmetry$)$

= 0.4332 + 0.5 (from the tables of standard normal variable)

= 0.9332

Illustration 3 : The number of students in classes of higher secondary schools of a city follows normal distribution. Average number of students in the classes is 50 and standard deviation is 15. If a class is selected at random then find the following probabilities (i) a class consists of more than 68 students (ii) a class consists of less than 32 students.

It is given here that the number of students in the class follows normal distribution.

Normal variable $X =$ number of students in a class

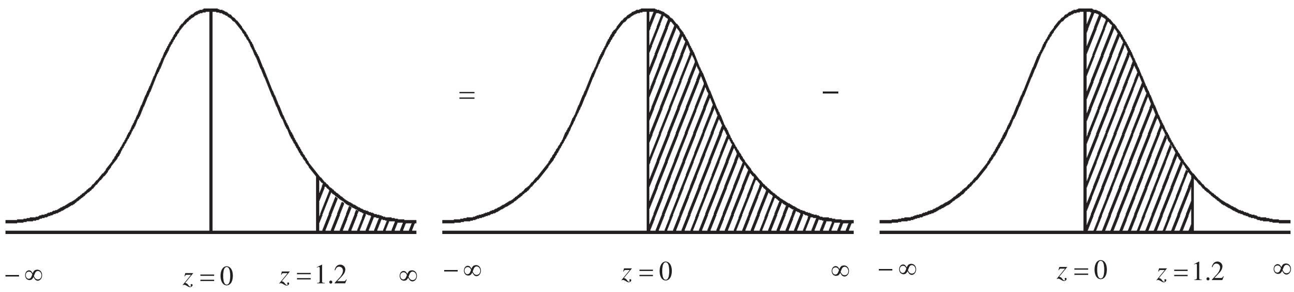
Also, mean $\mu = 50$ students and standard deviation $\sigma = 15$ students

(1) The probability that a randomly selected class consists of more than 68 students

$$= P(X \ge 68) = P\left(\frac{X-\mu}{\sigma} \ge \frac{68-\mu}{\sigma}\right)$$

$$= P\left(Z \ge \frac{68-50}{20}\right)$$

$$= P(Z \ge 1.2)$$



$$= P(0 \le Z < \infty) - P(0 \le Z \le 1.2)$$

= 0.5 − 0.3849 (from the tables of standard normal variable)

= 0.1151

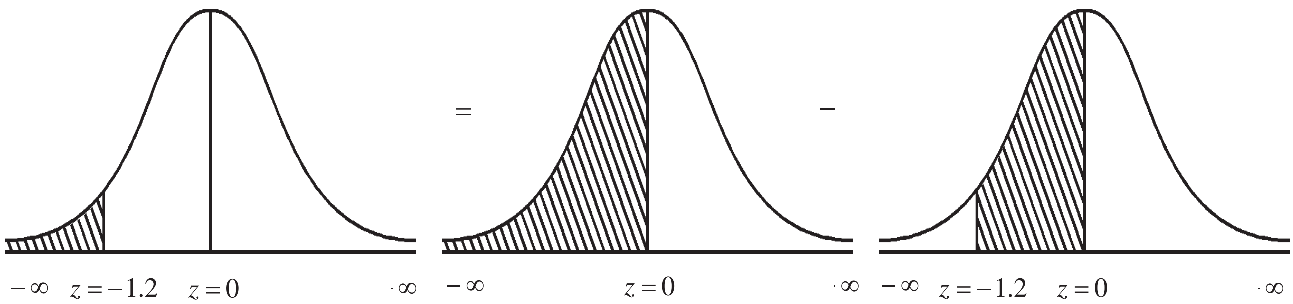
Thus, the probability that a randomly selected class consists of more than 68 students is 0.1151.

(2) The probability that a randomly selected class consists of less than 32 students

$$= P(X \le 32) = P\left(\frac{X-\mu}{\sigma} \le \frac{32-\mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{32-50}{20}\right)$$

$$= P(Z \le -1.2)$$



$$= P(-\infty < Z \le 0) - P(-1.2 \le Z \le 0)$$

$$= 0.5 - P(0 \le Z \le 1.2) \; (\because \text{ Symmetry})$$

= 0.5 − 0.3849 (from the tables of standard normal variable)

= 0.1151

Thus, the probability that a randomly selected class consists of less than 32 students is 0.1151

Illustration 4 : The average weight of grown up children living in a large society is 50 kg and its standard deviation is 5 kg. If their weight follows normal distribution and a grown up child is selected at random then find

(1) the probability that his weight is between 55 kg and 65 kg.

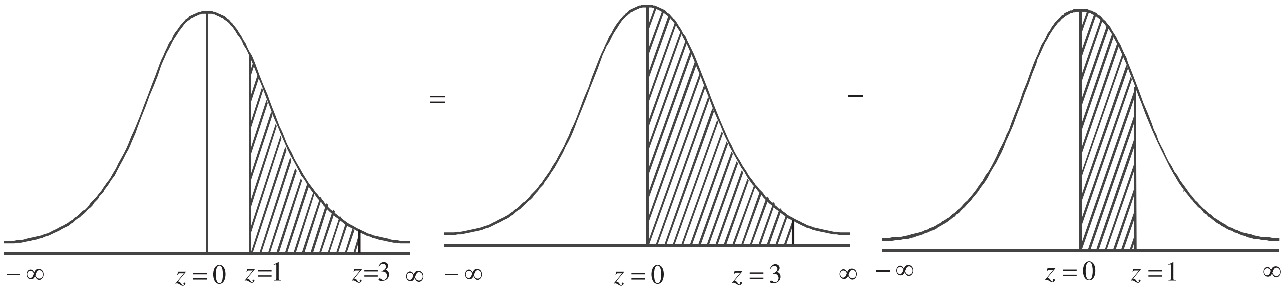
(2) the probability that his weight is between 35 kg and 45 kg.

Normal variable X = weight of a grown up child, average weight $\mu = 50$ kg and standard deviation $\sigma = 5$ kg.

(1) Probability that a randomly selected grown up child has weight between 55 kg to 65 kg

$$= P(55 \le X \le 65) = P\left(\frac{55-50}{5} \le \frac{X-\mu}{\sigma} \le \frac{65-50}{5}\right)$$

$$= P(1 \le Z \le 3)$$



$$= P(0 \le Z \le 3) - P(0 \le Z \le 1)$$

$$= 0.4987 - 0.3413$$

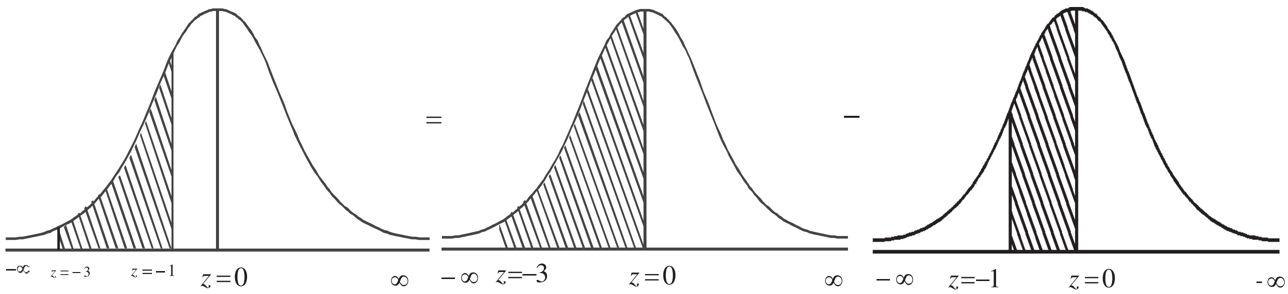
$$= 0.1574$$

Thus, the probability that a randomly selected grown up child has weight between 55 kg to 65 kg is 0.1574.

(2) Probability that a randomly selected grown up child has weight between 35 kg to 45 kg

$$= P(35 \le X \le 45) = P\left(\frac{35-50}{5} \le \frac{X-\mu}{\sigma} \le \frac{45-50}{5}\right)$$

$$= P(-3 \le Z \le -1)$$



$$= P(-3 \le Z \le 0) - P(-1 \le Z \le 0)$$

$$= P(0 \le Z \le 3) - P(0 \le Z \le 1) \quad (\because \text{ symmertry})$$

$$= 0.4987 - 0.3413$$

$$= 0.1574$$

Thus, the probability that a randomly selected grown up child has weight between 35 kg to 45 kg is 0.1574.

Note : From the above illustrations it is clear that the area under the normal curve for $Z = 0$ to $Z = a$ is equal to the area between $Z = -a$ to $Z = 0$. This is because the normal distribution is a symmetric distribution.

Illustration 5 : The monthly income of workers working in a production house follows normal distribution. Their average monthly income is ₹ 15,000 and standard deviation is ₹ 4000.

(1) If a worker is selected at random then find the probability that his monthly income is between ₹ 10,000 and ₹ 25,000.

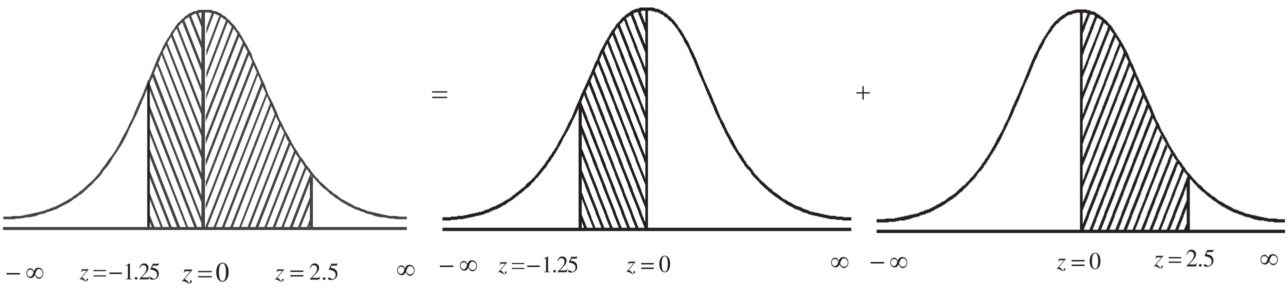
(2) Find the percentage of workers having monthly income between ₹ 12,000 and 22,000 in the production house.

Here, normal variable X = monthly income of worker, average income μ = ₹ 15,000 and standard deviation σ = ₹ 4000.

(1) The probability that a randomly selected worker has income between ₹ 10,000 and ₹ 25,000

$$= P(10000 \le X \le 25000) = P\left(\frac{10000-15000}{4000} \le \frac{X-\mu}{\sigma} \le \frac{25000-15000}{4000}\right)$$

$$= P(-1.25 \le Z \le 2.5)$$



$$= P(-1.25 \le Z \le 0) + P(0 \le Z \le 2.5)$$

$$= P(0 \le Z \le 1.25) + P(0 \le Z \le 2.5) \quad (\because \text{ symmertry})$$

$$= 0.3944 + 0.4938$$

$$= 0.8882$$

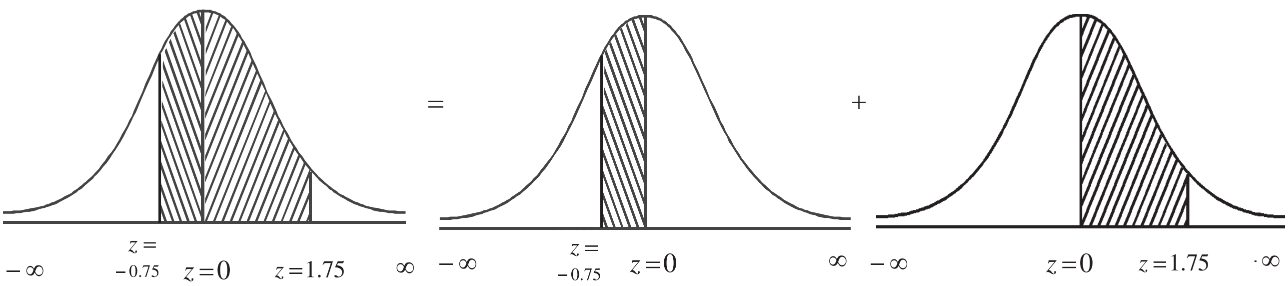
Thus, the probability that a randomly selected worker has monthly income between ₹ 10,000 and ₹ 25,000 is 0.8882.

(2) The probability that a randomly selected worker has income between ₹ 12,000 and ₹ 22,000

$$= P(12000 \le X \le 22000)$$

$$= P\left(\frac{12000-15000}{4000} \le \frac{X-\mu}{\sigma} \le \frac{22000-15000}{4000}\right)$$

$$= P(-0.75 \le Z \le 1.75)$$



$$= P(-0.75 \le Z \le 0) + P(0 \le Z \le 1.75)$$

$$= P(0 \le Z \le 0.75) + P(0 \le Z \le 1.75) \quad (\because \text{ symmertry})$$

$= 0.2734 + 0.4599$

$= 0.7333$

$\therefore$ The percentaged that a randomly selected worker has monthly income between ₹ 12,000 and ₹ 22,000

$= 0.7333 \times 100$

$= 73.33 \%$

Thus, 73.33 % of the wokers in the production house have monthly income between ₹ 12,000 and ₹ 22,000.

Note : In order to express probability in percentage, probability is multiplied by 100.

3.4 Properties of Normal Distribution

Some important properties of normal distribution are as under :

(1) It is a distribution of continuous random variable.

(2) The constants μ and σ are the parameters of distribution which indicate mean and variance respectively.

(3) The distribution is symmetric about μ and its skewness is zero (0).

(4) For this distribution, the value of mean, median and mode are same. In notation, $\mu = M = M_0$

(5) For this distribution, quartiles are equidistant from median i.e. $Q_3 - M = M - Q_1$ and $M = \frac{Q_3 + Q_1}{2}$

(6) The probability curve is completely bell shaped.

(7) Normal curve is asymptotic to X-axis. The tails never touch X-axis.

(8) The approximate value of quartiles of normal distribution can be obtained from the following formula

$Q_1 = \mu - 0.675\ \sigma$

$Q_3 = \mu + 0.675\ \sigma$

(9) For this distribution, quartile deviation = $\frac{2}{3}\sigma$ (approximately)

(10) For this distribution, mean deviation = $\frac{4}{5}\sigma$ (approximately)

(11) Important areas under normal curve are as below :

(i) Total area under normal curve is 1 and area on both the sides of perpendicular line at $X = \mu$ is 0.5

(ii) The area under the curve between the perpendicular lines at $\mu - \sigma$ and $\mu + \sigma$ is 0.6826 i.e. area under the normal curve between the perpendicular lines at $\mu \pm \sigma$ is 0.6826

(iii) The area under the curve between the perpendicular lines at $\mu - 2\sigma$ and $\mu + 2\sigma$ is 0.9545

(iv) The area under the curve between the perpendicular lines at $\mu - 3\sigma$ and $\mu + 3\sigma$ is 0.9973

(v) The area under the curve between the perpendicular lines at $\mu - 1.96\sigma$ and $\mu + 1.96\sigma$ is 0.95.

(vi) The area under the curve between the perpendicular lines at $\mu - 2.575\sigma$ and $\mu + 2.575\sigma$ is 0.99.

3.5 Properties of Standard Normal Distribution

Some important properties of standard normal distribution are as under :

(1) It is a distribution of continuous random variable.

(2) For this distribution, mean is zero (0) and its standard deviation is 1.

(3) The distribution is symmetric to $Z = 0$ and its skewness is zero (0).

(4) The probability curve is completely bell shaped and is asymptotic to X-axis.

(5) The approximate value of the first quartile of standard normal distribution is –0.675 and that of the third quartile is 0.675.

(6) For this distribution, quartile deviation $= \frac{2}{3}$ (approximately).

(7) For this distribution, mean deviation $= \frac{4}{5}$ (approximately).

(8) Important areas under normal curve are as below :

(i) Total area under normal curve is 1 and area on both the sides of perpendicular line at $Z = 0$ is 0.5

(ii) The area under the curve between the perpendicular lines at $Z = -1$ and $Z = +1$ is 0.6826 i.e. area under the normal curve between the perpendicular lines at $Z = \pm 1$ is 0.6826.

(iii) The area under the curve between the perpendicular lines at $Z = -2$ and $Z = +2$ is 0.9545.

(iv) The area under the curve between the perpendicular lines at $Z = -3$ and $Z = +3$ is 0.9973.

(v) The area under the curve between the perpendicular lines at $Z = -1.96$ and $Z = +1.96$ is 0.95

(vi) The area under the curve between the perpendicular lines at $Z = -2.575$ and $Z = +2.575$ is 0.99

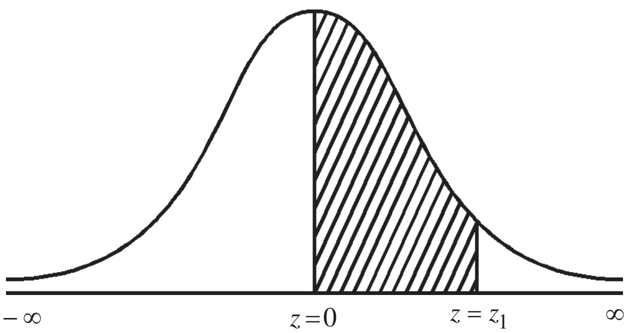
It should be noted here that the probability distribution of standard normal variable Z is a distribution of normal variable with mean zero and variance 1. Z is called standard score or Z-score and it is independent of unit of measurement.

We have seen earlier that when a value of normal variable X and values of parameters are known then corresponding value of Z-score is obtained and by using the table of standard normal variable, the respective probability can be obtained. Now, if the probability is known then to determine value of Z-score we shall study the following illustrations :

Illustration 6 : If the probability that value of standard normal variable Z lies between 0 and Z-score (z_1) is 0.3925 then obtain the possible values of Z-score (z_1).

The probability that value of standard normal variable Z lies between $Z = 0$ and $Z = z_1$ is 0.3925. This probability is equal to the area under the curve between $Z = 0$ and $Z = z_1$. The value of z_1 may be positive or negative.

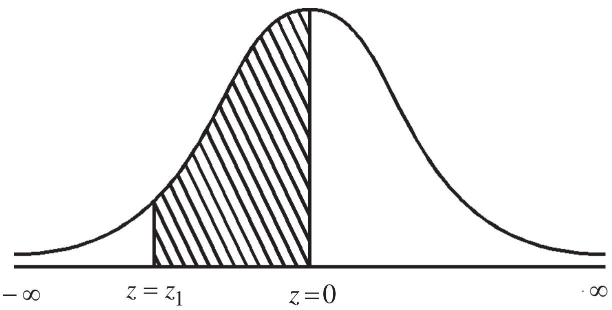
Suppose the value of z_1 is positive then $P(0 \le Z \le z_1) = 0.3925$.



For obtaining the value of z_1, see the first column of table of standard normal variable (Z-table). For $Z = 1.20$, the area is 0.384 which is less than 0.3925. Now, read the values in this row. For the value 0.3925, the corresponding value of Z is 1.24. Therefore, one possible value of Z-score is 1.24.

Now, suppose the value of z_1 is negative

$\therefore P(z_1 \le Z \le 0) = 0.3925$



Now, since the normal distribution is symmetric, $P(z_1 \le Z \le 0) = P(0 \le Z \le z_1) = 0.3925$. Thus, as above, $z_1 = 1.24$ but it can be seen in the above diagram that the perpendicular line at z_1 is to the left of $Z = 0$ hence Z-score $z_1 = -1.24$.

Thus, if the probability that the value of standard normal variable lies between $Z = 0$ and $Z = z_1$ is 0.3925 then the possible values of Z-score are ± 1.24.

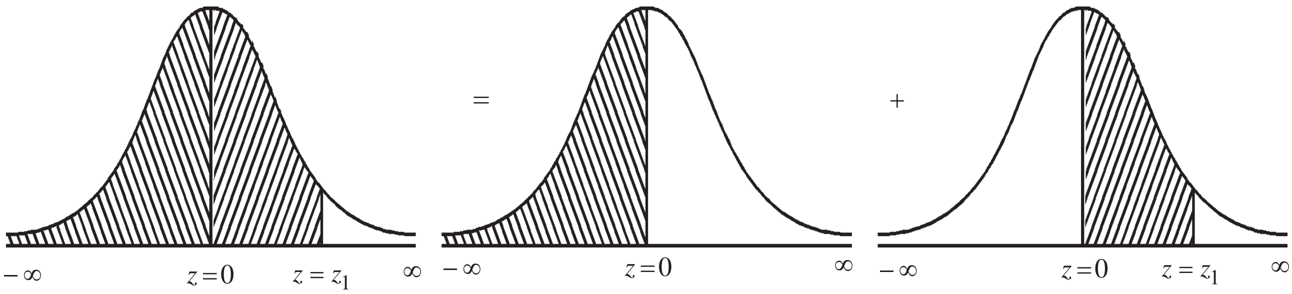
Thus, the sign of z_1 is positive if it is on right hand side of $Z = 0$ and negative if it is on left hand side of $Z = 0$.

Illustration 7 : If the probabilities for standard normal variable *Z* are as under then obtain the value of *Z*-socre (z_1) :

(1) Area to the left of $Z = z_1$ is 0.95

(2) Area to the right of $Z = z_1$ is 0.05.

(1) Area to the left of $Z = z_1$ is 0.95 i.e. $P(Z \le z_1) = 0.95$. In the curve of standard normal variable, the perpendicular line at $Z = z_1$ is to be drawn by moving from left to right so that the area under the curve is 0.95. The figure is as under :



$$\text{Thus } P(Z \le z_1) = P(-\infty < Z \le 0) + P(0 \le Z \le z_1) = 0.95$$

$$\therefore 0.5 + P(0 \le Z \le z_1) = 0.95$$

$$\therefore P(0 \le Z \le z_1) = 0.95 - 0.5$$

$$\therefore P(0 \le Z \le z_1) = 0.45$$

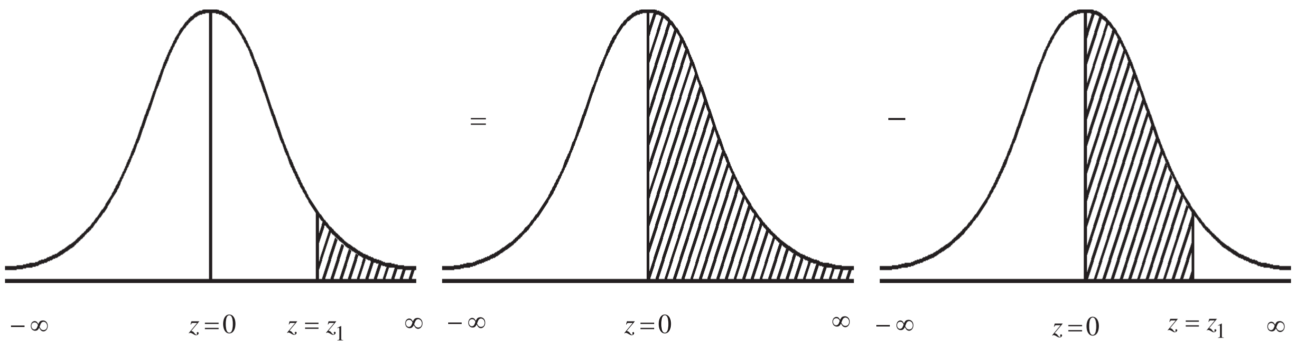
Now, corresponding to probability 0.45, it is not possible to obtain value of z_1 directly from the table of standard normal variable. Hence, the approximate value of z_1 is determined as follows

From table	Area	Z-score
Nearest value before 0.45	0.4495	1.64
Nearest value after 0.45	0.4505	1.65
Average value	0.4500	1.645

From the above table, it can be seen that $z_1 = 1.645$.

Thus, for $z_1 = 1.645$, $P(Z \le z_1) = 0.95$.

(2) Area to the right of $Z = z_1$ is 0.05 i.e. $P(Z \ge z_1) = 0.05$. In the curve of standard normal variable, the perpendicular line at $Z = z_1$ is to be drawn by moving from right to left so that the area under the curve is 0.05. The figure is as under



$$\therefore P(Z \ge z_1) = P(0 \le Z < \infty) - P(0 \le Z \le z_1) = 0.05$$

$$\therefore 0.5 - P(0 \le Z \le z_1) = 0.05$$

$$\therefore P(0 \le Z \le z_1) = 0.45$$

As calculated earlier, $z_1 = 1.645$.

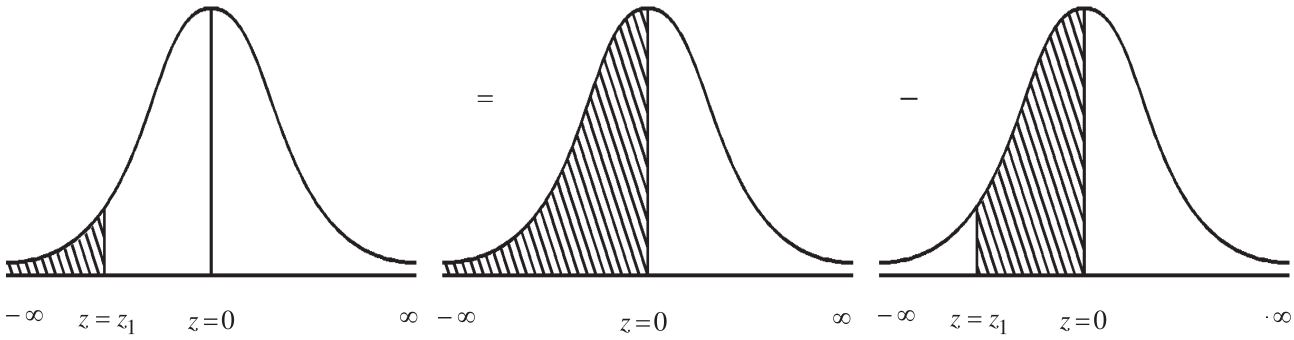
Thus, for $z_1 = 1.645$, $P(Z \ge z_1) = 0.05$.

Illustration 8 : If the probabilities for standard normal variable Z are as under then obtain the value of Z-score (z_1) :

(1) Area to the left of $Z = z_1$ is 0.10.

(2) Area to the right of $Z = z_1$ is 0.90.

(1) Area to the left of $Z = z_1$ is 0.10 i.e. $P(Z \le z_1) = 0.10$. In the curve of standard normal variable, the perpendicular line at $Z = z_1$ is to be drawn by moving from left to right so that the area under the curve is 0.10. The figure is as follows



$$\therefore\ P(Z \le z_1) = P(-\infty < Z \le 0) - P(z_1 \le Z \le 0) = 0.10$$

$$\therefore\ 0.50 - P(z_1 \le Z \le 0) = 0.10$$

$$\therefore\ P(z_1 \le Z \le 0) = 0.5 - 0.10$$

$$\therefore\ P(0 \le Z \le z_1) = 0.40 \quad (\because \text{ symmetry})$$

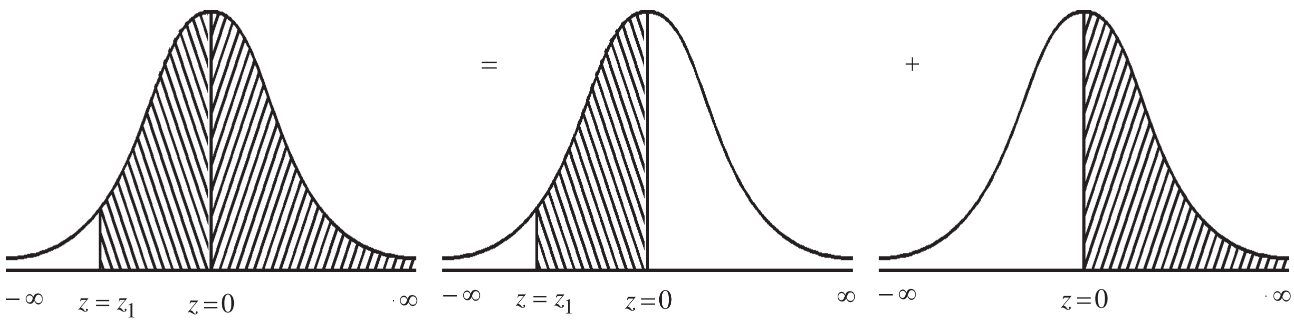
Now, corresponding to probability 0.4, it is not possible to obtain value of z_1 directly from the table of standard normal variable. Hence, the approximate value of z_1, is determined as under

From table	**Area**	**Z-score**
Nearest value before 0.40	0.3997	1.28
Nearest value after 0.40	0.4015	1.29
Average value	0.4006	1.285

From the above table, it can be seen that the nearest value to 0.40 is 0.3997 and the respective value of Z-score is 1.28. Also, z_1 is to the left of $Z = 0$, hence $z_1 = -1.28$.

Thus for $z_1 = -1.28$, $P(Z \le z_1) = 0.10$.

(2) Area to the right of $Z = z_1$ is 0.90 i.e $P(Z \ge z_1) = 0.90$. In the curve of standard normal variable, the perpendicular line at $Z = z_1$ is to be drawn by moving from right to left so that the area under the curve is 0.90. The figure is as under



$$P(Z \ge z_1) = P(z_1 \le Z \le 0) + P(0 \le Z < \infty) = 0.90$$

$$\therefore\ P(z_1 \le Z \le 0) + 0.50 = 0.90$$

$\therefore\ P(z_1 \le Z \le 0) = 0.40$

$\therefore\ P(0 \le Z \le z_1) = 0.40$ ($\because$ Symmetry)

As seen earlier $z_1 = -1.28$

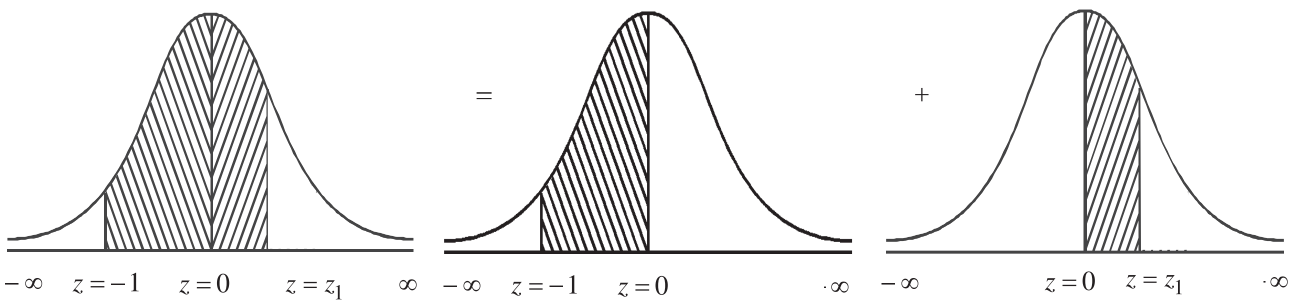
Thus, for $z_1 = -1.28$, $P(Z \ge z_1) = 0.90$

Illustration 9 : If Z is standard normal variable and z_1 is Z-score then obtain the values of z_1 satisfying the following conditions

(1) $P(-1 \le Z \le z_1) = 0.5255$

(2) $P(z_1 \le Z \le 2) = 0.7585$

(1) It is given that $P(-1 \le Z \le z_1) = 0.5255$. A perpendicular line at $Z = -1$ is drawn and then a perpendicular line at $Z = z_1$ is drawn to its right side so that the area between them is 0.5255. The figure is as under



$$P(-1 \le Z \le z_1) = P(-1 \le Z \le 0) + P(0 \le Z \le z_1) = 0.5255$$

$$\therefore\ P(0 \le Z \le 1) + P(0 \le Z \le z_1) = 0.5255 \quad (\because \text{ Symmetry})$$

$$\therefore\ 0.3413 + P(0 \le Z \le z_1) = 0.5255$$

$$\therefore\ P(0 \le Z \le z_1) = 0.5255 - 0.3413$$

$$\therefore\ P(0 \le Z \le z_1) = 0.1842$$

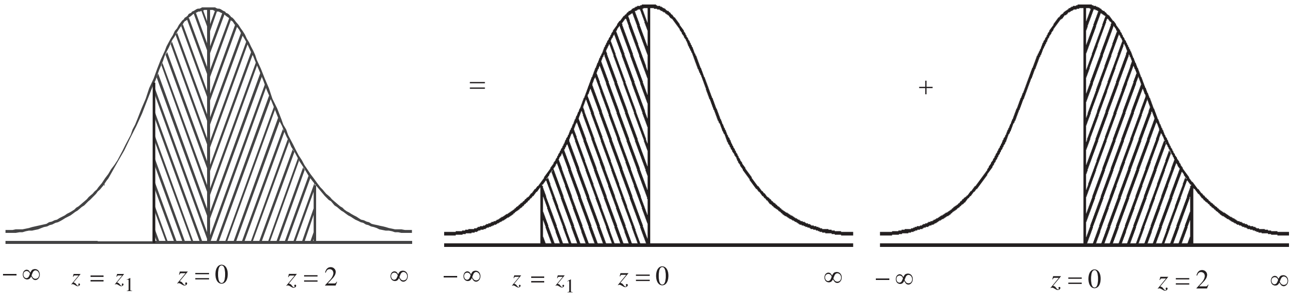
By using the table of standard normal variable the estimated value of Z-score, z_1 can be determined as under

From table	Area	Z-score
Nearest value before 0.1842	0.1808	0.47
Nearest value after 0.1842	0.1844	0.48
Average value	0.1826	**0.475**

From the above table, it can be seen that the nearest value to 0.1842 is 0.1844 and the respective value of *Z*-score is 0.48. Hence we take $z_1 = 0.48$.

Thus, for $z_1 = 0.48$, $P(-1 \le Z \le z_1) = 0.5255$

(2) It is given that $P(z_1 \le Z \le 2) = 0.7585$. A perpendicular line at $Z = 2$ is drawn and then a perpendicular line at $Z = z_1$ is drawn to its left so that the area between them is 0.7585. The figure is as under



$$\therefore P(z_1 \le Z \le 2) = P(z_1 \le Z \le 0) + P(0 \le Z \le 2) = 0.7585$$

$$\therefore P(z_1 \le Z \le 0) + 0.4772 = 0.7585$$

$$\therefore P(z_1 \le Z \le 0) = 0.7585 - 0.4772$$

$$\therefore P(0 \le Z \le z_1) = 0.2813 \quad (\because \text{ Symmetry})$$

By using the table of standard normal variable the estimated value of *Z*-score, z_1 can be determined as under

From table	Area	*Z*-score
Nearest value before 0.2813	0.2794	0.77
Nearest value after 0.2813	0.2823	0.78
Average value	0.2809	0.775

From the above table, it can be seen that the nearest value to 0.2813 is 0.2809 and the respective value of *Z*-score is 0.775. Since *Z*-score is to the left of $Z = 0$, therefore $z_1 = -0.775$.

Thus, for $z_1 = -0.775$, $P(z_1 \le Z \le 2) = 0.7585$

Activity

For 30 persons residing around your residence, collect information of their weight (in kg) and obtain its mean (in kg) and standard deviation (in kg). Assuming that the weight of selected person follows normal distribution with the obtained mean and standard deviation, estimate (1) minimum weight of 5% persons having maximum weight (2) maximum weight of 15% of persons having minimum weight.

3.6 Illustrations

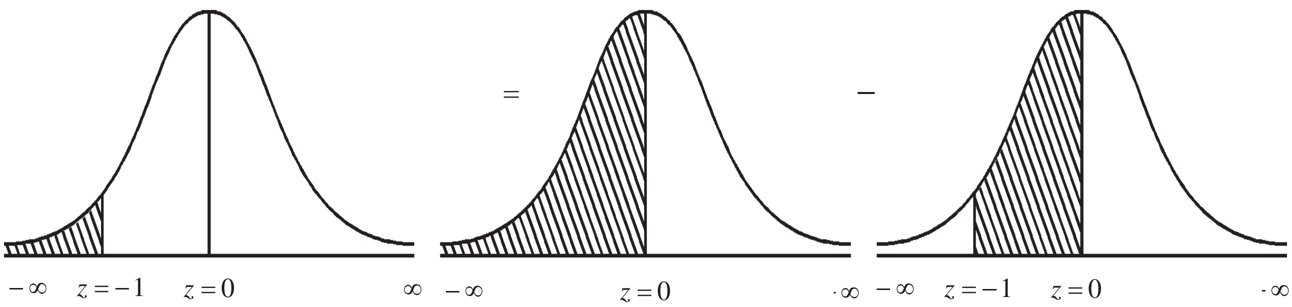
Illustration 10 : In a city, daily sale of petrol at a petrol pump follows normal distribution and its mean and standard deviation are 33,000 litre and 3000 litre respectively. (1) Obtain the percentage of days of a month during which the daily sales of petrol is less than 30,000 litre. (2) During the month of May, how many days are expected so that the sale of petrol is between 32,000 litre to 35,000 litre?

Here, X = daily sale of petrol at petrol pump (in litre). Also $\mu = 33,000$ litre and $\sigma = 3000$ litre.

(1) Probability that the sale of petrol is less than 30,000 litre

$$= P(X \le 30000) \quad = P\left(\frac{X-\mu}{\sigma} \le \frac{30000-33000}{3000}\right)$$

$$= P(Z \le -1)$$



$$= P(-\infty < Z \le 0) - P(-1 \le Z \le 0)$$

$$= 0.5 - P(0 \le Z \le 1) \; (\because \text{ Symmetry})$$

$$= 0.5 - 0.3413$$

$$= 0.1587$$

$\therefore$ Percentage of days during a certain month where the daily sale of petrol is less than 30,000 litre

$$= 0.1587 \times 100$$

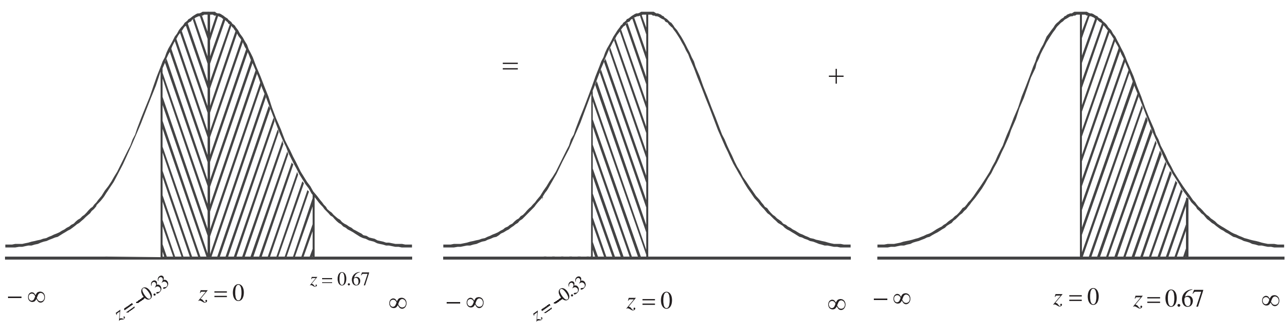
$$= 15.87 \ \%$$

Thus, during 15.87 % of the days of a month, the daily sale of petrol is less than 30,000 litres.

(2) The probability that during the month of May, the daily demand of petrol is between 32,000 and 35,000 litre

$$= P(32000 \le X \le 35000) \quad = P\left(\frac{32000-33000}{3000} \le \frac{X-\mu}{\sigma} \le \frac{35000-33000}{3000}\right)$$

$$= P(-0.33 \le Z \le 0.67)$$



$$= P(-0.33 \le Z \le 0) + P(0 \le Z \le 0.67)$$

$$= P(0 \le Z \le 0.33) + 0.2486 \qquad (\because \text{ Symmetry})$$

$$= 0.1293 + 0.2486$$

$$= 0.3779$$

The number of days in the month of May is N = 31. Therefore, the expected number of days in the month of May during which the daily sale of petrol is

between 32,000 litter to 35,000 litre $= 31 \times 0.3779$

$= 11.71$

$\simeq$ 12 Days (approximately)

Thus, in the month of May, approximately during 12 days the demand of petrol is between 32,000 litre and 35,000 litre.

Illustration 11 : 200 students are selected from all the students of a school and the marks obtained by them in an examination of 100 marks follows normal distribution. The mean marks of the distribution is 60 and its standard deviation is 8.

(1) If 70 or more marks are required for the special scholarship then obtain the number of students getting special scholarship.

(2) Obtain the minimum marks of 10% of the students getting maximum marks.

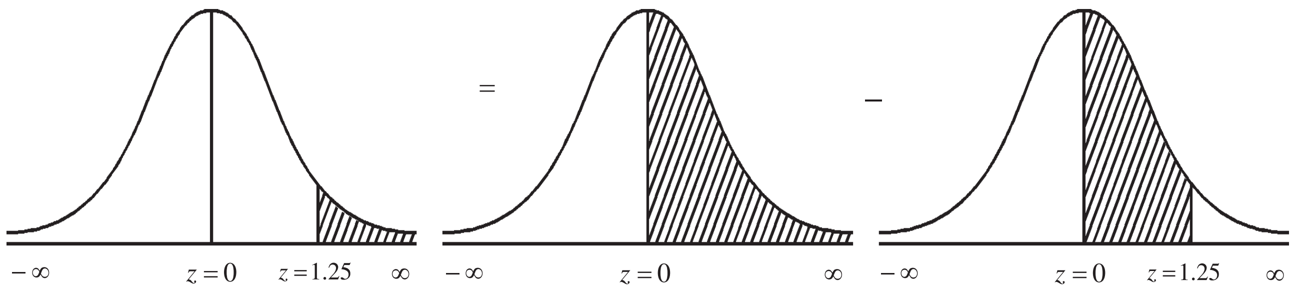
Here, X = marks obtained by a student

Also, N = 200, $\mu = 60$ and $\sigma = 8$.

(1) Probability that the marks of the student is 70 or more

$$= P(X \ge 70) = P\left(\frac{X-\mu}{\sigma} \ge \frac{70-60}{8}\right)$$

$$= P(Z \ge 1.25)$$



$$= P(0 \le Z < \infty) - P(0 \le Z \le 1.25)$$

$$= 0.5 - 0.3944$$

$$= 0.1056$$

$\therefore$ the expected number of students getting 70 or more marks

$$= 200 \times 0.1056$$

$$= 21.12$$

$$\simeq 21 \text{ (approximately)}$$

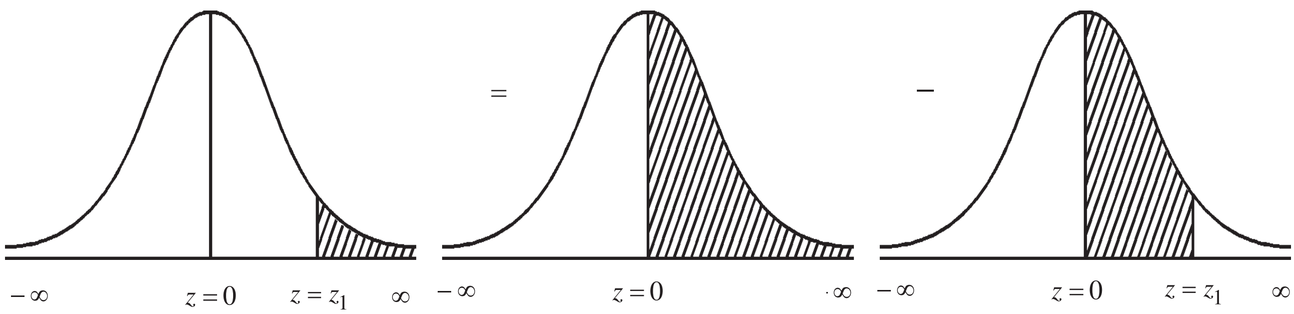
Thus, the approximate number of students getting special scholarship is 21.

(2) Suppose the minimum marks of 10% of the students getting maximum marks is x_1. The probability that a student gets x_1 or more marks is 0.10.

$\therefore \quad P(X \ge x_1) = 0.10$

$$\therefore \quad P\left(\frac{X-\mu}{\sigma} \geq \frac{x_1-60}{8}\right) = 0.10$$

$$\therefore \quad P(Z \geq z_1) = 0.10, \quad \text{where} \quad z_1 = \frac{x_1-60}{8}$$



$$\therefore 0.10 = P(0 \leq Z < \infty) - P(0 \leq Z \leq z_1)$$

$$\therefore 0.10 = 0.5 - P(0 \leq Z \leq z_1)$$

$$\therefore P(0 \leq Z \leq z_1) = 0.40$$

By using the table of standard normal variable the estimated value of z_1 can be determined as under

From table	**Area**	***Z*-score**
Nearest value before 0.40	0.3997	1.28
Nearest value after 0.40	0.4015	1.29
Average value	0.4006	1.285

From the above table it can be seen that the nearest value to 0.40 is 0.3997 and the respective value of Z-score is 1.28.

Therefore $z_1 = 1.28$

$$\therefore \frac{x_1-60}{8} = 1.28$$

$$\therefore x_1 - 60 = 10.24$$

$$\therefore x_1 = 70.24$$

Thus, the minimum marks of most intelligent 10% of the students is $70.24 \simeq 70$.

Illustration 12 : The monthly income of a group of employees follows normal distribution. The mean of the distribution is ₹ 15,000 and its standard deviation is ₹ 4000. From this information, (1) obtain range of monthly income for middle 60% of the employees. (2) if monthly income of 250 employees is between ₹ 15000 and certain fixed income ₹ x_1 then find the value of x_1.

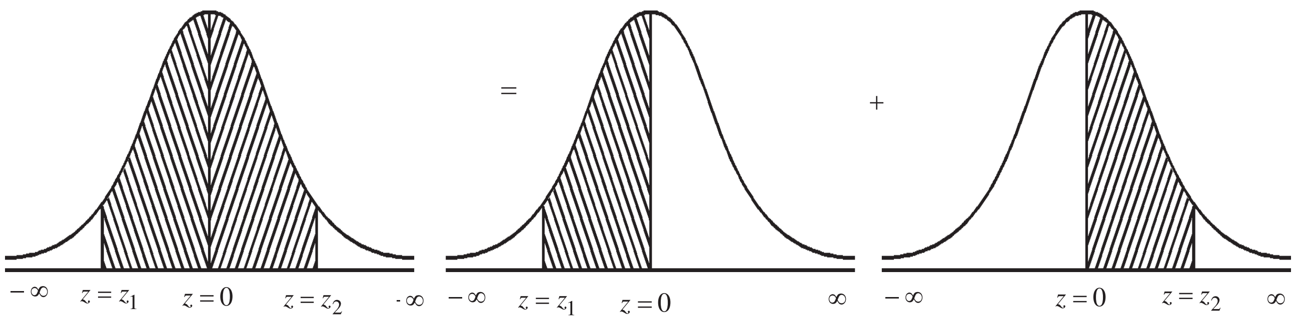
Here, X = monthly income of an employee. Also, N = 1000, $\mu = ₹\,15{,}000$ and $\sigma = ₹\,4000$.

(1) Suppose the range of monthly income of exactly middle 60% of employee is ₹ x_1 and ₹ x_2 where x_1 and x_2 are at equal distance from mean μ. Now the probability that the monthly income of employee is between x_1 and x_2 is 0.60.

i.e. $P(x_1 \le X \le x_2)$ = 0.60.

$\therefore\ P\left(\frac{x_1-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{x_2-\mu}{\sigma}\right) = 0.60$

$\therefore\ P(z_1 \le Z \le Z_2) = 0.60$ where $z_1 = \frac{x_1 - 15000}{4000}$ and $z_2 = \frac{x_2 - 15000}{4000}$



$$0.60 = P(z_1 \le Z \le 0) + P(0 \le Z \le z_2)$$

Now, since x_1 and x_2 are at equal distance from mean μ, the perpendicular line at Z = 0 divides the total area (probability) under the curve between $Z = z_1$ and $Z = z_2$ into two equal parts. So, $z_1 = -z_2$ and also $P(z_1 \le Z \le 0) = 0.30$ and $P(0 \le Z \le z_2) = 0.30$.

By using the table of standard normal variable the estimated value of z_1 and z_2 can be obtained as under

From table	Area	*Z*-score
Nearest value before 0.30	0.2995	0.84
Nearest value after 0.30	0.3023	0.85
Average value	0.3009	0.845

The nearest value to 0.30 is 0.2995 and the respective value of Z-score is 0.84.

$\therefore\ z_1 = -0.84$ and $z_2 = 0.84$

$\therefore\ \frac{x_1 - 15000}{4000} = -0.84$ and $\frac{x_2 - 15000}{4000} = 0.84$

$\therefore\ x_1 - 15000 = -3360$ and $\therefore\ x_2 - 15000 = 3360$

$\therefore\ x_1 = 11640$ and $\therefore\ x_2 = 18360$

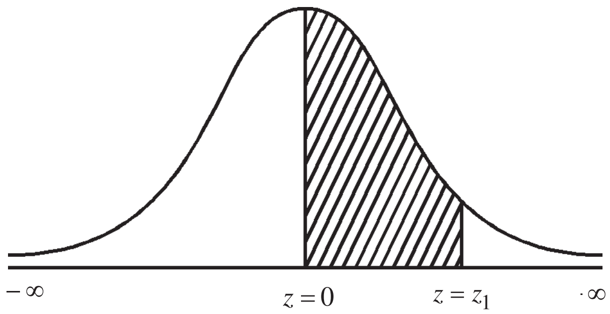
Thus, the range of monthly income for middle 60 % of the employees will be ₹ 11,640 to ₹ 18,360.

(2) Monthly income of 250 employees is between ₹ 15,000 and ₹ x_1

Therefore $P(15000 \le X \le x_1) = \frac{250}{1000}$

$\therefore\ P\left(\frac{15000-15000}{4000} \le \frac{X-\mu}{\sigma} \le \frac{x_1-15000}{4000}\right) = 0.25$

$\therefore\ P(0 \le Z \le z_1) = 0.25$ where $z_1 = \frac{x_1-15000}{4000}$



By using the table of standard normal variable the estimated value of z_1 can be obtained as under

From table	Area	Z-score
Nearest value before 0.25	0.2486	0.67
Nearest value after 0.25	0.2518	0.68
Average value	0.2502	0.675

It is clear from the above table that $z_1 = 0.675$

$\therefore\ \frac{x_1-15000}{4000} = 0.675$

$\therefore\ x_1 - 15000 = 2700$

$\therefore\ x_1 = 17700$

Thus, monthly income of 250 employees will be between ₹ 15,000 and ₹ 17,700.

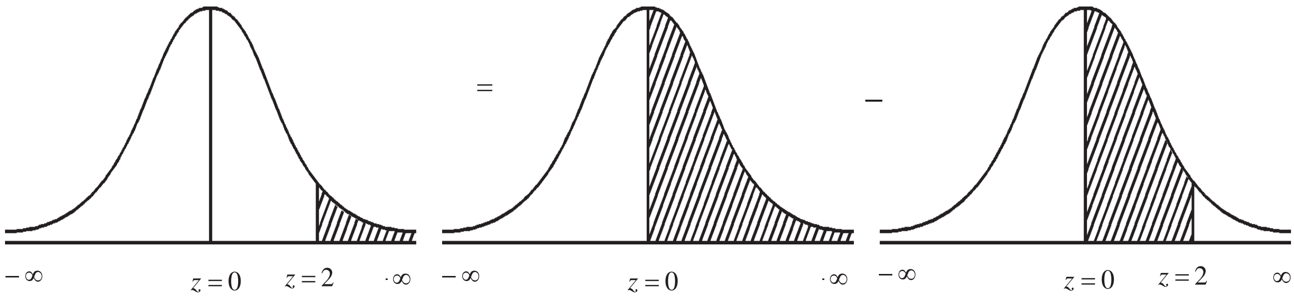
Illustration 13 : The bill amount of purchase by the customers in departmental store follows normal distribution and its mean is ₹ 800 and standard deviation is ₹ 200. On a day, 57 customers had the bill amount more than ₹ 1200. Estimate the number of customers who visited the store on that day.

Here $X =$ bill amount of purchase by the customer. $\mu = 800$ and $\sigma = 200$. Suppose N customers visited that departmental store during that day.

The probability that the bill amount of purchase by the customer is more than ₹ 1200

$$= P(X \geq 1200) \quad = P\left(\frac{X-\mu}{\sigma} \geq \frac{1200-800}{200}\right)$$

$$= P(Z \geq 2)$$



$$= P(0 \leq Z < \infty) - P(0 \leq Z \leq 2)$$

$= 0.5 - 0.4772$ (From Z-table)

$= 0.0228$

Now, the expected number of customer whose bill amount of purchase is more than ₹ 1200 $= N \times P(x \geq 1200)$

$$57 = N \times 0.0228$$

$$\therefore \quad N = \frac{57}{0.0228}$$

$$\therefore \quad N = 2500$$

$\therefore$ 2500 customers visited the store on that day.

Illustration 14 : For a group of 1000 persons, the average height is 165 cms and variance is 100 (cms)^2. The distribution of height of these persons follows normal distribution. From this information, determine the third decile and the 60th percentile and interpret it.

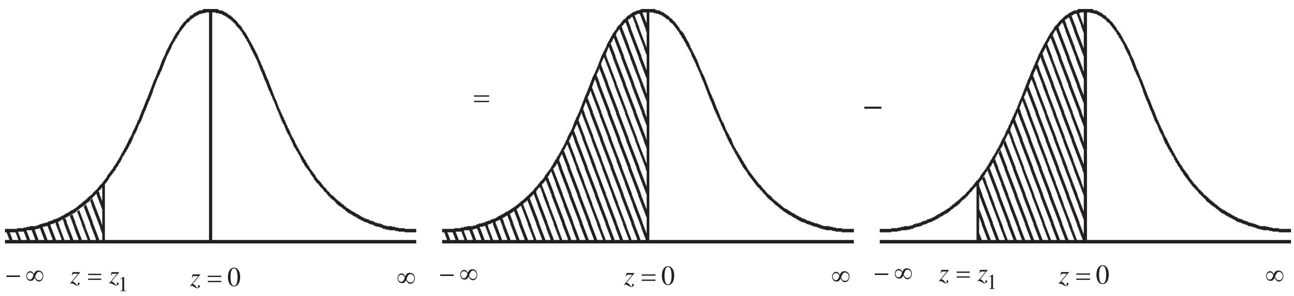
Here, $X =$ height of a person in the group. Also, $\mu = 165$ and $\sigma^2 = 100$ therefore $\sigma = 10$

The third decile (D_3) is to be determined. According to the definition of D_3, 30% of the observations in the data have the value less than or equal to D_3.

$$\therefore P(X \leq D_3) = \frac{30}{100}$$

$$\therefore P\left(\frac{X-\mu}{\sigma} \leq \frac{D_3-165}{10}\right) = 0.30$$

$$\therefore P(Z \leq z_1) = 0.30 \text{ where } z_1 = \frac{D_3-165}{10}$$



$$0.30 = P(-\infty < Z \le 0) - P(z_1 \le Z \le 0)$$

$$0.30 = 0.50 - P(z_1 \le Z \le 0)$$

$$\therefore P(z_1 \le Z \le 0) = 0.50 - 0.30$$

$$\therefore P(0 \le Z \le z_1) = 0.20 \quad (\because \text{ Symmetry})$$

By using the table of standard normal variable the estimated value of z_1 can be obtained as under

From table	**Area**	***Z*-score**
Nearest value before 0.2	0.1985	0.52
Nearest value after 0.2	0.2019	0.53
Average value	0.2002	0.525

The nearest value to 0.2 is 0.2002 and the respective value of *Z*-score is 0.525 and it is to the left of $Z = 0$

Therefore $z_1 = -0.525$

$$\therefore \quad \frac{D_3 - 165}{10} = -0.525$$

$$\therefore \quad D_3 - 165 = -5.25$$

$$\therefore \quad D_3 = 159.75$$

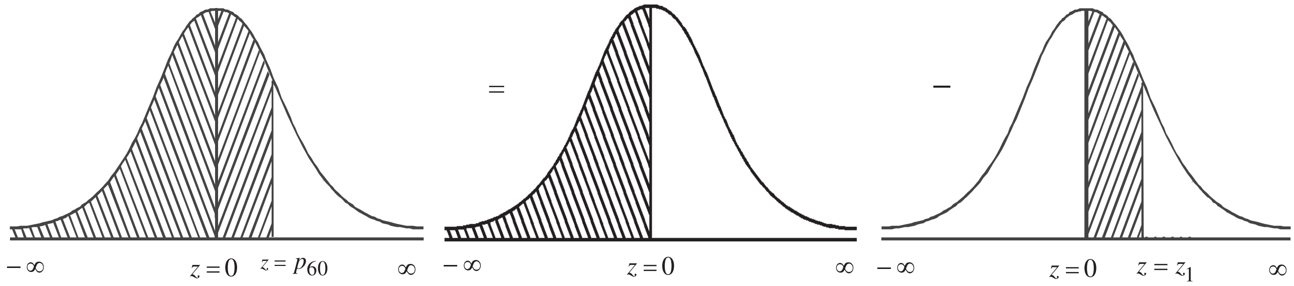
Thus, 30% of the persons in the group have height less than or equal to 159.75 cms.

Now, according to the definition of the 60th percentile (P_{60}), 60% of the observations in the given data have the value less than or equal to P_{60}.

$$\therefore P(X \le P_{60}) = \frac{60}{100}$$

$$\therefore P\left(\frac{X-\mu}{\sigma} \le \frac{P_{60}-165}{10}\right) = 0.60$$

$$\therefore P(Z \le z_1) = 0.60 \text{ where } z_1 = \frac{P_{60}-165}{10}$$



$$0.60 = P(-\infty < Z \le 0) + P(0 \le Z \le z_1)$$

$$0.60 = 0.50 + P(0 \le Z \le z_1)$$

$$\therefore P(0 \le Z \le z_1) = 0.10$$

By using the table of standard normal variable the estimated value of z_1 can be obtained as under

From table	Area	Z-score
Nearest value before 0.10	0.0987	0.25
Nearest value after 0.10	0.1026	0.26
Average value	0.10065	0.255

The nearest value to 0.10 is 0.10065 and the respective value of Z-score is 0.255.

$$\therefore z_1 = 0.255$$

$$\therefore \frac{P_{60}-165}{10} = 0.255$$

$$\therefore P_{60} - 165 = 2.55$$

$$\therefore P_{60} = 167.55$$

Thus, 60% of the persons in the group have height less than or equal to 167.55 cms.

Illustration 15 : A manufacturing company produces electric bulb and life of the electric bulb (in hours) follows normal distribution. Its average life is 2040 hours. If 3.36% of bulbs have life more than 2150 hours then find variance of the life of bulbs.

Here, X = life of electric bulb. Also, $\mu = 2040$. Suppose its variance is σ^2.

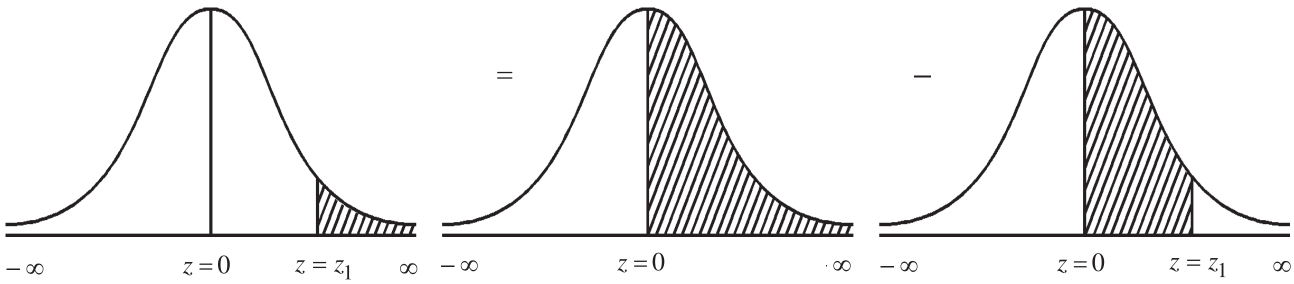
Now, 3.36% of the bulbs have life more than 2150 hours.

$$\therefore P(X \ge 2150) = \frac{3.36}{100}$$

$$\therefore P\left(\frac{X-\mu}{\sigma} \ge \frac{2150-2040}{\sigma}\right) = 0.0336$$

$$\therefore P\left(Z \ge \frac{110}{\sigma}\right) = 0.0336$$

$$\therefore P(Z \ge z_1) = 0.0336 \text{ where } z_1 = \frac{110}{\sigma}$$



$$0.0336 = P(0 \le Z < \infty) - P(0 \le Z \le z_1)$$

$$0.0336 = 0.5 - P(0 \le Z \le z_1)$$

$$\therefore P(0 \le Z \le z_1) = 0.5 - 0.0336$$

$$\therefore P(0 \le Z \le z_1) = 0.4664$$

From the table of standard normal variable, for Z-score 1.83, $P(0 \le Z \le 1.83) = 0.4664$

$$\therefore z_1 = 1.83$$

$$\therefore \frac{110}{\sigma} = 1.83$$

$$\therefore \sigma = \frac{110}{1.83}$$

$$\therefore \sigma = 60.11$$

$$\therefore \sigma^2 = 3613.21$$

Thus, the variance of the life of electric bulbs produced is 3613.21 $(\text{hours})^2$.

Illustration 16 : The profit in daily business of a businessman having grocery shop follows normal distribution. Variance of profit is 22500 $(₹)^2$, and the probability that the daily profit is less than Rs. 1000 is 0.0918. Find the average daily profit.

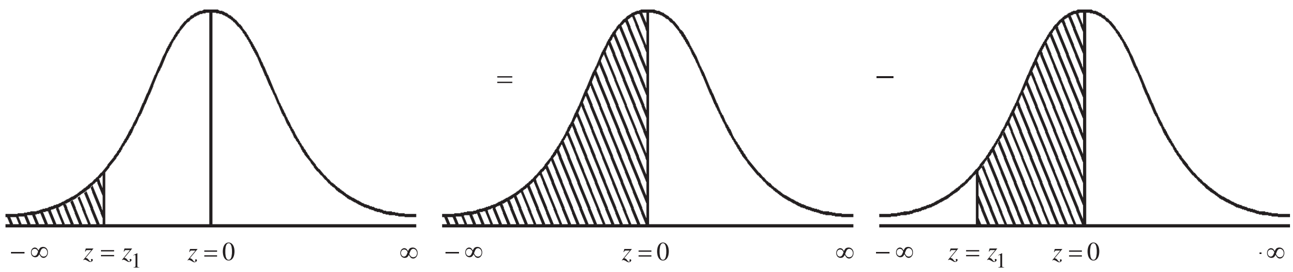
Here, X = daily profit of the businessman in his business. As $\sigma^2 = 22{,}500$, therefore $\sigma = 150$ and suppose the average profit is μ.

Now, the probability that the daily profit is less than ₹ 1000 = 0.0918

$$\therefore P(X \le 1000) = 0.0918$$

$$\therefore P\left(\frac{X-\mu}{\sigma} \le \frac{1000-\mu}{150}\right) = 0.0918$$

$$\therefore P(Z \le z_1) = 0.0918 \text{ where } z_1 = \frac{1000-\mu}{150}$$



$$0.0918 = P(-\infty < Z \le 0) - P(z_1 \le Z \le 0)$$

$$0.0918 = 0.5 - P(z_1 < Z \le 0)$$

$$\therefore\ P(z_1 \le Z \le 0) = 0.5 - 0.0918$$

$$\therefore\ P(0 \le Z \le z_1) = 0.4082 \quad (\because \text{ Symmetry})$$

From the table of standard normal variable, Z-score is 1.33.

$$\therefore\ z_1 = -1.33$$

$$\therefore\ \frac{1000-\mu}{150} = -1.33$$

$$\therefore\ \mu = 800.5$$

Thus, the average daily profit of the businessman in his business is ₹ 800.50.

Illustration 17 : The maximum temperature of a city during summer follows normal distribution. On a particular day, the probability that the maximum temperature of the city is more than 31° Celsius is 0.3085, whereas the probability that during some other day, the maximum temperature is less than 27° is 0.0668. Find mean and standard deviation of the maximum temperature of the city.

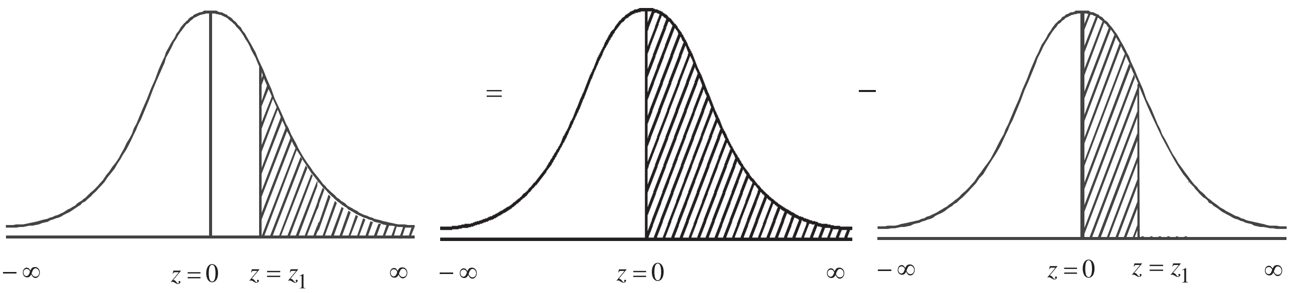
Here, X = maximum temperature (in Celsius) of the city. Suppose μ and σ are the mean and standard deviation.

Now, the probability that the maximum temperature is more than 31° Celsius = 0.3085

$$\therefore\ P(X \ge 31) = 0.3085$$

$$\therefore\ P\left(\frac{X-\mu}{\sigma} \ge \frac{31-\mu}{\sigma}\right) = 0.3085$$

$$\therefore\ P(Z \ge z_1) = 0.3085 \quad \text{where} \quad z_1 = \frac{31-\mu}{\sigma}$$



$$0.3085 = P(0 \le Z < \infty) - P(0 \le Z \le z_1)$$

$$0.3085 = 0.5 - P(0 \le Z \le z_1)$$

$$\therefore\ P(0 \le Z \le z_1) = 0.5 - 0.3085$$

$$\therefore\ P(0 \le Z \le z_1) = 0.1915$$

From the table of standard normal variable, Z-score is 0.5.

$$\therefore\ z_1 = 0.5$$

$$\therefore\ \frac{31-\mu}{\sigma} = 0.5$$

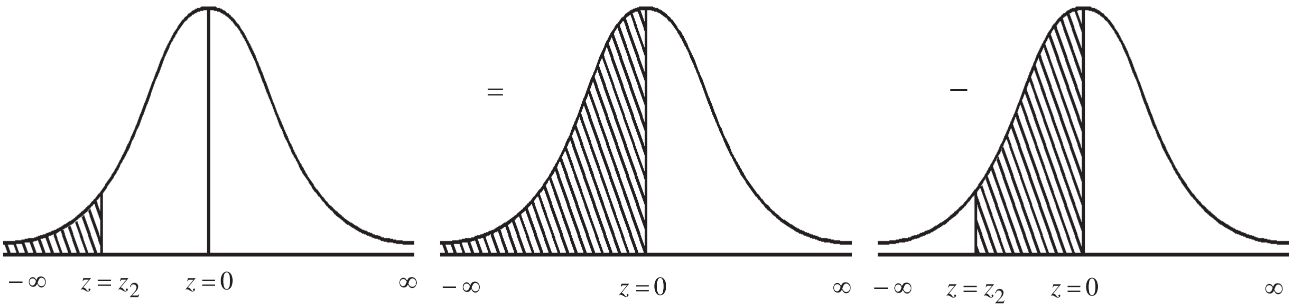
$$\therefore\ 31 - \mu = 0.5\,\sigma \quad .... (1)$$

The probability that the maximum temperature is less than 27° Celsius = 0.0668

$$\therefore\ P(X \le 27) = 0.0668$$

$$\therefore\ P\left(\frac{X-\mu}{\sigma} \le \frac{27-\mu}{\sigma}\right) = 0.0668$$

$$\therefore\ P(Z \le z_2) = 0.0668 \text{ where } z_2 = \frac{27-\mu}{\sigma}$$



$$0.0668 = P(-\infty < Z \le 0) - P(z_2 \le Z \le 0)$$

$$\therefore\ 0.0668 = 0.5 - P(z_2 \le Z \le 0)$$

$$\therefore\ P(z_2 \le Z \le 0) = 0.5 - 0.0668$$

$$\therefore\ P(0 \le Z \le z_2) = 0.4332 \ (\because \text{ Symmetry})$$

From the table of standard normal variable, Z-score is 1.5

$$\therefore\ z_2 = -1.5$$

$$\therefore\ \frac{27-\mu}{\sigma} = -1.5$$

$$\therefore\ 27 - \mu = -1.5\sigma \ \ldots\ldots\ldots\ldots\ldots\ (2)$$

Solving equations (1) and (2),

$$31 - \mu = 0.5\,\sigma$$

$$27 - \mu = -1.5\,\sigma$$

$$- \quad + \quad +$$

$$4 = 2\sigma$$

$$\therefore\ \sigma = 2$$

By substituting $\sigma = 2$ in equation (1),

$$31 - \mu = 0.5\,(2)$$

$$\therefore\ 31 - \mu = 1$$

$$\therefore\ \mu = 30$$

Thus, the mean of maximum temperature of a city is 30° Celsius and its standard deviation is 2° Celsius.

Illustration 18 : The probability density function of a normal variable is as under

$$f(x) = \frac{1}{4\sqrt{2\pi}} \quad e^{-\frac{1}{32}(x-50)^2} \quad ; \quad -\infty < x < \infty$$

Obtain parameters of this distribution and find the values of following :

(1) $P(52 \le X \le 58)$ (2) $P(|X-45| \le 4)$

By comparing the given probability density function with the probability density function of normal variable X

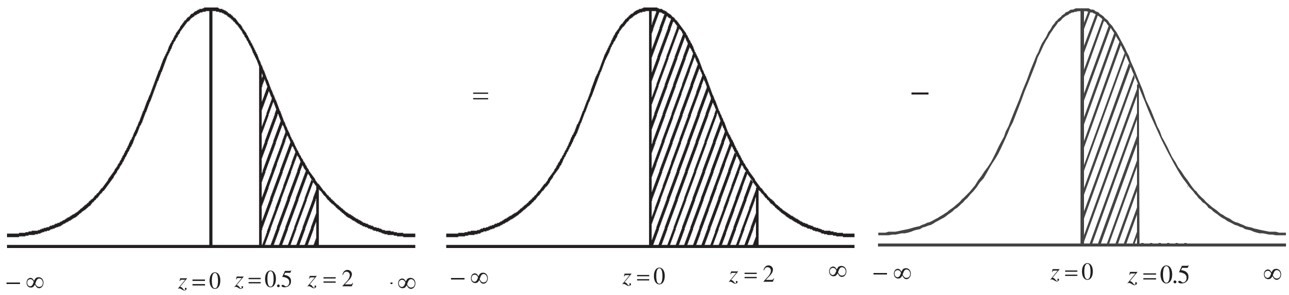
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \quad e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; \quad -\infty < x < \infty$$

Here, $\sigma\sqrt{2\pi} = 4\sqrt{2\pi}$ and $\mu = 50$

$\therefore \ \sigma = 4$

(1) $P(52 \le X \le 58) = P\left(\frac{52-50}{4} \le \frac{X-\mu}{\sigma} \le \frac{58-50}{4}\right)$

$= P(0.5 \le Z \le 2)$



$= P(0 \le Z \le 2) - P(0 \le Z \le 0.5)$

$= 0.4772 - 0.1915$

$= 0.2857$

Thus, $P(52 \le X \le 58) = 0.2857$

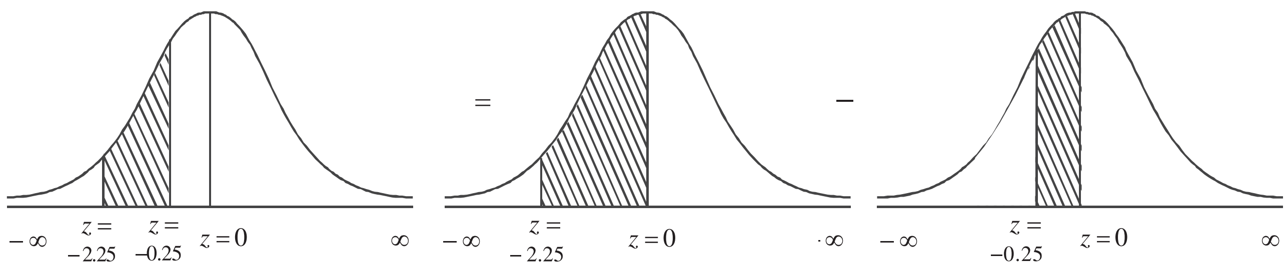
(2) $P(|X-45| \le 4) = P(-4 \le (X-45) \le 4)$ (definition of modulus)

$= P(-4+45 \le (X-45)+45 \le 4+45)$

$= P(41 \le X \le 49)$

$= P\left(\frac{41-50}{4} \le \frac{X-\mu}{\sigma} \le \frac{49-50}{4}\right)$

$= P(-2.25 \le Z \le -0.25)$



$= P(-2.25 \le Z \le 0) - P(-0.25 \le Z \le 0)$

$= P(0 \le Z \le 2.25) - P(0 \le Z \le 0.25)$ ($\because$ Symmetry)

$= 0.4878 - 0.0987$

$= 0.3891$

Thus, $P(|X - 45| \le 4) = 0.3891$.

Illustration 19 : The probability density function of a normal variable X is defined as under

$$f(x) = \text{constant} \bullet e^{-\frac{1}{2}\left(\frac{x-25}{10}\right)^2} \quad ; \quad -\infty < x < \infty$$

From this normal distribution estimate the values of the following :

(1) Third quartile (2) Quartile deviation (3) Mean deviation

By comparing the given probability density function with the probability density function of normal variable X,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; \quad -\infty < x < \infty$$

Here, $\mu = 25$ and $\sigma = 10$.

(1) Third quartile $Q_3 = \mu + 0.675\,\sigma$

$= 25 + 0.675\ (10)$

$= 25 + 6.75$

$= 31.75$

(2) Quartile deviation $= \frac{2}{3}\ \sigma$

$= \frac{2}{3}\ (10)$

$= \frac{20}{3}$

(3) Mean deviation $= \frac{4}{5}\ \sigma$

$= \frac{4}{5}\ (10)$

$= 8$

Thus, for the given normal distribution the estimates of the required values are 31.75, $\frac{20}{3}$ and 8 respectively.

Illustration 20 : The extreme quartiles for a normal distribution are 20 and 50 respectively. Obtain the limits which include 95% of the observations of the distribution.

Here, $Q_1 = 20$ and $Q_3 = 50$. For the normal distribution

Mean = Median = Mode = $\frac{Q_3+Q_1}{2}$

$$\therefore \ \mu = \frac{50+20}{2}$$

$$\therefore \ \mu = 35$$

Quartile deviation = $\frac{2}{3}\ \sigma$

$$\therefore \ \frac{Q_3-Q_1}{2} = \frac{2}{3}\ \sigma$$

$$\therefore \ \frac{50-20}{2} = \frac{2}{3}\ \sigma$$

$$\therefore \ \frac{50-20}{2} \times \frac{3}{2} = \sigma$$

$$\therefore \ \sigma = 22.5$$

For the normal distribution the limits including 95% of the observations are $\mu \pm 1.96\sigma$. Hence, the interval (limits) is

$$(\mu - 1.96\sigma,\ \mu + 1.96\sigma)$$

$$\therefore \ (35 - 1.96\,(22.5),\ 35 + 1.96\,(22.5))$$

$$\therefore \ (35 - 44.1,\ 35 + 44.1)$$

$$\therefore \ (-9.1,\ 79.1)$$

Thus, from the given information, the limits including 95% of the observations are –9.1 to 79.1.

Illustration 21 : For a normal distribution, the first quartile and the mean deviation are 20 and 24 respectively. Obtain an estimate of the value of mode.

Here, Q_1 = 20 and mean deviation = 24.

$$\therefore \ \frac{4}{5}\ \sigma = 24$$

$$\therefore \ \sigma = 24 \times \frac{5}{4}$$

$$\therefore \ \sigma = 30$$

Now, quartile deviation = $\frac{2}{3}\ \sigma$

$$\therefore \ \frac{Q_3-Q_1}{2} = \frac{2}{3}\ \sigma$$

$$\therefore \ \frac{Q_3-20}{2} = \frac{2}{3}\ (24)$$

$$\therefore \ Q_3 - 20 = 16 \times 2$$

$$\therefore \ Q_3 = 32 + 20$$

$$\therefore \ Q_3 = 52$$

Now, for normal distribution, Mean = Median = Mode $= \frac{Q_3+Q_1}{2}$

$$= \frac{52+20}{2}$$

$$= 36$$

Thus, from the given information the estimated value of mode is 36.

Illustration 22 : The number of vehicles arriving at toll station during busy hours of national highway follows normal distribution. The mean of this distribution is μ and its standard deviation is σ. The number of vehicles arriving at two different busy time periods are 88 and 64 and if the respective values of *Z*-score for these values are 0.8 and –0.4 then find mean and standard deviation of number of vehicles arriving at the toll station during busy period.

Here, X = number of vehicles arriving at toll station during busy period.

The mean of this distribution is μ and its standard deviation is σ

$$Z\text{-score} = \frac{X-\mu}{\sigma}$$

When $X = 88$ then $Z = 0.8$ therefore $0.8 = \frac{88-\mu}{\sigma}$

$$\therefore\ 0.8\sigma = 88-\mu \qquad (1)$$

When $X = 64$ then $Z = -0.4$ therefore $-0.4 = \frac{64-\mu}{\sigma}$

$$\therefore -0.4\sigma = 64-\mu \qquad (2)$$

Solving equations (1) and (2),

$$\begin{array}{rl} & 0.8\sigma = 88-\mu \\ - & 0.4\sigma = 64-\mu \\ + & \quad\ \ - \quad + \\ \hline & 1.2\sigma = 24 \end{array}$$

$$\therefore\ \sigma = 20$$

By putting $\sigma = 20$ in equation (1), $0.8(20) = 88-\mu$

$$\therefore\ 16 = 88-\mu$$

$$\therefore\ \mu = 72$$

Thus. the mean of given data is $\mu = 72$ vehicles and its standard deviation is $\sigma = 20$ vehicles.

Activity

Collect the information of average monthly expenses of 30 families residing around your residence. Assuming that the average monthly expense of these families follows normal distribution with the mean and standard deviation determined by you,

(1) Obtain the limits of average monthly income of middle 60% of the families.

(2) Find the percentage of observations lying between the range $\mu \pm \sigma$ from your data.

Summary

- A function for obtaining probability that a continuous random variable assumes value between specified interval is called probability density function of that variable.
- The probability for a definite value of continuous random variable X obtained by the probability density function is always zero (0).
- A curve drawn by considering different values of normal variable X and its respective values of probability density function $f\left(x\right)$ is called normal curve.
- Normal curve is completely bell shaped and its skewness is zero.
- If X is a random normal variable with mean μ and standard deviation σ then $Z = \frac{X-\mu}{\sigma}$ is called standard normal variable.
- Standard normal probability distribution is a probability distribution of normal variable with mean zero and standard deviation 1.
- The observed value of standard normal variable Z is called standard score or Z-score and it is independent of unit of measurement.
- Normal distribution is also defined as $N\left(\mu, \sigma^2\right)$ where μ and σ are parameters of the distribution which indicate its mean and variance respectively.
- In order to express probability in percentage, the probability is multiplied by 100.
- In order to obtain expected number of observations, the probability is multiplied by the total number of observations (N).

List of Formulae

If X is a normal variable with mean μ and standard deviation σ then

(1) Standard normal variable $Z = \frac{X-\mu}{\sigma}$

(2) Mean = Median = Mode = $\frac{Q_3+Q_1}{2}$

(3) Approximate value of the first quartile $Q_1 = \mu - 0.675\,\sigma$

(4) Approximate value of the third quartile $Q_3 = \mu + 0.675\,\sigma$

(5) Quartile deviation = $\frac{2}{3}$ σ (approximately)

(6) Mean deviation = $\frac{4}{5}$ σ (approximately)

EXERCISE 3

Section A

Find the correct option for the following multiple choice questions :

1. Which of the following is probability density function for normal variable X with mean μ and standard deviation σ ?

(a) $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)}$; $-\infty<x<\infty$ (b) $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\left(\frac{x-\mu}{\sigma}\right)^2}$; $-\infty<x<\infty$

(c) $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$; $-\infty<x<\infty$ (d) $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$; $0\le x<\infty$

2. For a normal variable X with mean μ and standard deviation σ, which of the following is standard normal variable Z for it ?

(a) $Z = \frac{x-\sigma}{\mu}$ (b) $Z = \frac{\sigma-x}{\mu}$ (c) $Z = \frac{\mu-x}{\sigma}$ (d) $Z = \frac{x-\mu}{\sigma}$

3. Which of the following is probability density function for standard normal variable?

(a) $f(z)= e^{-\frac{1}{2}z^2}$; $-\infty<z<\infty$ (b) $f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ $-\infty<z<\infty$

(c) $f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$; $0<z<\infty$ (d) $f(z)=\frac{1}{\sqrt{2\pi}}e^{-z^2}$ $-\infty<z<\infty$

4. Which of the following are mean and variance of standard normal variable?

(a) Mean = 0, Variance = 1 (b) Mean = 1, Variance = 0

(c) Mean = 0, Variance = 0 (d) Mean = 1, Variance = 1

5. What is the total area under normal curve among the following ?

(a) –1 (b) 0 (c) 1 (d) 0.5

6. What is the area under the normal curve to the right hand side of perpendicular line at $X = \mu$?

(a) 0 (b) 0.5 (c) 1 (d) –0.5

7. In normal distribution, usually which limits include 99 % of the observations?

(a) $\mu\pm1.96\sigma$ (b) $\mu\pm2\sigma$ (c) $\mu\pm3\sigma$ (d) $\mu\pm2.575\sigma$

8. In normal distribution, usually what percentage of the observations are included in the limits $\mu\pm\sigma$?

(a) 34.13 % (b) 95.45 % (c) 68.26 % (d) 50 %

9. Which of the following is approximate value of mean deviation for normal variable?

(a) $\frac{4}{5}\sigma$ (b) $\frac{4}{5}\mu$ (c) $\frac{2}{3}\sigma$ (d) $\frac{2}{3}\mu$

10. Which of the following is approximate value of quartile deviation for standard normal variable?

(a) $\frac{2}{3}\sigma$ (b) $\frac{2}{3}$ (c) $\frac{4}{5}\sigma$ (d) $\frac{4}{5}$

11. Mean and the first quartile for a normal distribution are 11 and 3 respectively. Which of the following is the value of the third quartile?

(a) 8 (b) 14 (c) 19 (d) 10

12. For a normal distribution, approximate value of mean deviation is 20. Which of the following is the value of quartile deviation?

(a) $\frac{25}{3}$ (b) $\frac{32}{3}$ (c) 24 (d) $\frac{50}{3}$

13. In usual notation of normal distribution, $x = 25$, $\mu = 20$ and $\sigma = 5$ then which of the following is the value of standard normal variable?

(a) 1 (b) –1 (c) 4 (d) $\frac{10}{3}$

14. Mean of a normal variable X is 50. If the value of Z–score is –2.5 for $x = 25$ then which of the following is a variance of the distribution ?

(a) 10 (b) 100 (c) 50 (d) 25

15. If the distribution of normal variable is shown as $N(20, 4)$ then which of the following intervals includes 99.73% of observations?

(a) (18, 22) (b) (16, 24) (c) (14, 26) (d) (12, 28)

Section B

Answer the following questions in one sentence :

1. Give the values of the constants used in probability density function of normal variable.

2. What is the probability that a continuous random variable takes definite value?

3. What is the shape of normal curve?

4. What is the skewness of normal distribution?

5. "Standard score is independent of unit of measurement". Is this statement true or false?

6. For which value of standard normal variable, the standard normal curve is symmetric on both the sides?

7. Which value of normal variable divides the area of normal curve in two equal parts?

8. What percentage of area is covered under the normal curve within the range $\mu - 2\sigma$ to $\mu + 2\sigma$?

9. Mean of a normal distribution is 13.25 and its standard deviation is 10. Estimate the value of its third quartile.

10. For a normal distribution having mean 10 and standard deviation 6, estimate the value of quartile deviation.

11. The approximate value of mean deviation for a normal distribution is 8. Find its standard deviation.

12. For a normal distribution, the estimated value of quartile deviation is 12. Find the value of its standard deviation.

13. For a probability distribution of standard normal variable, state the estimated limits for the middle 50 % observations.

14. The extreme quartiles of normal distribution are 20 and 30. Find its mean.

15. The monthly expense of a group of persons follows normal distribution with mean ₹ 10,000 and standard deviation ₹ 1000. A student has obtained a Z- score = ₹ 1 for randomly selected person having monthly expense more than 11,000. Is this calculation of Z- score true ? Give reason.

16. The age of a group of persons follows normal distribution with mean 45 years and standard deviation 10 years. Calculate Z-score for a randomly selected person having age 60 years.

17. Marks obtained by students of a school in Economics subject follows normal distribution with mean μ and standard deviation σ. The value of standard score that a randomly selected student obtained 60 marks is 1. If the variance of variable is 100 $(\text{marks})^2$ then find average marks.

Section C

Answer the following questions :

1. Define probability density function of continuous random variable.

2. Write the conditions for probability density function for continuous variable.

3. How is the normal curve drawn?

4. Define probability density function for normal variable.

5. What is the shape of standard normal curve? To which value of variable it is symmetric?

6. Define standard normal variable and write its probability density function.

7. A normal variable X has the probability density function as,

$$f(x) = \text{constant} \times e^{-\frac{1}{50}(x-10)^2} \quad ; \quad -\infty < x < \infty$$

Find the first quartile from this information.

8. The extreme quartiles of a normal variable are 10 and 30. Find its mean deviation.

9. For a normal variable, mean deviation is 48 and its third quartile is 120. Estimate its first quartile.

10. A normal variable X has the probability density function as,

$$f(x) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-100}{10}\right)^2} \quad ; \quad -\infty < x < \infty$$

For this distribution, obtain the limits which include middle 68.26% of the observations.

11. The probability that the value of standard normal variable lies between 0 and Z-score (z_1) is 0.4950. Find the possible values of Z-score.

Section D

Answer the following questions :

1. Define normal distribution and state the characteristics of normal curve.

2. State the properties of normal distribution

3. State the properties of standard normal distribution.

4. A normal distribution has mean 50 and variance 9. Find the probability that

(1) The value of normal variable X lies between 50 and 53.

(2) The value of normal variable X lies between 47 and 53.

5. If X is a normal variable with mean 100 and standard deviation 15 then find the percentage of observations

(1) Having value more than 85.

(2) Having value less than 130.

6. The weight of randomly selected 500 adult persons from a region of a city follows normal distribution. The average weight of these persons is 55 kg and its standard deviation is 7 kg.

(1) Estimate the number of persons having weight between 41 kg to 62 kg.

(2) Estimate the number of persons having weight less than 41 kg.

7. If probabilities for the value of standard normal variable Z are as under then estimate the value of Z-score (z_1) :

(1) Area to the left of $Z = z_1$ is 0.9928.

(2) Area to the rightt of $Z = z_1$ is 0.0250.

8. If Z is a standard normal variable then estimate the value of Z-score (z_1) such that the following conditions are satisfied :

(1) Area to the left of $Z = z_1$ is 0.15.

(2) Area to the rightt of $Z = z_1$ is 0.75.

9. If Z is a standard normal variable and z_1 represents the Z-score then estimate the value of z_1 so that the following conditions are satisfied :

(1) $P(-2 \le Z \le z_1) = 0.2857$ (2) $P(z_1 \le Z \le 1.75) = 0.10$

10. The monthly production of units in a factory is normally distributed with mean μ and standard deviation σ. The Z-scores corresponding to the production of 2400 units and 1800 units are 1 and –0.5 respectively. Find its mean and standard deviation.

11. A normal variable X has the following probability density function

$$f(x) = \frac{1}{6\sqrt{2\pi}} \cdot e^{-\frac{1}{72}(x-100)^2} \quad ; \quad -\infty < x < \infty$$

For this distribution, obtain the estimated limits for the exact middle 50% of the observations.

12. For a normal distribution, the first quartile is 35 and its third quartile is 65. Estimate the limits that includes exactly middle 95.45% of the observations.

13. For a normal distribution, the third quartile and quartile deviations are 36 and 24 respectively. Find the mean of the distribution.

14. A normal variable X has mean 200 and variance 100.

(1) Estimate the values of extreme quartiles.

(2) Find the approximate value of quartile deviation.

(3) Find the approximate value of mean deviation.

Section E

Solve the following :

1. An amount of purchase of a customer in a mall of a city follows normal distribution with mean ₹ 800 and standard deviation ₹ 200. If a customer is selected at random then find the probabilities for the following events :

(1) Amount of purchase made by him is in between ₹ 850 to ₹ 1200.

(2) Amount of purchase made by him is in between ₹ 600 to ₹ 750.

2. The average weight of 500 persons of age between 20 years and 26 years of certain area is 55 kg and its variance is 100 $(kg)^2$. The weight of these persons follows normal distribution. According to the weight of persons they can be categorized as under :

(1) Person having weight more than 70 kg is in the fat persons group

(2) Person having weight between 50 kg to 60 kg is in the healthy persons group.

(3) Person having weight less than 35 kg is in the physically weak person's group

From this information, estimate the number of fat persons, number of healthy persons and number of physically weak persons in that area.

3. The average monthly expense of students residing in university hostel is ₹ 2000 and its standard deviation is ₹ 500. If the monthly expense of a student follows normal distribution then

(1) Find percentage of students having expense between ₹ 750 and ₹ 1250.

(2) Find percentage of students having expense more than ₹ 1800.

(3) Find percentage of students having expense less than ₹ 2400.

4. The monthly average salary of workers working in a production house is ₹ 10,000 and its standard deviation is ₹ 2000. By assuming that the monthly salary of a worker follows normal distribution, estimate the maximum salary of 20% of the workers having lowest salary. Also estimate the minimum salary of 10% of the workers having highest salary.

5. A normal distribution has mean 52 and variance 64. Obtain estimated limits which include exactly middle 25% of the observations.

6. In a big showroom of electronic items, on an average 52 electronic units are sold every week and its variance is 9 $(\text{unit})^2$. Sale of electronic items follows normal distribution. The probability that the sale of electronic items during a week out of 52 weeks is from x_1 units to 61 units is 0.1574. Estimate the value of x_1. Also estimate the number of weeks during which the sale of electronic items is more than 55 units.

7. It is known that on an average a person spends 61 minutes in a painting exhibition. If this time is normally distributed and 20% of the persons spent less than 30 minutes in the exhibition then find variance of the distribution. Also determine the probability that a person spends more than 90 minutes in the exhibition.

8. If the diameter of pipes produced by a company manufacturing pipes is 20 mm to 22 mm then it is accepted by specified group of customers. The standard deviation of produced pipes is 4 mm and it is known that 70% of the pipes produced in the unit have diameter more than 19.05 mm. Find the average diameter of the produced pipes. Also find the percentage of pipes rejected by specified group of customers.

Note : The diameter of the produced pipes follows normal distribution.

9. A normal variable X has mean 400 and variance 900. Find the fourth decile and 90th percentile for this distribution and also interpret the values.

10. A normal variable X has the following density function :

$$f(x) = \frac{1}{50\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-150}{50}\right)^2} \quad ; \quad -\infty < x < \infty$$

For this distribution,

(i) If $P(x_1 \le x \le 250) = 0.4772$ then estimate x_1.

(ii) If $P(75 < x \le x_2) = 0.3539$ then estimate x_2.

Section F

Solve the following :

1. An intelligence test is conducted for 500 children and it is found that the average marks are 68 and standard deviations is 22. If the marks obtained by the children is normally distributed then (1) find the number of children getting marks more than 68. (2) Find the percentage of children getting marks between 70 and 90. (3) Find the minimum score of most intelligent 50 children.

2. Age of 500 employees working in a private company follows normal distribution with mean 40 years and standard deviation 6 years. The company wants to reduce its staff by 25% in the following manner :

(i) To retrench 5% of the employees having minimum age

(ii) After retrenching 5% of employees having minimum age, next 10% of the employees are to be transferred to another company.

(iii) To retire 10% of employees having maximum age.

From this information, find the age of employees who are to be retrenched, transferred and retired from the company.

3. An entrance test of 200 marks is conducted for higher study. 20,000 students remain presents in the examination and the marks obtained by them follows normal distribution with mean 120 and standard deviation 20. The rules for the result are as under :

(a) Students who acquire less than 40 percent marks are failed.

(b) An additional test is conducted for the students acquiring marks between 40 percent and 48 percent.

(c) Students who acquire mark between 48 percent and 75 percent are called for personal interview.

(d) Students who acquire marks more than 75 percent get direct admission for the higher studies.

Find approximate number of students who : (1) failed in test (2) appeared for additional 100 marks test (3) appeared for personal interview and (4) got direct admission for the higher studies.

4. The monthly income of a group of persons follows normal distribution with mean ₹ 20,000 and standard deviation ₹ 5000. If the minimum monthly income of 50 richest person is ₹ 31,625 then how many persons are in the group? Also, what is the maximum income of 50 persons having lowest monthly income?

5. Analysis of result of 12th standard students of a school is as under :

Pass with distinction	:	15 % of total students
Pass without distinction	:	75 % of total students
Fail	:	10 % of total students

For passing the examination, minimum 40 % of the total marks and for distinction minimum 80 % marks are required. If the percentage of result of the students follows normal distribution then find mean and standard deviation and by using it determine the percentage marks for which 75 % of the students have less than that percentage marks.

6. The monthly bill amount of regular customers of a provision store follows normal distribution. If 7.78 % customers have monthly bill amount less than ₹ 3590 and 94.52 % customers have bill amount less than ₹ 5100 then determine the parameters of the normal distribution. Also determine the interval for monthly bill amount of exactly middle 95% customers.

7. A normal variable X has following probability density function :

$$f(x) = \frac{1}{\sqrt{5000\pi}} \, e^{-\frac{1}{5000}(x-75)^2} \quad ; \quad -\infty \le x \le \infty$$

From this, answer the following questions :

(i) If $P(60 \le x \le x_2) = 0.5670$ then find x_2.

(ii) If $P(x_1 \le x \le 125) = 0.3979$ then find x_1.

(iii) Find $P(|x-50| \le 10)$.

8. A normal variable X has following probability density function :

$$f(x) = \text{constant} \cdot e^{-\frac{1}{200}(x-50)^2} \quad ; \quad -\infty < x < \infty$$

From this distribution, answer the following questions :

(1) Find median.

(2) Find estimated values of the extreme quartiles.

(3) Find approximate value of quartile deviation.

(4) Find approximate value of mean deviation.

Johann Carl Friedrich Gauss
(1777 – 1855)

Carl Friedrich Gauss was a German mathematician who contributed significantly to many fields, including number theory, algebra, statistics, analysis, differential geometry, geodesy, geophysics, mechanics, electrostatics astronomy, matrix theory, and optics. He was referred to as the Princeps mathematicorum. (Latin, "the foremost of mathematicians") and "greatest mathematician since antiquity". Gauss had an exceptional influence in many fields of mathematics and science and has several theories and results in his name.

In the area of probability and statistics, Gauss introduced what is now known as Gaussian or normal distribution, the Gaussian function and the Gaussian error curve. He showed how probability could be represented by a bell shaped or "normal" curve, which peaks around the mean or expected value and quickly falls off towards plus/minus infinity, which is basic to descriptions of statistically distributed data.

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4

Limit

Contents :

4.1 Introduction

We have studied function in 11th Standard. We studied in the chapter that when we substitute a particular value of a variable in the function, we got the corresponding value of the function. For example, if we substitute $x = 2$ in the function $f(x) = 2x + 3$, we get $f(2) = 7$. And if we substitute $x = 1$ in the function $f(x) = \frac{3-x}{3x+2}$, we get $f(1) = \frac{2}{5}$. But this is not possible for all functions and all values of x. Let us consider a function $f(x) = \frac{x^2-9}{x-3}$ and if we substitute $x = 3$ in $f(x)$, we get $f(3) = \frac{0}{0}$ which is an indeterminate value. To find approximate value of $f(3)$ for this function, we need to know the concept of limit of a function. So, limits can be used to approximate the value of a function when the value of the function is indeterminate.

We consider the following illustration to clarify the above concept.

Assume that we are watching a football game through internet. Unfortunately, the connection is choppy and we missed what happened at 14:00 (14 minutes after the commencement of match.)

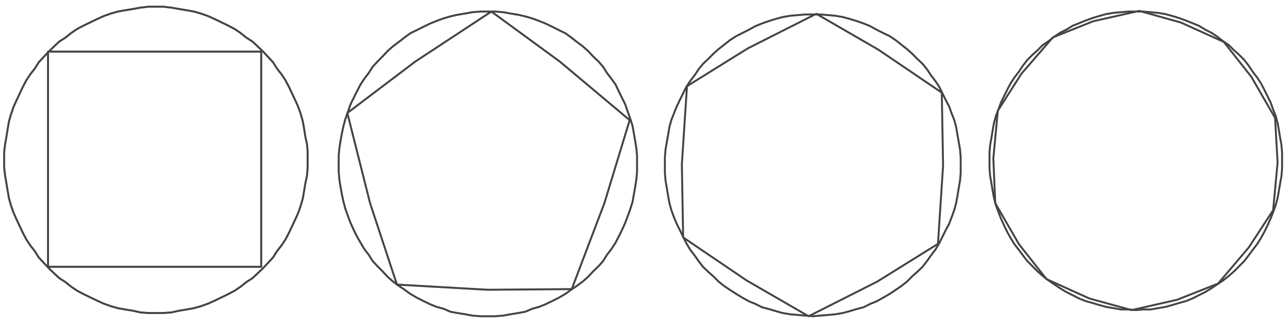
What would be the position of the ball at 14:00 ? We have seen the position of the ball at 13:58 (13 minutes and 58 seconds after the commencement of match), 13:59, 14:01, 14:02.

We will see the neighbouring instants of 14:00, (13:59 and 14:01) and estimate the position of the ball at 14:00. Our estimation is "At 14:00, the ball was somewhere between its position at 13:59 and 14:01." With a slow-motion camera, we might even say "At 14:00, the ball was somewhere between its positions at 13:59.99 and 14:00.01". It means that our estimation improves as we take closer and closer instants to 14:00. The approximate position of the ball thus obtained will be the limiting value of the position of the ball.

Thus, we can say that, "Limit is a method for finding confident approximate value."

We consider one more illustration.

Suppose we want to find the area of a circle. We can estimate the area of circle from the area of polygon drawn inside the circle.



4 side 5 side 6 side 10 side

We can see from the above figures that as the number of sides of polygon increases, area of the polygon approaches nearer the area of circle. The limiting value of the area of polygon is the best approximate value of the area of the circle.

Thus, limit can be used to approximate the unknown values by using its nearby values. Closer the neighbouring values, better is the approximation.

To understand the concept of limit, we shall understand the following basic terms.

4.2 Real Line and its Interval

Real line : The real line or real number line is a line where its points are the real numbers.

Interval : A set of real numbers between any two real numbers is an interval. We shall study different types of intervals.

Closed Interval : If $a \in R$, $b \in R$ and $a < b$ then the set of all real numbers between a and b including a and b is called a closed interval. It is denoted by $[a, b]$.

$$[a, b] = \{x \mid a \le x \le b, \ x \in R\}$$

Open Interval : If $a \in R$, $b \in R$ and $a < b$ then the set of all real numbers between a and b not including a and b is called an open interval. It is denoted by (a, b).

$$(a, b) = \{x \mid a < x < b, \ x \in R\}$$

Closed-Open Interval : If $a \in R$, $b \in R$ and $a < b$ then the set of all real numbers between a and b including a but not including b is called a closed open interval. It is denoted by $[a, b)$.

$[a, b) = \{x \mid a \le x < b, \ x \in R\}$

Open-Closed Interval : If $a \in R$, $b \in R$ and $a < b$ then set of all real numbers between a and b not including a but including b is called an open closed interval. It is denoted by $(a, b]$.

$(a, b] = \{x \mid a < x \le b, \ x \in R\}$

4.3 Modulus

If $x \in R$ then

$|x| = x$ if $x \ge 0$

$= -x$ if $x < 0$

Modulus of any real number is always non-negative.

e.g. $|3| = 3$, $|-4| = 4$, $|0| = 0$

Meaning of $|x - a| < \delta$ (*Delta*)

Using the definition of modulus

$|x-a| < \delta = (x-a) < \delta$ if $x \ge a$ **or** $x < a + \delta$ if $x \ge a$

$= (a-x) < \delta$ if $x < a$ **or** $x > a - \delta$ if $x < a$

$\therefore$ $|x-a| < \delta \Leftrightarrow x \in (a - \delta, a + \delta)$

4.4 Neighbourhood

Any open interval containing a, $a \in R$ is called a **neighbourhood** of a.

δ neighbourhood of a :

If $a \in R$ and δ is non-negative real number then the open interval $(a - \delta, a + \delta)$ is called δ neighbourhood of a. It is denoted by $N(a, \delta)$.

Here, it can be understood that

$N(a, \delta) = \{x \mid a - \delta < x < a + \delta, \ x \in R\}$

$= \{x \mid |x - a| < \delta, \ x \in R\}$

δ neighbourhood of a can be expressed in the following different ways.

Neighbourhood form	Modulus form	Interval form
$N(a, \delta)$	$|x-a| < \delta$	$(a - \delta, a + \delta)$

Illustration 1 : Express $N(5, 0.2)$ in modulus and interval form.

Comparing $N(5, 0.2)$ with $N(a, \delta)$, we get $a = 5$ and $\delta = 0.2$.

Modulus form : $|x-a| < \delta$

Putting $a = 5$ and $\delta = 0.2$,

$N(5, 0.2) = |x-5| < 0.2$

Interval form : $(a-\delta,\ a+\delta)$

Putting $a = 5$ and $\delta = 0.2$,

$$N(5, 0.2) = (5-0.2,\ 5+0.2)$$
$$= (4.8,\ 5.2)$$

Illustration 2 : Express 0.001 neighbourhood of 3 in modulus and interval form.

Comparing 0.001 neighbourhood of 3 with δ neighbourhood of a, we get a = 3 and δ = 0.001.

Modulus form : $|x-a| < \delta$

Putting $a = 3$ and $\delta = 0.001$,

0.001 neighbourhood of 3 = $|x-3| < 0.001$

Interval form : $(a-\delta,\ a+\delta)$

Putting $a = 3$ and $\delta = 0.001$,

0.001 neighbourhood of 3 $= (3-0.001,\ 3+0.001)$

$= (2.999,\ 3.001)$

Illustration 3 : Express $|x+1| < 0.5$ in neighbourhood and interval form.

Comparing $|x+1| < 0.5$ with $|x-a| < \delta$, we get $a = -1$ and $\delta = 0.5$.

Neighbourhood form : $N(a,\ \delta)$

Putting $a = -1$ and $\delta = 0.5$,

$|x+1| < 0.5 = N(-1,\ 0.5)$

Interval form : $(a-\delta,\ a+\delta)$

Putting $a = -1$ and $\delta = 0.5$,

$$|x+1| < 0.5 = (-1-0.5,\ -1+0.5)$$
$$= (-1.5,\ -0.5)$$

Illustration 4 : Express $(0.9,\ 1.1)$ in neighbourhood and modulus form.

Comparing $(0.9,\ 1.1)$ with $(a-\delta,\ a+\delta)$, we get $a-\delta = 0.9$ and $a+\delta = 1.1$.

Adding $a-\delta = 0.9$ and $a+\delta = 1.1$, we get $2a = 2$ $\therefore\ a = 1$.

Putting $a = 1$ in $a+\delta = 1.1$, we get $\delta = 0.1$.

Neighbourhood form : $N(a,\ \delta)$

Putting $a = 1$ and $\delta = 0.1$,

$(0.9, 1.1) = N(1, 0.1)$

Modulus form : $|x-a| < \delta$

Putting $a=1$ and $\delta = 0.1$,

$(0.9, 1.1) = |x-1| < 0.1$

Punctured δ neighbourhood of a :

If $a \in R$ and δ is a non-negative real number then the open interval $(a-\delta, a+\delta) - \{a\}$ is called punctured δ neighbourhood of *a*. It is denoted by $N^*(a, \delta)$.

Here, it can be understood that

$$N^*(a, \delta) = N(a, \delta) - \{a\}$$

$$= \{x \mid a-\delta < x < a+\delta, \ x \neq a, \ x \in R\}$$

$$= \{x \mid |x-a| < \delta, \ x \neq a, \ x \in R\}$$

e.g. $N^*(5, 2) = N(5, 2) - \{5\}$

$$= \{x \mid 3 < x < 7, x \neq 5, \ x \in R\}$$

$$= \{x \mid |x-5| < 2, \ x \neq 5, \ x \in R\}$$

EXERCISE 4.1

1. Express the following in modulus and interval form :

(1) 0.4 neighbourhood of 4 (2) 0.02 neighbourhood of 2

(3) 0.05 neighbourhood of 0 (4) 0.001 neighbourhood of –1

2. Express the following in interval and neighbourhood form :

(1) $|x-2| < 0.01$ (2) $|x+5| < 0.1$

(3) $|x| < \frac{1}{3}$ (4) $|x+3| < 0.15$

3. Express the following in modulus and neighbourhood form :

(1) $3.8 < x < 4.8$ (2) $1.95 < x < 2.05$

(3) $-0.4 < x < 1.4$ (4) $1.998 < x < 2.002$

4. Express $N(16, 0.5)$ in the interval and modulus form.

5. If $N(3, b) = (2.95, k)$ then find the values of b and k.

6. If $|x-10| < k_1 = (k_2, 10.01)$ then find the values of k_1 and k_2.

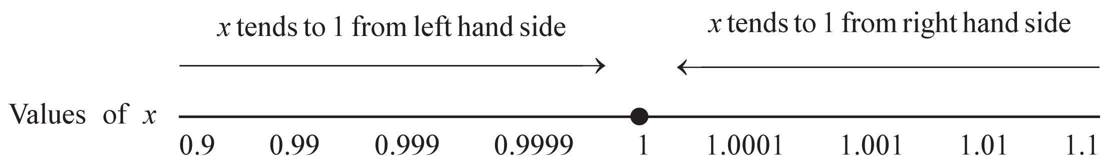
*

Meaning of $x \to a$:

If the value of variable x is brought very close to a number 'a' by increasing or decreasing its value then it can be said that x tends to a. It is denoted by $x \to a$.

It is necessary here to note that $x \to a$ means value of x approaches very close to a but it will not be equal to a.

e.g. Let us understand the meaning of $x \to 1$.

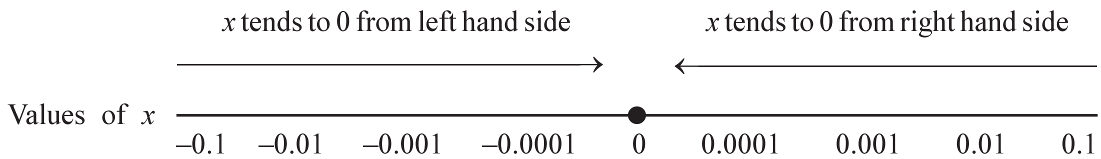


Meaning of $x \to 0$:

If by decreasing the positive value of a variable x or by increasing negative value of the variable x, the value of x is brought very close to '0' then it can be said that x tends to 0. It is denoted by $x \to 0$.

It is necessary here to note that $x \to 0$ means, the value of x approaches very close to 0 but it will not be equal to 0.

Let us understand the meaning of $x \to 0$.



4.5 Limit of a function

When the value of a variable x is brought closer and closer to a number 'a', the value of function $f(x)$ reaches closer and closer to a definite number 'l' then we can say that as x tends to a, $f(x)$ tends to l. i.e. as $x \to a$, $f(x) \to l$. Symbolically it can be written as $\lim\limits_{x \to a} f(x) = l$. l is called the limiting value of the function.

Definition : The function $f(x)$ has a limit l as x tends to 'a' if for each given predetermined $\varepsilon > 0$, however small, we can find a positive number δ such that $\left|f(x) - l\right| < \varepsilon$ (*Epsilon*) for all x such that $|x - a| < \delta$.

Now, we shall understand how limit of a function is obtained.

Suppose, we want to find the value of the function $f(x) = \frac{x^2-1}{x-1}$ at $x = 1$.

If we put $x = 1$ in $f(x) = \frac{x^2-1}{x-1}$ we get $f(1) = \frac{0}{0}$ which is indeterminate. So, we cannot find the value of $f(1)$ but assuming value of x very close to 1, we can approximate the value of $f(1)$. Let us see the changes in $f(x)$ as x tends to 1.

x (towards 1 from LHS of 1)	$f(x)$	x (towards 1 from RHS of 1)	$f(x)$
0.9	1.9	1.1	2.1
0.99	1.99	1.01	2.01
0.999	1.999	1.001	2.001
0.9999	1.9999	1.0001	2.0001
. . .	. . .	. . .	. . .

We can assume any value of x close to 1. Generally, we start with a value at a distance 0.1 on both sides of $x = 1$. i.e. we start with $x = 0.9$ and 1.1 and bring values of x closer to 1 from both the sides.

It is clear from the table that when the value of x is brought nearer to 1 by increasing or decreasing its values, the value of $f(x)$ approaches to 2.

This can symbollically be expressed as $\lim\limits_{x \to 1} \frac{x^2-1}{x-1} = 2$.

Limit of a function is obtained by putting different values of x in $f(x)$ and tabulating them. So, this method of obtaining the limit of a function is called a **tabular method.**

Illustration 5 : Find $\lim\limits_{x \to 3} 2x+5$ by tabular method.

We have $f(x) = 2x+5$. We shall take the values of x very near to 3 and prepare a table in the following way :

x	$f(x)$	x	$f(x)$
2.9	10.8	3.1	11.2
2.99	10.98	3.01	11.02
2.999	10.998	3.001	11.002
2.9999	10.9998	3.0001	11.0002
. . .	. . .	. . .	. . .

It is clear from the table that when the value of x is brought nearer to 3 by increasing or decreasing its values, the value of $f(x)$ approaches to 11. That is, when $x \to 3$, $f(x) \to 11$.

$\therefore \lim\limits_{x \to 3} 2x+5 = 11$

Illustration 6 : Find $\lim_{x \to -1} \frac{x^2-1}{x+1}$, $x \in R - \{-1\}$ by preparing table.

We have $f(x) = \frac{x^2-1}{x+1}$. We shall take the values of x very near to –1 and prepare a table in the following way :

x	$f(x)$	x	$f(x)$
–1.1	–2.1	–0.9	–1.9
–1.01	–2.01	–0.99	–1.99
–1.001	–2.001	–0.999	–1.999
–1.0001	–2.0001	–0.9999	–1.9999
. . .	. . .	. . .	. . .

It is clear from the table that when the value of x is brought nearer to –1 by increasing or decreasing its value, the value of $f(x)$ approaches to –2. That is, when $x \to -1$, $f(x) \to -2$.

$$\therefore \lim_{x \to -1} \frac{x^2-1}{x+1} = -2$$

Illustration 7 : Find $\lim_{x \to 0} \frac{2x^2+3x}{x}$, $x \in R - \{0\}$ using tabular method.

We have $f(x) = \frac{2x^2+3x}{x}$. We shall take the values of x very near to 0 and prepare a table in the following way :

x	$f(x)$	x	$f(x)$
–0.1	2.8	0.1	3.2
–0.01	2.98	0.01	3.02
–0.001	2.998	0.001	3.002
–0.0001	2.9998	0.0001	3.0002
. . .	. . .	. . .	. . .

It is clear from the table that when the value of x is brought nearer to 0 by increasing or decreasing its value, the value of $f(x)$ approaches to 3. That is, when $x \to 0$, $f(x) \to 3$.

$$\therefore \lim_{x \to 0} \frac{2x^2+3x}{x} = 3$$

Illustration 8 : Find $\lim\limits_{x \to 1} \frac{1}{x-1}$, $x \in R - \{1\}$ by tabular method.

We have $f(x) = \frac{1}{x-1}$. We shall take the values of x very near to 1 and prepare a table in the following way :

x	$f(x)$	x	$f(x)$
0.9	–10	1.1	10
0.99	–100	1.01	100
0.999	–1000	1.001	1000
0.9999	–10000	1.0001	10000
. . .	. . .	. . .	. . .

It is clear from the table that when the value of x is brought nearer to 1 by increasing or decreasing its value, the value of $f(x)$ does not approach to a particular value. That is, when $x \to 1$, $f(x)$ does not tend to a particular value. Thus, limit of the function does not exist.

$\therefore \lim\limits_{x \to 1} \frac{1}{x-1}$ does not exist.

Illustration 9 : Find $\lim\limits_{x \to 2} \frac{3x^2 - 4x - 4}{x^2 - 4}$, $x \in R - \{2\}$ by tabular method.

We have $f(x) = \frac{3x^2 - 4x - 4}{x^2 - 4}$. We can obtain the value of limit as calculated in previous illustrations. But for simplification we shall obtain the value of limit of $f(x)$ after eliminating the common factor $(x-2)$ from numerator and denominator.

$$\lim_{x \to 2} \frac{3x^2 - 4x - 4}{x^2 - 4} = \lim_{x \to 2} \frac{(x-2)(3x+2)}{(x-2)(x+2)}$$

$$= \lim_{x \to 2} \frac{3x+2}{x+2} \qquad (\because x - 2 \neq 0)$$

We shall take the values of x very near to 2 and prepare a table in the following way :

x	$f(x)$	x	$f(x)$
1.9	1.9744	2.1	2.02439
1.99	1.9975	2.01	2.002494
1.999	1.9997	2.001	2.0002499
1.9999	1.9999	2.0001	2.000025
. . .	. . .	. . .	. . .

It is clear from the table that when the value of x is brought very near to 2 by increasing or decreasing its value, the value of $f(x)$ approaches to 2. That is, when $x \to 2$, $f(x) \to 2$.

$$\therefore \lim_{x \to 2} \frac{3x^2 - 4x - 4}{x^2 - 4} = 2$$

EXERCISE 4.2

1. Find the values of the following using tabular method :

(1) $\lim_{x \to 1} 2x+1$

(2) $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3}$

(3) $\lim_{x \to 2} \frac{2x^2 + 3x - 14}{x - 2}$

(4) $\lim_{x \to -3} \frac{2x^2 + 9x + 9}{x + 3}$

(5) $\lim_{x \to 2} x$

2. Using tabular method, show that $\lim_{x \to 3} \frac{2}{x - 3}$ does not exist.

3. If $y = \frac{x^2 + x - 6}{x - 2}$, show that as $x \to 2$ then $y \to 5$ using tabular method.

4. If $y = 5 - 2x$, show that as $x \to -1$ then $y \to 7$ using tabular method.

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4.6 Working rules of limit

The following rules will be accepted without proof :

If $f(x)$ and $g(x)$ are two real valued functions of a real variable x and $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then

(1) $\lim_{x \to a} [f(x) \pm g(x)] = l \pm m$

The limit of the sum (or difference) of two functions is equal to the sum (or difference) of their limits.

(2) $\lim_{x \to a} [f(x) \times g(x)] = l \times m$

The limit of the product of two functions is equal to the product of their limits.

(3) $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{l}{m}, \quad m \neq 0$

The limit of the division of two functions is equal to the division of their limits, provided the limit of the function in denominator is not zero.

(4) $\lim_{x \to a} k\, f(x) = kl$, k is the constant.

The limit of the product of a function with a constant is equal to the product of the limit of the function with the same constant.

4.7 Standard forms of limit

(1) **Limit of a polynominl**

Let $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ then using the working rules of limit

$$\lim_{x \to b} f(x) = a_0 + a_1 b + a_2 b^2 + ... + a_n b^n$$

(2) $\lim_{x \to a} \left[\frac{x^n - a^n}{x - a} \right] = n a^{n-1}, \quad n \in Q$

We will see some illustrations based on the standard forms and working rules of limit.

Illustration 10 : Find the value of $\lim_{x \to 0} \frac{x^2 + 5x + 6}{x^2 + 2x + 3}$.

$$\lim_{x \to 0} \frac{x^2 + 5x + 6}{x^2 + 2x + 3} = \frac{(0)^2 + 5(0) + 6}{(0)^2 + 2(0) + 3}$$

$$= \frac{6}{3}$$

$$= 2$$

Illustration 11 : Find the value of $\lim_{x \to 2} \frac{2x + 3}{x - 1}$.

$$\lim_{x \to 2} \frac{2x + 3}{x - 1} = \frac{2(2) + 3}{2 - 1}$$

$$= \frac{7}{1}$$

$$= 7$$

Illustration 12 : Find the value of $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 5x + 6}$.

If we put $x = 3$ in the function $f(x)$, we get the value of the function as $\frac{0}{0}$, which is indeterminate. Hence, we shall factorize numerator and denominator. Since $x \to 3$, $(x - 3)$ will be a common factor of numerator and denominator.

Note : If we put $x = a$ in the given function and we get $\frac{0}{0}$ then $(x - a)$ will be the common factor of numerator and denominator.

Numerator $= x^2 - 2x - 3$

$= x^2 - 3x + x - 3$

$= x(x - 3) + 1(x - 3)$

$= (x - 3)(x + 1)$

Denominator $= x^2 - 5x + 6$

$= x^2 - 3x - 2x + 6$

$= x(x - 3) - 2(x - 3)$

$= (x - 3)(x - 2)$

Now, $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 5x + 6} = \lim_{x \to 3} \frac{(x - 3)(x + 1)}{(x - 3)(x - 2)}$

$$= \lim_{x \to 3} \frac{(x + 1)}{(x - 2)} \qquad (\because \; x - 3 \neq 0)$$

$$= \frac{3+1}{3-2}$$

$$= \frac{4}{1}$$

$$= 4$$

Illustration 13 : Find the value of $\lim\limits_{x \to 1} \dfrac{2x^2 + x - 3}{x^2 - 1}$.

If we put $x = 1$ in the function $f(x)$, we get the value of the function as $\frac{0}{0}$, which is indeterminate.

Numerator $= 2x^2 + x - 3$

$= 2x^2 + 3x - 2x - 3$

$= x(2x + 3) - 1(2x + 3)$

$= (2x + 3)\ (x - 1)$

Denominator $= x^2 - 1$

$= (x + 1)\ (x - 1)$

Now, $$\lim_{x \to 1} \frac{2x^2 + x - 3}{x^2 - 1} = \lim_{x \to 1} \frac{(2x+3)(x-1)}{(x+1)(x-1)}$$

$$= \lim_{x \to 1} \frac{2x+3}{x+1} \qquad (\because \ x - 1 \neq 0)$$

$$= \frac{2(1)+3}{1+1}$$

$$= \frac{5}{2}$$

Illustration 14 : Find the value of $\lim\limits_{x \to -3} \dfrac{2x^2 + 7x + 3}{3x^2 + 8x - 3}$.

If we put $x = -3$ in the function $f(x)$, we get the value of the function as $\frac{0}{0}$, which is indeterminate.

Numerator $= 2x^2 + 7x + 3$

$= 2x^2 + 6x + x + 3$

$= 2x(x + 3) + 1(x + 3)$

$= (x + 3)\ (2x + 1)$

Denominator $= 3x^2 + 8x - 3$

$= 3x^2 + 9x - x - 3$

$= 3x(x + 3) - 1(x + 3)$

$= (x + 3)\ (3x - 1)$

Now, $\lim_{x \to -3} \frac{2x^2 + 7x + 3}{3x^2 + 8x - 3} = \lim_{x \to -3} \frac{(x+3)(2x+1)}{(x+3)(3x-1)}$

$= \lim_{x \to -3} \frac{2x+1}{3x-1} \qquad (\because \quad x + 3 \neq 0)$

$= \frac{2(-3)+1}{3(-3)-1}$

$= \frac{-6 + 1}{-9 - 1}$

$= \frac{-5}{-10}$

$= \frac{1}{2}$

Illustration 15 : Find the value of $\lim_{x \to -\frac{1}{2}} \frac{2x^2 - x - 1}{4x^2 + 8x + 3}$**.**

If we put $x = -\frac{1}{2}$ in the function $f(x)$, we get the value of the function as $\frac{0}{0}$, which is indeterminate.

Numerator $= 2x^2 - x - 1$

$= 2x^2 - 2x + x - 1$

$= 2x(x-1) + 1(x-1)$

$= (x-1)(2x+1)$

Denominator $= 4x^2 + 8x + 3$

$= 4x^2 + 6x + 2x + 3$

$= 2x(2x+3) + 1(2x+3)$

$= (2x+3)(2x+1)$

Now, $\lim_{x \to -\frac{1}{2}} \frac{2x^2 - x - 1}{4x^2 + 8x + 3} = \lim_{x \to -\frac{1}{2}} \frac{(x-1)(2x+1)}{(2x+3)(2x+1)}$

$= \lim_{x \to -\frac{1}{2}} \frac{x-1}{2x+3} \qquad (\because \quad 2x + 1 \neq 0)$

$= \frac{-\frac{1}{2} - 1}{2\left(-\frac{1}{2}\right) + 3}$

$= \frac{-\frac{3}{2}}{-1+3}$

$= \frac{-\frac{3}{2}}{2}$

$= -\frac{3}{4}$

Illustration 16 : Find the value of $\lim\limits_{x \to 2} \left[\frac{1}{x-2} - \frac{2}{x^2-2x} \right]$.

$$\lim_{x \to 2} \left[\frac{1}{x-2} - \frac{2}{x^2-2x} \right] = \lim_{x \to 2} \left[\frac{1}{x-2} - \frac{2}{x(x-2)} \right]$$

$$= \lim_{x \to 2} \left[\frac{x-2}{x(x-2)} \right]$$

$$= \lim_{x \to 2} \frac{1}{x} \qquad (\because \ x-2 \neq 0)$$

$$= \frac{1}{2}$$

Illustration 17 : Find the value of $\lim\limits_{x \to 0} \frac{1}{x} \left[\frac{2x+3}{3x-5} + \frac{3}{5} \right]$.

$$\lim_{x \to 0} \frac{1}{x} \left[\frac{2x+3}{3x-5} + \frac{3}{5} \right] = \lim_{x \to 0} \frac{1}{x} \left[\frac{5(2x+3) + 3(3x-5)}{5(3x-5)} \right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\frac{10x + 15 + 9x - 15}{5(3x-5)} \right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\frac{19x}{5(3x-5)} \right]$$

$$= \lim_{x \to 0} \frac{19}{5(3x-5)} \qquad (\because \ x \neq 0)$$

$$= \frac{19}{5[3(0)-5]}$$

$$= \frac{19}{5(-5)}$$

$$= -\frac{19}{25}$$

Illustration 18 : If $f(x) = x^2 + x$ then find the value of $\lim\limits_{x \to 2} \frac{f(x) - f(2)}{x^2 - 4}$.

Here, $f(x) = x^2 + x$

$\therefore \ f(2) = (2)^2 + 2$

$= 4 + 2$

$= 6$

Now, $\lim_{x \to 2} \dfrac{f(x) - f(2)}{x^2 - 4} = \lim_{x \to 2} \dfrac{(x^2 + x) - 6}{x^2 - 4}$

Numerator $= x^2 + x - 6$

$= x^2 + 3x - 2x - 6$

$= x(x + 3) - 2(x + 3)$

$= (x + 3)(x - 2)$

Denominator $= x^2 - 4$

$= (x + 2)(x - 2)$

So, $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} \dfrac{(x + 3)(x - 2)}{(x + 2)(x - 2)}$

$= \lim_{x \to 2} \dfrac{x + 3}{x + 2}$ $\quad (\because \ x - 2 \neq 0)$

$= \dfrac{2 + 3}{2 + 2}$

$= \dfrac{5}{4}$

Illustration 19 : If $f(x) = x^3$ then find the value of $\lim_{h \to 0} \dfrac{f(3 + h) - f(3 - h)}{2h}$.

Here, $f(x) = x^3$

$\therefore \ f(3 + h) = (3 + h)^3$

$= 27 + 27h + 9h^2 + h^3$

and

$f(3 - h) = (3 - h)^3$

$= 27 - 27h + 9h^2 - h^3$

Now, $\lim_{h \to 0} \dfrac{f(3 + h) - f(3 - h)}{2h} = \lim_{h \to 0} \dfrac{(27 + 27h + 9h^2 + h^3) - (27 - 27h + 9h^2 - h^3)}{2h}$

$= \lim_{h \to 0} \dfrac{27 + 27h + 9h^2 + h^3 - 27 + 27h - 9h^2 + h^3}{2h}$

$$= \lim_{h \to 0} \frac{54h + 2h^3}{2h}$$

$$= \lim_{h \to 0} \frac{h\left(54 + 2h^2\right)}{2h}$$

$$= \lim_{h \to 0} \frac{54 + 2h^2}{2} \qquad (\because \ h \neq 0)$$

$$= \frac{54 + 2(0)^2}{2}$$

$$= \frac{54}{2}$$

$$= 27$$

Illustration 20 : Find the value of $\lim_{x \to 0} \frac{\sqrt{3 + x} - \sqrt{3}}{x}$.

$$\lim_{x \to 0} \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

(multiplying numerator and denominator by $\sqrt{3 + x} + \sqrt{3}$)

$$= \lim_{x \to 0} \frac{\sqrt{3 + x} - \sqrt{3}}{x} \times \frac{\sqrt{3 + x} + \sqrt{3}}{\sqrt{3 + x} + \sqrt{3}}$$

$$= \lim_{x \to 0} \frac{\left(\sqrt{3 + x}\right)^2 - \left(\sqrt{3}\right)^2}{x\left(\sqrt{3 + x} + \sqrt{3}\right)}$$

$$= \lim_{x \to 0} \frac{3 + x - 3}{x\left(\sqrt{3 + x} + \sqrt{3}\right)}$$

$$= \lim_{x \to 0} \frac{x}{x\left(\sqrt{3 + x} + \sqrt{3}\right)}$$

$$= \lim_{x \to 0} \frac{1}{\left(\sqrt{3 + x} + \sqrt{3}\right)} \qquad (\because \ x \neq 0)$$

$$= \frac{1}{\sqrt{3 + 0} + \sqrt{3}}$$

$$= \frac{1}{\sqrt{3} + \sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

Illustration 21 : Find the value of $\lim\limits_{x \to 2} \dfrac{\sqrt{x+7} - 3}{x-2}$.

$$\lim_{x \to 2} \frac{\sqrt{x+7} - 3}{x-2}$$

(multiplying numerator and denominator by $\sqrt{x+7} + 3$)

$$= \lim_{x \to 2} \frac{\sqrt{x+7} - 3}{x-2} \times \frac{\sqrt{x+7} + 3}{\sqrt{x+7} + 3}$$

$$= \lim_{x \to 2} \frac{\left(\sqrt{x+7}\right)^2 - (3)^2}{(x-2)\left(\sqrt{x+7} + 3\right)}$$

$$= \lim_{x \to 2} \frac{x+7-9}{(x-2)\left(\sqrt{x+7} + 3\right)}$$

$$= \lim_{x \to 2} \frac{x-2}{(x-2)\left(\sqrt{x+7} + 3\right)}$$

$$= \lim_{x \to 2} \frac{1}{\sqrt{x+7} + 3} \qquad (\because \ x - 2 \neq 0)$$

$$= \frac{1}{\sqrt{2+7} + 3}$$

$$= \frac{1}{\sqrt{9} + 3}$$

$$= \frac{1}{3+3}$$

$$= \frac{1}{6}$$

Illustration 22 : Find the value of $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ where $f(x) = \sqrt{x}$, $x > 0$.

$$f(x) = \sqrt{x}$$

$$\therefore \quad f(x+h) = \sqrt{x+h}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

(multiplying numerator and denominator by $\sqrt{x+h} + \sqrt{x}$)

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \qquad (\because \ h \neq 0)$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Illustration 23 : Find the value of $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{\sqrt{x} - \sqrt{2}}$.

$$\lim_{x \to 2} \frac{x^3 - 8}{\sqrt{x} - \sqrt{2}}$$

(multiplying numerator and denominator by $\sqrt{x} + \sqrt{2}$)

$$= \lim_{x \to 2} \frac{x^3 - 8}{\sqrt{x} - \sqrt{2}} \times \frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} + \sqrt{2}}$$

$$= \lim_{x \to 2} \frac{(x-2)\left(x^2 + 2x + 4\right)\left(\sqrt{x} + \sqrt{2}\right)}{\left(\sqrt{x}\right)^2 - \left(\sqrt{2}\right)^2}$$

$$= \lim_{x \to 2} \frac{(x-2)\left(x^2 + 2x + 4\right)\left(\sqrt{x} + \sqrt{2}\right)}{(x-2)}$$

$$= \lim_{x \to 2} \left(x^2 + 2x + 4\right)\left(\sqrt{x} + \sqrt{2}\right) \qquad (\because \ x - 2 \neq 0)$$

$$= \left[(2)^2 + 2(2) + 4\right]\left[\sqrt{2} + \sqrt{2}\right]$$

$$= (4 + 4 + 4)\left(2\sqrt{2}\right)$$

$$= 12\left(2\sqrt{2}\right)$$

$$= 24\sqrt{2}$$