Properties of Determinants

1 Mark Questions

1. Write the value of $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$. **All India 2014C**

Given, $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$

$= 2\begin{vmatrix} 8 & 75 \\ 9 & 86 \end{vmatrix} - 7\begin{vmatrix} 3 & 75 \\ 5 & 86 \end{vmatrix} + 65\begin{vmatrix} 3 & 8 \\ 5 & 9 \end{vmatrix}$

[expanding the determinant along R_1]

$= 2\,(688 - 675) - 7(258 - 375) + 65\,(27 - 40)$

$= 26 + 819 - 845$

$= 845 - 845 = 0$ **(1)**

2. Prove the following, using properties of determinants

$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$.

$= 2(a+b+c)^3$ **Delhi 2014**

$$\text{LHS} = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$$

On applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\text{LHS} = \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

On taking $2(a+b+c)$ common from C_1, we get

$$\text{LHS} = 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

(1/2)

On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\text{LHS} = 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

On taking $(a+b+c)$ common from R_2 and R_3, we get

$$\text{LHS} = 2(a+b+c)^3 \begin{vmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

On expanding along R_3, we get

$\text{LHS} = 2(a+b+c)^3\,[(1)\,(1-0)]$ **(1/2)**

$= 2(a+b+c)^3 = \text{RHS}$ **Hence proved.**

3. Using properties of determinants, prove that

$$\begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix} = 1 + x^2 + y^2 + z^2.$$

Delhi 2014

$$\text{LHS} = \begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix}$$

On taking common factors x, y and z from R_1, R_2 and R_3, we get

$$\text{LHS} = xyz \begin{vmatrix} x+\frac{1}{x} & y & z \\ x & y+\frac{1}{y} & z \\ x & y & z+\frac{1}{z} \end{vmatrix}$$

On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\text{LHS} = xyz \begin{vmatrix} x+\frac{1}{x} & y & z \\ -\frac{1}{x} & \frac{1}{y} & 0 \\ -\frac{1}{x} & 0 & \frac{1}{z} \end{vmatrix}$$ **(1/2)**

On multiplying and dividing C_1 by x, C_2 by y and C_3 by z and taking common $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ from C_1, C_2 and C_3, we get

$$\text{LHS} = xyz \times \frac{1}{xyz} \begin{vmatrix} x^2+1 & y^2 & z^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x^2+1 & y^2 & z^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

On expanding along R_3, we get

$$\text{LHS} = -1 \times (-z^2) + 1[1(x^2+1) + 1(y^2)]$$

$$= x^2 + y^2 + z^2 + 1 = \text{RHS}$$ **Hence proved.**

(1/2)

4. Write the value of the determinant.

$$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}.$$

Foreign 2012

Let $A = \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

$\Rightarrow |A| = 6\begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

[$\because$ taking 6 common from R_1]

$= 6 \times 0 = 0$

[$\because$ two rows (R_1 and R_3) are identical] **(1)**

5. What is the value of $\begin{vmatrix} 4 & a & b+c \\ 4 & b & c+a \\ 4 & c & a+b \end{vmatrix}$?

Foreign 2010

Let $\Delta = \begin{vmatrix} 4 & a & b+c \\ 4 & b & c+a \\ 4 & c & a+b \end{vmatrix}$

On applying $C_2 \to C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 4 & a+b+c & b+c \\ 4 & a+b+c & c+a \\ 4 & a+b+c & a+b \end{vmatrix}$$

Now, taking common 4 from C_1 and $(a + b + c)$ from C_2, we get

$$\Delta = 4(a+b+c)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}$$

$$= 4(a+b+c)(0) = 0 \qquad \textbf{(1)}$$

[$\because C_1$ and C_2 are identical]

6. Write the value of $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$.

Delhi 2009

Let $\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$

On taking $3x$ common from R_3, we get

$$\Delta = 3x\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 2 & 3 & 4 \end{vmatrix}$$

$\Rightarrow \quad \Delta = 0$

[$\because R_1$ and R_3 are identical] **(1)**

7. Write the value of $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$.

All India 2009

Let $\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$\Rightarrow \qquad \Delta = 0$ **(1)**

[$\because$ if a determinant has all elements zero in any of its rows or columns, then value of determinant is zero.]

4 Mark Questions

8. Using properties of determinants, prove that

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}.$$

All India 2014, 2010C; Delhi 2012, 2010, 2009C

> Firstly, split the determinant along their respective columns and replace determinants, having identical column with zero and arrange the remaining, to get the desired result.

Consider $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$

On splitting Δ along C_1, we get

$$\begin{vmatrix} b & c+a & a+b \end{vmatrix}$$

$$+\begin{vmatrix} c & c+a & a+b \\ r & r+p & p+q \\ z & z+x & x+y \end{vmatrix} \quad \textbf{(1)}$$

Again, on splitting both above determinants along their respective second columns, we get

$$\Delta = \begin{vmatrix} b & c & a+b \\ q & r & p+q \\ y & z & x+y \end{vmatrix} + \begin{vmatrix} b & a & a+b \\ q & p & p+q \\ y & x & x+y \end{vmatrix}$$

$$+\begin{vmatrix} c & c & a+b \\ r & r & p+q \\ z & z & x+y \end{vmatrix} + \begin{vmatrix} c & a & a+b \\ r & p & p+q \\ z & x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} b & c & a+b \\ q & r & p+q \\ y & z & x+y \end{vmatrix} + \begin{vmatrix} b & a & a+b \\ q & p & p+q \\ y & x & x+y \end{vmatrix}$$

$$+\begin{vmatrix} c & a & a+b \\ r & p & p+q \\ z & x & x+y \end{vmatrix}$$

$$\left[\because \begin{vmatrix} c & c & a+b \\ r & r & p+q \\ z & z & x+y \end{vmatrix} = 0, \text{ as } C_1 \text{ is identical to } C_2\right]$$

(1)

Similarly, on splitting all above determinants together, we get

$$\Delta = \begin{vmatrix} b & c & a \\ q & r & p \\ y & z & x \end{vmatrix} + \begin{vmatrix} b & c & b \\ q & r & q \\ y & z & y \end{vmatrix} + \begin{vmatrix} b & a & a \\ q & p & p \\ y & x & x \end{vmatrix}$$

$$+\begin{vmatrix} b & a & b \\ q & p & q \\ y & x & y \end{vmatrix} + \begin{vmatrix} c & a & a \\ r & p & p \\ z & x & x \end{vmatrix}$$

$$\begin{vmatrix} c & a & b \end{vmatrix}$$

$$+\begin{vmatrix} r & p & q \\ z & x & y \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} b & c & a \\ q & r & p \\ y & z & x \end{vmatrix} + \begin{vmatrix} c & a & b \\ r & p & q \\ z & x & y \end{vmatrix}$$

[∵ all other determinants have their two columns identical; so their value is 0.]

On applying $C_1 \leftrightarrow C_3$ in first and $C_1 \leftrightarrow C_2$ in second determinant, we get

$$\Delta = -\begin{vmatrix} a & c & b \\ p & r & q \\ x & z & y \end{vmatrix} - \begin{vmatrix} a & c & b \\ p & r & q \\ x & z & y \end{vmatrix}$$

[∵ when any two columns or rows of a determinant are interchanged, its value becomes negative]

On applying $C_2 \leftrightarrow C_3$ in both determinants, we get

$$\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \qquad \textbf{(1)}$$

Here, we have shown that

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

On taking transpose both sides, we get

$$\begin{vmatrix} b+c & q+r & y+z \end{vmatrix}$$

$$\begin{vmatrix} c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \quad \textbf{(1)}$$

Hence proved.

NOTE The determinant of matrix A or its transpose A' have same value, i.e. $|A| = |A'|$.

9. Using properties of determinants, prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + bc + ca + ab.$$

All India 2014

$$\text{Consider LHS} = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

On dividing R_1 by a, R_2 by b and R_3 by c and multiplying the determinant by abc, we get

$$\text{LHS} = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix} \quad (1)$$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\text{LHS} = abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 \\ \frac{1}{b} & \frac{1}{b}+1 \\ \frac{1}{\cdot} & \frac{1}{\cdot} \end{vmatrix}$$

$$\begin{vmatrix} & c & & c \\ & & & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 \\ & & & \frac{1}{b} \\ & & & \frac{1}{c}+1 \end{vmatrix}$$ **(1)**

On taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common from R_1, we get

$$\text{LHS} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

(1)

On applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get

$$\text{LHS} = (abc + bc + ca + ab)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$= (abc + bc + ca + ab)\,[1(1-0)]$

[expanding along R_1]

$= abc + bc + ca + ab = \text{RHS}$ **Hence proved.**

(1)

10. Using properties of determinants, prove that

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3.$$

All India 2014, 2009

To prove, $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$

$$\text{LHS} = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$$

On applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$\text{LHS} = \begin{vmatrix} x+y & x & x \\ 3x+2y & 2x & 0 \\ 7x+5y & 5x & 0 \end{vmatrix}$$ **(1½)**

On expanding along C_3, we get

$$\text{LHS} = x\begin{vmatrix} 3x+2y & 2x \\ 7x+5y & 5x \end{vmatrix}$$ **(1½)**

$$= x\,[5x\,(3x+2y) - 2x\,(7x+5y)]$$

$$= x\,[15x^2 + 10xy - (14x^2 + 10xy)] = x^3$$

$= \text{RHS}$ **(1)**

Hence proved.

11. Using properties of determinants, prove that

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2(a+x+y+z).$$

Foreign 2014

Firstly, we make $(a+x+y+z)$ a common factor in any row or column. Now, try to make two zeroes in that row or column and expand the determinant along that row or column.

Given to prove,

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2(a+x+y+z)$$

$$\text{LHS} = \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} \quad \textbf{(1)}$$

On applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\text{LHS} = \begin{vmatrix} a+x+y+z & y & z \\ a+x+y+z & a+y & z \\ a+x+y+z & y & a+z \end{vmatrix}$$

$$= (a+x+y+z)\begin{vmatrix} 1 & y & z \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix} \quad \textbf{(1)}$$

[taking common $(a+x+y+z)$ from C_1]

On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\text{LHS} = (a+x+y+z)\begin{vmatrix} 1 & y & z \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} \quad \textbf{(1)}$$

$$= (a+x+y+z)[1(a^2 - 0)]$$

$$= a^2(a+x+y+z) = \text{RHS} \quad \textbf{(1)}$$

Hence proved.

12. Using properties of determinants, prove that:

$$\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} = (5x+\lambda)(\lambda-x)^2.$$

Foreign 2014

To prove $\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix}$

$= (5x+\lambda)(\lambda-x)^2$

$$\text{LHS} = \begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix}$$

On applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\text{LHS} = \begin{vmatrix} 5x+\lambda & 2x & 2x \\ 5x+\lambda & x+\lambda & 2x \\ 5x+\lambda & 2x & x+\lambda \end{vmatrix} \quad \textbf{(1)}$$

$$= (5x+\lambda)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+\lambda & 2x \\ 1 & 2x & x+\lambda \end{vmatrix}$$

[taking $(5x+4)$ common from C_1] **(1)**

On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\text{LHS} = 5x+\lambda\begin{vmatrix} 1 & 2x & 2x \\ 0 & \lambda-x & 0 \\ 0 & 0 & \lambda-x \end{vmatrix} \quad \textbf{(1)}$$

On expanding along C_1, we get

$\text{LHS} = (5x+\lambda)[1(\lambda-x)^2 + 0 + 0]$

$= (5x+\lambda)(\lambda-x)^2 = \text{RHS}$ **(1)**

Hence proved.

13. Prove the following, using properties of determinants.

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$$

Foreign 2014

To prove, $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

$= 4a^2b^2c^2$

$$\text{LHS} = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

On taking common a from C_1, b from C_2 and c from C_3, we get

$$\text{LHS} = abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix} \quad \textbf{(1)}$$

On applying $C_1 \rightarrow C_1 + C_2 - C_3$ we get

$$\text{LHS} = abc \begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 2b & b+c & c \end{vmatrix} \quad \textbf{(1)}$$

Now, applying $R_2 \rightarrow R_2 - R_3$, we get

$$\text{LHS} = abc \begin{vmatrix} 0 & c & a+c \\ 0 & -c & a-c \\ 2b & b+c & c \end{vmatrix} \quad \textbf{(1)}$$

On expanding along C_1, we get

LHS $abc\,[2b\,\{c\,(a-c)+c(a+c)\}]$

$= 2(ab^2c)\,(2ac)$

$= 4a^2b^2c^2 = \text{RHS}$ **(1)**

Hence proved.

NOTE In this type of questions, we only use either row operations or column operations not both at same time.

14. Using properties of determinants, prove that

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(bc+ca+ab).$$

Delhi 2014C, 2011C

To prove, $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$ or $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

$$= (a-b)(b-c)(c-a)(ab+bc+ca)$$

$$\text{LHS} = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$$

On multiplying columns C_1, C_2 and C_3 by a, b and c, respectively and dividing the determinant by abc, we get **1**

$$\text{LHS} = \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix} \quad \textbf{(1)}$$

On taking abc common from R_3, we get

$$\text{LHS} = \frac{1}{abc} \cdot abc \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix} \quad \textbf{(1/2)}$$

Now, applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get

$$\text{LHS} = \begin{vmatrix} a^2-b^2 & b^2-c^2 & c^2 \\ a^3-b^3 & b^3-c^3 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (a-b)(a+b) & (b-c)(b+c) & c^2 \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

(1)

On taking $(a-b)$ and $(b-c)$ common from C_1 and C_2, respectively and then on expanding along R_3, we get

$$\text{LHS} = (a-b)(b-c)\cdot 1 \begin{vmatrix} a+b & b+c \\ a^2+ab+b^2 & b^2+bc+c^2 \end{vmatrix}$$ **(1/2)**

On applying $C_2 \rightarrow C_2 - C_1$, we get

$$\text{LHS} = (a-b)(b-c) \begin{vmatrix} a+b & c-a \\ a^2+ab+b^2 & (c^2-a^2)+b(c-a) \end{vmatrix}$$

On taking $(c-a)$ common from C_2, we get

$$\text{LHS} = (a-b)(b-c)(c-a) \begin{vmatrix} a+b & 1 \\ a^2+ab+b^2 & c+a+b \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)[(a^2+ab+ac+ab+b^2+bc)-(a^2+ab+b^2)]$$

$$= (a-b)(b-c)(c-a)(ab+bc+ca)$$

= RHS **(1)**

Hence proved.

15. Using properties of determinants, prove that

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2.$$

Delhi 2014C, 2013; Foreign 2009

Firstly, apply $C_1 \rightarrow C_1 + C_2 + C_3$ and then take a term common from C_1 and then solve it.

To prove, $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$

$$\begin{vmatrix} 1 & x & x^2 \end{vmatrix}$$

$$\text{LHS} = \begin{vmatrix} x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

On applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\text{LHS} = \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix} \qquad \textbf{(1/2)}$$

On taking common $(1+x+x^2)$ from C_1, we get

$$\text{LHS} = (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix} \qquad \textbf{(1/2)}$$

On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\text{LHS} = (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x\,(1-x) \\ 0 & x\,(x-1) & 1-x^2 \end{vmatrix} \qquad \textbf{(1)}$$

On expanding along C_1, we get

$$\text{LHS} = (1+x+x^2)\begin{vmatrix} 1-x & x\,(1-x) \\ -x\,(1-x) & (1-x^2) \end{vmatrix}$$

$$= (1+x+x^2)\begin{vmatrix} 1-x & x\,(1-x) \\ -x(1-x) & (1-x)\,(1+x) \end{vmatrix} \qquad \textbf{(1/2)}$$

On taking common $(1-x)$ from C_1 and C_2, we get

$$\text{LHS} = (1+x+x^2)\,(1-x)^2\begin{vmatrix} 1 & x \\ -x & 1+x \end{vmatrix} \qquad \textbf{(1/2)}$$

$$= (1+x+x^2)\,(1-x)^2\,(1+x+x^2)$$

$$= \{(1+x+x^2)(1-x)\}^2$$

$$= (1-x^3)^2 = \text{RHS}$$

$$[\because (a^2+b^2+ab)(a-b) = a^3-b^3] \qquad \textbf{(1)}$$

Hence proved.

16. Show that $\Delta = \Delta_1$, where

$$\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}, \Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix},$$

All India 2014C

Given, $\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}$

On taking common x, y and z from R_1, R_2 and R_3 respectively, we get

$$\Delta = xyz \begin{vmatrix} A & x & 1/x \\ B & y & 1/y \\ C & z & 1/z \end{vmatrix}$$

Now, on applying $C_3 \to xyzC_3$, we get

$$\Delta = \frac{xyz}{xyz} \begin{vmatrix} A & x & yz \\ B & y & zx \\ C & z & xy \end{vmatrix} = \begin{vmatrix} A & x & yz \\ B & Y & zx \\ C & z & xy \end{vmatrix}$$

Also, given $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}$ **(2)**

$$\Delta_1' = \begin{vmatrix} A & x & yz \\ B & y & zx \\ C & z & xy \end{vmatrix}$$

$\therefore \quad \Delta = \Delta_1' \Rightarrow \Delta = \Delta_1 \quad [\because |A'| = |A|]$

(2)

17. Using properties of determinants, prove that

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc.$$

All India 2014C, All India 2012

> Firstly, apply $R_1 \to R_1 + R_2 + R_3$ and then take a term common from R_1 and solve it.

To prove, $$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$$

$$\text{LHS} = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\text{LHS} = \begin{vmatrix} 2b+2c & 2a+2c & 2a+2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \quad \textbf{(1)}$$

$$= 2\begin{vmatrix} b+c & a+c & a+b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

[taking 2 common from R_1]

On applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\text{LHS} = 2\begin{vmatrix} c & 0 & a \\ b-c & a & -a \\ c & c & a+b \end{vmatrix} \qquad \textbf{(1)}$$

On expanding along R_1, we get

$\text{LHS} = 2\,[c\,(a^2 + ab + ac) + a\,(cb - c^2 - ac)]$

$= 2\,[ca^2 + abc + ac^2 + acb - ac^2 - a^2c]$ **(1)**

$= 2\,[2abc] = 4abc = \text{RHS}$ **(1)**

Hence proved.

18. Using properties of determinants, prove that

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$

Delhi 2013C, 2009C

Here, use row operations (or column operations) to make some factors common in one row or column. Then, take that factor outside the determinant and then expand the determinant.

Given, to prove $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$

$$= (a-b)(b-c)(c-a)(a+b+c)$$

$$\text{LHS} = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$$

On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\text{LHS} = \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & b^3 - a^3 \\ 0 & c-a & c^3 - a^3 \end{vmatrix} \quad \textbf{(1)}$$

$$= \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & (b-a)(b^2+a^2+ab) \\ 0 & c-a & (c-a)(c^2+a^2+ac) \end{vmatrix}$$

$$[\because x^3 - y^3 = (x-y)(x^2+y^2+xy)]$$

On taking $(b-a)$ and $(c-a)$ common from R_2 and R_3, respectfully, we get

$$\text{LHS} = (b-a)(c-a) \begin{vmatrix} 1 & a & a^3 \\ 0 & 1 & b^2+a^2+ab \\ 0 & 1 & c^2+a^2+ac \end{vmatrix} \quad \textbf{(1)}$$

$$= (b-a)(c-a)[1\{c^2+a^2+ac - (b^2+a^2+ab)\}]$$

[expanding along C_1]

$$= (b-a)(c-a)[c^2 - b^2 + ac - ab] \quad \textbf{(1)}$$

$= (b - a)(c - a)[(c - b)(c + b) + a(c - b)]$

$= (b - a)(c - a)(c - b)(c + b + a)$

$= (a - b)(b - c)(c - a)(a + b + c)$ **(1)**

$=$ RHS **Hence proved.**

Alternate Method

$$\text{LHS} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} \qquad [\because |A'| = |A|]$$

On applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get

$$\text{LHS} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3 - b^3 & b^3 - c^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a^2 + ab + b^2) & (b-c)(b^2 + bc + c^2) & c^3 \end{vmatrix}$$

(1½)

$[\because a^3 - b^3 = (a - b)(a^2 + ab + b^2)]$

On taking $(a - b)$ common from C_1 and $(b - c)$ from C_2, we get

$$\text{LHS} = (a - b)(b - c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2 + ab + b^2 & b^2 + bc + c^2 & c^3 \end{vmatrix}$$ **(1/2)**

On applying $C_1 \to C_1 - C_2$, we get

$$\text{LHS} = (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ (a^2 - c^2) + (ab - bc) & b^2 + bc + c^2 & c^3 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & c \\ (a-c)(a+b+c) & b^2+bc+c^2 & c^3 \end{vmatrix}$$

$$\left[\begin{aligned} \because\ & (a^2-c^2)+ab-bc \\ &= (a-c)(a+c)+b(a-c) \\ &= (a-c)(a+b+c) \end{aligned}\right]$$

On taking $(c-a)(a+b+c)$ common from C_1, we get

$$\text{LHS} = (a-b)(b-c)(c-a)(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ -1 & b^2+bc+c^2 & c^3 \end{vmatrix}$$ **(1)**

On expanding along column C_1, we get

$$= (a-b)(b-c)(c-a)(a+b+c)\left(-1\times\begin{vmatrix} 0 & 1 \\ 1 & c \end{vmatrix}\right)$$

$$= (a-b)(b-c)(c-a)(a+b+c)[-1(-1)]$$

$$= (a-b)(b-c)(c-a)(a+b+c)$$ **(1)**

= RHS **Hence proved.**

NOTE In this type of questions, we only use either row operations or column operations not both at same time.

19. Using properties of determinants, prove that

$$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix} = 3(x+y+z)(xy+yz+zx).$$ **All India 2013**

To prove, $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

$$= 3(x+y+z)(xy+yz+zx)$$

$$\text{LHS} = \begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$\text{LHS} = \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$ **(1)**

On taking common $(x+y+z)$ from C_1, we get

$$\text{LHS} = (x+y+z)\begin{vmatrix} 1 & -x+y & -x+y \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$ **(1)**

On applying $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$, we get

$$\text{LHS} = (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & x-y \\ 0 & x-z & 2z+x \end{vmatrix}$$ **(1)**

Now, on expanding along C_1, we get

$$\text{LHS} = (x+y+z)\cdot 1 \cdot \{(2y+x)(2z+x) - (x-y)(x-z)\}$$

$$= (x+y+z)\{4yz + 2xz + 2yx + x^2 - x^2 + xy + zx - yz\}$$

$$= (x+y+z)\cdot(3xy + 3yz + 3zx)$$

$$= 3(x+y+z)\cdot(xy + yz + zx) = \text{RHS}$$ **(1)**

Hence proved.

20. Using properties of determinants, prove that

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y).$$

All India 2013

To prove,
$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y)$$

$$\text{LHS} = \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

On applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$\text{LHS} = \begin{vmatrix} 3x+3y & 3x+3y & 3x+3y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} \quad \textbf{(1)}$$

On taking $(3x + 3y)$ common from R_1, we get

$$\text{LHS} = (3x+3y)\begin{vmatrix} 1 & 1 & 1 \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} \quad \textbf{(1)}$$

On applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we get

$$\text{LHS} = 3(x+y)\begin{vmatrix} 1 & 0 & 0 \\ x+2y & -2y & -y \\ x+y & y & -y \end{vmatrix} \quad \textbf{(1)}$$

Now, expanding along R_1, we get

$$\text{LHS} = 3(x+y)\cdot 1\cdot[(-2y)\cdot(-y) - (y)\cdot(-y)]$$
$$= 3(x+y)[2y^2 + y^2] = 3(x+y)(3y^2) \quad \textbf{(1)}$$
$$= 9y^2(x+y) = \text{RHS}$$

Hence proved.

21. Using properties of determinants, prove that

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$$

$$= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma).$$

Delhi 2012C, 2010C, 2008C

To prove, $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$

$$= (\alpha-\beta)\,(\beta-\gamma)\,(\gamma-\alpha)\,(\alpha+\beta+\gamma)$$

LHS $= \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$

On applying $R_3 \rightarrow R_3 + R_1$, we get

$$\text{LHS} = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \end{vmatrix}$$

$$= (\alpha + \beta + \gamma) \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix} \qquad \textbf{(1½)}$$

On applying $C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$, we get

$$\text{LHS} = (\alpha + \beta + \gamma) \begin{vmatrix} \alpha-\beta & \beta-\gamma & \gamma \\ \alpha^2-\beta^2 & \beta^2-\gamma^2 & \gamma^2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\alpha + \beta + \gamma) \begin{vmatrix} \alpha-\beta & \beta-\gamma & \gamma \\ (\alpha-\beta)(\alpha+\beta) & (\beta-\gamma)(\beta+\gamma) & \gamma^2 \\ 0 & 0 & 1 \end{vmatrix}$$

On taking $(\alpha - \beta)$ common from C_1 and $(\beta - \alpha)$ common from C_2, we get

$$\text{LHS} = (\alpha + \beta + \gamma)(\alpha - \beta)(\beta - \gamma) \begin{vmatrix} 1 & 1 & \gamma \\ \alpha+\beta & \beta+\gamma & \gamma^2 \\ 0 & 0 & 1 \end{vmatrix} \qquad (1½)$$

On expanding along R_3, we get

$$\text{LHS} = (\alpha + \beta + \gamma)(\alpha - \beta)(\beta - \gamma) \begin{vmatrix} 1 & 1 \\ \alpha+\beta & \beta+\gamma \end{vmatrix}$$

$$= (\alpha + \beta + \gamma)(\alpha - \beta)(\beta - \gamma)(\beta + \gamma - \alpha - \beta)$$

$$= (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha + \beta + \gamma) \qquad \textbf{(1)}$$

$= \text{RHS}$ **Hence proved.**

22. Using properties of determinants, prove that

$$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$$

$$=(a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2).$$

All India 2012C

To prove, $$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

$$\text{LHS} = \begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$$

On applying $C_2 \to C_2 - C_1$, we get

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$$\text{LHS} = \begin{vmatrix} a^2 & -(b-c)^2 & bc \\ b^2 & -(c-a)^2 & ca \\ c^2 & -(a-b)^2 & ab \end{vmatrix}$$

$$= - \begin{vmatrix} a^2 & b^2+c^2-2bc & bc \\ b^2 & c^2+a^2-2ac & ca \\ c^2 & a^2+b^2-2ab & ab \end{vmatrix} \quad \textbf{(1)}$$

[taking ' – ' common from C_1]

On applying $C_2 \to C_2 + C_1 + 2C_3$, we get

$$\text{LHS} = - \begin{vmatrix} a^2 & a^2+b^2+c^2 & bc \\ b^2 & a^2+b^2+c^2 & ca \\ c^2 & a^2+b^2+c^2 & ab \end{vmatrix}$$

$$= -(a^2+b^2+c^2) \begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}$$

[taking $(a^2+b^2+c^2)$ common from C_2] **(1)**

On applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\text{LHS} = -(a^2+b^2+c^2)\begin{vmatrix} a^2-b^2 & 0 & c\,(b-a) \\ b^2-c^2 & 0 & a\,(c-b) \\ c^2 & 1 & ab \end{vmatrix}$$

$$= -(a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} a+b & 0 & -c \\ b+c & 0 & -a \\ c^2 & 1 & ab \end{vmatrix}$$ **(1)**

2

[taking $(a-b)$ common from C_1 and $(b-c)$ from C_3]

On applying $C_1 \to C_1 - C_3$, we get

$$\text{LHS} = -(a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} a+b+c & 0 & -c \\ a+b+c & 0 & -a \\ c^2-ab & 1 & ab \end{vmatrix}$$

On expanding along C_2, we get

$$\text{LHS} = -(a^2+b^2+c^2)(a-b)(b-c)(-1)^{3+2}(a+b+c)(-a+c)$$

$$= (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$ **(1)**

$= \text{RHS}$ **Hence proved.**

23. Using properties of determinants, prove that

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2.$$

Delhi 2011; All India 2011C

To prove, $\begin{vmatrix} -a & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$

$$\text{LHS} = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

On taking a, b and c common from R_1, R_2 and R_3 respectively, we get

$$\text{LHS} = abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$ **(1)**

Again, on taking a, b and c common from C_1, C_2 and C_3 respectively, we get

$$\text{LHS} = a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$ **(1)**

On applying $C_1 \to C_1 + C_2$, we get

$$\text{LHS} = a^2b^2c^2 \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$ **(1)**

$= a^2b^2c^2\, 2(1+1)$ [expanding along C_1]

$= 4a^2b^2c^2 = \text{RHS}$ **(1)**

Hence proved.

24. Using properties of determinants, prove that

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x).$$

Delhi 2011, 2010C

To prove,

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz\,(x-y)(y-z)(z-x)$$

$$\text{LHS} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

(1/2)

[taking x, y and z common from C_1, C_2 and C_3, respectively]

On applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get

$$\text{LHS} = xyz \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix}$$ **(1½)**

On expanding along R_1, we get

$$\text{LHS} = xyz \begin{vmatrix} x-y & y-z \\ x^2-y^2 & y^2-z^2 \end{vmatrix}$$ **(1)**

On taking common $(x-y)$ from C_1 and $(y-z)$ from C_2, we get

$$\text{LHS} = xyz\,(x-y)(y-z) \begin{vmatrix} 1 & 1 \\ x+y & y+z \end{vmatrix}$$

$$= xyz\,(x-y)(y-z)(z-x)$$ **(1)**

$= \text{RHS}$ **Hence proved.**

25. Using properties of determinants, prove that

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2.$$

Delhi 2011, 2009

To prove,

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x^2)$$

$$\text{LHS} = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

LHS

$$= \begin{vmatrix} x+4+2x+2x & 2x+x+4+2x & 2x+2x+x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \quad \textbf{(1)}$$

$$= (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \quad \textbf{(1/2)}$$

[taking $5x + 4$ common from R_1]

Now, on applying $C_2 \to C_2 - C_1$, we get

$$= (5x+4)\begin{vmatrix} 1 & 0 & 1 \\ 2x & 4-x & 2x \\ 2x & 0 & 4+x \end{vmatrix} \quad \textbf{(1)}$$

On expanding along C_2, we get

$$= (5x+4)(4-x)\begin{vmatrix} 1 & 1 \\ 2x & 4+x \end{vmatrix} \quad \textbf{(1)}$$

$$= (5x+4)(4-x)(4+x-2x)$$

$$= (5x+4)(4-x)^2 = \text{RHS} \quad \textbf{(1/2)}$$

Hence proved.

26. Using properties of determinants, solve the following for x.

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

All India 2011; HOTS

> Firstly, apply some operations and use properties, so that when we expand the determinant, it is easy to simplify.

Given determinant

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

On applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\begin{vmatrix} 2 & 6 & 12 \\ 4 & 18 & 48 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0 \quad \textbf{(1½)}$$

On taking common 2 from R_1 and R_2, we get

$$2 \times 2 \begin{vmatrix} 1 & 3 & 6 \\ 2 & 9 & 24 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0 \quad \textbf{(1/2)}$$

On applying $R_2 \to R_2 - 2R_1$, we get

$$4 \begin{vmatrix} 1 & 3 & 6 \\ 0 & 3 & 12 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0 \quad \textbf{(1)}$$

On expanding along C_1, we get

$$4\,[3\,(3x-64) - 12\,(2x-27) + (x-8)\,(3 \times 12 - 3 \times 6)] = 0$$

$$\Rightarrow 4\,[9x - 192 - 24x + 324 + 18\,(x-8)] = 0$$

$$\Rightarrow \quad 4\,[3x - 12] = 0$$

$$\Rightarrow \quad 3x = 12$$

$$\therefore \quad x = 4 \quad \textbf{(1)}$$

27. Using properties of determinants, solve the following for x.

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

All India 2011

The given determinant equation is

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

On applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} = 0 \qquad \textbf{(1)}$$

On taking $(3a - x)$ common from C_1, we get

$$(3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix} = 0 \qquad \textbf{(1/2)}$$

On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$(3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} = 0 \qquad \textbf{(1½)}$$

On expanding along C_1, we get

$$(3a-x)\cdot 1 \cdot \begin{vmatrix} 2x & 0 \\ 0 & 2x \end{vmatrix} = 0$$

$$\Rightarrow \quad (3a-x)\cdot 2x \cdot 2x = 0$$

$$\Rightarrow \quad 4x^2(3a-x) = 0$$

$$\therefore \quad x = 0, 3a \qquad \textbf{(1)}$$

28. Using properties of determinants, solve the following for x.

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

All India 2011

The given determinant is

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

On applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 3x+a & x & x \\ 3x+a & x+a & x \\ 3x+a & x & x+a \end{vmatrix} = 0 \qquad \textbf{(1)}$$

On taking common $(3x + a)$ from C_1, we get

$$(3x+a)\begin{vmatrix} 1 & x & x \\ 1 & x+a & x \\ 1 & x & x+a \end{vmatrix} = 0 \qquad \textbf{(1/2)}$$

Now, on applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$(3x+a)\begin{vmatrix} 1 & x & x \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = 0 \qquad \textbf{(1½)}$$

On expanding along C_1, we get

$$(3x + a)(1 \cdot a \cdot a) = 0$$

$$\Rightarrow \quad a^2 (3x + a) = 0$$

$$\therefore \quad x = -\frac{a}{3} \qquad \textbf{(1)}$$

29. Prove, using properties of determinants

$$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k).$$

Foreign 2011

> Firstly, apply $R_1 \rightarrow R_1 + R_2 + R_3$ and then to take common from R_1 and then solve it.

To prove, $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$

$$\text{LHS} = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

On applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$\text{LHS} = \begin{vmatrix} 3y+k & 3y+k & 3y+k \\ y & y+k & y \\ y & y & y+k \end{vmatrix} \quad \textbf{(1)}$$

On taking $(3y+k)$ common from R_1, we get

$$\text{LHS} = (3y+k)\begin{vmatrix} 1 & 1 & 1 \\ y & y+k & y \\ y & y & y+k \end{vmatrix} \quad \textbf{(1/2)}$$

Now, on applying $C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$, we get

$$\text{LHS} = (3y+k)\begin{vmatrix} 0 & 0 & 1 \\ -k & k & y \\ 0 & -k & y+k \end{vmatrix} \quad \textbf{(1½)}$$

On expanding along R_1, we get

$\text{LHS} = (3y+k)\,[1(k^2)]$

$= k^2(3y+k) = \text{RHS}$ **(1)**

Hence proved.

30. Prove, using properties of determinants

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$$

Foreign 2011

To prove,

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

$$\text{LHS} = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\text{LHS} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

(1)

On taking $(a+b+c)$ common from R_1, we get

$$\text{LHS} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

(1/2)

Now, on applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b+c+a & -(a+b+c) & 2b \\ 0 & c+a+b & c-a-b \end{vmatrix}$$

(1½)

On taking $(a+b+c)$ common from C_1 and C_2, both, we get

$$\text{LHS} = (a+b+c)^3 \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$

On expanding along R_1, we get

$$\text{LHS} = (a+b+c)^3\,[1(1)]$$

$$= (a+b+c)^3 = \text{RHS}$$ **(1)**

Hence proved.

31. Prove, using properties of determinants

$$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3.$$

Foreign 2011; All India 2009C, 2008

To prove, $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$

$$= 2(x+y+z)^3$$

$$\text{LHS} = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

On applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\text{LHS} = \begin{vmatrix} 2x+2y+2z & x & y \\ 2x+2y+2z & y+z+2x & y \\ 2x+2y+2z & x & z+x+2y \end{vmatrix}$$

(1)

On taking $2(x+y+z)$ common from C_1, we get

$$\text{LHS} = 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$ **(1/2)**

On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\text{LHS} = 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 0 & y+z+x & 0 \\ 0 & 0 & z+x+y \end{vmatrix}$$

(1½)

On taking common $(x + y + z)$ from R_2 and R_3, both, we get

$$\text{LHS} = 2\,(x+y+z)^3 \begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

On expanding along C_1, we get

$$\text{LHS} = 2\,(x+y+z)^3\,[1(1-0) - 0 + 0]$$

$$= 2\,(x+y+z)^3 = \text{RHS} \qquad \textbf{(1)}$$

Hence proved.

32. Prove, using properties of determinants

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$$

All India 2011C; Foreign 2009

To prove,

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

$$\text{LHS} = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

On applying $R_1 \to \frac{1}{a}R_1, R_2 \to \frac{1}{b}R_2$ and $R_3 \to \frac{1}{c}R_3$, we get

$$\text{LHS} = abc \begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix} \quad \textbf{(1)}$$

On applying $C_1 \to aC_1, C_2 \to bC_2$ and $C_3 \to cC_3$, we get

$$\text{LHS} = \frac{abc}{abc} \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix} \quad \textbf{(1)}$$

On applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\text{LHS} = \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix} \quad \textbf{(1)}$$

[taking $(1+a^2+b^2+c^2)$ common from C_1]

On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\text{LHS} = (1 + a^2 + b^2 + c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$ **(1/2)**

On expanding along C_1, we get

$$\text{LHS} = (1 + a^2 + b^2 + c^2)\,[1(1 - 0)]$$

$$= 1 + a^2 + b^2 + c^2$$ **(1/2)**

$= \text{RHS}$ **Hence proved.**

33. Prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

Delhi 2010; All India 2010C

To prove, $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

$= 2abc\,(a+b+c)^3$

$$\text{LHS} = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (a+c)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

On applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\text{LHS} = \begin{vmatrix} (b+c)^2 & a^2-(b+c)^2 & a^2-(b+c)^2 \\ b^2 & (a+c)^2-b^2 & 0 \\ c^2 & 0 & (a+b)^2-c^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2 & (a+b+c)(a-b-c) & (a+b+c)(a-b-c) \\ b^2 & (a+c+b)(a+c-b) & 0 \\ c^2 & 0 & (a+b+c)(a+b-c) \end{vmatrix}$$ **(1½)**

$[\because x^2 - y^2 = (x-y)(x+y)]$

On taking $(a+b+c)$ common from C_2 and C_3, we get

$$\text{LHS} = (a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & a+c-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

On applying $R_1 \to R_1 - (R_2 + R_3)$, we get

$$\text{LHS} = (a+b+c)^2 \begin{vmatrix} 2bc & -2c & -2b \\ b^2 & a+c-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$ **(1/2)**

On applying $C_2 \to C_2 + \frac{1}{b} C_1$,

$C_3 \to C_3 + \frac{1}{c} C_1$, we get

$$\text{LHS} = (a+b+c)^2 \begin{vmatrix} 2bc & 0 & 0 \\ b^2 & a+c & b^2/c \\ c^2 & c^2/b & a+b \end{vmatrix}$$ **(1)**

On expanding along R_1, we get

$$\text{LHS} = (a+b+c)^2 [2bc(a^2 + ab + ac + bc - bc)]$$

$$= (a+b+c)^2 [2bc(a^2 + ab + ac)]$$

$$= (a+b+c)^2 \cdot 2abc(a+b+c)$$

$$= 2abc(a+b+c)^3 = \text{RHS}$$ **(1)**

Hence proved.

34. Using properties of determinants, prove that

$$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz + xy + yz + zx.$$

All India 2009, 2008

To prove, $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz + xy + yz + zx$

$$\text{LHS} = \begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$$

On dividing R_1 by x, R_2 by y and R_3 by z and multiplying the determinant by xyz, we get

$$\text{LHS} = xyz \begin{vmatrix} \frac{1}{x}+1 & \frac{1}{x} & \frac{1}{x} \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix}$$ **(1)**

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= xyz \begin{vmatrix} 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} & 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} & 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix}$$

(1)

On taking $\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$ common from R_1,

we get LHS $= xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1/y & (1/y)+1 & 1/y \\ 1/z & 1/z & (1/z)+1 \end{vmatrix} \quad \textbf{(1/2)}$$

On applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\text{LHS} = (xyz + yz + zx + xy)\begin{vmatrix} 1 & 0 & 0 \\ 1/y & 1 & 0 \\ 1/z & 0 & 1 \end{vmatrix} \quad \textbf{(1)}$$

$= (xyz + xy + yz + zx)\,[1\,(1-0)]$

[$\because$ expanding along R_1]

$= xyz + xy + yz + zx = \text{RHS}$ **(1/2)**

Hence proved.

35. Prove that $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1.$

All India 2009

To prove, $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1$

$$\text{LHS} = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$$

On applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, we get

$$\text{LHS} = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & p-1 \\ 0 & 3 & 3p-2 \end{vmatrix}$$ **(1½)**

Now, on expanding along C_1, we get

$$\text{LHS} = 1 \times \begin{vmatrix} 1 & p-1 \\ 3 & 3p-2 \end{vmatrix}$$

$= 1[(3p-2)-(3p-3)]$ **(1½)**

$= 3p - 2 - 3p + 3 = 1$ **(1)**

$=$ RHS **Hence proved.**

36. Using properties of determinants, prove that

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3.$$ Delhi 2009, 2008

To prove,

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$= (1+a^2+b^2)^3$

$$\text{LHS} = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

On applying $C_1 \to C_1 - bC_3$ and $C_2 \to C_2 + aC_3$, we get

$$\text{LHS} = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$ **(1½)**

On taking $(1+a^2+b^2)$ common from C_1 and C_2, we get

$$\text{LHS} = (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$ **(1)**

On expanding along R_1, we get

$$\text{LHS} = (1+a^2+b^2)^2 \times [1(1-a^2-b^2+2a^2) - 2b(0-b)]$$

$$= (1+a^2+b^2)^2 (1+a^2+b^2)$$

$$= (1+a^2+b^2)^3 = \text{RHS}$$ **(1½)**

Hence proved.

37. Prove that $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc.$ Delhi 2009

To prove, $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$

$= a^3 + b^3 + c^3 - 3abc$

$$\text{LHS} = \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

On applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\text{LHS} = \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix} \quad \textbf{(1)}$$

[$\because$ taking $(a+b+c)$ common from C_1]

On applying $R_3 \to R_3 - 2R_1$, we get

LHS = $(a+b+c)$

$$\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix}$$

On expanding along C_1, we get

$$\text{LHS} = (a+b+c)\cdot 1 \begin{vmatrix} b-c & c-a \\ c+a-2b & a+b-2c \end{vmatrix}$$

(1½)

On applying $R_2 \to R_2 + 2R_1$, we get

$$\text{LHS} = (a+b+c)\begin{vmatrix} b-c & c-a \\ a-c & b-a \end{vmatrix}$$

$= (a+b+c)[(b-c)(b-a) - (a-c)(c-a)]$

$= (a+b+c)[(b^2 - ab - bc + ac)$

$+ (a^2 + c^2 - 2ca)]$

$= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= a^3 + b^3 + c^3 - 3abc = \text{RHS}$ **(1½)**

Hence proved.

38. Show that, if $x \neq y \neq z$ and

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0, \text{ then } 1 + xyz = 0.$$

Delhi 2008C; HOTS

Firstly, apply properties of determinants in LHS and reduce it into simplest form. Then, equate the lowest term to zero and use the given fact that $x \neq y \neq z$ to get the desired result.

Given, $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ and $x \neq y \neq z$

The given determinant can be written as

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0 \textbf{(1/2)}$$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

[taking x, y and z common from R_1, R_2 and R_3, respectively in second determinant] **(1/2)**

On applying $C_2 \leftrightarrow C_3$ in first determinant, we get

$$-\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0 \quad \textbf{(1/2)}$$

On applying $C_1 \leftrightarrow C_2$ in first determinant, we get

$$(-)(-)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}(1+xyz)=0$$

$$\Rightarrow (1+xyz)\begin{vmatrix} 0 & x-y & (x-y)(x+y) \\ 0 & y-z & (y-z)(y+z) \\ 1 & z & z^2 \end{vmatrix}=0$$

On taking $(x-y)$ common from R_1 and $(y-z)$ from R_2, we get

$$(1+xyz)(x-y)(y-z)\begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix}=0$$

(1½)

On expanding along C_1, we get

$(1+xyz)(x-y)(y-z)[1\times(y+z)-(x+y)]=0$

$\Rightarrow (1+xyz)(x-y)(y-z)(z-x)=0$

$\Rightarrow$ Either $1+xyz=0$

or $x-y=y-z=z-x=0$

$\Rightarrow x=y=z$

But this is contradiction as given that

$x \neq y \neq z$

$\therefore \quad 1+xyz=0$ **(1)**

Hence proved.

39. Using properties of determinants, prove that

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}=(x+y+z)^3.$$

$$\text{LHS} = \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\text{LHS} = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

On taking $(x + y + z)$ common from R_1, we get

$$\text{LHS} = (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

(1)

On applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get

$$\text{LHS} = (x+y+z)\begin{vmatrix} 1 & 0 & 0 \\ 2z & 0 & -(x+y+z) \\ x-y-z & x+y+z & x+y+z \end{vmatrix}$$ **(1)**

On taking $(x + y + z)$ common from C_2 and C_3, both, we get

$$\text{LHS} = (x+y+z)^3\begin{vmatrix} 1 & 0 & 0 \\ 2z & 0 & -1 \\ x-y-z & 1 & 1 \end{vmatrix}$$ **(1)**

$$= (x+y+z)^3 [1(0+1)]$$

$$= (x+y+z)^3 = \text{RHS}$$ **(1)**

Hence proved.