Continuity and Differentiability

Case Study Based Questions

Case Study 1

Ms. Anika Jain, teacher at a well known reputed coaching institute is teaching. Derivatives of function in parametric forms to her students with simple method through video lecture.

Sometimes the relation between two variables is neither explicit nor implicit, but some link of a third variable with each of the two variables, separately, establishes a relation between the first two variables. In such a situation, we say that the relation between them is expressed via a third variable. The third variable is called the parameter, more precisely, a relation expressed between two variables x and y in the form x = f(t), y = g(t) is said to be parametric form with t is a parameter.

In order to find derivative of function in such form, we use chain rule.

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt} \Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}\left\{\text{provided } \frac{dx}{dt}\neq 0\right\}$$

Thus, $\dfrac{dy}{dx}=\dfrac{g'(t)}{f'(t)}$ $\left\{\text{as } \dfrac{dy}{dt}=g'(t) \text{ and } \dfrac{dx}{dt}=f'(t)\right\}$

provided $f'(t)\neq 0$.

Based on the given information, solve the following questions:

Q 1. If $x=t^2, y=t^3$, then $\dfrac{dy}{dx}$ is:

a. $\dfrac{3y}{2x}$ b. $\dfrac{2y}{3x}$ c. $\dfrac{3x}{2y}$ d. $\dfrac{2x}{3y}$

Q 2. If $x = t + \frac{1}{t}, y = t - \frac{1}{t}$, then $\frac{dy}{dx}$ is:

a. $\frac{y}{x}$ b. $\frac{x}{y}$ c. $-\frac{y}{x}$ d. $-\frac{x}{y}$

Q 3. If $x = a\sec^2\theta$ and $y = a\tan^2\theta$, then $\frac{dy}{dx}$ is:

a. $\sin\theta$ b. $\cos\theta$ c. 1 d. $-\cot\theta$

Q 4. If $\sin x = \frac{2t}{1+t^2}$, $\cos y = \frac{1-t^2}{1+t^2}$, then $\frac{dy}{dx}$ is:

a. $\frac{1}{t}$ b. $-t$ c. -1 d. 1

Q 5. If $x = 2\sin t + \sin 2t$ and $y = 2\cos t + \cos 2t$, then $\frac{dy}{dx}$ at $t = \frac{\pi}{6}$ is:

a. -1 b. $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ c. 1 d. $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

Solutions

1. Given, $x = t^2$ and $y = t^3$

Now differentiate both sides w.r.t. 't', we get

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = 3t^2$$

Now, $$y = t^3 = t^2 \cdot t = x \cdot t \qquad [\because x = t^2]$$

$\therefore$ $$t = \frac{y}{x}$$

We have, $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t = \frac{3}{2} \times \frac{y}{x}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{3y}{2x}$$

So, option (a) is correct.

2. Given, $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$

Now, differentiate both sides w.r.t. 't', we get

$$\frac{dx}{dt}=1-\frac{1}{t^2}=\frac{t^2-1}{t^2}=\frac{t^2-1}{t}\cdot\frac{1}{t}=\left(t-\frac{1}{t}\right)\cdot\frac{1}{t}$$

and $$\frac{dy}{dt}=1+\frac{1}{t^2}=\frac{dy}{dt}=\frac{t^2+1}{t^2}=\frac{t^2+1}{t}\cdot\frac{1}{t}=\left(t+\frac{1}{t}\right)\cdot\frac{1}{t}$$

$$\Rightarrow \qquad \frac{dx}{dt}=\frac{y}{t} \text{ and } \frac{dy}{dt}=\frac{x}{t}$$

$$\therefore \qquad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{x}{t}\times\frac{t}{y}=\frac{x}{y}.$$

So, option (b) is correct.

3. Given, $x=a\sec^2\theta$ and $y=a\tan^2\theta$

Now, differentiate both sides w.r.t. 'θ', we get

$$\frac{dx}{d\theta}=2a\sec\theta\cdot\frac{d}{d\theta}\sec\theta$$

$$=2a\sec\theta\cdot(\sec\theta\cdot\tan\theta)$$

$$=2a\sec^2\theta\cdot\tan\theta$$

and $$\frac{dy}{d\theta}=2a\tan\theta\cdot\frac{d}{d\theta}\tan\theta$$

$$=2a\tan\theta\cdot\sec^2\theta$$

$$\therefore \qquad \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{2a\tan\theta\cdot\sec^2\theta}{2a\sec^2\theta\cdot\tan\theta}=1$$

So, option (c) is correct.

4. Given, $\sin x=\dfrac{2t}{1+t^2}$ and $\cos y=\dfrac{1-t^2}{1+t^2}$

$$\Rightarrow \qquad x=\sin^{-1}\left(\frac{2t}{1+t^2}\right), \text{ and } y=\cos^{-1}\left(\frac{1-t^2}{1+t^2}\right)$$

$$\Rightarrow \qquad x=2\tan^{-1}t \text{ and } y=2\tan^{-1}t$$

Now, differentiate both sides w.r.t. 't', we get

$$\frac{dx}{dt}=\frac{2}{1+t^2} \text{ and } \frac{dy}{dt}=\frac{2}{1+t^2}$$

$$\therefore \qquad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2}{1+t^2}\times\frac{1+t^2}{2}=1$$

So, option (d) is correct.

5. Given, $x=2\sin t+\sin 2t$ and $y=2\cos t+\cos 2t$

Now, differentiate both sides w.r.t. 't', we get

$$\frac{dx}{dt}=2\cos t+\cos 2t\cdot\frac{d}{dt}(2t)$$

$$=2\cos t+2\cos 2t=2\,(\cos t+\cos 2t)$$

and $$\frac{dy}{dt}=-2\sin t-\sin 2t\,\frac{d}{dt}(2t)$$

$$=-2\sin t-\sin 2t\cdot 2$$

$$=-2\,(\sin t+\sin 2t)$$

$$\therefore \qquad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-2\,(\sin t+\sin 2t)}{2\,(\cos t+\cos 2t)}$$

$$=-\frac{(\sin t+\sin 2t)}{(\cos t+\cos 2t)}$$

$$\therefore \qquad \left[\frac{dy}{dx}\right]_{\text{at}\left(t=\frac{\pi}{6}\right)}=-\left\{\frac{\sin\left(\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{6}\right)+\cos\left(\frac{\pi}{3}\right)}\right\}$$

$$=-\frac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)}{\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)}=-1$$

So, option (a) is correct.

Case Study 2

A function is continuous at $x = c$ if the function is defined at $x = c$ and if the value of the function at $x = c$ equals the limit of the function at $x = c$.

i.e., $$\lim_{x \to c} f(x) = f(c)$$

If f is not continuous at c, we say f is discontinuous at c and c is called a point of discontinuity of f.

Based on the above information, solve the following questions:

Q 1. Suppose f and g be two real functions continuous at a real number 'c', then show that $f + g$ is continuous at $x = c$.

Q 2. Find the value of k so that the given function $f(x)$ is continuous at $x = 5$.

$$f(x) = \begin{cases} kx + 1; & x \le 5 \\ 3x - 5; & x > 5 \end{cases}$$

Solutions

1. $$\lim_{x \to c^-} [f(x) + g(x)] = \lim_{x \to c^-} f(x) + \lim_{x \to c^-} g(x)$$

$$= f(c) + g(c)$$

$$\lim_{x \to c^+} [f(x) + g(x)]$$

$$= \lim_{x \to c^+} f(x) + \lim_{x \to c^+} g(x) = f(c) + g(c)$$

$\because$ LHL = RHL

$\therefore$ $(f + g)$ is continuous at $x = c$. **Hence proved.**

2. Since, $f(x)$ is continuous at $x = 5$.

$$\therefore \quad f(5) = \lim_{x \to 5} f(x)$$

$$\Rightarrow \quad 5k + 1 = \lim_{x \to 5} 3x - 5$$

$$\Rightarrow \quad 5k + 1 = 3(5) - 5 = 10$$

$$\Rightarrow \qquad 5k = 9$$

$$\Rightarrow \qquad k = 9/5$$

Case Study 3

Let $f(x)$ be a real valued function. Then:

Left Hand Derivative (LHD):

$$Lf'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

Right Hand Derivative (RHD):

$$Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Also, a function $f(x)$ is said to be differentiable at $x = a$ if its LHD and RHD at $x = a$ exist and both are equal.

For the function $f(x) = \begin{cases} |x-3|, & x \geq 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$

Based on the above information, solve the following questions: (CBSE 2023)

Q1. What is RHD of ƒ(x) at x = 1?

Q2. What is LHD of ƒ(x) at x=1?

Q3. Check whether the function f(x) is differentiable at x = 1.

Or

Find ƒ' (2) and ƒ' (−1).

Solutions

1. Given, $f(x) = \begin{cases} |x-3|, & x \geq 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$

$$= \begin{cases} -(x-3), & 1 \leq x < 3 \\ x-3, & x \geq 3 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$$

At $x = 1$,

$$\text{RHD} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{-(1+h-3) + (1-3)}{h}$$

$$= \lim_{h \to 0} \frac{-h+2-2}{h} = -1$$

2. At $x = 1$,

$$\text{LHD} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{\frac{(1-h)^2}{4} - 3\frac{(1-h)}{2} + \frac{13}{4} + (1-3)}{-h}$$

$$= \lim_{h \to 0} \frac{\frac{(1^2 + h^2 - 2h)}{4} - \frac{6(1-h\} + 13}{4} - 2}{-h}$$

$$= \lim_{h \to 0} \frac{h^2 + 4h + 8 - 8}{-4h} = \lim_{h \to 0} \frac{h(h+4)}{-4h}$$

$$= \frac{4}{-4} = -1$$

3. $\because$ At $x = 1$, RHD $= -1$ and LHD $= -1$

$\therefore$ LHD $=$ RHD $= -1$

Hence, $f(x)$ is differentiable at $x = 1$.

Or

Since, 2 lies in the interval $[1, 3)$, so we consider the function

$$f(x) = -(x-3)$$

Then, $\quad f(x) = -x + 3$

$\Rightarrow \quad f'(x) = -1$

$\Rightarrow \quad f'(2) = -1$

Since, -1 lies in the interval $(\infty, 1)$, so we consider the function, $f(x) = \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}$

$$\Rightarrow \qquad f'(x) = \frac{2x}{4} - \frac{3}{2} + 0$$

$$\therefore \qquad f'(-1) = \frac{2(-1)}{4} - \frac{3}{2}$$

$$= -\frac{1}{2} - \frac{3}{2} = -\frac{4}{2} = -2$$

Case Study 4

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f[g(x)]$ is a differentiable function of x and $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. This rule is also known as CHAIN RULE.

Based on the above information, solve the following questions.

Q 1. Find the derivative of $\cos\sqrt{x}$ with respect to x.

Q 2. Find the derivative of $7^{x+\frac{1}{x}}$ with respect to x.

Q 3. Find the derivative of $\sqrt{\frac{1-\cos x}{1+\cos x}}$ with respect to x.

Q 4. Find the derivative of $\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) + \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ with respect to x.

Or

Find the derivative of $\sec^{-1} x + \operatorname{cosec}^{-1} \frac{x}{\sqrt{x^2-1}}$ with respect to x.

Solutions

1. Let $\qquad y = \cos\sqrt{x}$

$$\therefore \qquad \frac{dy}{dx} = \frac{d}{dx}(\cos\sqrt{x})$$

$$= -\sin\sqrt{x}\cdot\frac{d}{dx}(\sqrt{x})$$

$$= -\sin\sqrt{x}\times\frac{1}{2\sqrt{x}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

2. Let $\quad y = 7^{x+\frac{1}{x}}$

Then $\quad \dfrac{dy}{dx} = \dfrac{d}{dx}\left(7^{x+\frac{1}{x}}\right)$

$$= 7^{x+\frac{1}{x}}\cdot\log 7\cdot\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$= 7^{x+\frac{1}{x}}\cdot\log 7\cdot\left(1-\frac{1}{x^2}\right)$$

$$= \left(\frac{x^2-1}{x^2}\right)\cdot 7^{x+\frac{1}{x}}\cdot\log 7$$

3. Let $\quad y = \sqrt{\dfrac{1-\cos x}{1+\cos x}} = \sqrt{\dfrac{1-1+2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}-1+1}} = \tan\left(\dfrac{x}{2}\right)$

$\therefore \quad \dfrac{dy}{dx} = \sec^2\dfrac{x}{2}\cdot\dfrac{1}{2} = \dfrac{1}{2}\sec^2\dfrac{x}{2}$

4. Let $\quad y = \dfrac{1}{b}\tan^{-1}\left(\dfrac{x}{b}\right) + \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{b}\times\dfrac{1}{1+\dfrac{x^2}{b^2}}\times\dfrac{1}{b} + \dfrac{1}{a}\times\dfrac{1}{1+\dfrac{x^2}{a^2}}\times\dfrac{1}{a}$

$$= \frac{1}{b^2+x^2} + \frac{1}{a^2+x^2}$$

Or

Let $\; y = \sec^{-1}x + \operatorname{cosec}^{-1}\dfrac{x}{\sqrt{x^2-1}}$

Put $x = \sec\theta \Rightarrow \theta = \sec^{-1} x$

$\therefore$ $$y = \sec^{-1}(\sec\theta) + \operatorname{cosec}^{-1}\left(\frac{\sec\theta}{\sqrt{\sec^2\theta - 1}}\right)$$

$$= \theta + \operatorname{cosec}^{-1}\left(\frac{\sec\theta}{\tan\theta}\right)$$

$$= \theta + \operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta + \theta$$

$$= 2\theta = 2\sec^{-1} x$$

$\therefore$ $$\frac{dy}{dx} = 2\frac{d}{dx}(\sec^{-1} x)$$

$$= 2 \times \frac{1}{|x|\sqrt{x^2 - 1}} = \frac{2}{|x|\sqrt{x^2 - 1}}$$

Solutions for Questions 5 to 14 are Given Below

Case Study 5

Let $f(x)$ be a real valued function, then its

- Left Hand Derivative (L.H.D.) : $Lf'(a) = \lim_{h \to 0} \frac{f(a-h)-f(a)}{-h}$
- Right Hand Derivative (R.H.D.) : $Rf'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

Also, a function $f(x)$ is said to be differentiable at $x = a$ if its L.H.D. and R.H.D. at $x = a$ exist and are equal.

For the function $f(x) = \begin{cases} |x-3|, & x \geq 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$, answer the following questions.

(i) R.H.D. of $f(x)$ at $x = 1$ is

(a) 1 (b) -1 (c) 0 (d) 2

(ii) L.H.D. of $f(x)$ at $x = 1$ is

(a) 1 (b) -1 (c) 0 (d) 2

(iii) $f(x)$ is non-differentiable at

(a) $x = 1$ (b) $x = 2$ (c) $x = 3$ (d) $x = 4$

(iv) Find the value of $f'(2)$.

(a) 1 (b) 2 (c) 3 (d) -1

(v) The value of $f'(-1)$ is

(a) 2 (b) 1 (c) -2 (d) -1

Case Study 6

Let $x = f(t)$ and $y = g(t)$ be parametric forms with t as a parameter, then

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{g'(t)}{f'(t)}$, where $f'(t) \neq 0$.

On the basis of above information, answer the following questions.

(i) The derivative of $f(\tan x)$ w.r.t. $g(\sec x)$ at $x=\frac{\pi}{4}$, where $f'(1)=2$ and $g'(\sqrt{2})=4$, is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$ (c) 1 (d) 0

(ii) The derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is

(a) –1 (b) 1 (c) 2 (d) 4

(iii) The derivative of e^{x^3} with respect to $\log x$ is

(a) e^{x^3} (b) $3x^2 2e^{x^3}$ (c) $3x^3e^{x^3}$ (d) $3x^2e^{x^3}+3x$

(iv) The derivative of $\cos^{-1}(2x^2-1)$ w.r.t. $\cos^{-1}x$ is

(a) 2 (b) $\frac{-1}{2\sqrt{1-x^2}}$ (c) $\frac{2}{x}$ (d) $1-x^2$

(v) If $y=\frac{1}{4}u^4$ and $u=\frac{2}{3}x^3+5$, then $\frac{dy}{dx}=$

(a) $\frac{2}{27}x^2(2x^3+15)^3$ (b) $\frac{2}{7}x^2(2x^3+15)^3$ (c) $\frac{2}{27}x(2x^3+5)^3$ (d) $\frac{2}{7}(2x^3+15)^3$

Case Study 7

Let $f: A \to B$ and $g: B \to C$ be two functions defined on non-empty sets A, B, C, then $gof: A \to C$ be is called the composition of f and g defined as, $gof(x) = g\{f(x)\}\ \forall\ x \in A$.

Consider the functions $f(x)=\begin{cases}\sin x, & x\ge 0\\ 1-\cos x, & x\le 0\end{cases}$, $g(x)=e^x$ and then answer the following questions.

(i) The function $gof(x)$ is defined as

(a) $gof(x)=\begin{cases}e^x & ,x\ge 0\\ 1-e^{\cos x} & ,x\le 0\end{cases}$ (b) $gof(x)=\begin{cases}e^{\sin x} & ,x\le 0\\ e^{1-\cos x} & ,x\ge 0\end{cases}$

(c) $gof(x)=\begin{cases}e^{\sin x} & ,x\le 0\\ 1-e^{\cos x} & ,x\ge 0\end{cases}$ (d) $gof(x)=\begin{cases}e^{\sin x} & ,x\ge 0\\ e^{1-\cos x} & ,x\le 0\end{cases}$

(ii) $\frac{d}{dx}\{gof(x)\}=$

(a) $[gof(x)]'=\begin{cases}\cos x\cdot e^{\sin x} & ,x\ge 0\\ e^{1-\cos x}\cdot\sin x & ,x\le 0\end{cases}$ (b) $[gof(x)]'=\begin{cases}\cos x\cdot e^{\sin x} & ,x\ge 0\\ -\sin x\cdot e^{1-\cos x} & ,x\le 0\end{cases}$

(c) $[gof(x)]'=\begin{cases}\cos x\cdot e^{\sin x} & ,x\ge 0\\ \sin x\cdot(1-\cos x) & ,x\le 0\end{cases}$ (d) $[gof(x)]'=\begin{cases}\cos x\cdot e^{\sin x} & ,x\ge 0\\ (1-\sin x)\cdot e^{1-\cos x} & ,x\le 0\end{cases}$

(iii) R.H.D. of $gof(x)$ at $x = 0$ is

(a) 0 (b) 1 (c) –1 (d) 2

(iv) L.H.D. of $gof(x)$ at $x = 0$ is

(a) 0 (b) 1 (c) –1 (d) 2

(v) The value of $f'(x)$ at $x = \frac{\pi}{4}$ is

(a) 1/9 (b) $1/\sqrt{2}$ (c) 1/2 (d) not defined

Case Study 8

The function $f(x)$ will be discontinuous at $x = a$ if $f(x)$ has

- Discontinuity of first kind : $\lim_{h\to 0} f(a-h)$ and $\lim_{h\to 0} f(a+h)$ both exist but are not equal. If is also known as irremovable discontinuity.
- Discontinuity of second kind : If none of the limits $\lim_{h\to 0} f(a-h)$ and $\lim_{h\to 0} f(a+h)$ exist.
- Removable discontinuity : $\lim_{h\to 0} f(a-h)$ and $\lim_{h\to 0} f(a+h)$ both exist and equal but not equal to $f(a)$.

Based on the above information, answer the following questions.

(i) If $f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & \text{for } x \neq 3 \\ 4, & \text{for } x = 3 \end{cases}$, then at $x = 3$

(a) f has removable discontinuity (b) f is continuous

(c) f has irremovable discontinuity (d) none of these

(ii) Let $f(x) = \begin{cases} x+2, & \text{if } x \leq 4 \\ x+4, & \text{if } x > 4 \end{cases}$ then at $x = 4$

(a) f is continuous (b) f has removable discontinuity

(c) f has irremovable discontinuity (d) none of these

(iii) Consider the function $f(x)$ defined as $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{for } x \neq 2 \\ 5, & \text{for } x = 2 \end{cases}$, then at $x = 2$

(a) f has removable discontinuity (b) f has irremovable discontinuity

(c) f is continuous (d) f is continuous if $f(2) = 3$

(iv) If $f(x) = \begin{cases} \dfrac{x-|x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$, then at $x = 0$

(a) f is continuous (b) f has removable discontinuity

(c) f has irremovable discontinuity (d) none of these

(v) If $f(x) = \begin{cases} \dfrac{e^x-1}{\log(1+2x)}, & \text{if } x \neq 0 \\ 7, & \text{if } x = 0 \end{cases}$, then at $x = 0$

(a) f is continuous if $f(0) = 2$ (b) f is continuous

(c) f has irremovable discontinuity (d) f has removable discontinuity

Case Study 9

If a real valued function $f(x)$ is finitely derivable at any point of its domain, it is necessarily continuous at that point. But its converse need not be true.
For example, every polynomial, constant function are both continuous as well as differentiable and inverse trigonometric functions are continuous and differentiable in its domains etc.

Based on the above information, answer the following questions.

(i) If $f(x)=\begin{cases}x, \text{ for } x\le 0\\ 0, \text{ for } x>0\end{cases}$, then at $x=0$

(a) $f(x)$ is differentiable and continuous
(b) $f(x)$ is neither continuous nor differentiable
(c) $f(x)$ is continuous but not differentiable
(d) none of these

(ii) If $f(x) = |x - 1|, x \in R$, then at $x = 1$

(a) $f(x)$ is not continuous
(b) $f(x)$ is continuous but not differentiable
(c) $f(x)$ is continuous and differentiable
(d) none of these

(iii) $f(x) = x^3$ is

(a) continuous but not differentiable at $x = 3$
(b) continuous and differentiable at $x = 3$
(c) neither continuous nor differentiable at $x = 3$
(d) none of these

(iv) If $f(x) = [\sin x]$, then which of the following is true?

(a) $f(x)$ is continuous and differentiable at $x = 0$.
(b) $f(x)$ is discontinuous at $x = 0$.
(c) $f(x)$ is continuous at $x = 0$ but not differentiable.
(d) $f(x)$ is differentiable but not continuous at $x = \pi/2$.

(v) If $f(x) = \sin^{-1}x, -1 \le x \le 1$, then

(a) $f(x)$ is both continuous and differentiable
(b) $f(x)$ is neither continuous nor differentiable.
(c) $f(x)$ is continuous but not differentiable
(d) None of these

Case Study 10

Derivative of $y=f(x)$ w.r.t. x (if exists) is denoted by $\frac{dy}{dx}$ or $f'(x)$ and is called the first order derivative of y.

If we take derivative of $\frac{dy}{dx}$ again, then we get $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$ or $f''(x)$ and is called the second order derivative of y. Similarly, $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ is denoted and defined as $\frac{d^3y}{dx^3}$ or $f'''(x)$ and is known as third order derivative of y and so on.

Based on the above information, answer the following questions.

(i) If $y=\tan^{-1}\left(\frac{\log(e/x^2)}{\log(ex^2)}\right)+\tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$, then $\frac{d^2y}{dx^2}$ is equal to

(a) 2
(b) 1
(c) 0
(d) –1

(ii) If $u = x^2 + y^2$ and $x = s + 3t$, $y = 2s - t$, then $\frac{d^2u}{ds^2}$ is equal to

(a) 12
(b) 32
(c) 36
(d) 10

(iii) If $f(x) = 2 \log \sin x$, then $f''(x)$ is equal to

(a) $2\text{cosec}^3 x$
(b) $2\cot^2 x - 4x^2 \text{cosec}^2 x^2$
(c) $2x \cot x^2$
(d) $-2\text{cosec}^2 x$

(iv) If $f(x) = e^x \sin x$, then $f'''(x) =$

(a) $2e^x(\sin x + \cos x)$
(b) $2e^x(\cos x - \sin x)$
(c) $2e^x(\sin x - \cos x)$
(d) $2e^x \cos x$

(v) If $y^2 = ax^2 + bx + c$, then $\frac{d}{dx}\left(y^3 y_2\right)=$

(a) 1
(b) –1
(c) $\frac{4ac-b^2}{a^2}$
(d) 0

Case Study 11

- A function $f(x)$ is said to be continuous in an open interval (a, b), if it is continuous at every point in this interval.
- A function $f(x)$ is said to be continuous in the closed interval $[a, b]$, if $f(x)$ is continuous in (a, b) and $\lim_{h\to 0} f(a+h) = f(a)$ and $\lim_{h\to 0} f(b-h) = f(b)$

If function $f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & , x < 0 \\ c & , x = 0 \\ \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} & , x > 0 \end{cases}$ is continuous at $x = 0$, then answer the following questions.

(i) The value of a is

(a) $-3/2$ (b) 0 (c) 1/2 (d) $-1/2$

(ii) The value of b is

(a) 1 (b) -1 (c) 0 (d) any real number

(iii) The value of c is

(a) 1 (b) 1/2 (c) -1 (d) $-1/2$

(iv) The value of $a + c$ is

(a) 1 (b) 0 (c) -1 (d) -2

(v) The value of $c - a$ is

(a) 1 (b) 0 (c) -1 (d) 2

Case Study 12

Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = [u(x)]^{v(x)}$, where both $u(x)$ and $v(x)$ are differentiable functions and f and u need to be positive functions.

Let function $y = f(x) = (u(x))^{v(x)}$, then $y' = y\left[\dfrac{v(x)}{u(x)}u'(x) + v'(x)\cdot\log[u(x)]\right]$

On the basis of above information, answer the following questions.

(i) Differentiate x^x w.r.t. x

(a) $x^x(1 + \log x)$ (b) $x^x(1 - \log x)$ (c) $-x^x(1 + \log x)$ (d) $x^x \log x$

(ii) Differentiate $x^x + a^x + x^a + a^a$ w.r.t. x

(a) $(1 + \log x) + (a^x \log a + ax^{a-1})$ (b) $x^x(1 + \log x) + \log a + ax^{a-1}$

(c) $x^x(1 + \log x) + x^a \log x + ax^{a-1}$ (d) $x^x(1 + \log x) + a^x \log a + ax^{a-1}$

(iii) If $x = e^{x/y}$, then find $\dfrac{dy}{dx}$.

(a) $-\dfrac{(x+y)}{x\log x}$ (b) $-\dfrac{(x-y)}{x\log x}$ (c) $\dfrac{(x+y)}{x\log x}$ (d) $\dfrac{x-y}{x\log x}$

(iv) If $y = (2-x)^3(3+2x)^5$, then find $\dfrac{dy}{dx}$.

(a) $(2-x)^3(3+2x)^5\left[\dfrac{15}{3+2x} - \dfrac{8}{2-x}\right]$ (b) $(2-x)^3(3+2x)^5\left[\dfrac{15}{3+2x} + \dfrac{3}{2-x}\right]$

(c) $(2-x)^3(3+2x)^5\left[\frac{10}{3+2x}-\frac{3}{2-x}\right]$ (d) $(2-x)^3(3+2x)^5\cdot\left[\frac{10}{3+2x}+\frac{3}{2-x}\right]$

(v) If $y=x^x\cdot e^{(2x+5)}$, then find $\frac{dy}{dx}$.

(a) $x^x e^{2x+5}$ (b) $x^x e^{2x+5}(3-\log x)$ (c) $x^x e^{2x+5}(1-\log x)$ (d) $x^x e^{2x+5}.(3+\log x)$

Case Study 13

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x)]$ is a differentiable function of x and $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$. This rule is also known as CHAIN RULE.

Based on the above information, find the derivative of functions w.r.t. x in the following questions.

(i) $\cos\sqrt{x}$

(a) $\frac{-\sin\sqrt{x}}{2\sqrt{x}}$ (b) $\frac{\sin\sqrt{x}}{2\sqrt{x}}$ (c) $\sin\sqrt{x}$ (d) $-\sin\sqrt{x}$

(ii) $7^{x+\frac{1}{x}}$

(a) $\left(\frac{x^2-1}{x^2}\right).7^{x+\frac{1}{x}}\cdot\log 7$ (b) $\left(\frac{x^2+1}{x^2}\right).7^{x+\frac{1}{x}}\cdot\log 7$ (c) $\left(\frac{x^2-1}{x^2}\right).7^{x-\frac{1}{x}}\cdot\log 7$ (d) $\left(\frac{x^2+1}{x^2}\right).7^{x-\frac{1}{x}}\cdot\log 7$

(iii) $\sqrt{\frac{1-\cos x}{1+\cos x}}$

(a) $\frac{1}{2}\sec^2\frac{x}{2}$ (b) $-\frac{1}{2}\sec^2\frac{x}{2}$ (c) $\sec^2\frac{x}{2}$ (d) $-\sec^2\frac{x}{2}$

(iv) $\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)+\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

(a) $\frac{-1}{x^2+b^2}+\frac{1}{x^2+a^2}$ (b) $\frac{1}{x^2+b^2}+\frac{1}{x^2+a^2}$ (c) $\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}$ (d) none of these

(v) $\sec^{-1}x+\operatorname{cosec}^{-1}\frac{x}{\sqrt{x^2-1}}$

(a) $\frac{2}{\sqrt{x^2-1}}$ (b) $\frac{-2}{\sqrt{x^2-1}}$ (c) $\frac{1}{|x|\sqrt{x^2-1}}$ (d) $\frac{2}{|x|\sqrt{x^2-1}}$

Case Study 14

If a relation between x and y is such that y cannot be expressed in terms of x, then y is called an implicit function of x. When a given relation expresses y as an implicit function of x and we want to find $\frac{dy}{dx}$, then we differentiate every term of the given relation w.r.t. x, remembering that a term in y is first differentiated w.r.t. y and then multiplied by $\frac{dy}{dx}$.

Based on the above information, find the value of $\frac{dy}{dx}$ in each of the following questions.

(i) $x^3+x^2y+xy^2+y^3=81$

(a) $\frac{(3x^2+2xy+y^2)}{x^2+2xy+3y^2}$ (b) $\frac{-(3x^2+2xy+y^2)}{x^2+2xy+3y^2}$ (c) $\frac{(3x^2+2xy-y^2)}{x^2-2xy+3y^2}$ (d) $\frac{3x^2+xy+y^2}{x^2+xy+3y^2}$

(ii) $x^y = e^{x-y}$

(a) $\frac{x-y}{(1+\log x)}$ (b) $\frac{x+y}{(1+\log x)}$ (c) $\frac{x-y}{x(1+\log x)}$ (d) $\frac{x+y}{x(1+\log x)}$

(iii) $e^{\sin y} = xy$

(a) $\frac{-y}{x(y\cos y-1)}$ (b) $\frac{y}{y\cos y-1}$ (c) $\frac{y}{y\cos y+1}$ (d) $\frac{y}{x(y\cos y-1)}$

(iv) $\sin^2 x + \cos^2 y = 1$

(a) $\frac{\sin 2y}{\sin 2x}$ (b) $-\frac{\sin 2x}{\sin 2y}$ (c) $-\frac{\sin 2y}{\sin 2x}$ (d) $\frac{\sin 2x}{\sin 2y}$

(v) $y = (\sqrt{x})^{\sqrt{x}^{\sqrt{x}\ldots\infty}}$

(a) $\frac{-y^2}{x(2-y\log x)}$ (b) $\frac{y^2}{2+y\log x}$ (c) $\frac{y^2}{x(2+y\log x)}$ (d) $\frac{y^2}{x(2-y\log x)}$

HINTS & EXPLANATIONS

5. We have, $f(x)=\begin{cases} x-3 & ,x\geq 3 \\ 3-x & ,1\leq x<3 \\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4} & ,x<1 \end{cases}$

(i) (b): $Rf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0}\frac{3-(1+h)-2}{h} = \lim_{h\to 0} -\frac{h}{h} = -1$

(ii) (b): $Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$

$= \lim_{h\to 0}\frac{-1}{h}\left[\frac{(1-h)^2}{4}-\frac{3(1-h)}{2}+\frac{13}{4}-2\right]$

$= \lim_{h\to 0}\left(\frac{1+h^2-2h-6+6h+13-8}{-4h}\right)$

$= \lim_{h\to 0}\left(\frac{h^2+4h}{-4h}\right) = -1$

(iii) (c): Since, R.H.D. at $x = 3$ is 1 and L.H.D. at $x = 3$ is -1

$\therefore$ $f(x)$ is non-differentiable at $x = 3$.

(iv) (d)

(v) (c): From above, we have

$f'(x) = \frac{x}{2}-\frac{3}{2}, x<1$

$\therefore\ f'(-1) = \frac{-1}{2}-\frac{3}{2} = -2$

6. (i) (a): Now, $\frac{df(\tan x)}{dg(\sec x)} = \frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x\tan x}$

$= \frac{f'(\tan x)\sec x}{g'(\sec x)\tan x}$

$\therefore\ \left[\frac{df(\tan x)}{dg(\sec x)}\right]_{x=\pi/4} = \frac{f'(1)\sqrt{2}}{g'(\sqrt{2})\cdot 1} = \frac{2\sqrt{2}}{4\cdot 1} = \frac{1}{\sqrt{2}}$

(ii) (b)

(iii) (c): Let $y = e^{x^3}$, $z = \log x$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = e^{x^3}(3x^2) = 3x^2 e^{x^3}$ and $\frac{dz}{dx} = \frac{1}{x}$

$\therefore\ \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{3x^2e^{x^3}}{\left(\frac{1}{x}\right)} = 3x^3e^{x^3}$

(iv) (a): Let $y = \cos^{-1}(2x^2-1) = 2\cos^{-1}x$

Differentiating w.r.t. $\cos^{-1}x$, we get

$\frac{dy}{d(\cos^{-1}x)} = \frac{2d(\cos^{-1}x)}{d(\cos^{-1}x)} = 2$

(v) (a): We have, $y = \frac{1}{4}u^4 \Rightarrow \frac{dy}{du} = \frac{1}{4}\cdot 4u^3 = u^3$

and $u = \frac{2}{3}x^3+5 \Rightarrow \frac{du}{dx} = \frac{2}{3}\cdot 3x^2 = 2x^2$

$\therefore\ \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = u^3\cdot 2x^2 = \left(\frac{2}{3}x^3+5\right)^3(2x^2)$

$= \frac{2}{27}x^2(2x^3+15)^3$

7. (i) (d) (ii) (a) (iii) (b)

(iv) (a) (v) (b)

8. (i) (a) : $f(3) = 4$

$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{(x-3)}$

$= \lim_{x\to 3} (x+3) = 6 \because \lim_{x\to 3} f(x) \neq f(3)$

$\therefore$ $f(x)$ has removable discontinuity at $x = 3$.

(ii) (c) : $\lim_{x\to 4^-} f(x) = \lim_{x\to 4}(x+2) = 4+2 = 6$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4}(x+4) = 4+4 = 8$

$\therefore \lim_{x\to 4^-} f(x) \neq \lim_{x\to 4^+} f(x)$

$\therefore$ $f(x)$ has an irremovable discontinuity at $x = 4$.

(iii) (a) : $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x^2-4)}{(x-2)} = \lim_{x\to 2}(x+2) = 4$

and $f(2) = 5$ (given) $\therefore \lim_{x\to 2} f(x) \neq f(2)$

$\therefore$ $f(x)$ has removable discontinuity at $x = 2$.

(iv) (c) : $f(0) = 2$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} \frac{x+x}{x} = 2$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} \frac{x-x}{x} = 0$

$\because \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

$\therefore$ $f(x)$ has an irremovable discontinuity at $x = 0$.

(v) (d) : $f(0) = 7$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^x-1}{\log(1+2x)} = \lim_{x\to 0} \frac{\left(\frac{e^x-1}{x}\right)}{\frac{\log(1+2x)}{2x}\cdot 2} = \frac{1}{2}$$

$\because \lim_{x\to 0} f(x) \neq f(0)$

$\therefore$ $f(x)$ has removable discontinuity at $x = 0$.

9. (i) (c) (ii) (b) (iii) (b)

(iv) (b) (v) (a)

10. (i) (c) : Given, $y = \tan^{-1}\left(\frac{\log\left(\frac{e}{x^2}\right)}{\log ex^2}\right) + \tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$

$= \tan^{-1}\left(\frac{1-\log x^2}{1+\log x^2}\right) + \tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$

$= \tan^{-1}(1) - \tan^{-1}(2\log x) + \tan^{-1}(3) + \tan^{-1}(2\log x)$

$\Rightarrow y = \tan^{-1}(1) + \tan^{-1}(3)$

$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow \frac{d^2y}{dx^2} = 0$

(ii) (d) : Given, $x = s + 3t$, $y = 2s - t \Rightarrow \frac{dx}{ds} = 1, \frac{dy}{ds} = 2$

Now, $u = x^2 + y^2 \Rightarrow \frac{du}{ds} = 2x\frac{dx}{ds} + 2y\frac{dy}{ds} = 2x + 4y$

$\Rightarrow \frac{d^2u}{ds^2} = 2\left(\frac{dx}{ds}\right) + 4\left(\frac{dy}{ds}\right) \Rightarrow \frac{d^2u}{ds^2} = 2(1) + 4(2) = 10$

(iii) (d) : We have, $f(x) = 2\log\sin x$

$\Rightarrow f'(x) = 2\cdot\frac{1}{\sin x}\cdot\cos x = 2\cot x \Rightarrow f''(x) = -2\operatorname{cosec}^2 x$

(iv) (b) : We have, $f(x) = e^x \sin x$

$\Rightarrow f'(x) = e^x\cos x + e^x\sin x = e^x(\cos x + \sin x)$

$\Rightarrow f''(x) = e^x(\cos x - \sin x) + e^x(\cos x + \sin x) = 2e^x\cos x$

$\Rightarrow f'''(x) = 2[e^x\cos x - e^x\sin x] = 2e^x[\cos x - \sin x]$

(v) (d) : Given $y^2 = ax^2 + bx + c$

$\Rightarrow 2yy_1 = 2ax + b$ ...(i)

$\Rightarrow 2yy_2 + y_1(2y_1) = 2a$

$\Rightarrow yy_2 = a - y_1^2 \Rightarrow yy_2 = a - \left(\frac{2ax+b}{2y}\right)^2$ (Using (i))

$= \frac{4y^2a - (4a^2x^2 + b^2 + 4abx)}{4y^2}$

$\Rightarrow y^3y_2 = \frac{4a(ax^2+bx+c) - (4a^2x^2 + b^2 + 4abx)}{4}$

$= \frac{4ac - b^2}{4}$

$\Rightarrow \frac{d}{dx}(y^3y_2) = 0$

11. L.H.L. (at $x = 0$) $= \lim_{x\to 0} \frac{\sin(a+1)x + \sin x}{x}$ $\left(\frac{0}{0}\text{ form}\right)$

Using L' Hospital rule, we get

L.H.L. (at $x = 0$)

$= \lim_{x\to 0} (a+1)\cos(a+1)x + \cos x = a + 2$ ...(i)

R.H.L. (at $x = 0$) $= \lim_{x\to 0} \frac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} = \lim_{x\to 0} \frac{\sqrt{1+bx} - 1}{bx}$

$= \lim_{x\to 0} \frac{1}{\sqrt{1+bx}+1} = \frac{1}{2}$ ...(ii)

Since, $f(x)$ is continuous at $x = 0$.

$\therefore$ From (i) and (ii), we get

$a + 2 = c = \frac{1}{2} \Rightarrow a = -\frac{3}{2}, c = \frac{1}{2}$

Also, value of b does not affect the continuity of $f(x)$, so b can be any real number.

(i) (a) (ii) (d) (iii) (b)

(iv) (c) : $a + c = -\frac{3}{2} + \frac{1}{2} = -1$

(v) (d) : $c - a = \frac{1}{2} + \frac{3}{2} = 2$

12. (i) (a): Let $y = x^x \Rightarrow \log y = x \log x$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x \log x) \Rightarrow \frac{dy}{dx} = x^x[1 \times \log x + x \times \frac{1}{x}]$
$= x^x\,[1 + \log x]$

(ii) (d)

(iii) (d): Given $x = e^{x/y} \Rightarrow \log x = \frac{x}{y}\log e \Rightarrow y\log x = x$

$\Rightarrow y\frac{1}{x} + (\log x)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \left(1 - \frac{y}{x}\right)\frac{1}{\log x} \Rightarrow \frac{1}{x\log x}(x - y)$

(iv) (c): $y = (2 - x)^3\,(3 + 2x)^5$

$\Rightarrow \log y = \log(2 - x)^3 + \log(3 + 2x)^5$
$= 3\log(2 - x) + 5\log(3 + 2x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{3 \times (-1)}{2 - x} + \frac{5}{3 + 2x} \times (2)$

$\Rightarrow \frac{dy}{dx} = (2 - x)^3\,(3 + 2x)^5\left[\frac{10}{3 + 2x} - \frac{3}{2 - x}\right]$

(v) (d): $y = x^x.e^{(2x + 5)}$

$\Rightarrow \log y = x\log x + (2x + 5)$

$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \left(x\cdot\frac{1}{x} + \log x\right) + 2$

$\Rightarrow \frac{dy}{dx} = x^x\cdot e^{2x+5}\cdot(3 + \log x)$

13. (i) (a): Let $y = \cos\sqrt{x}$

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\cos\sqrt{x}) = -\sin\sqrt{x}\cdot\frac{d}{dx}(\sqrt{x})$

$= -\sin\sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$

(ii) (a): Let $y = 7^{x+\frac{1}{x}}$ $\therefore \frac{dy}{dx} = \frac{d}{dx}\left(7^{x+\frac{1}{x}}\right)$

$= 7^{x+\frac{1}{x}}\cdot\log 7\cdot\frac{d}{dx}\left(x + \frac{1}{x}\right) = 7^{x+\frac{1}{x}}\cdot\log 7\cdot\left(1 - \frac{1}{x^2}\right)$

$= \left(\frac{x^2 - 1}{x^2}\right)\cdot 7^{x+\frac{1}{x}}\cdot\log 7$

(iii) (a): Let $y = \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \sqrt{\frac{1 - 1 + 2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2} - 1 + 1}} = \tan\left(\frac{x}{2}\right)$

$\therefore \frac{dy}{dx} = \sec^2\frac{x}{2}.\frac{1}{2} = \frac{1}{2}\sec^2\frac{x}{2}$

(iv) (b): Let $y = \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) + \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$\therefore \frac{dy}{dx} = \frac{1}{b} \times \frac{1}{1 + \frac{x^2}{b^2}} \times \frac{1}{b} + \frac{1}{a} \times \frac{1}{1 + \frac{x^2}{a^2}} \times \frac{1}{a}$

$= \frac{1}{b^2 + x^2} + \frac{1}{a^2 + x^2}$

(v) (d): Let $y = \sec^{-1}x + \operatorname{cosec}^{-1}\frac{x}{\sqrt{x^2 - 1}}$

Put $x = \sec\theta \Rightarrow \theta = \sec^{-1}x$

$\therefore y = \sec^{-1}(\sec\theta) + \operatorname{cosec}^{-1}\left(\frac{\sec\theta}{\sqrt{\sec^2\theta - 1}}\right)$

$= \theta + \sin^{-1}\left[\sqrt{1 - \cos^2\theta}\right]$

$= \theta + \sin^{-1}(\sin\theta) = \theta + \theta = 2\theta = 2\sec^{-1}x$

$\therefore \frac{dy}{dx} = 2\frac{d}{dx}(\sec^{-1}x) = 2 \times \frac{1}{|x|\sqrt{x^2 - 1}} = \frac{2}{|x|\sqrt{x^2 - 1}}$

14. (i) (b): $x^3 + x^2y + xy^2 + y^3 = 81$

$\Rightarrow 3x^2 + x^2\frac{dy}{dx} + 2xy + 2xy\frac{dy}{dx} + y^2 + 3y^2\frac{dy}{dx} = 0$

$\Rightarrow (x^2 + 2xy + 3y^2)\frac{dy}{dx} = -3x^2 - 2xy - y^2$

$\Rightarrow \frac{dy}{dx} = \frac{-(3x^2 + 2xy + y^2)}{x^2 + 2xy + 3y^2}$

(ii) (c): $x^y = e^{x - y} \Rightarrow y\log x = x - y$

$\Rightarrow y \times \frac{1}{x} + \log x\cdot\frac{dy}{dx} = 1 - \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}[\log x + 1] = 1 - \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{x - y}{x[1 + \log x]}$

(iii) (d): $e^{\sin y} = xy \Rightarrow \sin y = \log x + \log y$

$\Rightarrow \cos y\frac{dy}{dx} = \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} \Rightarrow \frac{dy}{dx}\left[\cos y - \frac{1}{y}\right] = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x(y\cos y - 1)}$

(iv) (d): $\sin^2 x + \cos^2 y = 1$

$\Rightarrow 2\sin x\cos x + 2\cos y\left(-\sin y\frac{dy}{dx}\right) = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-\sin 2x}{-\sin 2y} = \frac{\sin 2x}{\sin 2y}$

(v) (d): $y = (\sqrt{x})^{\sqrt{x}^{\sqrt{x}\ldots\infty}} \Rightarrow y = (\sqrt{x})^y$

$\Rightarrow \log y = y(\log\sqrt{x}) \Rightarrow \log y = \frac{1}{2}(y\log x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[y \times \frac{1}{x} + \log x\left(\frac{dy}{dx}\right)\right]$

$\Rightarrow \frac{dy}{dx}\left\{\frac{1}{y} - \frac{1}{2}\log x\right\} = \frac{1}{2}\frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{2x} \times \frac{2y}{(2 - y\log x)} = \frac{y^2}{x(2 - y\log x)}$