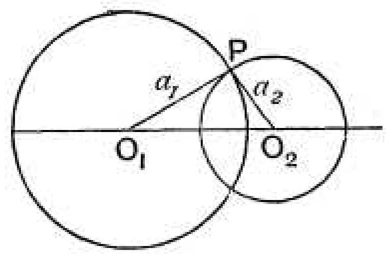
Chapter 9

SYSTEMS OF CIRCLES

182. Orthogonal Circles. Def. Two circles are said to intersect orthogonally when the tangents at their points of intersection are at right angles.



If the two circles intersect at P, the radii O_1P and O_2P, which are perpendicular to the tangents at P, must also be at right angles.

Hence $$O_1O_2^2 = O_1P^2 + O_2P^2,$$

i.e. the square of the distance between the centres must be equal to the sum of the squares of the radii.

Also the tangent from O_2 to the other circle is equal to the radius a_2, *i.e.* if two circles be orthogonal the length of the tangent drawn from the centre of one circle to the second circle is equal to the radius of the first.

Either of these two conditions will determine whether the circles are orthogonal.

The centres of the circles

$$x^2+y^2+2gx+2fy+c=0 \text{ and } x^2+y^2+2g'x+2f'y+c'=0,$$

are the points $(-g, -f)$ and $(-g', -f')$; also the squares of their radii are g^2+f^2-c and $g'^2+f'^2-c'$.

They therefore cut orthogonally if

$$(-g+g')^2+(-f+f')^2=g^2+f^2-c+g'^2+f'^2-c',$$

i.e. if $$2gg'+2ff'=c+c'.$$

183. Radical Axis. Def. The radical axis of two circles is the locus of a point which moves so that the lengths of the tangents drawn from it to the two circles are equal.

Let the equations to the circles be

$$x^2+y^2+2gx+2fy+c=0 \text{ ....(1)},$$

and $$x^2+y^2+2g_1x+2f_1y+c_1=0 \text{ ....(2)},$$

and let (x_1, y_1) be any point such that the tangents from it to these circles are equal.

By Art. 168, we have

$$x_1^2+y_1^2+2gx_1+2fy_1+c=x_1^2+y_1^2+2g_1x_1+2f_1y_1+c_1,$$

i.e. $$2x_1(g-g_1)+2y_1(f-f_1)+c-c_1=0.$$

But this is the condition that the point (x_1, y_1) should lie on the locus

$$2x(g-g_1)+2y(f-f_1)+c-c_1=0 \text{ ....(3)}.$$

This is therefore the equation to the radical axis, and it is clearly a straight line.

It is easily seen that the radical axis is perpendicular to the line joining the centres of the circles. For these centres are the points $(-g, -f)$ and $(-g_1, -f_1)$. The "m" of the line joining them is therefore $\dfrac{-f_1-(-f)}{-g_1-(-g)}$,

i.e. $\dfrac{f-f_1}{g-g_1}$.

The "m" of the line (3) is $-\dfrac{g-g_1}{f-f_1}$.

The product of these two "m's" is -1.

Hence, by Art. 69, the radical axis and the line joining the centres are perpendicular.

184. A **geometrical construction** can be given for the radical axis of two circles.

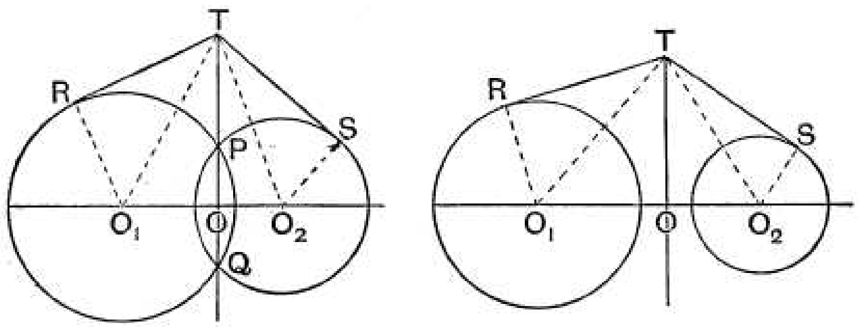


Fig. 1. Fig. 2.

If the circles intersect in real points, P and Q, as in Fig. 1, the radical axis is clearly the straight line PQ. For if T be any point on PQ and TR and TS be the tangents from it to the circles we have, by Euc. III. 36,

$$TR^2 = TP \cdot TQ = TS^2.$$

If they do not intersect in real points, as in the second figure, let their radii be a_1 and a_2, and let T be a point such that the tangents TR and TS are equal in length.

Draw TO perpendicular to O_1O_2.

Since $$TR^2 = TS^2,$$

we have $$TO_1^2 - O_1R^2 = TO_2^2 - O_2S^2,$$

i.e. $$TO^2 + O_1O^2 - a_1^2 = TO^2 + OO_2^2 - a_2^2,$$

i.e. $$O_1O^2 - OO_2^2 = a_1^2 - a_2^2,$$

i.e. $$(O_1O - OO_2)(O_1O + OO_2) = a_1^2 - a_2^2,$$

i.e. $$O_1O - OO_2 = \frac{a_1^2 - a_2^2}{O_1O_2} = \text{a constant quantity}.$$

Hence O is a fixed point, since it divides the fixed straight line O_1O_2 into parts whose difference is constant.

Therefore, since O_1OT is a right angle, the locus of T, *i.e.* the radical axis, is a fixed straight line perpendicular to the line joining the centres.

185. If the equations to the circles in Art. 183 be written in the form $S=0$ and $S'=0$, the equation (3) to the radical axis may be written $S-S'=0$, and therefore the radical axis passes through the common points, real or imaginary, of the circles $S=0$ and $S'=0$.

In the last article we saw that this was true geometrically for the case in which the circles meet in real points.

When the circles do not geometrically intersect, as in Fig. 2, we must then look upon the straight line TO as passing through the imaginary points of intersection of the two circles.

186. *The radical axes of three circles, taken in pairs, meet in a point.*

Let the equations to the three circles be

$$S=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$S'=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and

$$S''=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The radical axis of the circles (1) and (2) is the straight line

$$S-S'=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The radical axis of (2) and (3) is the straight line

$$S'-S''=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

If we add equation (5) to equation (4) we shall have the equation of a straight line through their points of intersection.

Hence

$$S-S''=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

is a straight line through the intersection of (4) and (5).

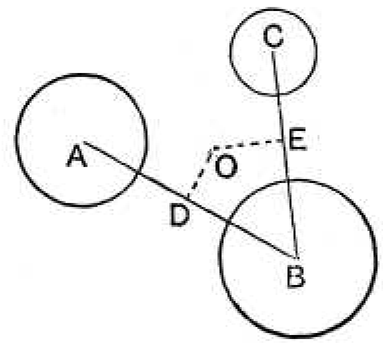
But (6) is the radical axis of the circles (3) and (1).

Hence the three radical axes of the three circles, taken in pairs, meet in a point.

This point is called the **Radical Centre** of the three circles.

This may also be easily proved geometrically. For let the three circles be called A, B, and C, and let the radical axis of A and B and that of B and C meet in a point O.

By the definition of the radical axis, the tangent from O to the circle A = the tangent from O to the circle B, and the tangent from O to the circle B = tangent from it to the circle C.



Hence the tangent from O to the circle A = the tangent from it to the circle C, *i.e.* O is also a point on the radical axis of the circles A and C.

187. *If $S=0$ and $S'=0$ be the equations of two circles, the equation of any circle through their points of intersection is $S=\lambda S'$. Also the equation to any circle, such that the radical axis of it and $S=0$ is $u=0$, is $S+\lambda u=0$.*

For wherever $S=0$ and $S'=0$ are *both* satisfied the equation $S=\lambda S'$ is clearly satisfied, so that $S=\lambda S'$ is some locus through the intersections of $S=0$ and $S'=0$.

Also in both S and S' the coefficients of x^2 and y^2 are equal and the coefficient of xy is zero. The same statement is therefore true for the equation $S=\lambda S'$. Hence the proposition.

Again, since u is only of the first degree, therefore in $S+\lambda u$ the coefficients of x^2 and y^2 are equal and the coefficient of xy is zero, so that $S+\lambda u=0$ is clearly a circle. Also it passes through the intersections of $S=0$ and $u=0$.

EXAMPLES XXIII

Prove that the following pairs of circles intersect orthogonally:

1. $x^2+y^2-2ax+c=0$ and $x^2+y^2+2by-c=0$.

2. $x^2+y^2-2ax+2by+c=0$ and $x^2+y^2+2bx+2ay-c=0$.

3. Find the equation to the circle which passes through the origin and cuts orthogonally each of the circles

$$x^2+y^2-6x+8=0 \text{ and } x^2+y^2-2x-2y=7.$$

Find the radical axis of the pairs of circles

4. $x^2+y^2=144$ and $x^2+y^2-15x+11y=0$.

5. $x^2+y^2-3x-4y+5=0$ and $3x^2+3y^2-7x+8y+11=0$.

6. $x^2+y^2-xy+6x-7y+8=0$ and $x^2+y^2-xy-4=0$, the axes being inclined at 120°.

Find the radical centre of the sets of circles

7. $x^2+y^2+x+2y+3=0, \qquad x^2+y^2+2x+4y+5=0,$
and
$$x^2+y^2-7x-8y-9=0.$$

8. $(x-2)^2+(y-3)^2=36, \qquad (x+3)^2+(y+2)^2=49,$
and
$$(x-4)^2+(y+5)^2=64.$$

9. Prove that the square of the tangent that can be drawn from any point on one circle to another circle is equal to twice the product of the perpendicular distance of the point from the radical axis of the two circles, and the distance between their centres.

10. Prove that a common tangent to two circles is bisected by the radical axis.

11. Find the general equation of all circles any pair of which have the same radical axis as the circles

$$x^2+y^2=4 \text{ and } x^2+y^2+2x+4y=6.$$

12. Find the equations to the straight lines joining the origin to the points of intersection of

$$x^2+y^2-4x-2y=4 \text{ and } x^2+y^2-2x-4y-4=0.$$

13. The polars of a point P with respect to two fixed circles meet in the point Q. Prove that the circle on PQ as diameter passes through two fixed points, and cuts both the given circles at right angles.

14. Prove that the two circles, which pass through the two points $(0, a)$ and $(0, -a)$ and touch the straight line $y=mx+c$, will cut orthogonally if $c^2=a^2(2+m^2)$.

15. Find the locus of the centre of the circle which cuts two given circles orthogonally.

16. If two circles cut orthogonally, prove that the polar of any point P on the first circle with respect to the second passes through the other end of the diameter of the first circle which goes through P.

Hence, (by considering the orthogonal circle of three circles as the locus of a point such that its polars with respect to the circles meet in a point) prove that the orthogonal circle of three circles, given by the general equation is

$$\begin{vmatrix} x+g_1, & y+f_1, & g_1x+f_1y+c_1 \\ x+g_2, & y+f_2, & g_2x+f_2y+c_2 \\ x+g_3, & y+f_3, & g_3x+f_3y+c_3 \end{vmatrix}=0.$$

ANSWERS

3. $3x^2+3y^2-8x+29y=0$. **4.** $15x-11y=144$.

5. $x+10y=2$. **6.** $6x-7y+12=0$. **7.** $(-\frac{2}{3}, -\frac{2}{3})$.

8. $(\frac{26}{25}, \frac{13}{50})$. **11.** $(\lambda+1)(x^2+y^2)+2\lambda(x+2y)=4+6\lambda$.

13. $(y-x)^2=0$.

SOLUTIONS/HINTS

1. Sum of squares of radii $=\{\sqrt{a^2-c}\}^2+\{\sqrt{b^2-c}\}^2$
$$=a^2+b^2=\{\text{join of centres}\}^2.$$

2. Sum of squares of radii
$$=\{\sqrt{a^2+b^2-c}\}^2+\{\sqrt{a^2+b^2+c}\}^2=2(a^2+b^2)$$
$$=(a+b)^2+(a-b)^2=\{\text{join of centres}\}^2.$$

3. Let $x^2+y^2+2gx+2fy=0$ be the equation of the circle. Then $-6g=8$, and $2g+2f=7$. [Art. 182.]
$$\therefore\ 2g=-\tfrac{8}{3},\ \text{and}\ 2f=\tfrac{29}{3}.$$
Hence the required circle is $3x^2+3y^2-8x+29y=0$.

4. Subtracting, $15x-11y=144$.

5. Multiply (1) by 3, subtract, and we have $x+10y=2$.

6. Subtracting, $6x-7y+12=0$.

7. The equations of two of the radical axes are $x+2y+2=0$, and $4x+5y+6=0$. Solving, $x=-\frac{2}{3}$, $y=-\frac{2}{3}$.

8. The equations of two of the radical axes are $10x+10y=13$, and $x-4y=0$. Solving, $x=\frac{26}{25}$, $y=\frac{13}{50}$.

9. Let
$$x^2+y^2-2\lambda_1 x=h^2 \quad \text{.................(i)}$$
and
$$x^2+y^2-2\lambda_2 x=h^2 \quad \text{.................(ii)}$$
be the equations of the two circles, so that the axis of y is the common radical axis, and (g, f) any point on (i), so that
$$g^2+f^2=2\lambda_1 g+h^2.$$
The square of the tangent from (g, f) to (ii)
$$=g^2+f^2-2\lambda_2 g-h^2=2(\lambda_1-\lambda_2)g. \quad \therefore \text{ etc.}$$

10. Taking a common tangent as axis of x, let

$$(x-a)^2+(y-b)^2=b^2, \text{ and } (x-c)^2+(y-d)^2=d^2$$

be the equations of the two circles.

The equation of the radical axis is

$$2(a-c)x+2(b-d)y=a^2-c^2.$$

This cuts $y=0$ where $x=\frac{1}{2}(a+c)$.

11. The equation of any circle passing through their intersection is of the form

$$x^2+y^2-4+\lambda(x^2+y^2+2x+4y-6)=0.$$

12. The equation of the common chord is $x-y=0$, which goes through the origin.

Hence the required lines are $(x-y)^2=0$.

13. Let $\qquad x^2+y^2-2\lambda_1 x+h^2=0,$

and $\qquad x^2+y^2-2\lambda_2 x+h^2=0$

be the equations of the two circles.

If the polars of $P(g, f)$ with regard to each circle pass through $Q(g', f')$, then

$$gg'+ff'-\lambda_1(g+g')+h^2=0,$$

and $$gg'+ff'-\lambda_2(g+g')+h^2=0, \quad \text{[Art. 163]},$$

whence $$g'=-g, \text{ and } f'=\frac{g^2-h^2}{f}.$$

Hence the equation of the circle on PQ as diameter is

$$(x-g)(x+g)+(y-f)\left(y-\frac{g^2-h^2}{f}\right)=0, \text{ [Art. 145]},$$

i.e. $$x^2+y^2-y\cdot\frac{f^2+g^2-h^2}{f}-h^2=0,$$

which passes through the fixed points $(\pm h, 0)$.

It is easily seen that the condition of Art. 182 for orthogonal intersection is satisfied.

14. Let $x^2+y^2-2h_1 x=a^2$, and $x^2+y^2-2h_2 x=a^2$, be the equations of the two circles.

Since $y=mx+c$ is a tangent, h_1 and h_2 are given by

$$(1+m^2)(c^2-a^2)=(mc-h)^2, \quad \text{[see Art. 153]}.$$

$$\therefore\ h_1 h_2=a^2(m^2+1)-c^2.$$

The circles cut orthogonally if $h_1 h_2 + a^2 = 0$, [Art. 182], *i.e.* if

$$a^2(m^2+2) = c^2.$$

15. If the circle $x^2+y^2+2gx+2fy+c=0$ cuts both $x^2+y^2+2g_1x+2f_1y+c_1=0$, and $x^2+y^2+2g_2x+2f_2y+c_2=0$ orthogonally, we have

$2gg_1+2ff_1=c+c_1$, and $2gg_2+2ff_2=c+c_2$. [Art. 182.]

Subtracting, we have $2g(g_1-g_2)+2f(f_1-f_2)=c_1-c_2$. Also the centre of the first circle is $(-g, -f)$. $\therefore$ the required locus is the straight line

$$2(g_1-g_2)x+2(f_1-f_2)y=c_2-c_1.$$

16. Let $\quad x^2+y^2=r^2, \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$

and $\quad x^2+y^2-2ax-2by+r^2=0 \ldots\ldots\ldots\ldots \text{(ii)}$

be the equations of two circles, which cut orthogonally.

The polar of $(r\cos\theta, r\sin\theta)$, any point on (i), with regard to (ii) is

$$xr\cos\theta+yr\sin\theta-a(x+r\cos\theta)-b(y+r\sin\theta)+r^2=0,$$

which passes through the point $(-r\cos\theta, -r\sin\theta)$.

The required equation is the condition that the lines

$$x(x'+g_1)+y(y'+f_1)+g_1x'+f_1y'+c_1=0,$$
$$x(x'+g_2)+y(y'+f_2)+g_2x'+f_2y'+c_2=0,$$
$$x(x'+g_3)+y(y'+f_3)+g_3x'+f_3y'+c_3=0,$$

should be concurrent. [See Arts. 163 and 79.]

188. Coaxal Circles. Def. A system of circles is said to be coaxal when they have a common radical axis, *i.e.* when the radical axis of each pair of circles of the system is the same.

To find the equation of a system of coaxal circles.

Since, by Art. 183, the radical axis of any pair of the circles is perpendicular to the line joining their centres, it follows that the centres of all the circles of a coaxal system must lie on a straight line which is perpendicular to the radical axis.

Take the line of centres as the axis of x and the radical axis as the axis of y (Figs. I. and II., Art. 190), so that O is the origin.

The equation to any circle with its centre on the axis of x is

$$x^2 + y^2 - 2gx + c = 0 \qquad \text{.................(1).}$$

Any point on the radical axis is $(0, y_1)$.

The square on the tangent from it to the circle (1) is, by Art. 168, $y_1^2 + c$.

Since this quantity is to be the same for all circles of the system it follows that c is the same for all such circles; the different circles are therefore obtained by giving different values to g in the equation (1).

The intersections of (1) with the radical axis are then obtained by putting $x = 0$ in equation (1), and we have

$$y = \pm\sqrt{-c}.$$

If c be negative, we have two real points of intersection as in Fig. I. of Art. 190. In such cases the circles are said to be of the Intersecting Species.

If c be positive, we have two imaginary points of intersection as in Fig. II.

189. Limiting points of a coaxal system.

The equation (1) of the previous article which gives any circle of the system may be written in the form

$$(x - g)^2 + y^2 = g^2 - c = [\sqrt{g^2 - c}]^2.$$

It therefore represents a circle whose centre is the point $(g, 0)$ and whose radius is $\sqrt{g^2 - c}$.

This radius vanishes, *i.e.* the circle becomes a point-circle, when $g^2 = c$, *i.e.* when $g = \pm\sqrt{c}$.

Hence at the particular points $(\pm\sqrt{c}, 0)$ we have point-circles which belong to the system. These point-circles are called the Limiting Points of the system.

If c be negative, these points are imaginary.

But it was shown in the last article that when c is negative the circles intersect in real points as in Fig. I., Art. 190.

If c be positive, the limiting points L_1 and L_2 (Fig. II.) are real, and in this case the circles intersect in imaginary points.

The limiting points are therefore real or imaginary according as the circles of the system intersect in imaginary or real points.

190. Orthogonal circles of a coaxal system.

Let T be *any* point on the common radical axis of a system of coaxal circles, and let TR be the tangent from it to any circle of the system.

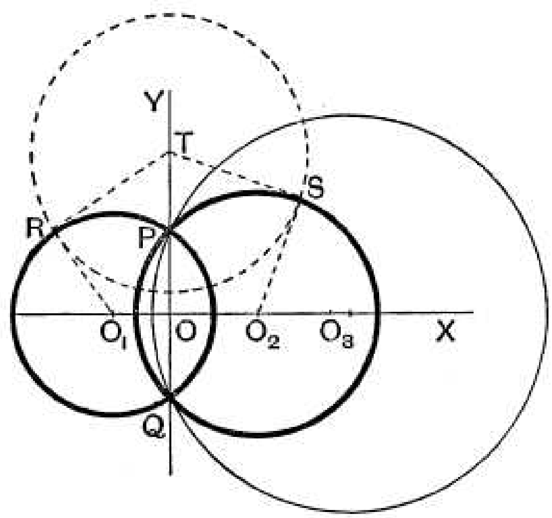


Fig. I.

Then a circle, whose centre is T and whose radius is TR, will cut each circle of the coaxal system orthogonally.

[For the radius TR of this circle is at right angles to the radius O_1R, and so for its intersection with *any* other circle of the system.]

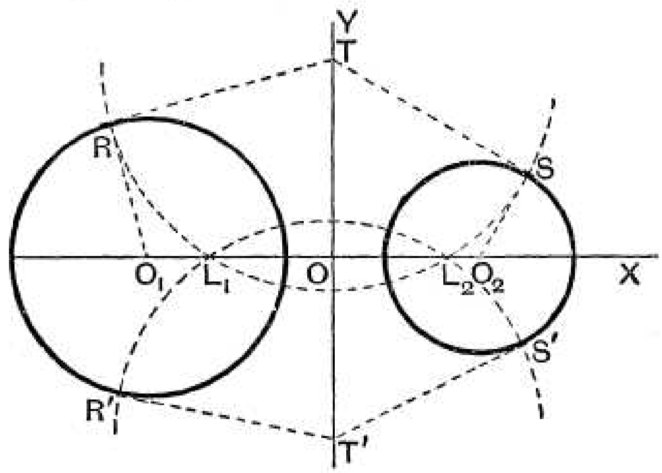


Fig. II.

Hence the limiting points (being point-*circles* of the system) are on this orthogonal circle.

The limiting points are therefore the intersections with the line of centres of *any* circle whose centre is on the common radical axis and whose radius is the tangent from it to any of the circles of the system.

Since, in Fig. I., the limiting points are imaginary these orthogonal circles do not meet the line of centres in real points.

In Fig. II. they pass through the limiting points L_1 and L_2.

These orthogonal circles (since they all pass through two points, real or imaginary) are therefore a coaxal system.

Also if the original circles, as in Fig. I., intersect in real points, the orthogonal circles intersect in imaginary points; in Fig. II. the original circles intersect in imaginary points, and the orthogonal circles in real points.

We therefore have the following theorem:

A set of coaxal circles can be cut orthogonally by another set of coaxal circles, the centres of each set lying on the radical axis of the other set; also one set is of the limiting-point species and the other set of the other species.

191. Without reference to the limiting points of the original system, it may be easily found whether or not the orthogonal circles meet the original line of centres.

For the circle, whose centre is T and whose radius is TR, meets or does not meet the line O_1O_2 according as TR^2 is $>$ or $<$ TO^2,

i.e. according as $\quad TO_1^2 - O_1R^2$ is $\gtrless TO^2$,

i.e. according as $\quad TO^2 + OO_1^2 - O_1R^2$ is $\gtrless TO^2$,

i.e. according as $\quad OO_1$ is $\gtrless O_1R$,

i.e. according as the radical axis is without, or within, each of the circles of the original system.

192. In the next article the above results will be proved analytically.

To find the equation to any circle which cuts two given circles orthogonally.

Take the radical axis of the two circles as the axis of y, so that their equations may be written in the form

$$x^2 + y^2 - 2gx + c = 0 \quad \text{.................(1)},$$

and

$$x^2 + y^2 - 2g_1x + c = 0 \quad \text{..............(2)},$$

the quantity c being the same for each.

Let the equation to any circle which cuts them orthogonally be

$$(x - A)^2 + (y - B)^2 = R^2 \quad \text{...............(3)}.$$

The equation (1) can be written in the form

$$(x - g)^2 + y^2 = [\sqrt{g^2 - c}]^2 \quad \text{...............(4)}.$$

The circles (3) and (4) cut orthogonally if the square of the distance between their centres is equal to the sum of the squares of their radii,

i.e. if $\quad (A - g)^2 + B^2 = R^2 + [\sqrt{g^2 - c}]^2$,

i.e. if $\quad A^2 + B^2 - 2Ag = R^2 - c \quad$ ....(5).

Similarly, (3) will cut (2) orthogonally if

$$A^2 + B^2 - 2Ag_1 = R^2 - c \quad \text{...............(6)}.$$

Subtracting (6) from (5), we have $A(g - g_1) = 0$.

Hence $A = 0$, and $R^2 = B^2 + c$.

Substituting these values in (3), the equation to the required orthogonal circle is

$$x^2 + y^2 - 2By - c = 0 \quad \text{.................(7)},$$

where B is any quantity whatever.

Whatever be the value of B the equation (7) represents a circle whose centre is on the axis of y and which passes through the points $(\pm\sqrt{c}, 0)$.

But the latter points are the limiting points of the coaxal system to which the two circles belong. [Art. 189.]

Hence any pair of circles belonging to a coaxal system is cut at right angles by any circle of another coaxal system; also the centres of the circles of the latter system lie on the common radical axis of the original system, and all the circles of the latter system pass through the limiting points (real or imaginary) of the first system.

Also the centre of the circle (7) is the point $(0, B)$ and its radius is $\sqrt{B^2 + c}$.

The square of the tangent drawn from $(0, B)$ to the circle (1) $= B^2 + c$ (by Art. 168).

Hence the radius of any circle of the second system is equal to the length of the tangent drawn from its centre to any circle of the first system.

193. The equation to the system of circles which cut a given coaxal system orthogonally may also be obtained by using the result of Art. 182.

For any circle of the coaxal system is, by Art. 188, given by

$$x^2 + y^2 - 2gx + c = 0 \quad \text{..................(1)},$$

where c is the same for all circles.

Any point on the radical axis is $(0, y')$.

The square on the tangent drawn from it to (1) is therefore $y'^2 + c$.

The equation to *any* circle cutting (1) orthogonally is therefore

$$x^2 + (y - y')^2 = y'^2 + c,$$

i.e.

$$x^2 + y^2 - 2yy' - c = 0.$$

Whatever be the value of y' this circle passes through the points $(\pm\sqrt{c}, 0)$, *i.e.* through the limiting points of the system of circles given by (1).

194. We can now deduce an easy construction for the circle that cuts any three circles orthogonally.

Consider the three circles in the figure of Art. 186.

By Art. 192 any circle cutting A and B orthogonally has its centre on their common radical axis, *i.e.* on the straight line OD.

Similarly any circle cutting B and C orthogonally has its centre on the radical axis OE.

Any circle cutting all three circles orthogonally must therefore have its centre at the intersection of OD and OE, *i.e.* at the radical centre O. Also its radius must be the length of the tangent drawn from the radical centre to any one of the three circles.

Ex. *Find the equation to the circle which cuts orthogonally each of the three circles*

$$x^2+y^2+2x+17y+4=0 \quad\ldots\ldots\ldots\ldots (1),$$

$$x^2+y^2+7x+6y+11=0 \quad\ldots\ldots\ldots\ldots (2),$$

$$x^2+y^2-x+22y+3=0 \quad\ldots\ldots\ldots\ldots (3).$$

The radical axis of (1) and (2) is

$$5x-11y+7=0.$$

The radical axis of (2) and (3) is

$$8x-16y+8=0.$$

These two straight lines meet in the point (3, 2) which is therefore the radical centre.

The square of the length of the tangent from the point (3, 2) to each of the given circles $=57$.

The required equation is therefore $(x-3)^2+(y-2)^2=57$,

i.e. $$x^2+y^2-6x-4y-44=0.$$

195. Ex. *Find the locus of a point which moves so that the length of the tangent drawn from it to one given circle is λ times the length of the tangent from it to another given circle.*

As in Art. 188 take as axes of x and y the line joining the centres of the two circles and the radical axis. The equations to the two circles are therefore

$$x^2+y^2-2g_1x+c=0 \quad\ldots\ldots\ldots\ldots (1),$$

and $$x^2+y^2-2g_2x+c=0 \quad\ldots\ldots\ldots\ldots (2).$$

Let (h, k) be a point such that the length of the tangent from it to (1) is always λ times the length of the tangent from it to (2).

Then $$h^2+k^2-2g_1h+c=\lambda^2[h^2+k^2-2g_2h+c].$$

Hence (h, k) always lies on the circle

$$x^2+y^2-2x\frac{g_2\lambda^2-g_1}{\lambda^2-1}+c=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

This circle is clearly a circle of the coaxal system to which (1) and (2) belong.

Again, the centre of (1) is the point $(g_1, 0)$, the centre of (2) is $(g_2, 0)$, whilst the centre of (3) is $\left(\frac{g_2\lambda^2-g_1}{\lambda^2-1}, 0\right)$.

Hence, if these three centres be called O_1, O_2, and O_3, we have

$$O_1O_3=\frac{g_2\lambda^2-g_1}{\lambda^2-1}-g_1=\frac{\lambda^2}{\lambda^2-1}(g_2-g_1),$$

and $$O_2O_3=\frac{g_2\lambda^2-g_1}{\lambda^2-1}-g_2=\frac{1}{\lambda^2-1}(g_2-g_1),$$

so that $O_1O_3 : O_2O_3 :: \lambda^2 : 1$.

The required locus is therefore a circle coaxal with the two given circles and whose centre divides externally, in the ratio $\lambda^2 : 1$, the line joining the centres of the two given circles.

EXAMPLES XXIV

1. Prove that a common tangent to two circles of a coaxal system subtends a right angle at either limiting point of the system.

2. Prove that the polar of a limiting point of a coaxal system with respect to any circle of the system is the same for all circles of the system.

3. Prove that the polars of any point with respect to a system of coaxal circles all pass through a fixed point, and that the two points are equidistant from the radical axis and subtend a right angle at a limiting point of the system. If the first point be one limiting point of the system prove that the second point is the other limiting point.

4. A fixed circle is cut by a series of circles all of which pass through two given points; prove that the straight line joining the intersections of the fixed circle with any circle of the system always passes through a fixed point.

5. Prove that tangents drawn from any point of a fixed circle of a coaxal system to two other fixed circles of the system are in a constant ratio.

6. Prove that a system of coaxal circles inverts with respect to either limiting point into a system of concentric circles and find the position of the common centre.

7. A straight line is drawn touching one of a system of coaxal circles in P and cutting another in Q and R. Shew that PQ and PR subtend equal or supplementary angles at one of the limiting points of the system.

8. Find the locus of the point of contact of parallel tangents which are drawn to each of a series of coaxal circles.

9. Prove that the circle of similitude of the two circles

$$x^2+y^2-2kx+\delta=0 \text{ and } x^2+y^2-2k'x+\delta=0$$

(*i.e.* the locus of the points at which the two circles subtend the same angle) is the coaxal circle

$$x^2+y^2-2\frac{kk'+\delta}{k+k'}x+\delta=0.$$

10. From the preceding question shew that the centres of similitude (*i.e.* the points in which the common tangents to two circles meet the line of centres) divide the line joining the centres internally and externally in the ratio of the radii.

11. If $x+y\sqrt{-1}=\tan(u+v\sqrt{-1})$, where x, y, u, and v are all real, prove that the curves $u=$constant give a family of coaxal circles passing through the points $(0, \pm 1)$, and that the curves $v=$constant give a system of circles cutting the first system orthogonally.

12. Find the equation to the circle which cuts orthogonally each of the circles

$$x^2+y^2+2gx+c=0, \qquad x^2+y^2+2g'x+c=0,$$

and $$x^2+y^2+2hx+2ky+a=0.$$

13. Find the equation to the circle cutting orthogonally the three circles

$$x^2+y^2=a^2, \quad (x-c)^2+y^2=a^2, \quad \text{and} \quad x^2+(y-b)^2=a^2.$$

14. Find the equation to the circle cutting orthogonally the three circles

$$x^2+y^2-2x+3y-7=0, \qquad x^2+y^2+5x-5y+9=0,$$

and $$x^2+y^2+7x-9y+29=0.$$

15. Shew that the equation to the circle cutting orthogonally the circles

$$(x-a)^2+(y-b)^2=b^2, \qquad (x-b)^2+(y-a)^2=a^2,$$

and $$(x-a-b-c)^2+y^2=ab+c^2,$$

is $$x^2+y^2-2x(a+b)-y(a+b)+a^2+3ab+b^2=0.$$

ANSWERS

8. $x^2 - y^2 + 2mxy = c$.

12. $k(x^2 + y^2) + (a - c)y - ck = 0$.

13. $x^2 + y^2 - cx - by + a^2 = 0$.

14. $x^2 + y^2 - 16x - 18y - 4 = 0$.

SOLUTIONS/HINTS

1. Let $(0, k)$ be any point O on the radical axis of the two circles $x^2 + y^2 - 2g_1x + c = 0$, and $x^2 + y^2 - 2g_2x + c = 0$. $\{\text{Tangent from } O \text{ to either circle}\}^2 = k^2 + c = OL_1^2$, where L_1 is a limiting point.

Taking O at the middle point of a common tangent, we have $OL_1 = \frac{1}{2}$ (either common tangent). $\therefore$ a common tangent subtends a right angle at L_1.

2. Let $x^2 + y^2 - 2\lambda x + c = 0$, for different values of λ, represent the coaxal system. (Art. 188.) The polar of $(\sqrt{c}, 0)$ is $x\sqrt{c} - \lambda(x + \sqrt{c}) + c = 0$, or $x + \sqrt{c} = 0$.

3. Let $x^2 + y^2 - 2\lambda x + c = 0$ represent the system. The polar of (h, k) is $xh + yk + c - \lambda(x + h) = 0$, which always passes through the intersection of the fixed lines $x + h = 0$, and $xh + yk + c = 0$, *i.e.* through the point

$$\left\{-h, \ \frac{h^2 - c}{k}\right\}.$$

Hence the two points are equally distant from the axis of y, and subtend a right angle at $(\sqrt{c}, 0)$, since

$$\frac{k}{h - \sqrt{c}} \cdot \frac{\frac{h^2 - c}{k}}{-h - \sqrt{c}} = -1.$$

If one point is $(\sqrt{c}, 0)$, the other is $(-\sqrt{c}, 0)$.

4. Let $x^2 + y^2 - 2\lambda x + a = 0$ be the equation of the coaxal system, and $x^2 + y^2 + 2gx + 2fy + c = 0$ that of the fixed circle.

The equation of a common chord is

$$2(g + \lambda)x + 2fy + c - a = 0,$$

which always passes through the intersection of

$$2gx + 2fy + c - a = 0 \quad \text{and} \quad x = 0,$$

i.e. through a point whose position does not depend on λ.

5. Let
$$x^2+y^2-2\lambda x+c=0, \quad \ldots\ldots\ldots\ldots\text{(i)}$$
$$x^2+y^2-2\lambda_1 x+c=0, \quad \ldots\ldots\ldots\ldots\text{(ii)}$$
$$x^2+y^2-2\lambda_2 x+c=0, \quad \ldots\ldots\ldots\ldots\text{(iii)}$$
be the equation of the three circles.

If (h, k) lies on (i), $h^2+k^2+c=2\lambda h$. The ratio of the squares of tangents from (h, k) to (ii) and (iii)
$$=\frac{h^2+k^2+c-2\lambda_1 h}{h^2+k^2+c-2\lambda_2 h}=\frac{2h(\lambda-\lambda_1)}{2h(\lambda-\lambda_2)}=\frac{\lambda-\lambda_1}{\lambda-\lambda_2}.$$

6. Take $x^2+y^2-2\lambda x+c=0$ for the equation of the system. Remove the origin to $(\sqrt{c}, 0)$, and the equation is
$$(x+\sqrt{c})^2+y^2-2\lambda(x+\sqrt{c})+c=0.$$

Change to polars; $\therefore\ r^2+2(\sqrt{c}-\lambda)(r\cos\theta+\sqrt{c})=0.$

Inverting, *i.e.* putting $\frac{a^2}{r}$ for r, the inverse is
$$a^4+2(\sqrt{c}-\lambda)(a^2 r\cos\theta+\sqrt{c}\,.\,r^2)=0,$$
which for different values of λ represents a concentric system of circles whose common centre is $\left(-\frac{a^2}{2\sqrt{c}}, 0\right)$.

7. Let P be on the circle $x^2+y^2-2\lambda_1 x+c=0$, and Q and R on the circle $x^2+y^2-2\lambda_2 x+c=0$. Let (h, k) be the coordinates of Q, so that $h^2+k^2+c=2\lambda_2 h$. If L_1 is a limiting point, then, since QP is a tangent at P to the first circle,
$$\therefore\ \frac{QP^2}{QL_1^{\,2}}=\frac{h^2+k^2-2\lambda_1 h+c}{(h-\sqrt{c})^2+k^2}=\frac{2h(\lambda_2-\lambda_1)}{2h(\lambda_2-\sqrt{c})}=\frac{\lambda_2-\lambda_1}{\lambda_2-\sqrt{c}}=\frac{RP^2}{RL_1^{\,2}},$$
similarly. $\therefore\ \frac{RP}{QP}=\pm\frac{RL_1}{QL_1}$; $\therefore\ PL_1$ bisects one or other of the angles between L_1Q and L_1R.

8. Let $x^2+y^2-2\lambda x+c^2=0$ be any circle of the system. The tangent at (h, k) is
$$xh+yk-\lambda(x+h)+c^2=0.$$
This will be parallel to $y=mx+d$ if $\lambda-h=km$.

But since (h, k) lies on the circle,

$$\therefore\ h^2+k^2+c^2=2\lambda h. \quad \therefore\ h^2+k^2+c^2=2h(h+km).$$

Hence the required locus is $x^2-y^2+2mxy=c^2$.

9. If (x, y) be any point on the circle of similitude, the tangents from (x, y) to the two circles are in the same ratio as the radii.

$$\therefore\ \frac{x^2+y^2-2kx+\delta}{x^2+y^2-2k'x+\delta}=\frac{k^2-\delta}{k'^2-\delta}. \quad \therefore\ x^2+y^2-2\frac{kk'+\delta}{k+k'}x+\delta=0.$$

10. Putting $y=0$,

$$x=\frac{kk'+\delta\pm\sqrt{(kk'+\delta)^2-\delta(k+k')^2}}{k+k'}=\frac{kk'+\delta\pm\sqrt{(k^2-\delta)(k'^2-\delta)}}{k+k'}$$

and $\dfrac{k\sqrt{k'^2-\delta}\pm k'\sqrt{k^2-\delta}}{\sqrt{k'^2-\delta}\pm\sqrt{k^2-\delta}}=$ the same expression, as would appear if the denominator were rationalized.

11. Since $x+yi=\tan(u+vi)$, $\therefore\ x-yi=\tan(u-vi)$.

$$\therefore\ \tan 2u=\tan[(u+vi)+(u-vi)]=\frac{(x+yi)+(x-yi)}{1-(x+yi)(x-yi)}$$

$$=\frac{2x}{1-(x^2+y^2)},$$

so that $$x^2+y^2+2x\cot 2u=1, \quad \ldots\ldots\ldots\ldots\text{(i)}$$

so $$\tan 2vi=\tan[(u+vi)-(u-vi)]=\frac{(x+yi)-(x-yi)}{1+(x+yi)(x-yi)}$$

$$\therefore\ i\tanh 2v=\frac{2yi}{1+(x^2+y^2)}.$$

$$\therefore\ x^2+y^2-2y\coth 2v=-1. \quad \ldots\ldots\ldots\text{(ii)}$$

The curve (i) for any constant value of u gives a circle through the points $(0, \pm 1)$. Also (ii) is similarly another set of coaxal circles through the imaginary points $(\pm\sqrt{-1}, 0)$. Also clearly (i) and (ii) satisfy the condition of Art. 182.

12. The equation of any circle cutting the first two circles orthogonally is, by Art. 192,

$$x^2 + y^2 - 2\lambda y - c = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

This cuts the third orthogonally if $-2\lambda k = a - c$. [Art. 182.]
Hence (i) becomes $k(x^2 + y^2) + (a - c)y - ck = 0$.

Aliter. The radical axis of the first pair of circles is $x = 0$, and that of the first and third is

$$(2g - 2h)x - 2ky + c - a = 0,$$

so that the radical centre is $\left(0, \frac{c-a}{2k}\right)$.

The equation to the circle required is therefore

$$(x - 0)^2 + \left(y - \frac{c-a}{2k}\right)^2 = \text{square of the tangent from the radical centre to either circle}$$

$$= 0 + \left(\frac{c-a}{2k}\right)^2 + 0 + c,$$

i.e. $$(x^2 + y^2)k + (a - c)y - ck = 0.$$

13. Let (h, k) be the centre and r the radius of the required circle.

Then $a^2 + r^2 = h^2 + k^2 = (h - c)^2 + k^2 = h^2 + (k - b)^2$,

whence $2h = c$, $2k = b$, and $r^2 = \frac{c^2}{4} + \frac{b^2}{4} - a^2$.

The required circle is thus $x^2 + y^2 - cx - by + a^2 = 0$.

14. Two of the radical axes are $3x - 4y + 12 = 0$, and $x - 2y + 10 = 0$. Whence, solving, the radical centre is $(8, 9)$.

The length of tangent to either circle from this point $= \sqrt{149}$. $\therefore$ the required equation is $(x - 8)^2 + (y - 9)^2 = 149$.

15. The radical axis of (i) and (iii) is

$$2(b + c)x - 2by = (b + 2c)(a + b),$$

and that of (ii) and (iii) is

$$2(a + c)x - 2ay = (a + 2c)(a + b).$$

Whence the centre is $\left(a + b, \frac{a+b}{2}\right)$, and radius $= \frac{a-b}{2}$ easily. Hence the equation required.