Chapter 4

Determinants

Types of Determinants and Their Properties

F. Determinant

(i) Submatrix : Let A be a given matrix. The matrix obtained by deleting some rows or columns of A is called as submatrix of A.

e.g. $A = \begin{bmatrix} a & b & c & d \\ x & y & z & w \\ p & q & r & s \end{bmatrix}$

Then $\begin{bmatrix} a & c \\ x & z \\ p & r \end{bmatrix}, \begin{bmatrix} a & b & d \\ p & q & s \end{bmatrix}, \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$ are all submatrices of A.

(ii) Determinant of A Square Matrix :

Let A $[a]_{1 \times 1}$ be a 1 x 1 matrix. Determinant A is defined as $|A| = a$ eg. $A = [-3]_{1 \times 1}$ $|A| = -3$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,then |A| is defined as ad –bc.

(iii)Minors & Cofactors : Let Δ be a determinant. Then minor of element a_{ij}, denoted by M_{ij} is defined as the determinant of the submatrix obtained by deleting i^{th} row & j^{th} column of Δ.

Cofactor of element a_{ij}, denoted by C_{ij} is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

eg. $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \Rightarrow M_{11} = s = C_{11}$

$M_{12} = c, C_{12} = -c$; $M_{21} = b, C_{21} = -b$; $M_{22} = a = C_{22}$

eg. $\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \Rightarrow M_{11} = \begin{vmatrix} p & r \\ y & z \end{vmatrix} = qz - yr = C_{11}$;

$M_{23} = \begin{vmatrix} a & b \\ x & y \end{vmatrix} = ay - bx$, $C_{23} = -(ay - bx) = bx - ay$ etc.

(iv)Determinant : Let $A = [a_{ij}]_n$ be a square matrix ($n > 1$). Determinant of A is defined as the sum of products of elements of any one row (or one column) with corresponding cofactors.

eg. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ (using first row)

$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$|A| = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$ (using second column)

$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

G. PROPERTIES OF DETERMINANTS

P- 1 : The value of a determinant remains unaltered , if the rows & columns are inter changed . e.g. If

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D' \Rightarrow D$ & D' are transpose of each other.

If $D' = -D$ then it is SKEW SYMMETRIC determinant but $D' = D \Rightarrow 2D = 0 \Rightarrow D = 0 \Rightarrow$ Skew symmetric determinant of third order has the value zero .

P-2 : If any two rows (or columns) of a determinant be interchanged , the value of determinant is changed in sign only . e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = -D$.

P-3 : If a determinant has any two rows (or columns) identical , then its value is zero.

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then it can be verified that $D = 0$.

P-4 : If all the elements of any row (or column) be multiplied by the same number then the determinant is multiplied by that number.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D' = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = KD$

P-5 : If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants.

e.g. $\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

P-6 : The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}$. Then $D' = D$.

Note that while applying this property atleast one row (or column) must remain unchanged.

P- 7 : If by putting x = a the value of a determinant vanishes then (x-a) is a factor of the determinant

Ex.17 Find the value of the determinant

$$\begin{vmatrix} {}^nC_{r-1} & {}^nC_r & (r+1)^{n+2}C_{r+1} \\ {}^nC_r & {}^nC_{r+1} & (r+2)^{n+2}C_{r+2} \\ {}^nC_{r+1} & {}^nC_{r+2} & (r+3)^{n+2}C_{r+3} \end{vmatrix}$$

Sol.

Operating $C_1 \to C_1 + C_2$ and using ${}^{n}C_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$ in C_3, we get

$$\begin{vmatrix} {}^{n+1}C_r & {}^{n}C_r & (n+2)\,{}^{n+1}C_r \\ {}^{n+1}C_{r+1} & {}^{n}C_{r+1} & (n+2)\,{}^{n+1}C_{r+1} \\ {}^{n+1}C_{r+2} & {}^{n}C_{r+2} & (n+2)\,{}^{n+1}C_{r+2} \end{vmatrix} = 0, \text{ as } C_1 \text{ and } C_3 \text{ are identical.}$$

Ex.18 **A is a n × n matrix (n > 2) [a_{ij}] where** $a_{ij} = \cos\left(\frac{(i+j)2\pi}{n}\right)$ **Find determinant A.**

Sol.

$$\Delta = \begin{vmatrix} \cos\frac{4\pi}{n} & \cos\frac{6\pi}{n} & \ldots\ldots & \cos\frac{(n+1)2\pi}{n} \\ \cos\frac{6\pi}{n} & \cos\frac{8\pi}{n} & \ldots\ldots & \cos\frac{(n+2)2\pi}{n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \cos\frac{(n+1)2\pi}{n} & \cos\frac{(n+2)2\pi}{n} & \ldots\ldots & \cos\frac{(n+n)2\pi}{n} \end{vmatrix}$$

$$= \begin{vmatrix} \sum_{j=1}^{n}\cos(j+1)\frac{2\pi}{n} & \cos\frac{6\pi}{n} & \ldots\ldots & \cos\frac{(n+1)2\pi}{n} \\ \sum_{j=1}^{n}\cos(j+2)\frac{2\pi}{n} & \cos\frac{8\pi}{n} & \ldots\ldots & \cos\frac{(n+2)2\pi}{n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \sum_{j=1}^{n}\cos(j+n)\frac{2\pi}{n} & \cos\frac{(n+2)2\pi}{n} & \ldots\ldots & \cos\frac{(n+n)2\pi}{n} \end{vmatrix}$$

Now, $\sum_{j=1}^{n}\cos(j+1)\frac{2\pi}{n} = \sum_{j=1}^{n}\cos(j+2)\frac{2\pi}{n} = \ldots\ldots = \sum_{j=1}^{n}\cos(j+n)\frac{2\pi}{n}$

$= 1 + \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \ldots. + \cos 2(n-1)\frac{\pi}{n} = 0$

⇒ value of determinant is zero.

H. MULTIPLICATION OF TWO DETERMINANTS

(i)

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$$

Similarly two determinants of order three are multiplied.

(ii) If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ then $D^2 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$ where A_i , B_i , C_i are cofactors

PROOF : Consider $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{vmatrix}$

Note : $a_1A_2 + b_1B_2 + c_1C_2 = 0$ etc.

therefore $D \times \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = D^3$

$$\Rightarrow \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = D^2 \ \textbf{or} \ \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ CA_3 & B_3 & C_3 \end{vmatrix} = D^2$$

Ex.19 Prove that

$$\Delta \equiv \begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & ca+bd & c^2a^2+b^2d^2 \\ 1 & ab+cd & a^2b^2+c^2d^2 \end{vmatrix} = (a - b)\ (a - c)\ (a - d)\ (b - c)\ (b - d)\ (c - d).$$

Sol.

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & (a-b)(c-d) & (a^2-b^2)(c^2-d^2) \\ 1 & (a-c)(b-d) & (a^2-c^2)(b^2-d^2) \end{vmatrix}$$

$$= \begin{vmatrix} (a-b)(c-d) & (a-b)(a+b)(c-d)(c+d) \\ (a-c)(b-d) & (a-c)(a+c)(b-d)(b+d) \end{vmatrix}$$

$$=(a - b)\ (c - d)\ (a - c)\ (b - d) \begin{vmatrix} 1 & (a+b)(c+d) \\ 1 & (a+c)(b+d) \end{vmatrix}$$

$= (a - b)\ (c - d)\ (a - c)\ (b - d)\ [(a + c)\ (b + d) - (a + b)\ (c + d)]$

$= (a - b)\ (c - d)\ (a - c)\ (b - d)\ (ab + cd - ac - bd) = (a - b)\ (a - c)\ (a - d)\ (b - c)\ (b - d)\ (c - d)$.

Alternatively :

Let $\left.\begin{matrix} bc+ad = x \\ ca+bd = y \\ ab+cd = z \end{matrix}\right]$ and using $c_3 \rightarrow c_3 + 2\,a\,b\,c\,d\,.\,c_3$

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x).$$

Ex.20 Show that

Show that $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-y^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$ (where $r^2 = x^2 + y^2 + z^2$ & $u^2 = x\,y + y\,z + z\,x$)

Sol. Consider the determinant , $\Delta = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$ We see that the L.H.S. determinant has its constituents
which are the co-factor of Δ. Hence L.H.S. determinant

$$= \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

$$= \begin{vmatrix} x^2+y^2+z^2 & xy+yz+zx & xy+yz+zx \\ xy+yz+zx & y^2+z^2+x^2 & yz+zx+xy \\ zx+xy+yz & yz+xz+xy & z^2+x^2+y^2 \end{vmatrix}$$

$$= \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$$

Ex.21 Without expanding, as for as possible, prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$$

Sol.

Let $D = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$ for x = y, D = 0 (since C_1 and C_2 are identical)

Hence (x – y) is a factor of D (y – z) and (z – x) are factors of D. But D is a homogeneous expression of the 4th degree is x, y, z.
∴ There must be one more factor of the 1st degree in x, y, z say k (x + y + z) where k is a constant.
Let D = k (x – y) (y – z) (z – x) (x + y + z),Putting x = 0, y = 1, z = 2

then $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 8 \end{vmatrix}$ = k (0 – 1) (1 – 2) (2 – 0) (0 + 1 + 2)

⇒ L(8 – 2) = k(–1) (–1) (2) (3) ∴ k = 1 ∴ D = (x - y) (y - z) (z - x) (x + y + z)

Ex.22 Prove that $\begin{vmatrix} x_1 & y_1 & 1 \\ ax_1+bx_2+cx_3 & ay_1+by_2+cy_3 & a+b+c \\ -ax_1+bx_2+cx_3 & -ay_1+by_2+cy_3 & -a+b+c \end{vmatrix} = 0.$

Sol.

Given that $\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ ax_1+bx_2+cx_3 & ay_1+by_2+cy_3 & a+b+c \\ -ax_1+bx_2+cx_3 & -ay_1+by_2+cy_3 & -a+b+c \end{vmatrix} = 0$

$= \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ -a & b & c \end{vmatrix}$

$= \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} \times 0 = 0.$

Ex.23 Express $\begin{vmatrix} (1+ax)^2 & (1+ay)^2 & (1+az)^2 \\ (1+bx)^2 & (1+by)^2 & (1+bz)^2 \\ (1+cx)^2 & (1+cy)^2 & (1+cz)^2 \end{vmatrix}$ **as product of two determinants.**

Sol. The given determinant is

$\begin{vmatrix} 1+2ax+a^2x^2 & 1+2ay+a^2y^2 & 1+2az+a^2z^2 \\ 1+2bx+b^2x^2 & 1+2by+b^2y^2 & 1+2bz+b^2z^2 \\ 1+2cx+c^2x^2 & 1+2cy+c^2y^2 & 1+2cz+c^2z^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix} \times \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix},$$

with the help of row–by–row multiplication rule.

Ex.24

$$\text{Let } D = \begin{vmatrix} 2a_1b_1 & a_1b_2 + a_2b_1 & a_1b_3 + a_3b_1 \\ a_1b_2 + a_2b_1 & 2a_2b_2 & a_2b_3 + a_3b_2 \\ a_1b_3 + a_3b_1 & a_3b_2 + a_2b_3 & 2a_3b_3 \end{vmatrix}$$

Express the determinant D as a product of two determinants. Hence or otherwise show that D = 0.

Sol.

$$\text{We have } D = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} b_1 & a_1 & 0 \\ b_2 & a_2 & 0 \\ b_3 & a_3 & 0 \end{vmatrix}$$

as can be seen by applying row–by–row multiplication rule., Hence D = 0.

Determinant- Formulas

1.The symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called the determinant of order two.
It's value is given by : $D = a_1 b_2 - a_2 b_1$

2.The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called the determinant of order three.

Its value can be found as : $D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

Or

$D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ .... and so on. In this manner we can expand a determinant in 6 ways using elements of ; R_1, R_2, R_3 or C_1, C_2, C_3.

3. Following examples of short hand writing large expressions are :

(i) The lines : $a_1x + b_1y + c_1 = 0$........ (1)
$a_2x + b_2y + c_2 = 0$........ (2)
$a_3x + b_3y + c_3 = 0$........ (3)

are concurrent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$. Condition for the consistency of three simultaneous linear equations in 2 variables.

(ii) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

(iii) Area of a triangle whose vertices are (x_r, y_r) ; $r = 1, 2, 3$ is : $D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ If $D = 0$ then the three points are collinear.

(iv) Equation of a straight line passing through (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

4. MINORS : The minor of a given element of a determinant is the determinant of the elements which remain after deleting the row & the column in which the given element stands For example, the minor of a_1 in (Key Concept 2) is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ & the minor of b_2 is

$\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$. Hence a determinant of order two will have "4 minors" & a determinant of order three will have "9 minors" .

5. COFACTOR : If M_{ij} represents the minor of some typical element then the cofactor is defined as :
$C_{ij} = (-1)^{i+j} \cdot M_{ij}$; Where i & j denotes the row & column in which the particular element lies. Note that the value of a determinant of order three in terms of 'Minor' & 'Cofactor' can be written as : $D = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$ OR $D = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ & so on ....

6. PROPERTIES OF DETERMINANTS :

P−1 : The value of a determinant remains unaltered, if the rows & columns are inter changed. e.g. if D = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$ D′ D & D′ are transpose of each other. If D′ = − D then it is SKEW SYMMETRIC determinant but D′ = D ⇒ 2 D = 0 ⇒ D = 0 ⇒ Skew symmetric determinant of third order has the value zero.

P−2 : If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let: $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then D′ = − D.

P−3 : If a determinant has any two rows (or columns) identical, then its value is zero. e.g. Let D = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then it can be verified that D = 0.

P−4 : If all the elements of any row (or column) be multiplied by the same number, then the determinant is multiplied by that number.

e.g. If D $= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then D′= KD

P−5 : If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants.

e.g. $\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

P−6 : The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).e.g. Let D $= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}$.

Then $D' = D$.
Note : that while applying this property ATLEAST ONE ROW (OR COLUMN) must remain unchanged.
P−7 : If by putting $x = a$ the value of a determinant vanishes then $(x - a)$ is a factor of the determinant.

7. MULTIPLICATION OF TWO DETERMINANTS:

(i) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$ Similarly two determinants of order three are multiplied.

(ii) If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ then , $D^2 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$ where A_i, B_i, C_i are cofactors

PROOF : Consider $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{vmatrix}$ Note : $a_1A_2 + b_1B_2 + c_1C_2 = 0$

etc. therefore, $D \times \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = D^3 \Rightarrow \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = D^2$ OR $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ CA_3 & B_3 & C_3 \end{vmatrix} = D^2$

8. SYSTEM OF LINEAR EQUATION (IN TWO VARIABLES) :

(i) Consistent Equations : Definite & unique solution. [intersecting lines]
(ii) Inconsistent Equation : No solution. [Parallel line]
(iii) Dependent equation : Infinite solutions. [Identical lines]
Let $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ then :
$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow$ Given equations are inconsistent & $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ Given equations are dependent

9. CRAMER'S RULE : [SIMULTANEOUS EQUATIONS INVOLVING THREE UNKNOWNS]

Let, $a_1x + b_1y + c_1z = d_1$ ...(I) ; $a_2x + b_2y + c_2z = d_2$ ... (II) ; $a_3x + b_3y + c_3z = d_3$ ...

(III)

Then, $x = \frac{D_1}{D}$, $Y = \frac{D_2}{D}$, $Z = \frac{D_3}{D}$.

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$; $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

NOTE : (a) If $D \neq 0$ and alteast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations are consistent and have unique non trivial solution.

(b) If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have trivial solution only.

(c) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations are consistentand have infinite solutions. In case $\left.\begin{matrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{matrix}\right\}$ represents these parallel planes then also $D = D_1 = D_2 = D_3 = 0$ but the system is inconsistent.

(d) If $D = 0$ but at least one of D_1, D_2, D_3 is not zero then the equations are in consistent and have no solution.

10. If x, y, z are not all zero, the condition for $a_1x + b_1y + c_1z = 0$; $a_2x + b_2y + c_2z = 0$ & $a_3x + b_3y + c_3z = 0$ to be consistent in x, y, z is that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$ Remember that if a given system of linear equations have Only Zero Solution for all its variables then the given equations are said to have TRIVIAL SOLUTION.