2.12 Laplace transforms

Laplace transform theorems

Definition[a]	$F(s)=\mathscr{L}\{f(t)\}=\int_0^{\infty} f(t)e^{-st}\,dt$ (2.514)	$\mathscr{L}\{\}$ Laplace transform
Convolution[b]	$F(s)\cdot G(s)=\mathscr{L}\left\{\int_0^{\infty} f(t-z)g(z)\,dz\right\}$ (2.515) $=\mathscr{L}\{f(t)*g(t)\}$ (2.516)	$F(s)$ $\mathscr{L}\{f(t)\}$ $G(s)$ $\mathscr{L}\{g(t)\}$ $*$ convolution
Inverse[c]	$f(t)=\frac{1}{2\pi\mathbf{i}}\int_{\gamma-\mathbf{i}\infty}^{\gamma+\mathbf{i}\infty} e^{st}F(s)\,ds$ (2.517) $=\sum$ residues (for $t>0$) (2.518)	γ constant
Transform of derivative	$\mathscr{L}\left\{\frac{d^n f(t)}{dt^n}\right\}=s^n\mathscr{L}\{f(t)\}-\sum_{r=0}^{n-1} s^{n-r-1}\frac{d^r f(t)}{dt^r}\bigg\rvert_{t=0}$ (2.519)	n integer >0
Derivative of transform	$\frac{d^n F(s)}{ds^n}=\mathscr{L}\{(-t)^n f(t)\}$ (2.520)	
Substitution	$F(s-a)=\mathscr{L}\{e^{at}f(t)\}$ (2.521)	a constant
Translation	$e^{-as}F(s)=\mathscr{L}\{u(t-a)f(t-a)\}$ (2.522) where $u(t)=\begin{cases}0 & (t<0)\\ 1 & (t>0)\end{cases}$ (2.523)	$u(t)$ unit step function

[a]If $\lvert e^{-s_0 t}f(t)\rvert$ is finite for sufficiently large t, the Laplace transform exists for $s>s_0$.
[b]Also known as the "faltung (or folding) theorem."
[c]Also known as the "Bromwich integral." γ is chosen so that the singularities in $F(s)$ are left of the integral line.

Laplace transform pairs

$$f(t) \Longrightarrow F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t \qquad (2.524)$$

$$\delta(t) \Longrightarrow 1 \qquad (2.525)$$

$$1 \Longrightarrow 1/s \qquad (s>0) \qquad (2.526)$$

$$t^n \Longrightarrow \frac{n!}{s^{n+1}} \qquad (s>0, n>-1) \qquad (2.527)$$

$$t^{1/2} \Longrightarrow \sqrt{\frac{\pi}{4s^3}} \qquad (2.528)$$

$$t^{-1/2} \Longrightarrow \sqrt{\frac{\pi}{s}} \qquad (2.529)$$

$$\mathrm{e}^{at} \Longrightarrow \frac{1}{s-a} \qquad (s>a) \qquad (2.530)$$

$$t\mathrm{e}^{at} \Longrightarrow \frac{1}{(s-a)^2} \qquad (s>a) \qquad (2.531)$$

$$(1-at)\mathrm{e}^{-at} \Longrightarrow \frac{s}{(s+a)^2} \qquad (2.532)$$

$$t^2\mathrm{e}^{-at} \Longrightarrow \frac{2}{(s+a)^3} \qquad (2.533)$$

$$\sin at \Longrightarrow \frac{a}{s^2+a^2} \qquad (s>0) \qquad (2.534)$$

$$\cos at \Longrightarrow \frac{s}{s^2+a^2} \qquad (s>0) \qquad (2.535)$$

$$\sinh at \Longrightarrow \frac{a}{s^2-a^2} \qquad (s>a) \qquad (2.536)$$

$$\cosh at \Longrightarrow \frac{s}{s^2-a^2} \qquad (s>a) \qquad (2.537)$$

$$e^{-bt}\sin at \Longrightarrow \frac{a}{(s+b)^2+a^2} \qquad (2.538)$$

$$e^{-bt}\cos at \Longrightarrow \frac{s+b}{(s+b)^2+a^2} \qquad (2.539)$$

$$\mathrm{e}^{-at}f(t) \Longrightarrow F(s+a) \qquad (2.540)$$