

Chapter 5. Continuity and Differentiability

Continuity

4 Marks Questions

1. Find the value of k , so that the function f defined below, is continuous at $x = 0$, where

$$f(x) = \begin{cases} \left(\frac{1 - \cos 4x}{8x^2} \right), & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases} \quad \text{All India 2014C}$$

Given function is

$$f(x) = \begin{cases} \left(\frac{1 - \cos 4x}{8x^2} \right), & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

Also, given $f(x)$ is continuous at $x = 0$.

$$\therefore (\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0) \quad \dots(i)$$

$$\text{Now,} \quad \text{LHL} = \lim_{x \rightarrow 0^-} f(x)$$

$$\begin{aligned} &= \lim_{x \rightarrow 0^-} \frac{1 - \cos 4x}{8x^2} \\ &= \lim_{h \rightarrow 0} \frac{1 - \cos(-4h)}{8h^2} \end{aligned}$$

$$[\text{put } x = 0 - h = -h, \text{ when } x \rightarrow 0, h \rightarrow 0](1)$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{1 - \cos 4h}{8h^2} \\ &= \lim_{h \rightarrow 0} \frac{2 \sin^2 2h}{8h^2} \end{aligned}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{\sin^2 2h}{4h^2} = \lim_{h \rightarrow 0} \left(\frac{\sin 2h}{2h} \right)^2 = 1 \quad (1)$$

At $x = 0$, $f(0) = k$

Now, from Eq. (i), we have

$$\text{LHL} = f(0)$$

$$\Rightarrow 1 \cdot 1 = k \Rightarrow k = 1 \quad (1)$$

Hence, for $k = 1$, the given function $f(x)$ is continuous at $x = 0$. (1)

$$2. \text{ If } f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2}, & \text{when } x < 0 \\ a, & \text{when } x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4}, & \text{when } x > 0 \end{cases}$$

and f is continuous at $x = 0$, then find the value of a . Delhi 2013C

Given,

$$f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2}, & \text{when } x < 0 \\ a, & \text{when } x = 0 \\ \frac{\sqrt{x}}{(\sqrt{16 + \sqrt{x}}) - 4}, & \text{when } x > 0 \end{cases}$$

Since, $f(x)$ is continuous at $x = 0$.

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

$$\begin{aligned} \Rightarrow \lim_{x \rightarrow 0^-} \frac{1 - \cos 4x}{x^2} \\ = \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{(\sqrt{16 + \sqrt{x}}) - 4} = f(0) \quad (1) \end{aligned}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{1 - \cos 4(0 - h)}{(0 - h)^2}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{0 + h}}{(\sqrt{16 + \sqrt{0 + h}} - 4)} = f(0)$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{1 - \cos 4h}{h^2}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{h}}{\sqrt{16 + \sqrt{h}} - 4} \times \frac{\sqrt{16 + \sqrt{h}} + 4}{\sqrt{16 + \sqrt{h}} + 4} = a \quad (1)$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{2 \sin^2 2h}{h^2} = \lim_{h \rightarrow 0} \frac{\sqrt{h}(\sqrt{16 + \sqrt{h}} + 4)}{(16 + \sqrt{h}) - 16} = a$$

$$[\because 1 - \cos 2\theta = 2 \sin^2 \theta \text{ and } (a + b)(a - b) = a^2 - b^2]$$

$$\Rightarrow 2 \lim_{h \rightarrow 0} \left(\frac{\sin 2h}{2h} \right)^2 \times 4$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{h}(\sqrt{16 + \sqrt{h}} + 4)}{\sqrt{h}} = a \quad (1)$$

$$\left[\because \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \right]$$

$$\Rightarrow 2 \times (1)^2 \times 4 = \sqrt{16 + \sqrt{0}} + 4 = a$$

$$\Rightarrow 8 = 4 + 4 = a$$

$$\therefore a = 8 \quad (1)$$

3. Find the value of k , for which

$$f(x) = \begin{cases} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if } -1 \leq x < 0 \\ \frac{2x+1}{x-1}, & \text{if } 0 \leq x < 1 \end{cases}$$

is continuous at $x=0$.

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$$\text{Given, } f(x) = \begin{cases} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if } -1 \leq x < 0 \\ \frac{2x+1}{x-1}, & \text{if } 0 \leq x < 1 \end{cases}$$

is continuous at $x = 0$.

$$\text{Now, } f(0) = \frac{2 \cdot 0 + 1}{0 - 1} = \frac{1}{-1} = -1 \quad (1)$$

$$\text{LHL} = \lim_{h \rightarrow 0} f(0 - h)$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{1-kh} - \sqrt{1+kh}}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{1-kh} - \sqrt{1+kh} \times (\sqrt{1-kh} + \sqrt{1+kh})}{-h(\sqrt{1-kh} + \sqrt{1+kh})} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{(1-kh) - (1+kh)}{-h(\sqrt{1-kh} + \sqrt{1+kh})}$$

$$[\because (a+b)(a-b) = a^2 - b^2]$$

$$= \lim_{h \rightarrow 0} \frac{-2kh}{-h(\sqrt{1-kh} + \sqrt{1+kh})} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{2k}{\sqrt{1-kh} + \sqrt{1+kh}} = \frac{2k}{1+1} = \frac{2k}{2} = k$$

$\therefore f(x)$ is continuous at $x = 0$.

$$\therefore f(0) = \text{LHL} \Rightarrow -1 = k$$

$$\Rightarrow k = -1 \quad (1)$$

4. Find the value of k , so that the function f defined by

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases} \text{ is continuous at } x = \frac{\pi}{2}.$$

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A function $f(x)$ is said to be continuous at point $x = a$, if $\text{LHL at } (x = a) = \text{RHL at } (x = a) = f(a)$. So, to find the value of k , we equate any one of LHL or RHL to $f(a)$ and simplify it.

$$\text{Given function is } f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$$

Also, given that function is continuous at $x = \pi/2$.

$$\therefore \text{At } x = \frac{\pi}{2}, \text{LHL} = \text{RHL} = f\left(\frac{\pi}{2}\right) \quad \dots(i) \quad (1)$$

$$\text{Now, LHL} = \lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{k \cos x}{\pi - 2x}$$

$$\text{LHL} = \lim_{h \rightarrow 0} \frac{k \cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim_{h \rightarrow 0} \frac{k \sin h}{\pi - \pi + 2h}$$

$$\left[\text{put } x = \frac{\pi}{2} - h, \text{ when } x \rightarrow \frac{\pi}{2}, \text{ then } h \rightarrow 0 \right]$$

$$\left[\because \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta \right]$$

$$= \lim_{h \rightarrow 0} \frac{k \sin h}{2h}$$

$$= \frac{k}{2} \lim_{h \rightarrow 0} \frac{\sin h}{h} = \frac{k}{2} \left[\because \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \right]$$

$$\Rightarrow \text{LHL} = k/2 \quad (1\frac{1}{2})$$

Also, from the given function, we get

$$f\left(\frac{\pi}{2}\right) = 3 \quad (1/2)$$

Now, from Eq. (i), we have

$$\text{LHL} = f\left(\frac{\pi}{2}\right) \Rightarrow \frac{k}{2} = 3$$

$$\text{Hence,} \quad k = 6 \quad (1)$$

5. Find the value of a for which the function f is

$$\text{defined as } f(x) = \begin{cases} a \sin \frac{\pi}{2} (x+1), & x \leq 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$

is continuous at $x = 0$.

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Given function is

$$f(x) = \begin{cases} a \sin \frac{\pi}{2} (x+1), & x \leq 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 0$.

$$\therefore \text{LHL} = \text{RHL} = f(0) \quad \dots(i) \quad (1)$$

$$\text{Now, LHL} = \lim_{x \rightarrow 0^-} a \sin \frac{\pi}{2} (x+1)$$

$$\text{LHL} = \lim_{h \rightarrow 0} a \sin \frac{\pi}{2} (-h+1)$$

[put $x = 0 - h = -h$]

$$= \lim_{h \rightarrow 0} \frac{a \sin \left[\frac{\pi (-h+1)}{2} \right]}{\frac{\pi (-h+1)}{2}} \times \frac{\pi (-h+1)}{2}$$

$$\left[\because \text{multiplying numerator and denominator by } \frac{\pi (-h+1)}{2} \right]$$

$$= \lim_{h \rightarrow 0} \frac{a \pi (-h+1)}{2} \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

$$\Rightarrow \text{LHL} = \frac{a\pi}{2} \quad [\text{put } h = 0] \quad (1)$$

$$\text{Now, RHL} = \lim_{x \rightarrow 0^+} \frac{\tan x - \sin x}{x^3}$$

Put $x = 0 + h = h$, we get

$$\text{RHL} = \lim_{h \rightarrow 0} \frac{\tan h - \sin h}{h^3} = \lim_{h \rightarrow 0} \frac{\frac{\sin h}{\cos h} - \sin h}{h^3}$$

$$= \lim_{h \rightarrow 0} \frac{\sinh h - \sinh h \cosh h}{h^3 \cosh h} = \lim_{h \rightarrow 0} \frac{\sinh h (1 - \cosh h)}{h^3 \cosh h}$$

$$= \lim_{h \rightarrow 0} \frac{\sinh h}{h} \cdot \lim_{h \rightarrow 0} \frac{1 - \cosh h}{h^2} \cdot \lim_{h \rightarrow 0} \frac{1}{\cosh h}$$

$$= 1 \times \lim_{h \rightarrow 0} \frac{1 - \cosh h}{h^2} \times 1$$

$$\left[\because \lim_{h \rightarrow 0} \frac{\sinh h}{h} = 1 \text{ and } \lim_{h \rightarrow 0} \frac{1}{\cosh h} = \frac{1}{\cosh 0} = \frac{1}{1} = 1 \right]$$

$$= \lim_{h \rightarrow 0} \frac{1 - \cosh h}{h^2}$$

$$= \lim_{h \rightarrow 0} \frac{2 \sin^2 \frac{h}{2}}{h^2} \left[\because 1 - \cos x = 2 \sin^2 \frac{x}{2} \right]$$

$$= \lim_{h \rightarrow 0} \frac{2 \times \sin^2 \frac{h}{2}}{\left(\frac{h^2}{4} \times 4 \right)} = \lim_{h \rightarrow 0} \frac{2}{4} \times \lim_{h \rightarrow 0} \frac{\sin^2 \frac{h}{2}}{\frac{h^2}{4}}$$

[$\because$ multiplying numerator and denominator by 4]

$$= \frac{1}{2} \times \lim_{h \rightarrow 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2 = \frac{1}{2} \times 1 \times 1$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \quad (1)$$

$$\Rightarrow \text{RHL} = \frac{1}{2}$$

Now, from Eq. (i), we have LHL = RHL

$$\therefore \frac{a\pi}{2} = \frac{1}{2} \Rightarrow a\pi = 1 \Rightarrow a = \frac{1}{\pi} \quad (1)$$

6. If the function $f(x)$ given by

$$f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$$

is continuous at $x = 1$, then find the values of a and b .
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The given function is

$$f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$$

Given that $f(x)$ is continuous at $x = 1$.

$$\therefore \text{LHL} = \text{RHL} = f(x) \quad \dots(i)$$

$$\begin{aligned} \text{Now, } \text{LHL} &= \lim_{x \rightarrow 1^-} f(x) \\ &= \lim_{x \rightarrow 1^-} (5ax - 2b) \quad (1) \end{aligned}$$

$$\begin{aligned} \text{LHL} &= \lim_{h \rightarrow 0} [5a(1-h) - 2b] \\ &\quad [\text{put } x = 1-h, \text{ when } x \rightarrow 1, h \rightarrow 0] \\ &= \lim_{h \rightarrow 0} (5a - 5ah - 2b) \end{aligned}$$

$$\Rightarrow \text{LHL} = 5a - 2b \quad [\text{put } h = 0] \quad (1)$$

$$\text{Now, } \text{RHL} = \lim_{x \rightarrow 1^+} (3ax + b)$$

$$\begin{aligned} \text{RHL} &= \lim_{h \rightarrow 0} [3a(1+h) + b] \\ &\quad [\text{put } x = 1+h, \text{ when } x \rightarrow 1, h \rightarrow 0] \\ &= \lim_{h \rightarrow 0} (3a + 3ah + b) \end{aligned}$$

$$\Rightarrow \text{RHL} = 3a + b \quad [\text{put } h = 0]$$

$$\text{Also, given that } f(1) = 11 \quad (1)$$

Now, from Eq. (i), we have

$$\text{RHL} = f(1)$$

$$\Rightarrow 3a + b = 11 \quad \dots(ii)$$

$$\text{and} \quad \text{LHL} = f(1)$$

$$\Rightarrow 5a - 2b = 11 \quad \dots(iii)$$

On multiplying Eq. (ii) by 5 and Eq. (iii) by 3 and then subtracting, we get

$$15a + 5b = 55$$

$$15a - 6b = 33$$

$$\begin{array}{r} - \quad + \quad - \\ \hline 11b = 22 \end{array}$$

$$\Rightarrow b = 2$$

On putting the value of b in Eq. (ii), we get

$$3a + 2 = 11 \Rightarrow 3a = 9 \Rightarrow a = 3$$

Hence, $a = 3$ and $b = 2$. (1)

7. Find the values of a and b such that the following function $f(x)$ is a continuous function.

$$f(x) = \begin{cases} 5, & x \leq 2 \\ ax + b, & 2 < x < 10 \\ 21, & x \geq 10 \end{cases}$$

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The given function is continuous that means $f(x)$ is continuous in its domain $[2, 10]$, so we take $x = 2$ and $x = 10$ to check continuity and then find a and b .

The given function is

$$f(x) = \begin{cases} 5, & x \leq 2 \\ ax + b, & 2 < x < 10 \\ 21, & x \geq 10 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 2$ and at $x = 10$.

$\therefore$ By definition,

$$\text{LHL} = \text{RHL} = f(2) \quad \dots(i)$$

$$\text{and} \quad \text{LHL} = \text{RHL} = f(10) \quad \dots(ii) \quad (1)$$

Now, first we calculate LHL and RHL at $x = 2$.

$$\text{LHL} = \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 5$$

$$\Rightarrow \text{LHL} = 5$$

$$\text{and} \quad \text{RHL} = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (ax + b)$$

$$\text{RHL} = \lim_{h \rightarrow 0} a(2 + h) + b = \lim_{h \rightarrow 0} (2a + ah + b)$$

$$[\text{put } x = 2 + h, \text{ when } x \rightarrow 2, h \rightarrow 0]$$

$$= 2a + b \quad [\text{put } h = 0]$$

From Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$$\Rightarrow 2a + b = 5 \quad \dots(iii) \quad (1)$$

Now, we find LHL and RHL at $x = 10$.

$$\text{LHL} = \lim_{x \rightarrow 10^-} f(x) = \lim_{x \rightarrow 10^-} (ax + b)$$

$$\text{LHL} = \lim_{h \rightarrow 0} [a(10 - h) + b]$$

$$[\text{put } x = 10 - h, \text{ when } x \rightarrow 10, h \rightarrow 0]$$

$$= \lim_{h \rightarrow 0} (10a - ah + b)$$

$$\Rightarrow \text{LHL} = 10a + b \quad [\text{put } h = 0]$$

$$\text{and RHL} = \lim_{x \rightarrow 10^+} f(x) = \lim_{x \rightarrow 10^+} 21 = 21$$

Now, from Eq. (ii), we have

$$\text{LHL} = \text{RHL}$$

$$\Rightarrow 10a + b = 21 \quad \dots(\text{iv}) \quad (1)$$

On subtracting Eq. (iv) from Eq. (iii), we get

$$-8a = -16 \Rightarrow a = 2$$

On putting $a = 2$ in Eq. (iv), we get

$$20 + b = 21$$

$$\therefore b = 1$$

Hence, $a = 2$ and $b = 1$. (1)

- 8.** Find the relationship between a and b , so that the function f defined by

$$f(x) = \begin{cases} ax + 1, & \text{if } x \leq 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$$

is continuous at $x = 3$.

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The given function is $f(x) = \begin{cases} ax + 1, & \text{if } x \leq 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$

Also, given that $f(x)$ is continuous at point $x = 3$.

$$\therefore \quad \text{LHL} = \text{RHL} = f(3) \quad \dots(i) \quad (1)$$

$$\text{Now,} \quad \text{LHL} = \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (ax + 1)$$

$$\text{LHL} = \lim_{h \rightarrow 0} [a(3 - h) + 1]$$

$$\begin{aligned} & [\text{put } x = 3 - h, \text{ when } x \rightarrow 3, h \rightarrow 0] \\ & = \lim_{h \rightarrow 0} (3a - ah + 1) \end{aligned}$$

$$\Rightarrow \quad \text{LHL} = 3a + 1 \quad [\text{put } h = 0] \quad (1)$$

$$\text{RHL} = \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (bx + 3)$$

$$\text{RHL} = \lim_{h \rightarrow 0} [b(3 + h) + 3]$$

$$\begin{aligned} & [\text{put } x = 3 + h, \text{ when } x \rightarrow 3, h \rightarrow 0] \\ & = \lim_{h \rightarrow 0} (3b + bh + 3) \end{aligned}$$

$$\Rightarrow \quad \text{RHL} = 3b + 3 \quad [\text{put } h = 0] \quad (1)$$

From Eq. (i), we have

$$\text{LHL} = \text{RHL} \Rightarrow 3a + 1 = 3b + 3$$

$\Rightarrow 3a - 3b = 2$, which is the required relation between a and b . (1)

9. Find the value of k , so that the function f

$$\text{defined by } f(x) = \begin{cases} kx + 1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$$

is continuous at $x = \pi$.

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$$\text{The given function is } f(x) = \begin{cases} kx + 1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$$

Also, given that $f(x)$ is continuous at $x = \pi$.

$$\therefore \quad \text{LHL} = \text{RHL} = f(\pi) \quad \dots(i) \quad (1)$$

$$\text{Now, LHL} = \lim_{x \rightarrow \pi^-} f(x) = \lim_{x \rightarrow \pi^-} (kx + 1)$$

$$\text{LHL} = \lim_{h \rightarrow 0} [k(\pi - h) + 1]$$

$$[\text{put } x = \pi - h, \text{ when } x \rightarrow \pi, h \rightarrow 0]$$

$$= \lim_{h \rightarrow 0} (k\pi - kh + 1)$$

$$= k\pi + 1 \quad [\text{put } h = 0] \quad (1)$$

$$\text{and RHL} = \lim_{x \rightarrow \pi^+} f(x) = \lim_{x \rightarrow \pi^+} \cos x$$

$$\text{RHL} = \lim_{h \rightarrow 0} \cos(\pi + h)$$

$$[\text{put } x = \pi + h, \text{ when } x \rightarrow \pi, h \rightarrow 0]$$

$$= \cos \pi \quad [\text{put } h = 0] \quad (1)$$

$$= -1 \quad [\because \cos \pi = -1]$$

Now, from Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$$\Rightarrow k\pi + 1 = -1$$

$$\Rightarrow k\pi = -2$$

$$\therefore k = \frac{-2}{\pi} \quad (1)$$

10. For what values of λ , is the function

$$f(x) = \begin{cases} \lambda (x^2 - 2x), & \text{if } x \leq 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

is continuous at $x = 0$?

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The given function is

$$f(x) = \begin{cases} \lambda (x^2 - 2x), & \text{if } x \leq 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 0$.

$$\therefore \text{LHL} = \text{RHL} = f(0) \quad \dots(i)$$

$$\text{Now, LHL} = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \lambda (x^2 - 2x) \quad (1)$$

$$\text{LHL} = \lim_{h \rightarrow 0} \lambda (h^2 + 2h)$$

$$\begin{aligned} & [\text{put } x = 0 - h = -h, \text{ when } x \rightarrow 0, h \rightarrow 0] \\ & = \lambda (0) \quad [\text{put } h = 0] \quad (1) \\ & = 0 \end{aligned}$$

$$\text{and RHL} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (4x + 1)$$

$$\text{RHL} = \lim_{h \rightarrow 0} (4h + 1)$$

$$\begin{aligned} & [\text{put } x = 0 + h = h, \text{ when } x \rightarrow 0, h \rightarrow 0] \\ \Rightarrow \text{RHL} &= 1 \quad [\text{put } h = 0] \quad (1) \end{aligned}$$

Thus, $\text{LHL} \neq \text{RHL}$

But it is given that $\text{LHL} = \text{RHL}$ [from Eq. (i)]

Therefore, we get a contradiction.

Hence, there doesn't exist any real value of λ for which $f(x)$ is continuous at $x = 0$. (1)

- 11.** Discuss the continuity of the function $f(x)$ at $x = 1/2$, when $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 1/2 + x, & 0 \leq x < 1/2 \\ 1, & x = 1/2 \\ 3/2 + x, & 1/2 < x \leq 1 \end{cases} \quad \text{Delhi 2011C}$$



Here, we find LHL, RHL and $f\left(\frac{1}{2}\right)$. If

$\text{LHL} = \text{RHL} = f\left(\frac{1}{2}\right)$, then we say that $f(x)$ is

continuous at $x = \frac{1}{2}$, otherwise $f(x)$ is

discontinuous at $x = \frac{1}{2}$.

The given function is

$$f(x) = \begin{cases} \frac{1}{2} + x, & 0 \leq x < \frac{1}{2} \\ 1, & x = \frac{1}{2} \\ \frac{3}{2} + x, & \frac{1}{2} < x \leq 1 \end{cases}$$

We have to check continuity of $f(x)$ at $x = \frac{1}{2}$.
(1)

$$\text{Now, LHL} = \lim_{x \rightarrow \frac{1}{2}^-} f(x) = \lim_{x \rightarrow \frac{1}{2}^-} \left(\frac{1}{2} + x \right)$$

$$\text{LHL} = \lim_{h \rightarrow 0} \left(\frac{1}{2} + \frac{1}{2} - h \right)$$

$$\left[\text{put } x = \frac{1}{2} - h, \text{ when } x \rightarrow \frac{1}{2}, h \rightarrow 0 \right]$$

$$= \lim_{h \rightarrow 0} (1 - h)$$

$$= 1 \quad [\text{put } h = 0] \quad (1)$$

$$\text{RHL} = \lim_{x \rightarrow \frac{1}{2}^+} f(x) = \lim_{x \rightarrow \frac{1}{2}^+} \left(\frac{3}{2} + x \right)$$

$$\text{RHL} = \lim_{h \rightarrow 0} \left(\frac{3}{2} + \frac{1}{2} + h \right)$$

$$\left[\text{put } x = \frac{1}{2} + h, \text{ when } x \rightarrow \frac{1}{2}, h \rightarrow 0 \right]$$

$$= \lim_{h \rightarrow 0} (2 + h) = 2 \quad [\text{put } h = 0] \quad (1)$$

Now, we know that a function $f(x)$ is said to be continuous at point $x = a$, if

$$\text{LHL} = \text{RHL} = f(a).$$

Here, $\text{LHL} \neq \text{RHL}$ at $x = 1/2$.

Hence, $f(x)$ is discontinuous at $x = \frac{1}{2}$. (1)

12. Find the value of a , if the function $f(x)$ defined by

$$f(x) = \begin{cases} 2x - 1, & x < 2 \\ a, & x = 2 \\ x + 1, & x > 2 \end{cases}$$

is continuous at $x = 2$.

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$$\text{Given, } f(x) = \begin{cases} 2x - 1, & x < 2 \\ a, & x = 2 \\ x + 1, & x > 2 \end{cases}$$

is continuous at point $x = 2$.

$\therefore$ By definition of continuity of a function at a point, we have

$$\text{LHL} = \text{RHL} = f(2) \quad \dots(i) \quad (1)$$

$$\text{Now, LHL} = \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2x - 1)$$

$$\Rightarrow \text{LHL} = \lim_{h \rightarrow 0} [2(2 - h) - 1]$$

$$[\text{put } x = 2 - h, \text{ when } x \rightarrow 2, h \rightarrow 0]$$

$$= \lim_{h \rightarrow 0} (4 - 2h - 1) = \lim_{h \rightarrow 0} (3 - 2h)$$

$$= 3 \quad [\text{put } h = 0] \quad (1\frac{1}{2})$$

Also, from the given function, we have

$$f(2) = a \quad (1/2)$$

On putting the values of $f(2)$ and LHL in Eq. (i), we get

$$3 = a$$

$$\Rightarrow a = 3 \quad (1)$$

13. Find the values of a and b such that the function defined as follows is continuous.

$$f(x) = \begin{cases} x + 2, & x \leq 2 \\ ax + b, & 2 < x < 5 \\ 3x - 2, & x \geq 5 \end{cases}$$

Delhi 2010, 2010C

• Given function is $f(x) = \begin{cases} x + 2, & x \leq 2 \\ ax + b, & 2 < x < 5 \\ 3x - 2, & x \geq 5 \end{cases}$

Also, given that $f(x)$ is continuous at $x = 2$ and $x = 5$.

∴ By definition of continuity, we get

$$\text{LHL} = \text{RHL} = f(2) \quad \dots(i)$$

and $\text{LHL} = \text{RHL} = f(5) \quad \dots(ii)$

First, we find LHL and RHL at $x = 2$. (1)

$$\text{LHL} = \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x + 2)$$

$$\Rightarrow \text{LHL} = \lim_{h \rightarrow 0} (2 - h + 2)$$

$$[\text{put } x = 2 - h, \text{ when } x \rightarrow 2, h \rightarrow 0]$$

$$= \lim_{h \rightarrow 0} (4 - h)$$

$$\Rightarrow \text{LHL} = 4 \quad [\text{put } h = 0]$$

$$\text{Now, RHL} = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (ax + b)$$

$$\Rightarrow \text{RHL} = \lim_{h \rightarrow 0} [a(2 + h) + b]$$

$$[\text{put } x = 2 + h, \text{ when } x \rightarrow 2, h \rightarrow 0]$$

$$= \lim_{h \rightarrow 0} (2a + ah + b)$$

$$\Rightarrow \text{RHL} = 2a + b \quad [\text{put } h = 0]$$

From Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$$\therefore 2a + b = 4 \quad \dots(\text{iii}) \quad (1)$$

Now, we find LHL and RHL at $x = 5$.

$$\text{LHL} = \lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^-} (ax + b)$$

$$\Rightarrow \text{LHL} = \lim_{h \rightarrow 0} [a(5 - h) + b]$$

$$\begin{aligned} & [\text{put } x = 5 - h, \text{ when } x \rightarrow 5, h \rightarrow 0] \\ &= \lim_{h \rightarrow 0} (5a - ah + b) \end{aligned}$$

$$\Rightarrow \text{LHL} = 5a + b \quad [\text{put } h = 0]$$

$$\text{and } \text{RHL} = \lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5^+} (3x - 2)$$

$$= \lim_{h \rightarrow 0} [3(5 + h) - 2]$$

$$\begin{aligned} & [\text{put } x = 5 + h, \text{ when } x \rightarrow 5, h \rightarrow 0] \\ &= \lim_{h \rightarrow 0} (15 + 3h - 2) \end{aligned}$$

$$\Rightarrow \text{RHL} = 13 \quad [\text{put } h = 0]$$

$\therefore$ From Eq. (ii), we have

$$\text{LHL} = \text{RHL}$$

$$\Rightarrow 5a + b = 13 \quad \dots(\text{iv}) \quad (1)$$

On subtracting Eq. (iv) from Eq. (iii), we get

$$-3a = -9 \Rightarrow a = 3$$

Put $a = 3$ in Eq. (iv), we get

$$15 + b = 13 \Rightarrow b = -2$$

$$\text{Hence, } a = 3 \text{ and } b = -2 \quad (1)$$

14. For what value of k , is the function defined by

$$f(x) = \begin{cases} k(x^2 + 2), & \text{if } x \leq 0 \\ 3x + 1, & \text{if } x > 0 \end{cases}, \text{ continuous at } x = 0?$$

Also, write whether the function is

continuous at $x = 1$.

Delhi 2010, 2010C

The given function is

$$f(x) = \begin{cases} k(x^2 + 2), & \text{if } x \leq 0 \\ 3x + 1, & \text{if } x > 0 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 0$.

$$\therefore \quad \text{LHL} = \text{RHL} = f(0) \quad \dots(i) \quad (1/2)$$

$$\text{Now,} \quad \text{LHL} = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} k(x^2 + 2)$$

$$\Rightarrow \quad \text{LHL} = \lim_{h \rightarrow 0} k(h^2 + 2)$$

$$\begin{aligned} & [\text{put } x = 0 - h = -h, \text{ when } x \rightarrow 0, h \rightarrow 0] \\ & = \lim_{h \rightarrow 0} (kh^2 + 2k) \end{aligned}$$

$$\Rightarrow \quad \text{LHL} = 2k \quad [\text{put } h = 0] \quad (1/2)$$

$$\text{and} \quad \text{RHL} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (3x + 1)$$

$$\Rightarrow \quad \text{RHL} = \lim_{h \rightarrow 0} (3h + 1)$$

$$[\text{put } x = 0 + h = h, \text{ when } x \rightarrow 0, h \rightarrow 0]$$

$$\Rightarrow \quad \text{RHL} = 1 \quad [\text{put } h = 0] \quad (1/2)$$

Now, from Eq. (i), we have

$$\begin{aligned} \text{LHL} = \text{RHL} &\Rightarrow 2k = 1 \\ \Rightarrow k &= \frac{1}{2} \end{aligned} \quad (1/2)$$

Now, we check the continuity of the given function $f(x)$ at $x = 1$.

$$\begin{aligned} \text{LHL} &= \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3x + 1) \\ \Rightarrow \text{LHL} &= \lim_{h \rightarrow 0} [3(1 - h) + 1] \\ &\quad [\text{put } x = 1 - h, \text{ when } x \rightarrow 1, h \rightarrow 0] \\ &= \lim_{h \rightarrow 0} (3 - 3h + 1) \\ &= 4 \quad [\text{put } h = 0] \quad (1/2) \end{aligned}$$

$$\begin{aligned} \text{RHL} &= \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (3x + 1) \\ \Rightarrow \text{RHL} &= \lim_{h \rightarrow 0} [3(1 + h) + 1] \\ &\quad [\text{put } x = 1 + h, \text{ when } x \rightarrow 1, h \rightarrow 0] \\ &= \lim_{h \rightarrow 0} [3 + 3h + 1] \\ &= 4 \quad [\text{put } h = 0] \quad (1/2) \end{aligned}$$

and $f(1) = 4$ from given function.

$\therefore$ At $x = 1$, $\text{LHL} = \text{RHL} = f(1)$

Hence, $f(x)$ is continuous at $x = 1$. (1)

- 15.** Find all points of discontinuity of f , where f is defined as follows:

$$f(x) = \begin{cases} |x| + 3, & x \leq -3 \\ -2x, & -3 < x < 3 \\ 6x + 2, & x \geq 3 \end{cases}$$

Delhi 2010



Firstly, verify continuity of the given function at $x = -3$ and $x = 3$. Then, point at which the given function is discontinuous will be the point of discontinuity.

$$\text{The given function is } f(x) = \begin{cases} |x| + 3, & x \leq -3 \\ -2x, & -3 < x < 3 \\ 6x + 2, & x \geq 3 \end{cases}$$

First, we verify continuity at $x = -3$ and then at $x = 3$.

Continuity at $x = -3$

$$\text{LHL} = \lim_{x \rightarrow (-3)^-} f(x) = \lim_{x \rightarrow (-3)^-} (|x| + 3)$$

$$\Rightarrow \text{LHL} = \lim_{h \rightarrow 0} (|-3 - h| + 3)$$

$$\begin{aligned} & \text{[put } x = -3 - h, \\ & \text{when } x \rightarrow -3, h \rightarrow 0] \end{aligned}$$

$$= |-3| + 3$$

$$= 3 + 3$$

$$= 6$$

$$\text{[put } h = 0]$$

$$[\because |-x| = x, \forall x \in \mathbb{R}]$$

$$(1/2)$$

$$\text{RHL} = \lim_{x \rightarrow (-3)^+} f(x) = \lim_{x \rightarrow (-3)^+} (-2x)$$

$$\Rightarrow \text{RHL} = \lim_{h \rightarrow 0} [-2(-3 + h)]$$

$$\begin{aligned} & [\text{put } x = -3 + h, \text{ when } x \rightarrow -3, h \rightarrow 0] \\ & = \lim_{h \rightarrow 0} (6 - 2h) \end{aligned}$$

$$\Rightarrow \text{RHL} = 6 \quad [\text{put } h = 0] \text{ (1/2)}$$

Also, $f(-3)$ = value of $f(x)$ at $x = -3$

$$= |-3| + 3$$

$$= 3 + 3 = 6 [\because |-x| = x, \forall x \in R] \text{ (1/2)}$$

$$\therefore \text{LHL} = \text{RHL} = f(-3)$$

$\therefore f(x)$ is continuous at $x = -3$. So, $x = -3$ is the point of continuity. (1/2)

Continuity at $x = 3$

$$\text{LHL} = \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} -2x$$

$$\Rightarrow \text{LHL} = \lim_{h \rightarrow 0} -2(3 - h)$$

$$[\text{put } x = 3 - h, \text{ when } x \rightarrow 3, h \rightarrow 0]$$

$$= \lim_{h \rightarrow 0} (-6 + 2h)$$

$$\Rightarrow \text{LHL} = -6 \quad [\text{put } h = 0] \text{ (1/2)}$$

$$\text{and } \text{RHL} = \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (6x + 2)$$

$$\Rightarrow \text{RHL} = \lim_{h \rightarrow 0} [6(3 + h) + 2]$$

$$[\text{put } x = 3 + h, \text{ when } x \rightarrow 3, h \rightarrow 0]$$

$$= \lim_{h \rightarrow 0} (18 + 6h + 2)$$

$$\Rightarrow \text{RHL} = 20 \quad [\text{put } h = 0] \text{ (1/2)}$$

$$\therefore \text{LHL} \neq \text{RHL}$$

As $f(x)$ is a polynomial function in a given interval, so it is continuous in a given interval but $f(x)$ is not continuous at $x = 3$. So, $x = 3$ is the point of discontinuity of $f(x)$. (1)

16. Show that the function $f(x)$ defined by

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & x > 0 \\ 2, & x = 0 \\ \frac{4(1 - \sqrt{1-x})}{x}, & x < 0 \end{cases}$$

is continuous at $x = 0$.

All India 2009C

To show that the given function is continuous at $x = 0$, we show that

$$(\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0) \quad \dots(\text{i}) \quad (1)$$

Now, given function is

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & x > 0 \\ 2, & x = 0 \\ \frac{4(1 - \sqrt{1-x})}{x}, & x < 0 \end{cases}$$

$$\text{LHL} = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{4(1 - \sqrt{1-x})}{x}$$

$$\begin{aligned}
\Rightarrow \text{LHL} &= \lim_{h \rightarrow 0} \frac{4[1 - \sqrt{1 - (0 - h)}]}{0 - h} \\
&\quad [\text{put } x = 0 - h, \text{ when } x \rightarrow 0, h \rightarrow 0] \\
&= \lim_{h \rightarrow 0} \frac{4[1 - \sqrt{1 + h}]}{-h} \\
&= \lim_{h \rightarrow 0} \frac{4[1 - \sqrt{1 + h}]}{-h} \times \frac{1 + \sqrt{1 + h}}{1 + \sqrt{1 + h}} \\
&= \lim_{h \rightarrow 0} \frac{4[(1)^2 - (\sqrt{1 + h})^2]}{-h[1 + \sqrt{1 + h}]} \\
&\quad [\because (a - b)(a + b) = a^2 - b^2] \\
&= \lim_{h \rightarrow 0} \frac{4[1 - (1 + h)]}{-h[1 + \sqrt{1 + h}]} \\
&= \lim_{h \rightarrow 0} \frac{-h \times 4}{-h[1 + \sqrt{1 + h}]} = \lim_{h \rightarrow 0} \frac{4}{1 + \sqrt{1 + h}} \\
&= \frac{4}{1 + \sqrt{1}} = \frac{4}{2} = 2 \quad [\text{put } h = 0]
\end{aligned}$$

$$\Rightarrow \text{LHL} = 2 \quad (1)$$

$$\text{Now, RHL} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(\frac{\sin x}{x} + \cos x \right)$$

$$\begin{aligned}
\Rightarrow \text{RHL} &= \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} + \cos h \right) \\
&\quad [\text{put } x = 0 + h = h, \text{ when } x \rightarrow 0, h \rightarrow 0] \\
&= \lim_{h \rightarrow 0} \frac{\sin h}{h} + \lim_{h \rightarrow 0} \cos h \quad [\text{put } h = 0] \\
&= 1 + \cos 0^\circ \quad \left[\because \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \right] \\
&= 1 + 1 \quad [\because \cos 0 = 1] \\
&= 2 \quad (1)
\end{aligned}$$

Also, given that at $x = 0$, $f(x) = 2 \Rightarrow f(0) = 2$

Since, $(\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0) = 2$

Hence, $f(x)$ is continuous at $x = 0$. (1)

- 17.** For what value of k , is the following function continuous at $x = 2$?

$$f(x) = \begin{cases} 2x + 1, & x < 2 \\ k, & x = 2 \\ 3x - 1, & x > 2 \end{cases}$$

Delhi 2008

Do same as Que 12.

[Ans. $k = 5$]

- 18.** If $f(x)$ defined by the following, is continuous at $x = 0$, then find the values of a, b and c .

$$f(x) = \begin{cases} \frac{\sin(a+1)x + \sin x}{x}, & \text{if } x < 0 \\ c, & \text{if } x = 0 \\ \frac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}}, & \text{if } x > 0 \end{cases}$$

HOTS; All India 2008

Given function is

$$f(x) = \begin{cases} \frac{\sin(a+1)x + \sin x}{x}, & \text{if } x < 0 \\ c, & \text{if } x = 0 \\ \frac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}}, & \text{if } x > 0 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 0$.

$$\therefore (\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0) \quad \dots(i)$$

$$\text{Now, LHL} = \lim_{x \rightarrow 0^-} f(x)$$

$$= \lim_{x \rightarrow 0^-} \frac{\sin(a+1)x + \sin x}{x}$$

$$\Rightarrow \text{LHL} = \lim_{h \rightarrow 0} \frac{\sin[-(a+1)h] + \sin(-h)}{-h}$$

$$[\text{put } x = 0 - h = -h, \text{ when } x \rightarrow 0, h \rightarrow 0]$$

$$= \lim_{h \rightarrow 0} \frac{-\sin(a+1)h - \sin h}{-h}$$

$$[\because \sin(-\theta) = -\sin \theta]$$

$$= \lim_{h \rightarrow 0} \frac{\sin(a+1)h + \sin h}{h}$$

$$2 \sin \left[\frac{(a+1)h + h}{2} \right] \cos \left[\frac{(a+1)h - h}{2} \right]$$

$$= \lim_{h \rightarrow 0} \frac{\sin C + \sin D}{h}$$

$$\left[\because \sin C + \sin D = 2 \sin \frac{(C+D)}{2} \cdot \cos \frac{(C-D)}{2} \right]$$

$$= \lim_{h \rightarrow 0} \frac{2 \sin \left[\frac{(a+1)h + h}{2} \right] \cos \left[\frac{(a+1)h - h}{2} \right]}{\left[\frac{(a+1)h + h}{2} \right] \times h}$$

$$\times \frac{(a+1)h + h}{2}$$

$$\left[\text{multiplying and dividing by } \frac{(a+1)h + h}{2} \right]$$

$$= \lim_{h \rightarrow 0} \frac{2 \cos \left[\frac{(a+1)h - h}{2} \right] \times \frac{h[a+1+1]}{2}}{h}$$

$$\left[\begin{array}{l} \because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \\ \therefore \lim_{h \rightarrow 0} \frac{\sin[(a+1)h + h]}{\frac{(a+1)h + h}{2}} = 1 \end{array} \right]$$

$$= \lim_{h \rightarrow 0} \frac{2(a+2)}{2} \cos \left[\frac{(a+1)h - h}{2} \right]$$

$$= (a+2) \cos 0^\circ \quad [\text{put } h = 0]$$

$$= (a+2) \times 1 \quad [\because \cos 0^\circ = 1]$$

$$= a+2 \quad (1\frac{1}{2})$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}}$$

$$\Rightarrow \text{RHL} = \lim_{h \rightarrow 0} \frac{\sqrt{h + bh^2} - \sqrt{h}}{bh^{3/2}}$$

$$[\text{put } x = 0 + h = h, \text{ when } x \rightarrow 0, h \rightarrow 0]$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{h(1+bh)} - \sqrt{h}}{bh^{3/2}}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{h} [\sqrt{1+bh} - 1]}{b\sqrt{h} \cdot h} [\because h^{3/2} = \sqrt{h} \cdot h]$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{1+bh} - 1}{bh}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{1+bh} - 1}{bh} \times \frac{\sqrt{1+bh} + 1}{\sqrt{1+bh} + 1}$$

[multiplying and dividing by $\sqrt{1+bh} + 1$]

$$= \lim_{h \rightarrow 0} \frac{(\sqrt{1+bh})^2 - (1)^2}{bh[\sqrt{1+bh} + 1]}$$

$$[\because (a+b)(a-b) = a^2 - b^2]$$

$$= \lim_{h \rightarrow 0} \frac{1+bh-1}{bh[\sqrt{1+bh} + 1]}$$

$$= \lim_{h \rightarrow 0} \frac{bh}{bh[\sqrt{1+bh} + 1]}$$

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{1+bh} + 1} \quad (1\frac{1}{2})$$

$$\Rightarrow \text{RHL} = \frac{1}{1+1} = \frac{1}{2} \quad [\text{put } h = 0]$$

Now, from Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$$\Rightarrow a + 2 = \frac{1}{2}$$

$$\Rightarrow a = \frac{1}{2} - 2 \Rightarrow a = -\frac{3}{2}$$

Also, given that $f(0) = c$

Again from Eq. (i), we have

$$\text{RHL} = f(0)$$

$$\Rightarrow c = 1/2$$

Hence, we get $a = -\frac{3}{2}$, $c = \frac{1}{2}$ and b may take

any real value. (1)

NOTE Here, we cannot find any real and unique value of b that means b may take any real value i.e.

value of b that means b may take any real value, i.e.
 $b \in \mathbb{R}$.

- 19.** Show that $f(x) = \begin{cases} 5x - 4, & \text{when } 0 < x \leq 1 \\ 4x^3 - 3x, & \text{when } 1 < x < 2 \end{cases}$
 is continuous at $x = 1$. Delhi 2008C

Given function is

$$f(x) = \begin{cases} 5x - 4, & \text{when } 0 < x \leq 1 \\ 4x^3 - 3x, & \text{when } 1 < x < 2 \end{cases}$$

To show that $f(x)$ is continuous at $x = 1$, we need to prove

$$\text{LHL}_{x=1} = \text{RHL}_{x=1} = f(1) \quad \dots(i) \quad (1)$$

$$\text{Now, LHL} = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (5x - 4)$$

$$\Rightarrow \text{LHL} = \lim_{h \rightarrow 0} [5(1-h) - 4]$$

$$[\text{put } x = 1 - h, \text{ when } x \rightarrow 1, h \rightarrow 0]$$

$$= \lim_{h \rightarrow 0} (5 - 5h - 4) = \lim_{h \rightarrow 0} (1 - 5h)$$

$$\text{LHL} = 1 \quad [\text{put } h = 0] \quad (1)$$

$$\text{RHL} = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (4x^3 - 3x)$$

$$\Rightarrow \text{RHL} = \lim_{h \rightarrow 0} [4(1+h)^3 - 3(1+h)]$$

$$[\text{put } x = 1 + h, \text{ when } x \rightarrow 1, h \rightarrow 0]$$

$$= 4(1)^3 - 3(1) \quad [\text{put } h = 0]$$

$$\Rightarrow \text{RHL} = 4 - 3 = 1 \quad (1)$$

Also, from given function,

$$f(1) = \text{Value of } f(x) \text{ at } x = 1$$

$$\Rightarrow f(1) = 5(1) - 4$$

$$[\text{put } x = 1 \text{ in } f(x) = 5x - 4]$$

$$= 5 - 4 = 1$$

$$\text{Here, } (\text{LHL})_{x=1} = (\text{RHL})_{x=1} = f(1) \quad (1)$$

Hence, $f(1)$ is continuous at $x = 1$.

- 20.** If the following function $f(x)$ is continuous at $x = 0$, then find the value of k .

$$f(x) = \begin{cases} \frac{1 - \cos 2x}{2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}.$$

All India 2008C

Do same as Que 1.

[Ans. $k = 1$]

Differentiability

1 Mark Questions

1. Write the derivative of $\sin x$ with respect to $\cos x$. Delhi 2014C

Let $u = \sin x$

On differentiating u w.r.t. x , we get

$$\frac{du}{dx} = \cos x \quad \dots(i)$$

and $v = \cos x$

On differentiating v w.r.t. x , we get

$$\frac{dv}{dx} = -\sin x \quad \dots(ii)$$

Now,

$$\begin{aligned} \frac{du}{dv} &= \frac{du}{dx} \times \frac{dx}{dv} \\ &= -\frac{\cos x}{\sin x} \text{ [from Eqs. (i) and (ii)]} \end{aligned}$$

$$\Rightarrow \frac{du}{dv} = -\cot x \quad (1)$$

2. If $\cos y = x \cos (a + y)$, where $\cos a \neq \pm 1$,
prove that $\frac{dy}{dx} = \frac{\cos^2(a + y)}{\sin a}$. Foreign 2014

Given, $\cos y = x \cos(a + y)$

$$\Rightarrow x = \frac{\cos y}{\cos(a + y)}$$

On differentiating both sides w.r.t. y , we get

$$\frac{dx}{dy} = \frac{\cos(a + y) \frac{d}{dy}(\cos y) - \cos y \frac{d}{dy}[\cos(a + y)]}{\cos^2(a + y)}$$

[by using quotient rule]

$$\begin{aligned}\Rightarrow \frac{dx}{dy} &= \frac{\cos(a + y)(-\sin y) - \cos y[-\sin(a + y)]}{\cos^2(a + y)} \\ &= \frac{\cos y \sin(a + y) - \cos(a + y) \sin y}{\cos^2(a + y)} \\ &= \frac{\sin(a + y - y)}{\cos^2(a + y)}\end{aligned}$$

$$\begin{aligned}& [\because \sin A \cos B - \cos A \sin B = \sin(A - B)] \\ \Rightarrow \frac{dx}{dy} &= \frac{\sin a}{\cos^2(a + y)} \\ \Rightarrow \frac{dy}{dx} &= \frac{\cos^2(a + y)}{\sin a} \quad (1)\end{aligned}$$

Hence proved.

3. If $y = \sin^{-1} \{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\}$ and

$0 < x < 1$, then find $\frac{dy}{dx}$.

All India 2014C; Delhi 2010



Firstly, convert the given expression in $\sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ form and then put $x = \sin \phi$ and $y = \sin \theta$. Now, simplify the resulting expression and differentiate it.

$$\text{Given, } y = \sin^{-1} [x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$$

Above equation can be rewritten as

$$y = \sin^{-1} [x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}]$$

$$[\because x = (\sqrt{x})^2]$$

Now, put $\sqrt{x} = \sin \theta$ and $x = \sin \phi$, so that

$$\theta = \sin^{-1} \sqrt{x} \text{ and } \phi = \sin^{-1} x, \text{ we get}$$

$$y = \sin^{-1} [\sin \phi \sqrt{1 - \sin^2 \theta} - \sin \theta \sqrt{1 - \sin^2 \phi}]$$

$$\Rightarrow y = \sin^{-1} [\sin \phi \cos \theta - \sin \theta \cos \phi]$$

$$[\because \sqrt{1 - \sin^2 x} = \cos x]$$

$$\Rightarrow y = \sin^{-1} \sin (\phi - \theta)$$

$$[\because \sin \phi \cos \theta - \sin \theta \cos \phi = \sin (\phi - \theta)]$$

$$\Rightarrow y = \phi - \theta \quad [\because \sin^{-1} \sin \theta = \theta]$$

$$\Rightarrow y = \sin^{-1} x - \sin^{-1} \sqrt{x}$$

$$[\because \phi = \sin^{-1} x \text{ and } \theta = \sin^{-1} \sqrt{x}]$$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{d}{dx}(\sqrt{x})$$

$$\left[\because \frac{d}{d\theta}(\sin^{-1} \theta) = \frac{1}{\sqrt{1-\theta^2}} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\left[\because \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} \right]$$

$$\text{Hence, } \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x-x^2}} \quad (1)$$

Alternate Method



Use the formula,

$$\sin^{-1} x - \sin^{-1} y = \sin^{-1} [x\sqrt{1-y^2} - y\sqrt{1-x^2}]$$

and then differentiate with respect to x to get the required value.

$$\therefore y = \sin^{-1} [x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$$

$$\Rightarrow y = \sin^{-1} [x \sqrt{1 - (\sqrt{x})^2} - \sqrt{x} \sqrt{1 - x^2}]$$

$$[\because \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\} = \sin^{-1} x - \sin^{-1} y]$$

$$\text{Here, } x = x \text{ and } y = \sqrt{x}$$

$$\therefore y = \sin^{-1} x - \sin^{-1} \sqrt{x}$$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{d}{dx} (\sin^{-1} x) - \frac{d}{dx} (\sin^{-1} \sqrt{x})$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx} (\sqrt{x})$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x-x^2}} \quad (1)$$

4. If $e^x + e^y = e^{x+y}$, prove that $\frac{dy}{dx} + e^{y-x} = 0$.

Foregin 2014

$$\text{Given, } e^x + e^y = e^{x+y} \quad \dots(i)$$

On dividing Eq. (i) by e^{x+y} , we get

$$e^{-y} + e^{-x} = 1 \quad \dots(ii)$$

On differentiating both sides of Eq. (ii) w.r.t. x , we get

$$e^{-y} \cdot \left(\frac{-dy}{dx} \right) + e^{-x}(-1) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-e^{-x}}{e^{-y}} = -e^{(y-x)}$$

$$\Rightarrow \frac{dy}{dx} + e^{(y-x)} = 0 \quad (1)$$

5. Find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$, if

$$x = ae^{\theta} (\sin \theta - \cos \theta) \text{ and } y = ae^{\theta} (\sin \theta + \cos \theta).$$

All India 2014

Given, $x = ae^{\theta} (\sin \theta - \cos \theta)$

On differentiating w.r.t. θ , we get

$$\begin{aligned}\frac{dx}{d\theta} &= a \frac{d}{d\theta} [e^{\theta} \sin \theta - e^{\theta} \cos \theta] \\&= a \left[\frac{d}{d\theta} (e^{\theta} \sin \theta) - \frac{d}{d\theta} (e^{\theta} \cos \theta) \right] \\&= a \left[e^{\theta} \frac{d}{d\theta} (\sin \theta) + \sin \theta \frac{d}{d\theta} (e^{\theta}) \right. \\&\quad \left. - e^{\theta} \frac{d}{d\theta} (\cos \theta) - \cos \theta \frac{d}{d\theta} (e^{\theta}) \right] \\&= a [e^{\theta} \cos \theta + e^{\theta} \sin \theta - e^{\theta} (-\sin \theta) - e^{\theta} \cos \theta] \\&= a [e^{\theta} \cos \theta + e^{\theta} \sin \theta + e^{\theta} \sin \theta - e^{\theta} \cos \theta] \\&\Rightarrow \frac{dx}{d\theta} = a [2e^{\theta} \sin \theta] = 2ae^{\theta} \sin \theta \quad \dots(i)\end{aligned}$$

Also, $y = ae^{\theta} (\sin \theta + \cos \theta)$

On differentiating w.r.t. θ , we get

$$\frac{dy}{d\theta} = a \left[\frac{d}{d\theta} (e^{\theta} \sin \theta) + \frac{d}{d\theta} (e^{\theta} \cos \theta) \right]$$

$$= a \left[e^{\theta} \frac{d}{d\theta} (\sin \theta) + \sin \theta \frac{d}{d\theta} (e^{\theta}) + e^{\theta} \frac{d}{d\theta} (\cos \theta) + \cos \theta \frac{d}{d\theta} (e^{\theta}) \right]$$

$$= a[e^{\theta} \cos \theta + e^{\theta} \sin \theta - e^{\theta} \sin \theta + e^{\theta} \cos \theta]$$

$$= a[2e^{\theta} \cos \theta]$$

$$\Rightarrow \frac{dy}{d\theta} = 2ae^{\theta} \cos \theta \quad \dots(ii)$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{2ae^{\theta} \cos \theta}{2ae^{\theta} \sin \theta}$$

[from Eqs. (i) and (ii)]

$$= \cot \theta$$

$$\text{At } \theta = \frac{\pi}{4}, \frac{dy}{dx} = \cot \frac{\pi}{4}$$

$$\text{Hence, } \frac{dy}{dx} = 1 \quad \left[\because \cot \frac{\pi}{4} = 1 \right] (1)$$

6. If $x = a \left(\cos t + \log \tan \frac{t}{2} \right)$, $y = a \sin t$, then

evaluate $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{3}$.

Delhi 2014C

$$\text{Given, } x = a \left(\cos t + \log \tan \frac{t}{2} \right)$$

$$\text{and } y = a \sin t. \quad \dots(i)$$

$$\text{Now, } x = a \left(\cos t + \log \tan \frac{t}{2} \right)$$

On differentiating both sides w.r.t. t , we get

$$\begin{aligned} \frac{dx}{dt} &= a \left[\frac{d}{dt} (\cos t) + \frac{d}{dt} \log \tan \frac{t}{2} \right] \\ &= a \left[-\sin t + \frac{1}{\tan \frac{t}{2}} \frac{d}{dt} \left(\tan \frac{t}{2} \right) \right] \\ &\quad \left[\because \frac{d}{dx} (u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx} \right] \end{aligned}$$

$$= a \left[-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \sec^2 \frac{t}{2} \frac{d}{dt} \left(\frac{t}{2} \right) \right]$$

$$= a \left[-\sin t + \frac{1}{\tan \frac{t}{2}} \times \sec^2 \frac{t}{2} \times \frac{1}{2} \right]$$

$$= a \left[-\sin t + \frac{1}{\frac{\sin t/2}{\cos t/2}} \times \frac{1}{\cos^2 \frac{t}{2}} \times \frac{1}{2} \right]$$

$$= a \left[-\sin t + \frac{1}{2 \sin \frac{t}{2} \cdot \cos \frac{t}{2}} \right]$$

$$= a \left[-\sin t + \frac{1}{\sin t} \right] \quad [\because \sin 2\theta = 2 \sin \theta \cos \theta]$$

$$= a \left(\frac{1 - \sin^2 t}{\sin t} \right) \Rightarrow \frac{dx}{dt} = a \left(\frac{\cos^2 t}{\sin t} \right) \quad \dots(ii)$$

$$[\because 1 - \sin^2 \theta = \cos^2 \theta]$$

and $y = a \sin t$

On differentiating both sides w.r.t. t , we get

$$\frac{dy}{dt} = a \cos t \quad \dots(\text{iii})$$

$$\begin{aligned} \text{Now, } \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a \cos t}{a \left(\frac{\cos^2 t}{\sin t} \right)} \\ &\quad \text{[from Eqs. (ii) and (iii)]} \\ &= \frac{a \cos t}{a \cos^2 t} \times \sin t = \tan t \end{aligned}$$

On differentiating both sides of above equation w.r.t. x , we get

$$\begin{aligned} \frac{d}{dx} \left(\frac{dy}{dx} \right) &= \frac{d}{dx} (\tan t) \\ &= \frac{d}{dt} (\tan t) \frac{dt}{dx} \left[\because \frac{d}{dx} f(t) = \frac{d}{dt} f(t) \cdot \frac{dt}{dx} \right] \end{aligned}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \sec^2 t \times \frac{\sin t}{a \cos^2 t}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{\sin t \sec^4 t}{a}$$

Now, on putting $t = \frac{\pi}{3}$, we get

$$\left[\frac{d^2 y}{dx^2} \right]_{t=\frac{\pi}{3}} = \frac{\sin \frac{\pi}{3} \times \sec^4 \frac{\pi}{3}}{a} = \frac{\frac{\sqrt{3}}{2} \times (2)^4}{a}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{\frac{\sqrt{3}}{2} \times 16}{a}$$

$$\text{Hence, } \frac{d^2 y}{dx^2} = \frac{8\sqrt{3}}{a} \quad (1)$$

7. If $x^m y^n = (x + y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.
Foreign 2014

Given, $x^m y^n = (x + y)^{m+n}$

On taking log both sides, we get

$$\log(x^m y^n) = \log(x + y)^{m+n}$$

$$\Rightarrow \log(x^m) + \log(y^n) = (m + n) \log(x + y)$$

$$\Rightarrow m \log x + n \log y = (m + n) \log(x + y)$$

On differentiating w.r.t. x , we get

$$\frac{m}{x} + \frac{n}{y} \frac{dy}{dx} = \frac{m+n}{x+y} \left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{m}{x} - \frac{(m+n)}{x+y} = \left(\frac{m+n}{x+y} - \frac{n}{y}\right) \frac{dy}{dx}$$

$$\Rightarrow \left[\frac{my + ny - nx - ny}{y(x+y)} \right] \frac{dy}{dx} = \frac{mx + my - mx - nx}{x(x+y)}$$

$$\Rightarrow \frac{dy}{dx} \left[\frac{my - nx}{y} \right] = \frac{my - nx}{x}$$

Hence, $\frac{dy}{dx} = \frac{y}{x}$ (1)

8. If $x = a \cos \theta + b \sin \theta$ and $y = a \sin \theta - b \cos \theta$,

show that $y^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = 0$.
Foreign 2014

Given, $x = a \cos \theta + b \sin \theta$... (i)

and $y = a \sin \theta - b \cos \theta$... (ii)

On differentiating Eq. (i) w.r.t. θ , we get

$$\frac{dx}{d\theta} = -a \sin \theta + b \cos \theta$$

On differentiating Eq. (ii) w.r.t. θ , we get

$$\frac{dy}{d\theta} = a \cos \theta + b \sin \theta$$

Now, $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{a \cos \theta + b \sin \theta}{-a \sin \theta + b \cos \theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{-y} \left[\begin{array}{l} \because x = a \cos \theta + b \sin \theta, \\ y = a \sin \theta - b \cos \theta \end{array} \right]$$

$$\Rightarrow y \frac{dy}{dx} = -x \Rightarrow y \frac{dy}{dx} + x = 0 \quad \dots (iii)$$

On differentiating Eq. (iii) w.r.t. x , we get

$$y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 + 1 = 0$$

$$\Rightarrow y \frac{d^2y}{dx^2} + \frac{dy}{dx} \left(\frac{-x}{y} \right) + 1 = 0 \left[\because \frac{dy}{dx} = \frac{-x}{y} \right]$$

$$\Rightarrow y^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 0 \quad (1)$$

Hence proved.

9. Differentiate $\tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right)$ w.r.t.

$$\cos^{-1}(2x \sqrt{1-x^2}), \text{ when } x \neq 0.$$

Delhi 2014

• Let $u = \tan^{-1} \left[\frac{\sqrt{1-x^2}}{x} \right]$

Put $x = \cos \theta \Rightarrow \theta = \cos^{-1} x$

Then, $u = \tan^{-1} \left[\frac{\sqrt{1-\cos^2 \theta}}{\cos \theta} \right]$

$$\begin{aligned}
 & \left[\frac{\cos \theta}{\sqrt{\sin^2 \theta}} \right] \\
 &= \tan^{-1} \left[\frac{\sqrt{\sin^2 \theta}}{\cos \theta} \right] = \tan^{-1} \left[\frac{\sin \theta}{\cos \theta} \right] \\
 &= \tan^{-1} [\tan \theta] = \theta
 \end{aligned}$$

$$\Rightarrow u = \cos^{-1} x \quad [\because x = \cos \theta]$$

On differentiating w.r.t. x , we get

$$\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Again, let $v = \cos^{-1} (2x \sqrt{1-x^2})$

$$\text{Put } x = \cos \theta \Rightarrow \theta = \cos^{-1} x$$

$$\begin{aligned}
 \text{Then, } v &= \cos^{-1} [2 \cos \theta \sqrt{1 - \cos^2 \theta}] \\
 &= \cos^{-1} [2 \cos \theta \sin \theta] \\
 &\quad \left[\begin{array}{l} \because \cos^2 \theta + \sin^2 \theta = 1 \\ \sin \theta = \sqrt{1 - \cos^2 \theta} \end{array} \right] \\
 &= \cos^{-1} [\sin 2\theta] \\
 &= \cos^{-1} \left[\cos \left(\frac{\pi}{2} - 2\theta \right) \right] = \frac{\pi}{2} - 2\theta
 \end{aligned}$$

$$\Rightarrow v = \frac{\pi}{2} - 2 \cos^{-1} x$$

On differentiating w.r.t. x , we get

$$\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$$

$$\begin{aligned}
 \text{Now, } \frac{du}{dv} &= \frac{du}{dx} \times \frac{dx}{dv} \\
 &= -\frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{2} = -\frac{1}{2} \quad (1)
 \end{aligned}$$

10. If $y = x^x$, then prove that

$$\frac{d^2 y}{dx^2} - \frac{1}{y} \left(\frac{dy}{dx} \right)^2 - \frac{y}{x} = 0.$$

Delhi 2014

Given, $y = x^x$

On taking log both sides, we get

$$\log y = \log x^x$$

$$\Rightarrow \log y = x \log x$$

On differentiating w.r.t. x , we get

$$\frac{1}{y} \frac{dy}{dx} = x \frac{d}{dx} (\log x) + \log x \frac{d}{dx} (x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = x \times \frac{1}{x} + \log x$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = (1 + \log x)$$

$$\Rightarrow \frac{dy}{dx} = y (1 + \log x) \quad \dots(i)$$

Again, differentiating w.r.t. x , we get

$$\frac{d^2y}{dx^2} = y \frac{d}{dx} (1 + \log x) + (1 + \log x) \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = y \times \frac{1}{x} + (1 + \log x) \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{y}{x} + (1 + \log x) \frac{dy}{dx} \quad \dots(ii)$$

Now, we have to prove that

$$\frac{d^2y}{dx^2} - \frac{1}{y} \left(\frac{dy}{dx} \right)^2 - \frac{y}{x} = 0$$

$$\begin{aligned} \text{LHS} &= \frac{d^2y}{dx^2} - \frac{1}{y} \left(\frac{dy}{dx} \right)^2 - \frac{y}{x} \\ &= \frac{y}{x} + (1 + \log x) \frac{dy}{dx} - \frac{1}{y} [y (1 + \log x)]^2 - \frac{y}{x} \\ &\quad \text{[from Eqs. (i) and (ii)]} \\ &= \frac{y}{x} + (1 + \log x) y (1 + \log x) \\ &\quad - \frac{1}{y} [y^2 (1 + \log x)^2] - \frac{y}{x} \quad \text{[from Eq. (i)]} \\ &= y(1 + \log x)^2 - y(1 + \log x)^2 = 0 = \text{RHS} \end{aligned}$$

Hence proved. (1)

11. Differentiate $\tan^{-1} \left(\frac{x}{\sqrt{1-x^2}} \right)$ w.r.t.

$$\sin^{-1} (2x \sqrt{1-x^2}).$$

Delhi 2014

$$\text{Let } u = \tan^{-1} \left(\frac{x}{\sqrt{1-x^2}} \right)$$

Put $x = \sin \theta \Rightarrow \theta = \sin^{-1} x$, then

$$u = \tan^{-1} \left[\frac{\sin \theta}{\sqrt{1-\sin^2 \theta}} \right]$$

$$\Rightarrow u = \tan^{-1} \left[\frac{\sin \theta}{\cos \theta} \right] \left[\begin{array}{l} \because \sin^2 \theta + \cos^2 \theta = 1 \\ \Rightarrow \cos \theta = \sqrt{1-\sin^2 \theta} \end{array} \right]$$

$$\Rightarrow u = \tan^{-1} (\tan \theta) \Rightarrow u = \theta \Rightarrow u = \sin^{-1} x$$

On differentiating w.r.t. x , we get

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \dots(i)$$

Again, let $v = \sin^{-1}(2x\sqrt{1-x^2})$

Put $x = \sin \theta \Rightarrow \theta = \sin^{-1} x$, then

$$v = \sin^{-1}(2 \sin \theta \sqrt{1 - \sin^2 \theta})$$

$$\Rightarrow v = \sin^{-1}(2 \sin \theta \cos \theta)$$

$$\Rightarrow v = \sin^{-1}(\sin 2\theta) \Rightarrow v = 2\theta$$

$$\Rightarrow v = 2 \sin^{-1} x$$

On differentiating w.r.t. x , we get

$$\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}} \quad \dots(ii)$$

$$\text{Now,} \quad \frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv}$$

$$\Rightarrow \frac{du}{dv} = \frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{2}$$

$$\therefore \frac{du}{dv} = \frac{1}{2} \quad (1)$$

12. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t.

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right), \text{ when } x \neq 0.$$

Delhi 2014

$$\text{Let } u = \tan^{-1} \left(\frac{\sqrt{1+x^2} - 1}{x} \right)$$

Put $x = \cot \theta \Rightarrow \theta = \cot^{-1} x$, then

$$\begin{aligned} u &= \tan^{-1} \left[\frac{\sqrt{1 + \cot^2 \theta} - 1}{\cot \theta} \right] \\ &= \tan^{-1} \left[\frac{\sqrt{\operatorname{cosec}^2 \theta} - 1}{\cot \theta} \right] \\ &= \tan^{-1} \left[\frac{\operatorname{cosec} \theta - 1}{\cot \theta} \right] = \tan^{-1} \left[\frac{1 - \sin \theta}{\sin \theta} \right] \\ &= \tan^{-1} \left[\frac{\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} - 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}} \right] \\ &\quad [\because \sin^2 x + \cos^2 x = 1] \\ &= \tan^{-1} \left[\frac{\left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right)^2}{\left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right) \left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right)} \right] \\ &\quad [\because a^2 - b^2 = (a + b)(a - b)] \end{aligned}$$

$$= \tan^{-1} \left[\frac{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}} \right] = \tan^{-1} \left[\frac{1 - \tan \frac{\theta}{2}}{1 + \tan \frac{\theta}{2}} \right]$$

$[\because \text{dividing numerator and denominator by } \cos \theta/2]$

$$= \tan^{-1} \left[\tan \left(\frac{\pi}{4} - \frac{\theta}{2} \right) \right]$$

$$\Rightarrow u = \frac{\pi}{4} - \frac{\theta}{2} = \frac{\pi}{4} - \frac{\cot^{-1} x}{2}$$

On differentiating w.r.t. x , we get

$$\frac{du}{dx} = \frac{1}{2(1+x^2)} \quad \dots(i)$$

$$\left[\because \frac{d}{dx} (\cot^{-1} x) = \frac{-1}{1+x^2} \right]$$

Again, let $v = \sin^{-1} \left(\frac{2x}{1+x^2} \right)$

Put $x = \tan \theta \Rightarrow \theta = \tan^{-1} x$, then we get

$$v = \sin^{-1} \left[\frac{2 \tan \theta}{1 + \tan^2 \theta} \right]$$

$$\Rightarrow v = \sin^{-1} [\sin 2\theta] \Rightarrow v = 2\theta \Rightarrow v = 2 \tan^{-1} x$$

On differentiating w.r.t. x , we get

$$\frac{dv}{dx} = \frac{2}{1+x^2} \quad \dots(ii)$$

$$\text{Now, } \frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv} = \frac{1}{2(1+x^2)} \times \frac{(1+x^2)}{2}$$

[from Eqs. (i) and (ii)]

$$\Rightarrow \frac{du}{dv} = \frac{1}{4} \quad (1)$$

13. If $y = Pe^{ax} + Qe^{bx}$, then show that

$$\frac{d^2y}{dx^2} - (a+b) \frac{dy}{dx} + aby = 0.$$

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$$\text{Given, } y = Pe^{ax} + Qe^{bx} \quad \dots(i)$$

On differentiating w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= P \frac{d}{dx} (e^{ax}) + Q \frac{d}{dx} (e^{bx}) \\ \Rightarrow \frac{dy}{dx} &= Pa e^{ax} + Qb e^{bx} \quad \dots(ii) \end{aligned}$$

Again, differentiating w.r.t. x , we get

$$\begin{aligned} \frac{d^2y}{dx^2} &= Pa \frac{d}{dx} (e^{ax}) + bQ \frac{d}{dx} (e^{bx}) \\ &= Pa (a e^{ax}) + bQ (b e^{bx}) \\ &= a^2P e^{ax} + b^2Q e^{bx} \quad \dots(iii) \end{aligned}$$

$$\text{Now, LHS} = \frac{d^2y}{dx^2} - (a + b) \frac{dy}{dx} + aby$$

On putting values from Eqs. (i), (ii) and (iii), we get

$$\begin{aligned} \text{LHS} &= a^2P e^{ax} + b^2Q e^{bx} \\ &\quad - (a + b) (aP e^{ax} + bQ e^{bx}) + ab (P e^{ax} + Q e^{bx}) \\ &= a^2P e^{ax} + b^2Q e^{bx} - a^2P e^{ax} - abQ e^{bx} \\ &\quad - abP e^{ax} - b^2Q e^{bx} + abP e^{ax} + abQ e^{bx} \\ &= 0 = \text{RHS} \quad (1) \end{aligned}$$

Hence proved.

14. If $x = \cos t (3 - 2 \cos^2 t)$ and

$y = \sin t (3 - 2 \sin^2 t)$, then find the value of $\frac{dy}{dx}$

$$\text{at } t = \frac{\pi}{4}.$$

All India 2014

Given, $x = \cos t (3 - 2 \cos^2 t)$

$\Rightarrow x = 3 \cos t - 2 \cos^3 t$

On differentiating w.r.t. t , we get

$$\frac{dx}{dt} = 3(-\sin t) - 2(3) \cos^2 t (-\sin t)$$

$\Rightarrow \frac{dx}{dt} = -3 \sin t + 6 \cos^2 t \sin t \quad \dots(i)$

Also, $y = \sin t (3 - 2 \sin^2 t)$

$\Rightarrow y = 3 \sin t - 2 \sin^3 t$

On differentiating w.r.t. t , we get

$$\frac{dy}{dt} = 3 \cos t - 2 \times 3 \times \sin^2 t \cos t$$

$\Rightarrow \frac{dy}{dt} = 3 \cos t - 6 \sin^2 t \cos t \quad \dots(ii)$

Now, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3 \cos t - 6 \cos t \sin^2 t}{-3 \sin t + 6 \cos^2 t \sin t}$

[from Eqs. (i) and (ii)]

$$= \frac{\cos t - 2 \cos t \sin^2 t}{-\sin t + 2 \cos^2 t \sin t}$$

At $t = \frac{\pi}{4}$,

$$\begin{aligned} \left[\frac{dy}{dx} \right]_{t=\frac{\pi}{4}} &= \frac{\cos \frac{\pi}{4} - 2 \cos \frac{\pi}{4} \sin^2 \frac{\pi}{4}}{-\sin \frac{\pi}{4} + 2 \cos^2 \frac{\pi}{4} \sin \frac{\pi}{4}} \\ &= \frac{\frac{1}{\sqrt{2}} - 2 \left(\frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} \right)^2}{-\frac{1}{\sqrt{2}} + 2 \left(\frac{1}{\sqrt{2}} \right)^2 \left(\frac{1}{\sqrt{2}} \right)} = \frac{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} = 0 \end{aligned}$$

15. If $(x - y) e^{\frac{x}{x-y}} = a$. Prove that $y \frac{dy}{dx} + x = 2y$.

Delhi 2014C

Given, $(x - y) \cdot e^{\frac{x}{x-y}} = a$

On taking log both sides, we get

$$\log \left[(x - y) \cdot e^{\frac{x}{x-y}} \right] = \log a$$

$$\Rightarrow \log(x - y) + \log e^{\frac{x}{x-y}} = \log a$$

[$\because \log(mn) = \log m + \log n$]

$$\Rightarrow \log(x - y) + \frac{x}{x - y} \log_e e = \log a$$

$$\Rightarrow \log(x - y) + \frac{x}{x - y} = \log a$$

[$\because \log_e = 1$]

On differentiating w.r.t. x , we get

$$\frac{d}{dx} [\log(x - y)] + \frac{d}{dx} \left(\frac{x}{x - y} \right) = \frac{d}{dx} (\log a)$$

$$\Rightarrow \frac{1}{x - y} \frac{d}{dx} (x - y)$$

$$+ \frac{(x - y) \frac{d}{dx} (x) - x \frac{d}{dx} (x - y)}{(x - y)^2} = 0$$

$$\left[\because \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{u \frac{d}{dx} (v) - v \frac{d}{dx} (u)}{v^2} \right]$$

$$\Rightarrow \frac{1}{x - y} \cdot (1 - y') + \frac{(x - y) - x(1 - y')}{(x - y)^2} = 0$$

where, $y' = dy/dx$

$$\Rightarrow (x - y)(1 - y') + x - y - x(1 - y') = 0$$

$$\Rightarrow yy' + x - 2y = 0$$

$$\Rightarrow y \frac{dy}{dx} + x = 2y \quad (1)$$

Hence proved.

16. If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$,
then find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. **Delhi 2014C**

Given, $x = a(\cos t + t \sin t)$

On differentiating both sides w.r.t. t , we get

$$\begin{aligned}\frac{dx}{dt} &= a(-\sin t + 1 \cdot \sin t + t \cos t) \\ \Rightarrow \frac{dx}{dt} &= a t \cos t\end{aligned}$$

and $y = a(\sin t - t \cos t)$

On differentiating both sides w.r.t. t , we get

$$\frac{dy}{dt} = a(\cos t - \cos t \cdot 1 + t \sin t) = a t \sin t \quad \dots(i)$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{a t \sin t}{a t \cos t} = \tan t$$

Again differentiating both sides w.r.t. x , we get

$$\begin{aligned}&= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} (\tan t) \frac{dt}{dx} \\ \Rightarrow \frac{d^2y}{dx^2} &= \sec^2 t \cdot \frac{1}{\frac{dx}{dt}} = \frac{\sec^2 t}{a t \cos t} = \frac{\sec^3 t}{a t}\end{aligned}$$

$$\text{At } t = \frac{\pi}{4},$$

$$\begin{aligned}\therefore \left(\frac{d^2y}{dx^2} \right)_{t=\frac{\pi}{4}} &= \frac{\sec^3 \left(\frac{\pi}{4} \right)}{a \left(\frac{\pi}{4} \right)} = \frac{(\sqrt{2})^3 \cdot 4}{2 \pi} \\ &= \frac{8\sqrt{2}}{a \pi}\end{aligned} \quad (1)$$

17. If $y = \tan^{-1} \left(\frac{a}{x} \right) + \log \sqrt{\frac{x-a}{x+a}}$, prove that

$$\frac{dy}{dx} = \frac{2a^3}{x^4 - a^4}.$$

All India 2014C

Given, $y = \tan^{-1} \left(\frac{a}{x} \right) + \log \sqrt{\frac{x-a}{x+a}}$

$$= \tan^{-1} \left(\frac{a}{x} \right) + \log \left(\frac{x-a}{x+a} \right)^{1/2}$$

$$\Rightarrow y = \tan^{-1} \left(\frac{a}{x} \right) + \frac{1}{2} [\log(x-a) - \log(x+a)]$$

$$\left[\because \log \frac{m}{n} = \log m - \log n \right]$$

On differentiating w.r.t. x , we get

$$\frac{dy}{dx} = \frac{1}{1 + \frac{a^2}{x^2}} \cdot \left(\frac{-a}{x^2} \right) + \frac{1}{2} \left[\frac{1}{x-a} - \frac{1}{x+a} \right]$$

$$\left[\because \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2} \text{ and } \frac{d}{dx} (\log x) = \frac{1}{x} \right]$$

$$= \frac{-a}{x^2 + a^2} + \frac{a}{x^2 - a^2}$$

$$= \frac{-x^2a + a^3 + x^2a + a^3}{x^4 - a^4}$$

$$[\because (a+b)(a-b) = a^2 - b^2]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a^3}{x^4 - a^4} \quad (1)$$

Hence proved.

18. If $x = a \sin 2t (1 + \cos 2t)$ and

$y = b \cos 2t (1 - \cos 2t)$, then show that at

$$t = \frac{\pi}{4}, \left(\frac{dy}{dx} \right) = \frac{b}{a}.$$

All India 2014

Given $x = a \sin 2t (1 + \cos 2t)$

$$\text{Given, } x = a \sin 2t (1 + \cos 2t)$$

$$\Rightarrow x = a \sin 2t (2 \cos^2 t)$$

$$[\because \cos 2\theta = 2 \cos^2 \theta - 1]$$

$$\Rightarrow x = 2a \sin 2t \cos^2 t$$

On differentiating x w.r.t. t , we get

$$\begin{aligned} \frac{dx}{dt} &= 2a \left[\sin 2t \frac{d}{dt} (\cos^2 t) + \cos^2 t \frac{d}{dt} (\sin 2t) \right] \\ &= 2a [\sin 2t \{2 \cos t (-\sin t)\} \\ &\quad + 2 \cos^2 t (\cos 2t)] \\ &= 2a [-\sin^2 2t + 2 \cos^2 t \cos 2t] \\ &\quad [\because 2 \sin \theta \cos \theta = \sin 2\theta] \end{aligned}$$

$$\begin{aligned} \text{Also, } y &= b \cos 2t (1 - \cos 2t) \\ &= b \cos 2t (2 \sin^2 t) \\ &\quad [\because \cos 2\theta = 1 - 2 \sin^2 \theta] \\ &= 2b \cos 2t \sin^2 t \end{aligned}$$

On differentiating y w.r.t. t , we get

$$\begin{aligned} \frac{dy}{dt} &= 2b \left[\cos 2t \frac{d}{dt} (\sin^2 t) + \sin^2 t \frac{d}{dt} (\cos 2t) \right] \\ &= 2b [\cos 2t (2 \sin t \cos t) \\ &\quad + \sin^2 t (-\sin 2t) \cdot 2] \\ &= 2b [\cos 2t \sin 2t - 2 \sin^2 t \sin 2t] \end{aligned}$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2b [\cos 2t \sin 2t - 2 \sin^2 t \sin 2t]}{2a [2 \cos^2 t \cos 2t - \sin^2 2t]}$$

$$\text{At } t = \frac{\pi}{4},$$

$$\begin{aligned} \left[\frac{dy}{dx} \right]_{t=\frac{\pi}{4}} &= \frac{b}{a} \frac{\begin{bmatrix} \cos 2\left(\frac{\pi}{4}\right) \sin 2\left(\frac{\pi}{4}\right) \\ -2 \sin^2\left(\frac{\pi}{4}\right) \sin 2\left(\frac{\pi}{4}\right) \end{bmatrix}}{\begin{bmatrix} 2 \cos^2\left(\frac{\pi}{4}\right) \cos 2\left(\frac{\pi}{4}\right) - \sin^2 2\left(\frac{\pi}{4}\right) \end{bmatrix}} \\ &= \frac{b}{a} \frac{\begin{bmatrix} \cos\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right) - 2 \times \left(\frac{1}{\sqrt{2}}\right)^2 \sin\left(\frac{\pi}{2}\right) \end{bmatrix}}{\begin{bmatrix} 2 \times \left(\frac{1}{\sqrt{2}}\right)^2 \cos\left(\frac{\pi}{2}\right) - \sin^2\left(\frac{\pi}{2}\right) \end{bmatrix}} \end{aligned}$$

$$\begin{aligned}
 & \left[\frac{1}{\sqrt{2}} \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) \right] \\
 & \left[\because \sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \right] \\
 & = \frac{b}{a} \frac{[0 \times 1 - 1 \times 1]}{[1 \times 0 - 1]} \\
 & = \frac{b}{a} \left(\frac{-1}{-1} \right) \quad \left[\because \sin \frac{\pi}{2} = 1, \cos \frac{\pi}{2} = 0 \right] \\
 \Rightarrow \quad \left[\frac{dy}{dx} \right]_{t=\frac{\pi}{4}} &= \frac{b}{a} \quad (1)
 \end{aligned}$$

Hence proved.

19. If $(\tan^{-1} x)^y + y^{\cot x} = 1$, then find dy/dx .

All India 2014C

Let $u = (\tan^{-1} x)^y$ and $v = y^{\cot x}$

Then, given equation becomes $u + v = 1$

On differentiating both sides w.r.t. x , we get

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \quad \dots(i)$$

Now, $u = (\tan^{-1} x)^y$

On taking log both sides, we get

$$\log u = y \log(\tan^{-1} x)$$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned}
 \frac{1}{u} \frac{du}{dx} &= \frac{dy}{dx} \log(\tan^{-1} x) + \frac{y}{\tan^{-1} x (1+x^2)} \\
 \Rightarrow \quad \frac{du}{dx} &= (\tan^{-1} x)^y \left[\frac{dy}{dx} \log(\tan^{-1} x) + \frac{y}{\tan^{-1} x (1+x^2)} \right] \dots(ii)
 \end{aligned}$$

Also, $v = y^{\cot x}$

On taking log both sides, we get

$$\log v = \cot x \log y$$

On differentiating w.r.t. x , we get

$$\frac{1}{v} \frac{dv}{dx} = -\operatorname{cosec}^2 x \log y + \frac{\cot x}{y} \frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{dx} = y^{\cot x} \left[-\operatorname{cosec}^2 x \log y + \frac{\cot x}{y} \frac{dy}{dx} \right] \dots (iii)$$

On putting values from Eqs. (ii) and (iii) in Eq. (i), we get

$$(\tan^{-1} x)^y \left[\frac{dv}{dx} \log (\tan^{-1} x) + \frac{y}{\tan^{-1} x (1+x^2)} \right] + y^{\cot x} \left[-\operatorname{cosec}^2 x \log y + \frac{\cot x}{y} \frac{dv}{dx} \right] = 0$$

$$\Rightarrow \frac{dy}{dx} [(\tan^{-1} x)^y \log (\tan^{-1} x) + \cot x y^{\cot x - 1}] = - \left[\frac{y}{1+x^2} (\tan^{-1} x)^{y-1} - y^{\cot x} \operatorname{cosec}^2 x \log y \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{- \left[\frac{y}{1+x^2} (\tan^{-1} x)^{y-1} - y^{\cot x} \cdot \operatorname{cosec}^2 x \log y \right]}{[(\tan^{-1} x)^y \log (\tan^{-1} x) + \cot x y^{\cot x - 1}]}$$

$$\Rightarrow \frac{dy}{dx} = \frac{- \left[\frac{y}{1+x^2} (\tan^{-1} x)^{y-1} - y^{\cot x} \operatorname{cosec}^2 x \log y \right]}{[(\tan^{-1} x)^y \log (\tan^{-1} x) + \cot x y^{\cot x - 1}]} \quad (1)$$

4 marks Questions

20. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$,

then prove that $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$.

Delhi 2013C

To prove, $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$

Given, $x = 2 \cos \theta - \cos 2 \theta$

and $y = 2 \sin \theta - \sin 2 \theta$

On differentiating both sides w.r.t. θ , we get

$$\frac{dx}{d\theta} = -2 \sin \theta + 2 \sin 2 \theta$$

and $\frac{dy}{d\theta} = 2 \cos \theta - 2 \cos 2 \theta$ (1)

$$\therefore \frac{dy}{dx} = \frac{dy / d\theta}{dx / d\theta} = \frac{2 (\cos \theta - \cos 2 \theta)}{2 (-\sin \theta + \sin 2 \theta)} \quad (1)$$

$$= \frac{2 \sin\left(\frac{\theta + 2\theta}{2}\right) \sin\left(\frac{2\theta - \theta}{2}\right)}{2 \left[\cos\left(\frac{2\theta + \theta}{2}\right) \sin\left(\frac{2\theta - \theta}{2}\right) \right]} \quad (1)$$

$$\left[\begin{array}{l} \because \cos C - \cos D = 2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{D-C}{2}\right) \\ \text{and } \sin C - \sin D = 2 \cos\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right) \end{array} \right]$$

$$= \frac{\sin\left(\frac{3\theta}{2}\right) \sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{3\theta}{2}\right) \sin\left(\frac{\theta}{2}\right)} = \tan\left(\frac{3\theta}{2}\right) = \text{RHS} \quad (1)$$

Hence proved.

21. If $y = (\sin x)^x + \sin^{-1} \sqrt{x}$, then find $\frac{dy}{dx}$.

Delhi 2013C, 2009; All India 2009C

Given, $y = (\sin x)^x + \sin^{-1} \sqrt{x} \quad \dots(i)$

Let $u = (\sin x)^x \quad \dots(ii)$

Then, Eq. (i) becomes, $y = u + \sin^{-1} \sqrt{x} \quad \dots(iii)$

On taking log both sides of Eq. (ii), we get

$$\log u = x \log \sin x \quad (1)$$

On differentiating both sides w.r.t. x , we get

$$\frac{1}{u} \frac{du}{dx} = x \frac{d}{dx} (\log \sin x) + \log \sin x \frac{d}{dx} (x)$$

[by product rule]

$$\Rightarrow \frac{du}{dx} = u \left[x \times \frac{1}{\sin x} \frac{d}{dx} (\sin x) + \log \sin x (1) \right] \quad (1)$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x \left[\frac{x}{\sin x} \times \cos x + \log \sin x \right]$$

[from Eq. (ii)]

$$\Rightarrow \frac{du}{dx} = (\sin x)^x [x \cot x + \log \sin x] \quad \dots (iv)$$

On differentiating both sides of Eq.(iii) w.r.t. x , we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{1}{\sqrt{1 - (\sqrt{x})^2}} \frac{d}{dx} (\sqrt{x}) \quad (1)$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^x [x \cot x + \log \sin x]$$

$$+ \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} \quad [\text{from Eq. (iv)}] \quad (1)$$

22. If $y = x \log \left(\frac{x}{a+bx} \right)$, then prove that

$$x^3 \frac{d^2 y}{dx^2} = \left(x \frac{dy}{dx} - y \right)^2. \quad \text{Delhi 2013C}$$

To prove, $x^3 \frac{d^2y}{dx^2} = \left(x \frac{dy}{dx} - y \right)^2$

Given, $y = x \log \left(\frac{x}{a + bx} \right)$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = x \frac{d}{dx} \log \left(\frac{x}{a + bx} \right) + \log \left(\frac{x}{a + bx} \right) \frac{d}{dx} (x)$$

[by product rule]

$$= x \left(\frac{1}{\frac{x}{a + bx}} \right) \frac{d}{dx} \left(\frac{x}{a + bx} \right) + \log \left(\frac{x}{a + bx} \right) \cdot 1$$

$$\left[\because \frac{d}{dx} (\log x) = \frac{1}{x} \right]$$

$$= (a + bx) \left[\frac{(a + bx)(1) - x(b)}{(a + bx)^2} \right] + \log \left(\frac{x}{a + bx} \right)$$

[by quotient rule]

$$= (a + bx) \left[\frac{a}{(a + bx)^2} \right] + \log \left(\frac{x}{a + bx} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{a}{a + bx} + \log \left(\frac{x}{a + bx} \right) \quad \dots(i) \quad (1)$$

Now, $x \frac{dy}{dx} - y = \frac{ax}{a + bx} + x \log \left(\frac{x}{a + bx} \right) - y$

$$= \frac{ax}{a + bx} + x \log \left(\frac{x}{a + bx} \right) - x \log \left(\frac{x}{a + bx} \right)$$

$$\Rightarrow x \frac{dy}{dx} - y = \frac{ax}{a + bx} \quad \dots(ii) \quad (1)$$

From Eq. (i), we have

$$\frac{dy}{dx} = \frac{a}{a + bx} + \log \left(\frac{x}{a + bx} \right)$$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned}
 \frac{d^2y}{dx^2} &= \frac{-a}{(a+bx)^2} \cdot b \\
 &\quad + \frac{1}{\left(\frac{x}{a+bx}\right)} \cdot \left\{ \frac{(a+bx) \cdot 1 - x \cdot b}{(a+bx)^2} \right\} \\
 &= \frac{-ab}{(a+bx)^2} + \frac{a+bx}{x} \left\{ \frac{a+bx-bx}{(a+bx)^2} \right\} \\
 &= \frac{-ab}{(a+bx)^2} + \frac{a+bx}{x} \times \frac{a}{(a+bx)^2} \\
 &= \frac{-ab}{(a+bx)^2} + \frac{a}{x(a+bx)} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow x^3 \frac{d^2y}{dx^2} &= -\frac{abx^3}{(a+bx)^2} + \frac{ax^2}{(a+bx)} \\
 &\quad [\text{multiplying both sides by } x^3] \\
 &= \frac{ax^2}{(a+bx)^2} \{-bx + (a+bx)\} \\
 &= \frac{a^2x^2}{(a+bx)^2}
 \end{aligned}$$

$$\Rightarrow x^3 \frac{d^2y}{dx^2} = \left(\frac{ax}{a+bx} \right)^2 = \left(x \frac{dy}{dx} - y \right)^2$$

[from Eq. (ii)] (1)

Hence proved.

23. Differentiate the following function with respect to x .

$$(\log x)^x + x^{\log x}$$

Delhi 2013

Let $y = (\log x)^x + x^{\log x}$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \{(\log x)^x + x^{\log x}\} \\
 &= \frac{d}{dx} (\log x)^x + \frac{d}{dx} (x^{\log x})
 \end{aligned}$$

$$\begin{aligned}
&= (\log x)^x \frac{d}{dx} [\{x \log(\log x)\}] \\
&\quad + x^{\log x} \frac{d}{dx} (\log x \log x) \\
&\quad \left[\because \frac{d}{dx} (u^v) = u^v \frac{d}{dx} (v \log u) \right] \quad (2) \\
&= (\log x)^x \left\{ x \left(\frac{1}{\log x} \right) \frac{1}{x} + \log(\log x) \right\} \\
&\quad + x^{\log x} \left[2(\log x) \frac{1}{x} \right] \\
&\quad \left[\because \frac{d}{dx} \log(\log x) = \frac{1}{\log x} \times \frac{1}{x}, \right. \\
&\quad \left. \frac{d}{dx} (\log x \log x) = \frac{d}{dx} \{(\log x)^2\} = 2(\log x) \frac{1}{x} \right] \\
&= (\log x)^x \left\{ \frac{1}{\log x} + \log(\log x) \right\} \\
&\quad + 2 \left(\frac{\log x}{x} \right) x^{\log x} \quad (2)
\end{aligned}$$

24. If $y = \log[x + \sqrt{x^2 + a^2}]$, then show that

$$(x^2 + a^2) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = 0.$$

Delhi 2013

Given, $y = \log[x + \sqrt{x^2 + a^2}]$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}} \frac{d}{dx} \left(x + \sqrt{x^2 + a^2} \right)$$

$$\left[\because \frac{d}{dx} (\log x) = \frac{1}{x} \frac{d}{dx} (x) \right] \quad (1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}} \left(1 + \frac{2x}{2\sqrt{x^2 + a^2}} \right)$$

$$\left[\because \frac{d}{dx} (\sqrt{x^2 + a^2}) = \frac{1 \times 2x}{2\sqrt{x^2 + a^2}} \right]$$

$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \left(\frac{x + \sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}} \left(\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}} \right)$$

$$\Rightarrow \frac{dy}{dx} (\sqrt{x^2 + a^2}) = 1 \quad (1)$$

Again on differentiating both sides w.r.t. to x , we get

$$\sqrt{x^2 + a^2} \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx} (\sqrt{x^2 + a^2}) = \frac{d(1)}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} (\sqrt{x^2 + a^2}) + \frac{1 \cdot 2x \cdot \frac{dy}{dx}}{2\sqrt{x^2 + a^2}} = 0 \quad (1)$$

$$\left[\because \frac{d}{dx} (u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \right]$$

$$\Rightarrow (x^2 + a^2) + \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 0 \quad (1)$$

Hence proved.

- 25.** Show that the function $f(x)=|x-3|$, $x \in R$, is continuous but not differentiable at $x=3$.
Delhi 2013

💡 Firstly, to check the differentiability of the function $f(x)$ at $x=3$. Find LHD and RHD, if $LHD \neq RHD$, then function is not differentiable and then we check continuity of the function at $x=3$ by showing $LHL = RHL = f(3)$.

Given, $f(x)=|x-3|$

First, we check the differentiability of the given function $f(x)$ at $x=3$.

$$\text{LHD} = f'(3^-) = \lim_{h \rightarrow 0} \frac{f(3-h) - f(3)}{-h}$$

$$\left[\because Lf'(a) = \lim_{h \rightarrow 0} \frac{f(a-h) - f(a)}{-h} \right]$$

$$= \lim_{h \rightarrow 0} \frac{|3-h-3| - |3-3|}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{|-h|}{-h} = \lim_{h \rightarrow 0} \frac{h}{-h} = -1 \quad [\because |-x| = x] \quad (1)$$

$$\text{RHD} = f'(3^+) = \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$$

$$\left[\because Rf'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \right]$$

$$= \lim_{h \rightarrow 0} \frac{|3+h-3| - |3-3|}{h}$$

$$= \lim_{h \rightarrow 0} \frac{|h|}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1 \quad [\because |-x| = x] \quad (1)$$

Since, $LHD \neq RHD$ at $x=3$

So, f is not differentiable.

Now, we check the continuity of the given function $f(x)$ at $x=3$

$$\begin{aligned}\therefore \text{LHL} &= \lim_{x \rightarrow 3^-} |x-3| \\ &= \lim_{h \rightarrow 0} |3-h-3| \quad [\text{put } x = 3-h] \\ &= \lim_{h \rightarrow 0} |-h| = 0 \quad (1)\end{aligned}$$

$$\begin{aligned}\text{RHL} &= \lim_{x \rightarrow 3^+} |x-3| \\ &= \lim_{h \rightarrow 0} |3+h-3| = \lim_{h \rightarrow 0} |h| = 0 \quad [\text{put } x = x+h]\end{aligned}$$

$$\text{and } f(3) = |3-3| = 0$$

$$\text{Thus, } \text{LHL} = \text{RHL} = f(3)$$

$$\text{Hence, } f \text{ is continuous at } x=3. \quad (1)$$

26. If $x=a \sin t$ and $y=a(\cos t + \log \tan(t/2))$, then

$$\text{find } \frac{d^2y}{dx^2}.$$

Delhi 2013

Given, $y = a \left(\cos t + \log \tan \frac{t}{2} \right)$ and $x = a \sin t$

On differentiating both sides w.r.t. t , we get

$$\begin{aligned}\frac{dy}{dt} &= a \left[\frac{d}{dt}(\cos t) + \frac{d}{dt} \left(\log \tan \frac{t}{2} \right) \right] \\ \Rightarrow \frac{dy}{dt} &= a \left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)} \times \frac{d}{dt} \left(\tan \frac{t}{2} \right) \right] \\ \Rightarrow \frac{dy}{dt} &= a \left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)} \times \sec^2\left(\frac{t}{2}\right) \times \frac{1}{2} \right] \quad (1) \\ &= a \left[-\sin t + \frac{1}{2} \times \frac{\cos\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)} \times \frac{1}{\cos^2\left(\frac{t}{2}\right)} \right] \\ &= a \left[-\sin t + \frac{1}{2 \sin \frac{t}{2} \cdot \cos \frac{t}{2}} \right] \\ &= a \left[-\sin t + \frac{1}{\sin t} \right] \quad \left[\because \sin t = 2 \sin \frac{t}{2} \cdot \cos \frac{t}{2} \right] \\ &= a \left[\frac{1 - \sin^2 t}{\sin t} \right] = a \frac{\cos^2 t}{\sin t}\end{aligned}$$

$$[\because \sin^2 t + \cos^2 t = 1 \Rightarrow 1 - \sin^2 t = \cos^2 t] \quad (1)$$

$$\text{and } \frac{dx}{dt} = \frac{d}{dt}(a \sin t) = a \cos t \quad (1/2)$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left[\frac{a \cos^2 t}{\sin t} \right]}{a \cos t} = \cot t$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{dy}{dx} \right] = \frac{d}{dx} (\cot t) \quad (1/2)$$

$$= \frac{d}{dt} (\cot t) \times \left(\frac{dt}{dx} \right) = (-\operatorname{cosec}^2 t) \left(\frac{dt}{dx} \right)$$

$$= -(\operatorname{cosec}^2 t) \cdot \frac{1}{a \cos t}$$

$$= -\frac{\operatorname{cosec}^2 t}{a \cos t} \quad (1)$$

27. Differentiate the following with respect to x

$$\sin^{-1} \left[\frac{2^{x+1} \cdot 3^x}{1 + (36)^x} \right]. \quad \text{HOTS; All India 2013}$$



Firstly, put 6^x equal to $\tan \theta$, so that it becomes to some standard trigonometric function. Then, simplify the expression and then differentiate by chain rule.

$$\begin{aligned}\text{Let } y &= \sin^{-1} \left[\frac{2^{x+1} \cdot 3^x}{1 + (36)^x} \right] \\ &= \sin^{-1} \left(\frac{2 \cdot 2^x \cdot 3^x}{1 + (6^2)^x} \right) \\ &= \sin^{-1} \left[\frac{2 \cdot 6^x}{1 + (6^x)^2} \right] \quad (1)\end{aligned}$$

$$\text{Put } \tan \theta = 6^x \Rightarrow \theta = \tan^{-1}(6^x)$$

$$\begin{aligned}\text{Then, } y &= \sin^{-1} \left(\frac{2 \cdot \tan \theta}{1 + \tan^2 \theta} \right) \\ &= \sin^{-1}(\sin 2\theta) \left[\because \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} \right] \quad (1) \\ &= 2\theta\end{aligned}$$

$$\Rightarrow y = 2 \tan^{-1}(6^x)$$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{2}{1 + (6^x)^2} \frac{d}{dx}(6^x) \left[\because \frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2} \right] \\ &\Rightarrow \frac{dy}{dx} = \frac{2}{1 + (6^x)^2} \cdot 6^x \cdot \log 6 \\ &= \left[\frac{2^{x+1} \cdot 3^x}{1 + (36)^x} \right] \log 6 \quad (1)\end{aligned}$$

28. If $x = a \cos^3 \theta$ and $y = a \sin^3 \theta$, then find the

$$\text{value of } \frac{d^2 y}{dx^2} \text{ at } \theta = \frac{\pi}{6}.$$

All India 2013

$$\text{Given, } x = a \cos^3 \theta \text{ and } y = a \sin^3 \theta$$

On differentiating both sides w.r.t. θ , we get

$$\begin{aligned}
 \frac{dx}{d\theta} &= 3a \cos^2 \theta \frac{d}{d\theta} (\cos \theta) \\
 &= 3a \cos^2 \theta \cdot (-\sin \theta) \\
 &= -3a \cos^2 \theta \cdot \sin \theta \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \frac{dy}{d\theta} &= 3a \sin^2 \theta \frac{d}{d\theta} (\sin \theta) \\
 &= 3a \sin^2 \theta \cdot (\cos \theta) = 3a \sin^2 \theta \cdot \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } \frac{dy}{dx} &= \left(\frac{dy/d\theta}{dx/d\theta} \right) \\
 &= \frac{3a \sin^2 \theta \cdot \cos \theta}{-3a \cos^2 \theta \cdot \sin \theta} = -\tan \theta \quad (1)
 \end{aligned}$$

Again, on differentiating both sides w.r.t. x , we get

$$\begin{aligned}
 \frac{d^2y}{dx^2} &= -\frac{d}{d\theta} (\tan \theta) \frac{d\theta}{dx} \\
 &= -\sec^2 \theta \cdot \frac{d\theta}{dx} \\
 &= -\sec^2 \theta \cdot \left(\frac{-1}{3a \cos^2 \theta \cdot \sin \theta} \right)
 \end{aligned}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{3a \cos^4 \theta \cdot \sin \theta} \quad (1)$$

At

$$\begin{aligned}
 \theta = \frac{\pi}{6}, \left(\frac{d^2y}{dx^2} \right)_{\left(\theta = \frac{\pi}{6} \right)} &= \frac{1}{3a \left(\cos \frac{\pi}{6} \right)^4 \left(\sin \frac{\pi}{6} \right)} \\
 &= \frac{1}{3a \left(\frac{\sqrt{3}}{2} \right)^4 \left(\frac{1}{2} \right)} \\
 &= \frac{1}{3a \left(\frac{9}{16} \right) \left(\frac{1}{2} \right)} = \frac{32}{27a} \quad (1)
 \end{aligned}$$

29. If $x \sin(a + y) + \sin a \cos(a + y) = 0$, then prove

$$\text{that } \frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}.$$

All India 2013

$$\text{To prove, } \frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$$

$$\text{Given, } x \sin(a + y) + \sin a \cos(a + y) = 0$$

$$\Rightarrow x = \frac{-\sin a \cos(a + y)}{\sin(a + y)} \quad (1)$$

On differentiating both sides w.r.t. y , we get

$$\frac{dx}{dy} = \frac{- \left[\sin(a + y) \frac{d}{dy} \{ \sin a \cos(a + y) \} - \sin a \cos(a + y) \frac{d}{dy} \{ \sin(a + y) \} \right]}{\sin^2(a + y)}$$

[using quotient rule]

$$= \left\{ \frac{\sin(a + y) \cdot \sin a \sin(a + y) + \sin a \cos(a + y) \cos(a + y)}{\sin^2(a + y)} \right\} (1)$$

$$= \frac{\sin a}{\sin^2(a + y)} \{ \sin^2(a + y) + \cos^2(a + y) \}$$

$$= \frac{\sin a}{\sin^2(a + y)} \cdot 1 \quad [\because \sin^2 \theta + \cos^2 \theta = 1] (1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$$

Hence proved. (1)

30. If $x^y = e^{x-y}$, then prove that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$ or

$$\frac{dy}{dx} = \frac{\log x}{\{\log(xe)\}^2}.$$

All India 2013, 2011, 2010



Firstly, take log on both sides and convert it into $y = f(x)$ form. Then, differentiate both sides by quotient rule to get required result.

To prove, $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$

Given $x^y = e^{x-y}$

On taking log both sides, we get

$$y \log_e x = (x - y) \log_e e \quad (1)$$

$$\Rightarrow y \log_e x = x - y \quad [\because \log_e e = 1]$$

$$\Rightarrow y(1 + \log x) = x$$

$$\Rightarrow y = \frac{x}{1 + \log x} \quad (1)$$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1 + \log x) \frac{d}{dx}(x) - x \frac{d}{dx}(1 + \log x)}{(1 + \log x)^2} \\ &\quad \left[\because \frac{d}{dx} \left(\frac{v}{u} \right) = \frac{u \frac{dv}{dx} - v \frac{du}{dx}}{u^2} \right] (1) \\ &= \frac{1 + \log x - x \cdot \frac{1}{x}}{(1 + \log x)^2} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1 + \log x - 1}{(1 + \log x)^2} \\
 &= \frac{\log x}{(1 + \log x)^2} \quad \text{Hence proved. (1)}
 \end{aligned}$$

Also, it can be written as

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{\log x}{(\log_e e + \log x)^2} \quad [-1 \log_e e = 1] \\
 \Rightarrow \frac{dy}{dx} &= \frac{\log x}{\{\log(ex)\}^2}
 \end{aligned}$$

31. If $y^x = e^{y-x}$, then prove that $\frac{dy}{dx} = \frac{(1 + \log y)^2}{\log y}$.

All India 2013

To prove, $\frac{dy}{dx} = \frac{(1 + \log y)^2}{\log y}$

Given that $y^x = e^{y-x}$

On taking log both sides, we get

$$\log y^x = \log e^{(y-x)}$$

$$\Rightarrow x \log y = (y-x) \log e$$

$$\Rightarrow x \log y = y - x \quad [\because \log e = 1] \dots(i) \quad (1)$$

On differentiating both sides w.r.t. x , we get

$$x \cdot \frac{d}{dx} (\log y) + \log y \frac{d}{dx} (x) = \frac{d}{dx} (y) - \frac{d}{dx} (x)$$

$$\Rightarrow x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + \log y \cdot 1 = \frac{dy}{dx} - 1$$

$$\Rightarrow (1 + \log y) = \frac{dy}{dx} \left(1 - \frac{x}{y} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(1 + \log y)}{(y - x)} \dots(ii) \quad (1)$$

On putting the value of x from Eq. (i) in Eq. (ii), we get

$$\frac{dy}{dx} = \frac{y(1 + \log y)}{y - \left(\frac{y}{1 + \log y} \right)} = \frac{y(1 + \log y)^2}{(y + y \log y - y)} \quad (1)$$

$$= \frac{y(1 + \log y)^2}{y \log y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1 + \log y)^2}{\log y} \quad (1)$$

Hence proved.

32. If $(\cos x)^y = (\cos y)^x$, then find $\frac{dy}{dx}$.

HOTS; Delhi 2012



Firstly, take log on both sides, then differentiate both sides by product rule.

Given, $(\cos x)^y = (\cos y)^x$

On taking log both sides, we get

$$\log (\cos x)^y = \log (\cos y)^x$$

$$\Rightarrow y \log (\cos x) = x \log (\cos y)$$

$$[\because \log x^n = n \log x] \quad (1)$$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned} y \cdot \frac{d}{dx} \log (\cos x) + \log (\cos x) \cdot \frac{d}{dx} (y) \\ = x \frac{d}{dx} \log (\cos y) + \log (\cos y) \frac{d}{dx} (x) \\ \left[\because \frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx} \right] \quad (1) \end{aligned}$$

$$\begin{aligned} \Rightarrow y \cdot \frac{1}{\cos x} \frac{d}{dx} (\cos x) + \log (\cos x) \frac{dy}{dx} \\ = x \cdot \frac{1}{\cos y} \frac{d}{dx} (\cos y) + \log (\cos y) \cdot 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow y \cdot \frac{1}{\cos x} (-\sin x) + \log (\cos x) \cdot \frac{dy}{dx} \\ = x \frac{1}{\cos y} (-\sin y) \cdot \frac{dy}{dx} + \log (\cos y) \cdot 1 \quad (1) \end{aligned}$$

$$\begin{aligned} \Rightarrow -y \tan x + \log (\cos x) \frac{dy}{dx} = -x \tan y \frac{dy}{dx} \\ + \log (\cos y) \end{aligned}$$

$$\Rightarrow [x \tan y + \log (\cos x)] \frac{dy}{dx} = \log (\cos y) + y \tan x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log (\cos y) + y \tan x}{x \tan y + \log (\cos x)} \quad (1)$$

33. If $\sin y = x \sin(a + y)$, then prove that

$$\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a} \quad \text{HOTS; Delhi 2012}$$



In the given expression, we collect all the terms of y on RHS and a term x on LHS and then differentiate with respect to y on both sides to get required result.

Given, $\sin y = x \sin(a + y)$

$$\Rightarrow x = \frac{\sin y}{\sin(a + y)}$$

On differentiating both sides w.r.t. y , we get

$$\begin{aligned} \frac{dx}{dy} &= \frac{\sin(a + y) \cdot \frac{d}{dy}(\sin y) - \sin y \cdot \frac{d}{dy} \sin(a + y)}{\sin^2(a + y)} \\ &\quad \text{[by using quotient rule] (1½)} \\ &= \frac{\sin(a + y) \cos y - \sin y \cos(a + y)}{\sin^2(a + y)} \\ &= \frac{\sin(a + y - y)}{\sin^2(a + y)} \quad (1½) \\ &\quad [\because \sin A \cos B - \cos A \sin B = \sin(A - B)] \end{aligned}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\sin a}{\sin^2(a + y)}$$

$$\text{Hence, } \frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a} \quad (1)$$

Hence proved.

NOTE As the result is in y form, so we consider here x as a dependent variable.

34. If $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$, then show that $\frac{dy}{dx} = \frac{-y}{x}$.

All India 2012

Given, $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$

To show, $\frac{dy}{dx} = \frac{-y}{x}$

Now $x = (a^{\sin^{-1} t})^{1/2}$

Now, $x = (a^{\sin^{-1} t})^{1/2}$

On differentiating both sides w.r.t. t , we get

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{2} (a^{\sin^{-1} t})^{-1/2} \frac{d}{dt} (a^{\sin^{-1} t}) \\ &\quad \left[\because \frac{d}{dx} (x^n) = nx^{n-1} \right] \\ &= \frac{1}{2} (a^{\sin^{-1} t})^{-1/2} a^{\sin^{-1} t} \log a \frac{d}{dt} (\sin^{-1} t) \\ &\quad \left[\because \frac{d}{dx} (a^x) = a^x \log a \right] \\ &= \frac{1}{2} (a^{\sin^{-1} t})^{-1/2} a^{\sin^{-1} t} \log a \cdot \frac{1}{\sqrt{1-t^2}} \\ &= \frac{1}{2} (a^{\sin^{-1} t})^{1/2} \log a \cdot \frac{1}{\sqrt{1-t^2}} \\ \Rightarrow \frac{dx}{dt} &= \frac{\frac{1}{2} \sqrt{a^{\sin^{-1} t}} \cdot \log a}{\sqrt{1-t^2}} \quad \dots(i) \quad (1\frac{1}{2})\end{aligned}$$

And $y = (a^{\cos^{-1} t})^{1/2}$

On differentiating both sides w.r.t. t , we get

$$\begin{aligned}\frac{dy}{dt} &= \frac{1}{2} (a^{\cos^{-1} t})^{-1/2} \frac{d}{dt} (a^{\cos^{-1} t}) \\ &\quad \left[\because \frac{d}{dx} x^n = nx^{n-1} \right] \\ &= \frac{1}{2} (a^{\cos^{-1} t})^{-1/2} a^{\cos^{-1} t} \log a \frac{d}{dt} (\cos^{-1} t) \\ &= \frac{1}{2} (a^{\cos^{-1} t})^{1/2} \log a \cdot \frac{(-1)}{\sqrt{1-t^2}} \\ \Rightarrow \frac{dy}{dt} &= \frac{-\frac{1}{2} \sqrt{a^{\cos^{-1} t}} \cdot \log a}{\sqrt{1-t^2}} \quad \dots(ii) \quad (1\frac{1}{2})\end{aligned}$$

On dividing Eq. (ii) by Eq. (i), we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-\frac{1}{2}\sqrt{a^{\cos^{-1}t}} \log a}{\sqrt{1-t^2}}\right)}{\left(\frac{\frac{1}{2}\sqrt{a^{\sin^{-1}t}} \log a}{\sqrt{1-t^2}}\right)} \\ &= -\frac{\sqrt{a^{\cos^{-1}t}}}{\sqrt{a^{\sin^{-1}t}}} = -\frac{y}{x} \quad (1) \\ [\because \text{ given } \sqrt{a^{\cos^{-1}t}} = y \text{ and } \sqrt{a^{\sin^{-1}t}} = x]\end{aligned}$$

Hence proved.

35. Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ w.r.t. x .

HOTS; All India 2012

💡 Firstly, put $x = \tan \theta$ and convert y in terms of θ , then put $\theta = \tan^{-1} x$ and differentiate w.r.t. x .

$$\text{Let } y = \tan^{-1} \left[\frac{\sqrt{1+x^2} - 1}{x} \right]$$

Put $x = \tan \theta \Rightarrow \theta = \tan^{-1} x$, we get

$$y = \tan^{-1} \frac{\sqrt{1+\tan^2 \theta} - 1}{\tan \theta} \quad (1)$$

$$y = \tan^{-1} \left[\frac{\sec \theta - 1}{\tan \theta} \right] \quad [\because 1 + \tan^2 \theta = \sec^2 \theta]$$

$$= \tan^{-1} \left[\frac{\frac{1}{\cos \theta} - 1}{\frac{\sin \theta}{\cos \theta}} \right]$$

$$\left[\because \sec \theta = \frac{1}{\cos \theta} \text{ and } \tan \theta = \frac{\sin \theta}{\cos \theta} \right]$$

$$= \tan^{-1} \left[\frac{1 - \cos \theta}{\sin \theta} \right]$$

$$= \tan^{-1} \left[\frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right]$$

$$\left[\because 1 - \cos x = 2 \sin^2 \frac{x}{2} \text{ and } \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \right]$$

$$= \tan^{-1} \left(\tan \frac{\theta}{2} \right) = \frac{\theta}{2} \quad [\because \tan^{-1} (\tan \phi) = \phi]$$

$$\Rightarrow y = \frac{\tan^{-1} x}{2} \quad [\because \theta = \tan^{-1} x] \quad (1\frac{1}{2})$$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1+x^2} \left[\because \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2} \right]$$

$$\text{Hence, } \frac{dy}{dx} = \frac{1}{2(1+x^2)} \quad (1\frac{1}{2})$$

36. If $y = (\tan^{-1} x)^2$, then show that

$$(x^2 + 1)^2 \frac{d^2 y}{dx^2} + 2x(x^2 + 1) \frac{dy}{dx} = 2.$$

Delhi 2012

Given, $y = (\tan^{-1} x)^2$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = 2 \tan^{-1} x \cdot \frac{1}{1+x^2} \left[\because \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2 \tan^{-1} x}{1+x^2}$$

$$\Rightarrow (1+x^2) \frac{dy}{dx} = 2 \tan^{-1} x \quad (1\frac{1}{2})$$

Again, differentiating both sides w.r.t. x , we get

$$\begin{aligned} (1+x^2) \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx} (1+x^2) \\ = \frac{d}{dx} (2 \tan^{-1} x) \quad (1) \end{aligned}$$

$$\begin{aligned} \Rightarrow (1+x^2) \cdot \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot 2x &= \frac{2}{1+x^2} \\ \left[\because \frac{d}{dx} (\tan^{-1} x) &= \frac{1}{1+x^2} \right] \end{aligned}$$

$$\Rightarrow (1+x^2)^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} (1+x^2) = 2$$

$$\Rightarrow (1+x^2)^2 \frac{d^2 y}{dx^2} + 2x(1+x^2) \frac{dy}{dx} = 2 \quad (1\frac{1}{2})$$

Hence proved.

37. If $y = x^{\sin x - \cos x} + \frac{x^2 - 1}{x^2 + 1}$, then find $\frac{dy}{dx}$.

Delhi 2012C

Given, $y = x^{\sin x - \cos x} + \frac{x^2 - 1}{x^2 + 1}$

Let $u = x^{\sin x - \cos x}$

and $v = \frac{x^2 - 1}{x^2 + 1}$

Consider $u = x^{\sin x - \cos x}$

On taking log both sides, we get

$$\log u = (\sin x - \cos x) \cdot \log x$$

On differentiating both sides w.r.t. x , we get

$$\frac{1}{u} \frac{du}{dx} = (\sin x - \cos x) \cdot \frac{1}{x} + \log x \cdot (\cos x + \sin x)$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x - \cos x}$$

$$\left[\frac{\sin x - \cos x}{x} + \log x \cdot (\cos x + \sin x) \right]$$

(1½)

Now, consider $v = \frac{x^2 - 1}{x^2 + 1} = 1 - \frac{2}{x^2 + 1}$

On differentiating both sides w.r.t. x , we get

$$\frac{dv}{dx} = 0 - \frac{(x^2 + 1) \frac{d}{dx} (2) - 2 \frac{d}{dx} (x^2 + 1)}{(x^2 + 1)^2}$$

$$\Rightarrow \frac{dv}{dx} = - \left[\frac{0 - 2 \cdot 2x}{(x^2 + 1)^2} \right] = \frac{4x}{(x^2 + 1)^2} \quad (1½)$$

Now, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$$= x^{\sin x - \cos x} \left[\frac{\sin x - \cos x}{x} \right]$$

$$+ \log x (\cos x + \sin x) + \frac{4x}{(x^2 + 1)^2} \quad (1)$$

38. If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$, then

find $\frac{d^2y}{dx^2}$ and $\frac{d^2y}{dt^2}$.

Delhi 2012C

Given, $x = a(\cos t + t \sin t)$

On differentiating both sides w.r.t. t , we get

$$\begin{aligned}\frac{dx}{dt} &= a(-\sin t + 1 \cdot \sin t + t \cos t) \quad (1) \\ &= at \cos t\end{aligned}$$

Also given, $y = a(\sin t - t \cos t)$

On differentiating both sides w.r.t. t , we get

$$\frac{dy}{dt} = a(\cos t - \cos t \cdot 1 + t \sin t) = at \sin t \quad \dots(i) \quad (1)$$

Now,
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at \sin t}{at \cos t} = \tan t$$

Again, differentiating both sides w.r.t. x , we get

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} (\tan t) \frac{dt}{dx} = \sec^2 t \frac{1}{dx/dt} \\ &= \frac{\sec^2 t}{at \cos t} = \frac{\sec^3 t}{at} \quad (1)\end{aligned}$$

Also,

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{d}{dt} (at \sin t) \\ &= a(\sin t + t \cos t) \quad (1)\end{aligned}$$

39. Find $\frac{dy}{dx}$, when $y = x^{\cot x} + \frac{2x^2 - 3}{x^2 + x + 2}$.

All India 2012C

Given, $y = x^{\cot x} + \frac{2x^2 - 3}{x^2 + x + 2}$

Let $u = x^{\cot x}$ and $v = \frac{2x^2 - 3}{x^2 + x + 2}$

Consider $u = x^{\cot x}$

Consider $u = x^{\cot x}$

On taking log both sides, we get

$$\log u = \cot x \log x$$

On differentiating both sides w.r.t. x , we get

$$\frac{1}{u} \frac{du}{dx} = \cot x \cdot \frac{1}{x} - \operatorname{cosec}^2 x \cdot \log x$$

$$\begin{aligned} \Rightarrow \frac{du}{dx} &= u \left(\frac{\cot x}{x} - \operatorname{cosec}^2 x \cdot \log x \right) \\ &= x^{\cot x} \left(\frac{\cot x}{x} - \operatorname{cosec}^2 x \cdot \log x \right) \quad (1\frac{1}{2}) \end{aligned}$$

$$\text{Now, consider } v = \frac{2x^2 - 3}{x^2 + x + 2}$$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned} \frac{dv}{dx} &= \frac{(x^2 + x + 2)(4x) - (2x^2 - 3)(2x + 1)}{(x^2 + x + 2)^2} \\ &= \frac{4x^3 + 4x^2 + 8x - 4x^3 - 2x^2 + 6x + 3}{(x^2 + x + 2)^2} \\ &= \frac{2x^2 + 14x + 3}{(x^2 + x + 2)^2} \quad (1\frac{1}{2}) \end{aligned}$$

$$\begin{aligned} \text{Now, } \frac{dy}{dx} &= \frac{du}{dx} + \frac{dv}{dx} \\ &= x^{\cot x} \left(\frac{\cot x}{x} - \operatorname{cosec}^2 x \cdot \log x \right) \\ &\quad + \frac{2x^2 + 14x + 3}{(x^2 + x + 2)^2} \quad (1) \end{aligned}$$

40. If $x = \cos t + \log \tan \frac{t}{2}$ and $y = \sin t$, then find the

values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. All India 2012C

Given, $x = \cos t + \log \tan\left(\frac{t}{2}\right)$

On differentiating both sides w.r.t. t , we get

$$\frac{dx}{dt} = -\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2}$$

$$= -\sin t + \frac{\cos(t/2)}{\sin(t/2)} \cdot \frac{1}{\cos^2\left(\frac{t}{2}\right)} \cdot \frac{1}{2}$$

$$\begin{aligned}
 &= -\sin t + \frac{1}{2 \sin\left(\frac{t}{2}\right) \cos\left(\frac{t}{2}\right)} \\
 &= -\sin t + \frac{1}{\sin t} = \frac{-\sin^2 t + 1}{\sin t} = \frac{\cos^2 t}{\sin t} \quad (1) \\
 &\quad \left[\because 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \sin \theta \right]
 \end{aligned}$$

Also given, $y = \sin t$

On differentiating both sides w.r.t. t , we get

$$\frac{dy}{dt} = \cos t$$

Again, differentiating both sides w.r.t. t , we get

$$\frac{d^2y}{dt^2} = -\sin t \quad (1)$$

$$\text{Now, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{\cos^2 t} = \frac{\sin t}{\cos t} = \tan t$$

On differentiating both sides w.r.t. x , we get

$$\frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx} = \sec^2 t \cdot \frac{\sin t}{\cos^2 t} = \sec^4 t \sin t$$

$$\text{At } t = \frac{\pi}{4}, \quad \frac{d^2y}{dt^2} = -\sin \frac{\pi}{4} = -\frac{1}{\sqrt{2}} \quad (1)$$

$$\begin{aligned}
 \text{At } t = \frac{\pi}{4}, \quad \frac{d^2y}{dx^2} &= \sec^4 \frac{\pi}{4} \cdot \sin \frac{\pi}{4} \\
 &= (\sqrt{2})^4 \cdot \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \quad (1)
 \end{aligned}$$

- 41.** If $x\sqrt{1+y} + y\sqrt{1+x} = 0, (x \neq y)$, then prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

HOTS; Foreign 2012; Delhi 2011C



Firstly, solve the given equation and convert it into $y = f(x)$ form. Then, differentiate to get the required result.

To prove, $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

Given equation is $x\sqrt{1+y} + y\sqrt{1+x} = 0$,
where $x \neq y$, we first convert the given equation into $y = f(x)$ form.

So, $x\sqrt{1+y} = -y\sqrt{1+x}$

On squaring both sides, we get

$$x^2(1+y) = y^2(1+x)$$

$$\Rightarrow x^2 + x^2y = y^2 + y^2x$$

$$\Rightarrow x^2 - y^2 = y^2x - x^2y$$

$$\Rightarrow (x - y)(x + y) = -xy(x - y)$$

$$[\because a^2 - b^2 = (a - b)(a + b)]$$

$$\Rightarrow (x - y)(x + y) + xy(x - y) = 0$$

$$\Rightarrow (x - y)(x + y + xy) = 0$$

$$\therefore \text{ Either } x - y = 0 \text{ or } x + y + xy = 0$$

$$\text{Now, } x - y = 0 \Rightarrow x = y$$

But it is given that $x \neq y$.

So, we get a contradiction.

$$\Rightarrow x - y = 0 \text{ is rejected.} \quad (1)$$

$$\therefore y + xy + x = 0 \Rightarrow y(1 + x) = -x$$

$$\Rightarrow y = \frac{-x}{1 + x} \quad \dots(i) \quad (1)$$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{(1 + x) \times \frac{d}{dx}(-x) - (-x) \times \frac{d}{dx}(1 + x)}{(1 + x)^2} \quad (1)$$

$$\left[\because \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1 + x)(-1) + x(1)}{(1 + x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1 - x + x}{(1 + x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{(1 + x)^2} \quad (1)$$

Hence proved

42. If $x = \tan \left(\frac{1}{a} \log y \right)$, then show that

$$(1 + x^2) \frac{d^2 y}{dx^2} + (2x - a) \frac{dy}{dx} = 0.$$

All India 2011

$$\text{Given, } x = \tan\left(\frac{1}{a} \log y\right)$$

$$\Rightarrow \tan^{-1} x = \frac{1}{a} \log y$$

$$[\because \tan \theta = a \Rightarrow \theta = \tan^{-1} a]$$

$$\Rightarrow a \tan^{-1} x = \log y$$

On differentiating both sides w.r.t. x , we get

$$a \times \frac{1}{1+x^2} = \frac{1}{y} \cdot \frac{dy}{dx} \quad (1)$$

$$\left[\because \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2} \right]$$

$$\Rightarrow (1+x^2) \frac{dy}{dx} = ay \quad (1)$$

Again, on differentiating both sides w.r.t. x , we get

$$(1+x^2) \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \frac{d}{dx} (1+x^2) = \frac{d}{dx} (ay)$$

$$\left[\because \frac{d}{dx} (u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx} \right] \quad (1)$$

$$\Rightarrow (1+x^2) \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot (2x) = a \cdot \frac{dy}{dx}$$

$$\Rightarrow (1+x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - a \frac{dy}{dx} = 0$$

$$\Rightarrow (1+x^2) \frac{d^2y}{dx^2} + (2x-a) \frac{dy}{dx} = 0 \quad (1)$$

43. Differentiate $x^{x \cos x} + \frac{x^2+1}{x^2-1}$ w.r.t. x .

Delhi 2011

Let $u = x^{x \cos x}$ and $v = \frac{x^2 + 1}{x^2 - 1}$, then $y = u + v$.

Now, find $\frac{du}{dx}$ and $\frac{dv}{dx}$. Then, put these values in

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

Let $y = x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$

Again let $u = x^{x \cos x}$ and $v = \frac{x^2 + 1}{x^2 - 1}$

Then, $y = u + v$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(i) \text{ (1)}$$

Now, $u = x^{x \cos x}$

On taking log both sides, we get

$$\log u = \log x^{x \cos x}$$

$$\Rightarrow \log u = (x \cos x) \cdot \log x \quad [\because \log m^n = n \log m]$$

On differentiating both sides w.r.t. x , we get

$$\frac{1}{u} \frac{du}{dx} = (x \cos x) \times \frac{d}{dx} (\log x) + \log x \times \frac{d}{dx} (x \cos x)$$

[by product rule]

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = x \cos x \times \frac{1}{x} + \log x \cdot [-x \sin x + \cos x]$$

$$\left[\begin{aligned} \because \frac{d}{dx} (x \cos x) &= x \times \frac{d}{dx} \cos x + \cos x \frac{d}{dx} x \\ &= x (-\sin x) + \cos x \cdot 1 \\ &= -x \sin x + \cos x \end{aligned} \right]$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = \cos x - x \log x \sin x + \log x \cos x$$

$$\Rightarrow \frac{du}{dx} = u [\cos x - x \log x \sin x + \log x \cos x]$$

$$\Rightarrow \frac{du}{dx} = x^{x \cos x} [\cos x - x \log x \sin x + \log x \cos x] [\because u = x^{x \cos x}] \quad (1\frac{1}{2})$$

$$\text{and } v = \frac{x^2 + 1}{x^2 - 1}$$

On differentiating w.r.t. x , we get

$$\frac{dv}{dx} = \frac{(x^2 - 1) \times \frac{d}{dx}(x^2 + 1) - (x^2 + 1) \times \frac{d}{dx}(x^2 - 1)}{(x^2 - 1)^2}$$

[by quotient rule]

$$\Rightarrow \frac{dv}{dx} = \frac{(x^2 - 1) \cdot 2x - (x^2 + 1) \cdot 2x}{(x^2 - 1)^2}$$

$$= \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2 - 1)^2}$$

$$\Rightarrow \frac{dv}{dx} = \frac{-4x}{(x^2 - 1)^2}$$

On putting values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in Eq. (i), we get

$$\frac{dy}{dx} = x^{x \cos x} [\cos x - x \log x \sin x + \log x \cos x] - \frac{4x}{(x^2 - 1)^2} \quad (1\frac{1}{2})$$

44. If $x = a(\theta - \sin \theta)$, $y = a(1 + \cos \theta)$, then find

$$\frac{d^2y}{dx^2}.$$

Delhi 2011



Here, we use chain rule, i.e. if $y = f_1(\theta)$ and $x = f_2(\theta)$, then $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$ to get required value.

Given, $x = a(\theta - \sin \theta)$ and $y = a(1 + \cos \theta)$

On differentiating both sides w.r.t. θ , we get

$$\frac{dx}{d\theta} = a(1 - \cos \theta) \quad \left[\because \frac{d}{d\theta} \sin \theta = \cos \theta \right]$$

$$\text{and } \frac{dy}{d\theta} = -a \sin \theta \quad \left[\because \frac{d}{d\theta} \cos \theta = -\sin \theta \right] \quad (1)$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\left(\frac{dy}{d\theta} \right)}{\left(\frac{dx}{d\theta} \right)} \\ &= \frac{-a \sin \theta}{a(1 - \cos \theta)} = \frac{-\sin \theta}{1 - \cos \theta} \end{aligned}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} = -\cot \frac{\theta}{2} \quad (1)$$

$$\left[\begin{array}{l} \because \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\ \text{and } 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2} \end{array} \right]$$

Again differentiating w.r.t. x , we get

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(-\cot \frac{\theta}{2} \right) \\ &= \frac{d}{d\theta} \left(-\cot \frac{\theta}{2} \right) \times \frac{d\theta}{dx} \\ &\quad \left[\because \frac{d}{dx} [f(\theta)] = \frac{d}{d\theta} f(\theta) \times \frac{d\theta}{dx} \right] \\ &= \frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2} \times \frac{1}{a(1 - \cos \theta)} \quad (1) \\ &\quad \left[\because \frac{d}{d\theta} \cot \theta = -\operatorname{cosec}^2 \theta \right] \\ &= \frac{1}{2a} \operatorname{cosec}^2 \frac{\theta}{2} \times \frac{1}{2 \sin^2 \frac{\theta}{2}} \\ &\quad \left[\because 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2} \right] \\ &= \frac{1}{4a} \operatorname{cosec}^4 \frac{\theta}{2} \quad (1) \end{aligned}$$

45. Prove that

$$\frac{d}{dx} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) \right] = \sqrt{a^2 - x^2}.$$

Foreign 2011

$$\text{To prove } \frac{d}{dx} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) \right]$$

$$= \sqrt{a^2 - x^2}$$

$$\text{LHS} = \frac{d}{dx} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]$$

$$= \left[\frac{x}{2} \times \frac{d}{dx} \sqrt{a^2 - x^2} + \sqrt{a^2 - x^2} \right.$$

$$\left. \times \frac{d}{dx} \left(\frac{x}{2} \right) + \frac{a^2}{2} \times \frac{d}{dx} \sin^{-1} \frac{x}{a} \right] \quad (1)$$

$$\left[\because \frac{d}{dx} (u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx} \right]$$

$$= \frac{x}{2} \cdot \frac{1}{2 \sqrt{a^2 - x^2}} \frac{d}{dx} (a^2 - x^2)$$

$$+ \sqrt{a^2 - x^2} \cdot \frac{1}{2} + \frac{a^2}{2} \cdot \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} \cdot \frac{d}{dx} \left(\frac{x}{a} \right)$$

$$= \left[\frac{x}{2} \cdot \frac{-2x}{2 \sqrt{a^2 - x^2}} + \sqrt{a^2 - x^2} \cdot \frac{1}{2} \right.$$

$$\left. + \frac{a^2}{2} \times \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} \cdot \frac{1}{a} \right] \quad (1)$$

$$= \frac{-x^2}{2 \sqrt{a^2 - x^2}} + \frac{\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2a} \cdot \frac{1}{\sqrt{\frac{a^2 - x^2}{a^2}}}$$

$$= \frac{-x^2}{2 \sqrt{a^2 - x^2}} + \frac{\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2a} \cdot \frac{1}{\sqrt{\frac{a^2 - x^2}{a^2}}}$$

$$\begin{aligned}
&= \frac{-x^2}{2\sqrt{a^2 - x^2}} + \frac{\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2\sqrt{a^2 - x^2}} \quad (1) \\
&= \frac{-x^2 + (a^2 - x^2) + a^2}{2\sqrt{a^2 - x^2}} \\
&= \frac{2a^2 - 2x^2}{2\sqrt{a^2 - x^2}} = \frac{2(a^2 - x^2)}{2\sqrt{a^2 - x^2}} \\
&= \frac{a^2 - x^2}{(a^2 - x^2)^{1/2}} = (a^2 - x^2)^{1/2} = \sqrt{a^2 - x^2} \\
&= \text{RHS} \qquad \qquad \qquad \text{Hence proved. (1)}
\end{aligned}$$

46. If $y = \log [x + \sqrt{x^2 + 1}]$, then prove that

$$(x^2 + 1) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = 0.$$

Foreign 2011

Do same as Que 24.

[Hint put $a = 1$ in Que. 24]

47. If $\log (\sqrt{1+x^2} - x) = y \sqrt{1+x^2}$, then show

$$\text{that } (1+x^2) \frac{dy}{dx} + xy + 1 = 0.$$

All India 2011C

$$\text{To prove, } (1+x^2) \frac{dy}{dx} + xy + 1 = 0$$

$$\text{Given, } \log (\sqrt{1+x^2} - x) = y \sqrt{1+x^2} \quad \dots(i)$$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned}
&\frac{1}{\sqrt{1+x^2} - x} \frac{d}{dx} [\sqrt{1+x^2} - x] \\
&= y \frac{d}{dx} \sqrt{1+x^2} + \sqrt{1+x^2} \frac{dy}{dx} \quad (1)
\end{aligned}$$

[by chain rule]

$$\begin{aligned}
&\Rightarrow \frac{1}{\sqrt{1+x^2} - x} \left[\frac{1}{2\sqrt{1+x^2}} \frac{d}{dx}(x^2) - 1 \right] \\
&\quad = \frac{y}{2\sqrt{1+x^2}} \frac{d}{dx}(1+x^2) + \sqrt{1+x^2} \frac{dy}{dx} \\
&\Rightarrow \frac{1}{\sqrt{1+x^2} - x} \left[\frac{2x}{2\sqrt{1+x^2}} - 1 \right] \\
&\quad = y \times \frac{2x}{2\sqrt{1+x^2}} + \sqrt{1+x^2} \cdot \frac{dy}{dx} \quad (1\frac{1}{2}) \\
&\Rightarrow \frac{1}{\sqrt{1+x^2} - x} \left[\frac{x - \sqrt{1+x^2}}{\sqrt{1+x^2}} \right] \\
&\quad = \frac{xy}{\sqrt{1+x^2}} + \sqrt{1+x^2} \cdot \frac{dy}{dx} \\
&\Rightarrow \frac{-1}{\sqrt{1+x^2}} = \frac{xy + (1+x^2) \frac{dy}{dx}}{\sqrt{1+x^2}} \\
&\Rightarrow -1 = xy + (1+x^2) \frac{dy}{dx} \\
&\Rightarrow (1+x^2) \frac{dy}{dx} + xy + 1 = 0 \quad (1\frac{1}{2})
\end{aligned}$$

Hence proved.

48. If $x = a(\theta + \sin \theta)$ and $y = a(1 - \cos \theta)$, then

find $\frac{d^2y}{dx^2}$.

All India 2011C

Given,

$$x = a(\theta + \sin \theta) \text{ and } y = a(1 - \cos \theta) \quad \dots(i)$$

On differentiating both sides w.r.t. x , we get

$$\frac{dx}{d\theta} = a(1 + \cos \theta) \left[\because \frac{d}{d\theta} \sin \theta = \cos \theta \right]$$

$$\text{and } \frac{dy}{d\theta} = a \sin \theta \left[\because \frac{d}{d\theta} \cos \theta = -\sin \theta \right]$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a \sin \theta}{a(1 + \cos \theta)}$$

$$= \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}$$

$$\left[\begin{array}{l} \because \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \\ \text{and } 1 + \cos x = 2 \cos^2 \frac{x}{2} \end{array} \right]$$

$$\Rightarrow \frac{dy}{dx} = \tan \frac{\theta}{2} \quad (1)$$

Again, differentiating w.r.t. x , we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\tan \frac{\theta}{2} \right) \quad (1/2)$$

$$\left[\because \frac{d}{dx} f(\theta) = \frac{d}{d\theta} f(\theta) \cdot \frac{d\theta}{dx} \right]$$

$$= \frac{d}{d\theta} \left(\tan \frac{\theta}{2} \right) \times \frac{d\theta}{dx} = \sec^2 \frac{\theta}{2} \cdot \frac{d}{d\theta} \left(\frac{\theta}{2} \right) \cdot \frac{d\theta}{dx}$$

$$= \frac{1}{2} \sec^2 \frac{\theta}{2} \times \frac{1}{a(1 + \cos \theta)} \quad (1)$$

$$= \frac{1}{2} \sec^2 \frac{\theta}{2} \times \frac{1}{a \times 2 \cos^2 \frac{\theta}{2}}$$

$$\left[1 + \cos x = 2 \cos^2 \frac{x}{2} \right]$$

$$= \frac{1}{4a} \sec^4 \frac{\theta}{2} \quad (1\frac{1}{2})$$

49. If $y = a \sin x + b \cos x$, then prove that

$$y^2 + \left(\frac{dy}{dx} \right)^2 = a^2 + b^2.$$

All India 2011C; HOTS



Firstly, we differentiate the given expression with respect to x and get first derivative of y . Now, put the value of y and first derivative of y in LHS of given expression and then solve it and get required RHS.

To prove, $y^2 + \left(\frac{dy}{dx}\right)^2 = a^2 + b^2$... (i)

Given, $y = a \sin x + b \cos x$... (ii)

On differentiating both sides of Eq. (ii) w.r.t. x , we get

$$\frac{dy}{dx} = a \cos x - b \sin x \quad (1)$$

Now, we take LHS of Eq. (i), we get

$$\text{LHS} = y^2 + \left(\frac{dy}{dx}\right)^2$$

On putting the value of y and dy/dx , we get

$$\begin{aligned} \text{LHS} &= (a \sin x + b \cos x)^2 + (a \cos x - b \sin x)^2 \\ &= a^2 \sin^2 x + b^2 \cos^2 x + 2ab \sin x \cos x \\ &\quad + a^2 \cos^2 x + b^2 \sin^2 x \\ &\quad - 2ab \sin x \cos x \end{aligned} \quad (1\frac{1}{2})$$

$$[\because (a \pm b)^2 = a^2 + b^2 \pm 2ab]$$

$$= a^2 \sin^2 x + b^2 \cos^2 x + a^2 \cos^2 x + b^2 \sin^2 x$$

$$= a^2 (\sin^2 x + \cos^2 x) + b^2 (\sin^2 x + \cos^2 x)$$

$$= a^2 + b^2 \quad [\because \sin^2 x + \cos^2 x = 1]$$

$$= \text{RHS} \quad \text{Hence proved. (1\frac{1}{2})}$$

50. If $x = a (\cos \theta + \theta \sin \theta)$

and $y = a(\sin \theta - \theta \cos \theta)$, then find $\frac{d^2y}{dx^2}$.

All India 2011C, 2008

Do same as Que 38

$$\left[\text{Ans. } 7 \frac{\sec^3 \theta}{a\theta} \right]$$

51. If $x = a(\theta - \sin \theta)$ and $y = a(1 + \cos \theta)$, then

$$\text{find } \frac{dy}{dx} \text{ at } \theta = \frac{\pi}{3}.$$

Delhi 2011C

$$\text{Given, } x = a(\theta - \sin \theta)$$

$$\text{and } y = a(1 + \cos \theta)$$

On differentiating both sides w.r.t. θ , we get

$$\frac{dx}{d\theta} = a(1 - \cos \theta) \quad \text{and} \quad \frac{dy}{d\theta} = -a \sin \theta \quad (1)$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a \sin \theta}{a(1 - \cos \theta)} \quad (1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2a \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{a \times 2 \sin^2 \frac{\theta}{2}} \quad \left[\begin{array}{l} \because \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \\ \text{and } 1 - \cos x = 2 \sin^2 \frac{x}{2} \end{array} \right] \quad (1)$$

$$\Rightarrow \frac{dy}{dx} = -\cot \frac{\theta}{2}$$

On putting $\theta = \frac{\pi}{3}$, we get

$$\left[\frac{dy}{dx} \right]_{\theta = \frac{\pi}{3}} = -\cot \frac{\pi}{6} = -\sqrt{3}$$

$$\left[\because \cot \frac{\pi}{6} = \sqrt{3} \right] \quad (1)$$

Hence, $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$ is $-\sqrt{3}$.

52. If $y = (\sin x - \cos x)^{(\sin x - \cos x)}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$,

$$\text{then find } \frac{dy}{dx}.$$

All India 2010C



Firstly, take log on both sides and then differentiate to get the value of $\frac{dy}{dx}$.

Given, $y = (\sin x - \cos x)^{(\sin x - \cos x)}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$

On taking log both sides, we get

$$\log y = \log (\sin x - \cos x)^{(\sin x - \cos x)}$$

$$\Rightarrow \log y = (\sin x - \cos x) \cdot \log (\sin x - \cos x)$$

$$[\because \log m^n = n \log m] \quad (1)$$

On differentiating both sides w.r.t. x , we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = (\sin x - \cos x) \times \frac{d}{dx} \log (\sin x - \cos x)$$

$$+ \log (\sin x - \cos x) \times \frac{d}{dx} (\sin x - \cos x)$$

[by product rule]

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = (\sin x - \cos x) \cdot \frac{1}{(\sin x - \cos x)} \cdot \frac{d}{dx} (\sin x - \cos x) + \log (\sin x - \cos x) \cdot (\cos x + \sin x) \quad (1)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = (\sin x - \cos x) \frac{1}{(\sin x - \cos x)} (\cos x + \sin x) + \log (\sin x - \cos x) \cdot (\cos x + \sin x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = (\cos x + \sin x) + (\cos x + \sin x) \cdot [\log (\sin x - \cos x)]$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = (\cos x + \sin x) [1 + \log (\sin x - \cos x)] \quad (1)$$

$$\Rightarrow \frac{dy}{dx} = y (\cos x + \sin x) [1 + \log (\sin x - \cos x)]$$

$$\therefore \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)} \cdot (\cos x + \sin x) [1 + \log (\sin x - \cos x)] \quad (1)$$

53. If $y = \cos^{-1} \left[\frac{2x - 3\sqrt{1-x^2}}{\sqrt{13}} \right]$, then find $\frac{dy}{dx}$.

All India 2010C



In the given expression, put $x = \sin \theta$ and simplify the resulting expression, then differentiate it.

$$\text{Given, } y = \cos^{-1} \left[\frac{2x - 3\sqrt{1-x^2}}{\sqrt{13}} \right]$$

$$\text{Put } x = \sin \theta, \text{ then } \theta = \sin^{-1} x$$

$$\left[\begin{array}{l} \therefore \text{ for } \sqrt{a^2 - x^2}, \text{ we put } x = a \sin \theta \\ \therefore \text{ for } \sqrt{1-x^2}, \text{ we put } x = \sin \theta \end{array} \right] \quad (1)$$

$$\therefore y = \cos^{-1} \left[\frac{2 \sin \theta - 3\sqrt{1-\sin^2 \theta}}{\sqrt{13}} \right]$$

$$\Rightarrow y = \cos^{-1} \left[\frac{2 \sin \theta - 3 \cos \theta}{\sqrt{13}} \right]$$

$$[\because \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta]$$

$$\Rightarrow y = \cos^{-1} \left[\frac{2}{\sqrt{13}} \sin \theta - \frac{3}{\sqrt{13}} \cos \theta \right]$$

Now, let $\frac{2}{\sqrt{13}} = \cos \alpha$ and $\frac{3}{\sqrt{13}} = \sin \alpha$ (1½)

$$\left[\because \sin^2 \alpha + \cos^2 \alpha = \left(\frac{3}{\sqrt{13}} \right)^2 + \left(\frac{2}{\sqrt{13}} \right)^2 \right. \\ \left. = \frac{9}{13} + \frac{4}{13} = \frac{13}{13} = 1 \right]$$

$$\therefore y = \cos^{-1} [\sin \theta \cos \alpha - \cos \theta \sin \alpha]$$

$$\Rightarrow y = \cos^{-1} \sin (\theta - \alpha)$$

$$[\because \sin \theta \cos \alpha - \cos \theta \sin \alpha = \sin (\theta - \alpha)]$$

$$\Rightarrow y = \cos^{-1} \cos \left[\frac{\pi}{2} - (\theta - \alpha) \right]$$

$$\left[\because \sin x = \cos \left(\frac{\pi}{2} - x \right) \right]$$

$$[\text{here, } x = \theta - \alpha]$$

$$\Rightarrow y = \frac{\pi}{2} - \theta + \alpha \quad [\because \cos^{-1} (\cos \theta) = \theta]$$

$$\Rightarrow y = \frac{\pi}{2} - \sin^{-1} x + \alpha \quad [\because \theta = \sin^{-1} x]$$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = 0 - \frac{1}{\sqrt{1-x^2}} + 0$$

$$\left[\because \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} \quad (1\frac{1}{2})$$

54. If $y = (\cot^{-1} x)^2$, then show that

$$(x^2 + 1)^2 \frac{d^2 y}{dx^2} + 2x(x^2 + 1) \frac{dy}{dx} = 2.$$

Delhi 2010C

To show, $(x^2 + 1)^2 \frac{d^2y}{dx^2} + 2x(x^2 + 1) \frac{dy}{dx} = 2$

Given, $y = (\cot^{-1} x)^2$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= 2 \cdot \cot^{-1} x \cdot \frac{d}{dx} (\cot^{-1} x) \\ \Rightarrow \frac{dy}{dx} &= 2 \cot^{-1} x \times \frac{-1}{1+x^2} \\ &\quad \left[\because \frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2} \right] \end{aligned}$$

$$\Rightarrow (1+x^2) \frac{dy}{dx} = -2 \cot^{-1} x \quad (1\frac{1}{2})$$

Again, differentiating both sides w.r.t. x , we get

$$\begin{aligned} (1+x^2) \times \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \times \frac{d}{dx} (1+x^2) \\ = \frac{d}{dx} (-2 \cot^{-1} x) \\ \left[\because \frac{d}{dx} (u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx} \right] \\ \Rightarrow (1+x^2) \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 2x = \frac{-2 \times (-1)}{1+x^2} \quad (1\frac{1}{2}) \\ \left[\because \frac{d}{dx} (\cot^{-1} x) = \frac{-1}{1+x^2} \right] \end{aligned}$$

On multiplying both sides by $(1+x^2)$, we get

$$(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2) \frac{dy}{dx} = 2 \quad (1)$$

Hence proved.

55. If $y = \operatorname{cosec}^{-1} x, x > 1$, then show that

$$x(x^2 - 1) \frac{d^2 y}{dx^2} + (2x^2 - 1) \frac{dy}{dx} = 0. \quad \text{All India 2010}$$

$$\text{To show, } x(x^2 - 1) \frac{d^2 y}{dx^2} + (2x^2 - 1) \frac{dy}{dx} = 0$$

$$\text{Given, } y = \operatorname{cosec}^{-1} x; x > 1$$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2 - 1}} \Rightarrow x\sqrt{x^2 - 1} \cdot \frac{dy}{dx} = -1 \quad (1)$$

Again, differentiating both sides w.r.t. x , we get

$$(x\sqrt{x^2 - 1}) \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx} (x\sqrt{x^2 - 1}) = \frac{d}{dx} (-1)$$

$$\left[\because \frac{d}{dx} (u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \right] \quad (1)$$

$$\Rightarrow x\sqrt{x^2 - 1} \cdot \frac{d^2 y}{dx^2} + \frac{dy}{dx} \left\{ x \times \frac{d}{dx} \sqrt{x^2 - 1} + \sqrt{x^2 - 1} \times \frac{d}{dx} (x) \right\} = 0$$

$$\Rightarrow x\sqrt{x^2 - 1} \frac{d^2 y}{dx^2} + \frac{dy}{dx} \left\{ \frac{x}{2\sqrt{x^2 - 1}} \frac{d}{dx} (x^2 - 1) + \sqrt{x^2 - 1} \times 1 \right\} = 0$$

$$\Rightarrow x\sqrt{x^2 - 1} \frac{d^2 y}{dx^2} + \frac{dy}{dx} \left\{ \frac{x \cdot 2x}{2\sqrt{x^2 - 1}} + \sqrt{x^2 - 1} \right\} = 0$$

$$\Rightarrow x \sqrt{x^2 - 1} \frac{d^2 y}{dx^2} + \frac{dy}{dx} \left\{ \frac{x^2}{\sqrt{x^2 - 1}} + \sqrt{x^2 - 1} \right\} = 0$$

$$\Rightarrow x \sqrt{x^2 - 1} \frac{d^2 y}{dx^2} + \frac{x^2}{\sqrt{x^2 - 1}} \frac{dy}{dx} + \sqrt{x^2 - 1} \frac{dy}{dx} = 0$$

(1)

On multiplying both sides by $\sqrt{x^2 - 1}$, we get

$$x(x^2 - 1) \frac{d^2 y}{dx^2} + x^2 \frac{dy}{dx} + (x^2 - 1) \frac{dy}{dx} = 0$$

$$\Rightarrow x(x^2 - 1) \frac{d^2 y}{dx^2} + (x^2 + x^2 - 1) \frac{dy}{dx} = 0$$

$$\Rightarrow x(x^2 - 1) \frac{d^2 y}{dx^2} + (2x^2 - 1) \frac{dy}{dx} = 0 \quad (1)$$

Hence proved.

56. If $y = \cos^{-1} \left(\frac{3x + 4\sqrt{1-x^2}}{5} \right)$, then find $\frac{dy}{dx}$.
All India 2010

$$\text{Given, } y = \cos^{-1} \left[\frac{3x + 4\sqrt{1-x^2}}{5} \right]$$

Put $x = \sin \theta \Rightarrow \theta = \sin^{-1} x$, we get (1)

$$\therefore y = \cos^{-1} \left[\frac{3 \sin \theta + 4 \sqrt{1 - \sin^2 \theta}}{5} \right]$$

$$\Rightarrow y = \cos^{-1} \left[\frac{3 \sin \theta + 4 \cos \theta}{5} \right]$$

$$[\because \sqrt{1 - \sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta]$$

$$\Rightarrow y = \cos^{-1} \left[\frac{3}{5} \sin \theta + \frac{4}{5} \cos \theta \right] \quad (1)$$

Now, let $\cos \alpha = \frac{4}{5}$ and $\sin \alpha = \frac{3}{5}$

$$\left[\begin{aligned} \because \sin^2 \alpha + \cos^2 \alpha &= \left(\frac{3}{5} \right)^2 + \left(\frac{4}{5} \right)^2 \\ &= \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1 \end{aligned} \right]$$

Then, we get

$$y = \cos^{-1} [\sin \theta \sin \alpha + \cos \theta \cos \alpha]$$

$$\Rightarrow y = \cos^{-1} \cos (\theta - \alpha)$$

$$[\because \cos \theta \cos \alpha + \sin \theta \sin \alpha = \cos (\theta - \alpha)]$$

$$\Rightarrow y = \theta - \alpha$$

$$\Rightarrow y = \sin^{-1} x - \alpha \quad [\because \theta = \sin^{-1} x] \quad (1)$$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - 0 \quad \left[\because \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \right]$$

$$\text{Hence, } \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \quad (1)$$

- 57.** Show that the function defined as follows, is continuous at $x = 1$, $x = 2$ but not differentiable at $x = 2$

$$f(x) = \begin{cases} 3x - 2, & 0 < x \leq 1 \\ 2x^2 - x, & 1 < x \leq 2. \\ 5x - 4, & x > 2 \end{cases} \quad \text{Delhi 2010}$$



Firstly, we will check the continuity of the given function at $x=1, 2$ and then to check the differentiability of the function $f(x)$ at these points, find LHD and RHD. If $LHD \neq RHD$, then function is not differentiable.

The given function is $f(x) = \begin{cases} 3x - 2, & 0 < x \leq 1 \\ 2x^2 - x, & 1 < x \leq 2 \\ 5x - 4, & x > 2 \end{cases}$

First, we show the continuity of above function at $x = 1$ and at $x = 2$.

Continuity at $x = 1$

$$LHL = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3x - 2)$$

$$[\text{put } x = 1 - h, \text{ when } x \rightarrow 1, h \rightarrow 0]$$

$$\begin{aligned} \Rightarrow LHL &= \lim_{h \rightarrow 0} [3(1 - h) - 2] \\ &= \lim_{h \rightarrow 0} (3 - 3h - 2) = 1 \end{aligned}$$

$$RHL = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2x^2 - x)$$

$$[\text{put } x = 1 + h, \text{ when } x \rightarrow 1, h \rightarrow 0]$$

$$\begin{aligned} \Rightarrow RHL &= \lim_{h \rightarrow 0} [2(1 + h)^2 - (1 + h)] \\ &= \lim_{h \rightarrow 0} [2(1 + h^2 + 2h) - (1 + h)] \\ &= \lim_{h \rightarrow 0} [2 + 2h^2 + 4h - 1 - h] \\ &= \lim_{h \rightarrow 0} (2h^2 + 3h + 1) \Rightarrow RHL = 1 \end{aligned}$$

Also, from the given function, at $x = 1$

$$f(1) = 3(1) - 2 = 3 - 2 = 1$$

Since, $\text{LHL} = \text{RHL} = f(1)$

Hence, $f(x)$ is continuous at $x = 1$. (1)

Continuity at $x = 2$

$$\text{LHL} = \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2x^2 - x)$$

[put $x = 2 - h$, when $x \rightarrow 2, h \rightarrow 0$]

$$\begin{aligned}\Rightarrow \text{LHL} &= \lim_{h \rightarrow 0} [2(2 - h)^2 - (2 - h)] \\ &= \lim_{h \rightarrow 0} [2(4 + h^2 - 4h) - (2 - h)] \\ &= \lim_{h \rightarrow 0} (8 + 2h^2 - 8h - 2 + h)\end{aligned}$$

$$\Rightarrow \text{LHL} = 8 - 2 = 6 \quad [\text{put } h = 0]$$

$$\text{and } \text{RHL} = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (5x - 4)$$

[put $x = 2 + h$, when $x \rightarrow 2, h \rightarrow 0$]

$$\begin{aligned}\Rightarrow \text{RHL} &= \lim_{h \rightarrow 0} [5(2 + h) - 4] \\ &= \lim_{h \rightarrow 0} (10 + 5h - 4) = \lim_{h \rightarrow 0} (5h + 6)\end{aligned}$$

$$\Rightarrow \text{RHL} = 6$$

Also, from the given function, at $x = 2$.

$$f(2) = 2(2)^2 - 2$$

[for $f(2)$, put $x = 2$ in $f(x) = 2x^2 - x$]

$$= 8 - 2 = 6$$

Since, $\text{LHL} = \text{RHL} = f(2)$

So, $f(x)$ is continuous at $x = 2$. (1)

Hence, $f(x)$ is continuous at all indicated points.

Now, we verify differentiability of the given function at $x = 1$ and $x = 2$.

Differentiability at $x = 1$

$$\begin{aligned}\text{LHD} &= \lim_{h \rightarrow 0} \frac{f(1 - h) - f(1)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{[3(1 - h) - 2] - [3 - 2]}{-h} \\ &= \frac{(1 - 3h) - (1)}{-h} = \frac{-3h}{-h}\end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{2}{1+h} - \frac{2}{1}}{-h} = \lim_{h \rightarrow 0} \frac{\frac{2}{1+h} - \frac{2}{1}}{-h}$$

$$\Rightarrow \text{LHD} = 3$$

$$\text{RHD} = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[2(1+h)^2 - (1+h)] - [2 - 1]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[2 + 2h^2 + 4h - 1 - h] - 1}{h}$$

$$[\because (a+b)^2 = a^2 + b^2 + 2ab]$$

$$= \lim_{h \rightarrow 0} \frac{2h^2 + 3h}{h} = \lim_{h \rightarrow 0} (2h + 3)$$

$$\Rightarrow \text{RHD} = 3 \quad [\text{put } h = 0]$$

Since, LHD = RHD

So, $f(x)$ is differentiable at $x = 1$. (1)

Differentiability at $x = 2$

$$\text{LHD} = \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{-h}$$

$$\Rightarrow \text{LHD} = \lim_{h \rightarrow 0} \frac{[2(2-h)^2 - (2-h)] - [8-2]}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{2(4 + h^2 - 4h) - (2-h) - 6}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{h(2h-7)}{-h}$$

$$\Rightarrow \text{LHD} = 7$$

$$\text{RHD} = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[5(2+h) - 4] - [8-2]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(6+5h) - (6)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{5h}{h}$$

$$\Rightarrow \text{RHD} = 5$$

Since, $\text{LHD} \neq \text{RHD}$

So, $f(x)$ is not differentiable at $x = 2$. (1)

Hence, $f(x)$ is continuous at $x = 1$ and $x = 2$ but not differentiable at $x = 2$.

58. If $y = e^{a \cos^{-1} x}$, $-1 \leq x \leq 1$, then show that

$$(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0.$$

All India 2010

To show, $(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2y = 0$

Given, $y = e^{a \cos^{-1} x}, -1 \leq x \leq 1$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = e^{a \cos^{-1} x} \cdot \frac{d}{dx} (a \cos^{-1} x)$$

[by chain rule]

$$\Rightarrow \frac{dy}{dx} = e^{a \cos^{-1} x} \times \frac{-a}{\sqrt{1-x^2}}$$

$$\left[\because \frac{d}{dx} e^x = e^x, \frac{d}{dx} (\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \right]$$

$$\Rightarrow \sqrt{1-x^2} \frac{dy}{dx} = -ae^{a \cos^{-1} x}$$

$$\Rightarrow \sqrt{1-x^2} \frac{dy}{dx} = -ay \quad \dots(i)$$

$$[\because e^{a \cos^{-1} x} = y, \text{ given}] \quad (1)$$

Again, differentiating both sides of Eq. (i)

w.r.t. x , we get

$$\begin{aligned} \sqrt{1-x^2} \times \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx} (\sqrt{1-x^2}) \\ = \frac{d}{dx} (-ay) \quad [\because \text{by product rule}] \\ \Rightarrow \sqrt{1-x^2} \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx} (1-x^2) \\ = -a \cdot \frac{dy}{dx} \end{aligned}$$

$$\begin{aligned} \Rightarrow \sqrt{1-x^2} \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{-2x}{2\sqrt{1-x^2}} = -a \cdot \frac{dy}{dx} \\ \Rightarrow \sqrt{1-x^2} \frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}} \cdot \frac{dy}{dx} = -a \cdot \frac{dy}{dx} \quad (1\frac{1}{2}) \end{aligned}$$

On multiplying both sides by $\sqrt{1-x^2}$, we get

$$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = -a \sqrt{1-x^2} \cdot \frac{dy}{dx} \quad \dots(ii)$$

But from Eq. (i), we have

$$\sqrt{1-x^2} \frac{dy}{dx} = -ay \Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}} \quad \dots(iii) \quad (1/2)$$

On putting the value of $\frac{dy}{dx}$ from Eq. (iii) in

Eq. (ii), we get

$$\begin{aligned} (1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} &= -a \sqrt{1-x^2} \times \frac{-ay}{\sqrt{1-x^2}} \\ \Rightarrow (1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} &= a^2y \\ \therefore (1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2y &= 0 \quad (1) \end{aligned}$$

Hence proved.

59. Find $\frac{dy}{dx}$, if $y = (\cos x)^x + (\sin x)^{1/x}$.

Delhi 2010

💡 Let $u = (\cos x)^x$ and $v = (\sin x)^{1/x}$. Now, take log on both sides of u and v and then differentiate with respect to x to get $\frac{du}{dx}$ and $\frac{dv}{dx}$. Further, put these values in equation $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

Given, $y = (\cos x)^x + (\sin x)^{1/x}$

Let $u = (\cos x)^x$ and $v = (\sin x)^{1/x}$

Then, given equation becomes,

$$\begin{aligned} y &= u + v \\ \Rightarrow \frac{dy}{dx} &= \frac{du}{dx} + \frac{dv}{dx} \quad \dots(i) \end{aligned}$$

Now, $u = (\cos x)^x$

On taking log both sides, we get

$$\begin{aligned} \log u &= \log (\cos x)^x \\ \Rightarrow \log u &= x \log (\cos x) \\ &[\because \log m^n = n \log m] \end{aligned}$$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned} \frac{1}{u} \frac{du}{dx} &= x \cdot \frac{d}{dx} \log (\cos x) + \log (\cos x) \cdot \frac{d}{dx} (x) \\ &\left[\because \frac{d}{dx} (\log u) = \frac{1}{u} \frac{du}{dx} \right] \end{aligned}$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = x \cdot \frac{1}{\cos x} (-\sin x) + \log \cos x \cdot 1$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = -x \tan x + \log (\cos x)$$

$$\Rightarrow \frac{du}{dx} = u [-x \tan x + \log \cos x]$$

$$\Rightarrow \frac{du}{dx} = (\cos x)^x$$

$$[-x \tan x + \log \cos x] \quad \dots(ii) \text{ (1)}$$

Also $v = (\sin x)^{1/x}$

$$\text{Also, } v = (\sin x)^{1/x}$$

On taking log both sides, we get

$$\log v = \log (\sin x)^{1/x}$$

$$\Rightarrow \log v = \frac{1}{x} \log \sin x$$

$$[\because \log m^n = n \log m]$$

Again, differentiating both sides w.r.t. x , we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{1}{x} \cdot \frac{d}{dx} (\log \sin x) + \log \sin x \cdot \frac{d}{dx} \left(\frac{1}{x} \right)$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = \frac{1}{x} \cdot \frac{1}{\sin x} \cdot \cos x + \log \sin x \left(-\frac{1}{x^2} \right)$$

$$\left[\begin{array}{l} \because \frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2} \\ \text{and } \frac{d}{dx} (\log v) = \frac{1}{v} \frac{dv}{dx} \end{array} \right]$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = \frac{\cot x}{x} - \frac{\log (\sin x)}{x^2}$$

$$\Rightarrow \frac{dv}{dx} = v \left(\frac{\cot x}{x} - \frac{\log (\sin x)}{x^2} \right)$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{1/x} \left[\frac{\cot x}{x} - \frac{\log (\sin x)}{x^2} \right] \dots \text{(iii)}$$

(1½)

On putting the value of $\frac{du}{dx}$ from Eq. (ii) and

that of $\frac{dv}{dx}$ from Eq. (iii) in Eq. (i), we get

$$\frac{dy}{dx} = (\cos x)^x [-x \tan x + \log \cos x]$$

$$+ (\sin x)^{1/x} \left[\frac{\cot x}{x} - \frac{\log (\sin x)}{x^2} \right] \text{(1½)}$$

60. If $y = e^x \sin x$, then prove that

$$\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0.$$

All India 2010C



Firstly, we find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and then put their values along with value of y in LHS of proven expression.

Given, $y = e^x \sin x$... (i)

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = e^x \cdot \frac{d}{dx} (\sin x) + \sin x \cdot \frac{d}{dx} (e^x)$$

[by using product rule]

$$\Rightarrow \frac{dy}{dx} = e^x \cdot \cos x + \sin x \cdot e^x \quad (1)$$

$$\Rightarrow \frac{dy}{dx} = e^x (\cos x + \sin x) \quad \dots (ii)$$

Again, differentiating both sides w.r.t. x , we get

$$\frac{d^2y}{dx^2} = e^x \cdot \frac{d}{dx} (\cos x + \sin x)$$
$$+ (\cos x + \sin x) \cdot \frac{d}{dx} (e^x)$$

[by using product rule]

$$\Rightarrow \frac{d^2y}{dx^2} = e^x (-\sin x + \cos x)$$
$$+ (\cos x + \sin x) \cdot e^x$$
$$= e^x [-\sin x + \cos x + \cos x + \sin x]$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \cos x e^x \quad \dots (iii) \quad (1)$$

Now, we have to show that

$$\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$$

On putting the values of $\frac{d^2y}{dx^2}$ from Eq. (iii), $\frac{dy}{dx}$ from Eq. (ii) and that of y from Eq. (i) on LHS, we get

$$\text{LHS} = \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y \quad (1)$$

$$= 2e^x \cos x - 2e^x (\cos x + \sin x) + 2e^x \sin x$$

$$= 2e^x \cos x - 2e^x \cos x - 2e^x \sin x + 2e^x \sin x$$

$$= 0 = \text{RHS} \quad (1)$$

Hence proved.

61. If $y = (x)^x + (\sin x)^x$, then find $\frac{dy}{dx}$.

All India 2010C

Given, $y = (x)^x + (\sin x)^x$

Let $u = (x)^x$ and $v = (\sin x)^x$

$\therefore$ Given equation becomes, $y = u + v$

On differentiating w.r.t. x , we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(i)$$

Now, $u = x^x$ (1)

On taking log both sides, we get

$$\log u = \log x^x \Rightarrow \log u = x \log x$$
$$[\because \log m^n = n \log m]$$

On differentiating both sides w.r.t. x , we get

$$\frac{1}{u} \frac{du}{dx} = x \cdot \frac{d}{dx} (\log x) + \log x \cdot \frac{d}{dx} (x)$$
$$\Rightarrow \frac{1}{u} \frac{du}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$
$$\Rightarrow \frac{1}{u} \frac{du}{dx} = 1 + \log x$$
$$\Rightarrow \frac{du}{dx} = u (1 + \log x)$$
$$\Rightarrow \frac{du}{dx} = x^x (1 + \log x) \quad [\because u = x^x] \dots(ii) \quad (1)$$

Also, $v = (\sin x)^x$

On taking log both sides, we get

$$\log v = \log (\sin x)^x$$
$$\Rightarrow \log v = x \log (\sin x) \quad [\because \log m^n = n \log m]$$

On differentiating both sides w.r.t. x , we get

$$\frac{1}{v} \frac{dv}{dx} = x \cdot \frac{d}{dx} \log(\sin x) + \log(\sin x) \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) + \log \sin x$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = x \cdot \frac{1}{\sin x} \cdot \cos x + \log \sin x$$

$$\left[\because \frac{d}{dx}(\log v) = \frac{1}{v} \frac{dv}{dx} \right]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = x \cot x + \log \sin x$$

$$\Rightarrow \frac{dv}{dx} = v(x \cot x + \log \sin x)$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^x (x \cot x + \log \sin x) \dots \text{(iii) (1)}$$

On putting the values of $\frac{du}{dx}$ from Eq. (ii) and

$\frac{dv}{dx}$ from Eq. (iii) in Eq. (i), we get

$$\frac{dy}{dx} = x^x (1 + \log x)$$

$$+ (\sin x)^x (x \cot x + \log \sin x) \text{ (1)}$$

62. If $y = 3 \cos(\log x) + 4 \sin(\log x)$, then show

$$\text{that } x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0.$$

Delhi 2009, 2009C

To show, $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$

Given, $y = 3 \cos(\log x) + 4 \sin(\log x)$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= -3 \sin(\log x) \frac{d}{dx}(\log x) \\ &\quad + 4 \cos(\log x) \frac{d}{dx}(\log x) \end{aligned}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-3 \sin(\log x)}{x} + \frac{4 \cos(\log x)}{x}$$

$$\Rightarrow x \frac{dy}{dx} = -3 \sin(\log x) + 4 \cos(\log x)$$

Again, differentiating both sides w.r.t. x , we get

$$\begin{aligned} &x \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x) \\ &= \frac{d}{dx} [-3 \sin(\log x) + 4 \cos(\log x)] \quad (1\frac{1}{2}) \end{aligned}$$

$$\Rightarrow x \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 1 = -3 \cos(\log x) \frac{d}{dx}(\log x) - 4 \sin(\log x) \frac{d}{dx}(\log x)$$

$$\Rightarrow x \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 1 = \frac{-3 \cos(\log x)}{x} - \frac{4 \sin(\log x)}{x} \quad (1\frac{1}{2})$$

$$\Rightarrow x \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-[3 \cos(\log x) + 4 \sin(\log x)]}{x}$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = -[3 \cos(\log x) + 4 \sin(\log x)]$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = -y$$

$[\because 3 \cos(\log x) + 4 \sin(\log x) = y, \text{ given}]$

$$\text{Hence, } x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0 \quad (1)$$

Hence proved.

63. If $y = (x)^{\sin x} + (\log x)^x$, then find $\frac{dy}{dx}$. Delhi 2009

Do same as Que 61.

$$\left[\text{Ans. } \frac{dy}{dx} = x^{\sin x - 1} [\sin x + x \log x \cos x] + (\log x)^{x-1} [1 + \log x - \log(\log x)] \right]$$

64. If $x = a \left(\cos \theta + \log \tan \frac{\theta}{2} \right)$ and $y = a \sin \theta$,

then find the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$.
Delhi 2009C

Do same as Que 6.

$$\left[\text{Ans. } \left(\frac{d^2y}{dx^2} \right)_{\theta=\frac{\pi}{4}} = \frac{2\sqrt{2}}{a} \right]$$

65. If $y = (\log x)^x + (x)^{\cos x}$, then find $\frac{dy}{dx}$.

Delhi 2009C

Do same as Que 61.

$$\left[\text{Ans. } (\log x)^{x-1}[1 + \log x \cdot \log(\log x)] \right. \\ \left. + x \cos^{x-1}[\cos x - x \log x \sin x] \right]$$

66. If $y = e^x (\sin x + \cos x)$, then show that

$$\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0.$$

All India 2009

$$\text{Given, } y = e^x (\sin x + \cos x) \quad \dots(i)$$

$$\text{To show, } \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0 \quad \dots(ii)$$

On differentiating both sides of Eq. (i) w.r.t. x , we get

$$\frac{dy}{dx} = e^x \cdot \frac{d}{dx} (\sin x + \cos x)$$

$$+ (\sin x + \cos x) \cdot \frac{d}{dx} (e^x)$$

$$= e^x (\cos x - \sin x) + (\sin x + \cos x) \cdot e^x$$

$$\begin{aligned}
&= e^x [\cos x - \sin x + \sin x + \cos x] \\
&= e^x (2 \cos x) \\
\Rightarrow \quad \frac{dy}{dx} &= 2e^x \cos x \quad \dots(\text{iii}) \quad (1\frac{1}{2})
\end{aligned}$$

Again differentiating both sides of Eq. (iii) w.r.t. x , we get

$$\begin{aligned}
\frac{d^2y}{dx^2} &= (2e^x) \cdot \frac{d}{dx} (\cos x) + \cos x \cdot \frac{d}{dx} (2e^x) \\
&= 2e^x (-\sin x) + \cos x \cdot 2e^x \\
&= 2e^x \cos x - 2e^x \sin x \quad \dots(\text{iv}) \quad (1\frac{1}{2})
\end{aligned}$$

Now, we put the values of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ from

Eqs. (iv) and (iii) along with value of y in LHS of Eq. (ii), we get

$$\begin{aligned}
\text{LHS} &= \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y \\
&= (2e^x \cos x - 2e^x \sin x) \\
&\quad - 2(2e^x \cos x) + 2e^x (\sin x + \cos x) \\
&= 2e^x \cos x - 2e^x \sin x \\
&\quad - 4e^x \cos x + 2e^x \sin x + 2e^x \cos x \\
&= 4e^x \cos x - 4e^x \cos x = 0 = \text{RHS} \quad (1)
\end{aligned}$$

Hence proved.

67. If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, then show that

$$(1-x^2) \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} - y = 0.$$

All India 2009

Given, $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sqrt{1-x^2} \times \frac{d}{dx} (\sin^{-1} x) - (\sin^{-1} x) \times \frac{d}{dx} \sqrt{1-x^2}}{(\sqrt{1-x^2})^2} \\ &= \frac{\left[\sqrt{1-x^2} \times \frac{1}{\sqrt{1-x^2}} - \sin^{-1} x \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx} (1-x^2) \right]}{(\sqrt{1-x^2})^2} \quad (1) \\ &= \frac{\left[\sqrt{1-x^2} \cdot \frac{1}{\sqrt{1-x^2}} - (\sin^{-1} x) \cdot \frac{-2x}{2\sqrt{1-x^2}} \right]}{1-x^2} \end{aligned}$$

$$= \frac{1 + \frac{x \sin^{-1} x}{\sqrt{1-x^2}}}{(1-x^2)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1+xy}{1-x^2} \quad \left[\because \frac{\sin^{-1} x}{\sqrt{1-x^2}} = y, \text{ given} \right]$$

$$\Rightarrow (1-x^2) \frac{dy}{dx} = 1+xy \quad (1)$$

Again differentiating above equation both sides w.r.t. x , we get

$$(1-x^2) \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx} (1-x^2) = \frac{d}{dx} (1+xy) \quad (1)$$

$$\Rightarrow (1-x^2) \frac{d^2y}{dx^2} + \frac{dy}{dx} (-2x) = x \frac{dy}{dx} + y \cdot 1$$

$$\Rightarrow (1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - x \frac{dy}{dx} - y = 0$$

$$\Rightarrow (1-x^2) \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} - y = 0 \quad (1)$$

Hence proved.

68. Differentiate the following function w.r.t. x .

$$(x)^{\cos x} + (\sin x)^{\tan x} \quad \text{Delhi 2009}$$

Do same as Que 61.

$$\left[\text{Ans. } x^{\cos x - 1} [\cos x - x \log \sin x] \right. \\ \left. + (\sin x)^{\tan x} [1 + \sec^2 x \log \sin x] \right]$$

69. Differentiate the following function w.r.t. x .

$$x^{\sin x} + (\sin x)^{\cos x} \quad \text{Delhi 2009}$$

Do same as Que 61.

$$\left[\text{Ans. } x^{\sin x - 1} [\sin x + x \log x \cos x] \right. \\ \left. + \sin x^{\cos x} [\cos x \cot x - \sin x \log(\sin x)] \right]$$

70. If $y = x^{\cot x} + (\sin x)^x$, then find $\frac{dy}{dx}$.

All India 2008C

Do same as Que 61.

$$\left[\text{Ans. } x^{\cot x-1} [\cot x - x \log x \operatorname{cosec}^2 x] \right. \\ \left. + (\sin x)^x [x \cot x + \log x \sin x] \right]$$

71. If $xy + y^2 = \tan x + y$, then find $\frac{dy}{dx}$.

All India 2008C



Firstly, differentiate the given expression with respect to x and then collect all the first derivative of y on one side to get the required result.

Given equation is $xy + y^2 = \tan x + y$

On differentiating both sides w.r.t. x , we get

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x + y) \quad (1/2)$$

$$\Rightarrow \left[x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(x) \right] + 2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \quad (1\frac{1}{2})$$

$$\Rightarrow x \cdot \frac{dy}{dx} + y \cdot 1 + 2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \quad (1)$$

$$\Rightarrow (x + 2y - 1) \frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1} \quad (1)$$

72. If $y = (\log x)^{\cos x} + \frac{x^2 + 1}{x^2 - 1}$, then find $\frac{dy}{dx}$.

Delhi 2008C

Do same as Que 37.

$$\left[\text{Ans. } (\log x)^{\cos x} \left[\frac{\cos x}{x \log x} - \sin x \log(\log x) \right. \right. \\ \left. \left. - \frac{4x}{(x^2 - 1)^2} \right] \right]$$

73. If $y = \sin^{-1} \left[\frac{5x + 12\sqrt{1-x^2}}{13} \right]$, then find $\frac{dy}{dx}$.

HOTS; All India 2008

Do same as Que 56.

Ans. $\frac{1}{\sqrt{1-x^2}}$

74. If $y = \cot^{-1} \left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right]$,

$0 < x < \frac{\pi}{2}$, then find $\frac{dy}{dx}$.

HOTS; Delhi 2008



Firstly, convert the given function into simplest form by using

$$1 + \sin x = \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$= \left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^2$$

$$\text{and } 1 - \sin x = \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

and then differentiate.

$$\text{Given, } y = \cot^{-1} \left[\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}} \right] \dots (i)$$

Now, on putting

$$1 + \sin x = \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$\text{and } 1 - \sin x = \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}$$

$$- 2 \sin \frac{x}{2} \cos \frac{x}{2} \text{ in Eq. (i), we get (1)}$$

$$y = \cot^{-1} \left[\frac{\sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}}{\sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}}} \right]$$

$$+ \frac{\sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}}}{\sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}}$$

$$\Rightarrow y = \cot^{-1} \left[\frac{\sqrt{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^2} + \sqrt{\left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)^2}}{\sqrt{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^2} - \sqrt{\left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)^2}} \right] \quad (1)$$

$$\left[\because a^2 + b^2 + 2ab = (a + b)^2 \right]$$

$$\Rightarrow y = \cot^{-1} \left[\frac{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right) + \left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)}{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right) - \left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)} \right]$$

$$\Rightarrow y = \cot^{-1} \left[\frac{2 \cos \frac{x}{2}}{2 \sin \frac{x}{2}} \right] \quad (1)$$

$$\Rightarrow y = \cot^{-1} \cot \frac{x}{2} \quad \left[\because \cot \theta = \frac{\cos \theta}{\sin \theta} \right]$$

$$\Rightarrow y = \frac{x}{2} \quad [\because \cot^{-1} \cot \theta = \theta]$$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{1}{2} \quad (1)$$

NOTE (i) When $0 < x < \frac{\pi}{2}$, then we consider

$$\begin{aligned} & \sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}} \\ &= \sqrt{\left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)^2} \end{aligned}$$

(ii) When $\frac{\pi}{2} < x < \pi$, then we consider

$$\begin{aligned} & \sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}} \\ &= \sqrt{\left(\sin \frac{x}{2} - \cos \frac{x}{2} \right)^2} \end{aligned}$$

75. If $y = \sqrt{x^2 + 1} - \log \left(\frac{1}{x} + \sqrt{1 + \frac{1}{x^2}} \right)$, then find

$$\frac{dy}{dx}$$

Delhi 2008

Given, $y = \sqrt{x^2 + 1} - \log \left(\frac{1}{x} + \sqrt{1 + \frac{1}{x^2}} \right)$

It can be written as

$$y = \sqrt{x^2 + 1} - \log \left(\frac{1}{x} + \sqrt{\frac{x^2 + 1}{x^2}} \right)$$

$$\Rightarrow y = \sqrt{x^2 + 1} - \log \left(\frac{\sqrt{x^2 + 1} + 1}{x} \right) \quad (1/2)$$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\sqrt{x^2 + 1} \right) - \frac{d}{dx} \log \left(\frac{\sqrt{x^2 + 1} + 1}{x} \right) \\ &= \frac{1}{2\sqrt{x^2 + 1}} \frac{d}{dx} (x^2 + 1) - \frac{d}{dx} \log \left(\frac{\sqrt{x^2 + 1} + 1}{x} \right) \\ &= \frac{2x}{2\sqrt{x^2 + 1}} - \frac{1}{\left(\frac{\sqrt{x^2 + 1} + 1}{x} \right)} \\ &\quad \times \frac{d}{dx} \left(\frac{\sqrt{x^2 + 1} + 1}{x} \right) \quad (1) \\ &= \frac{x}{\sqrt{x^2 + 1}} - \frac{x}{\sqrt{x^2 + 1} + 1} \\ &\quad \times \frac{x \cdot \frac{d}{dx} (\sqrt{x^2 + 1} + 1) - (\sqrt{x^2 + 1} + 1) \frac{d}{dx} (x)}{x^2} \\ &\quad \left[\because \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right] \end{aligned}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}+1}$$

$$\times \frac{x \left(\frac{2x}{2\sqrt{x^2+1}} \right) - (\sqrt{x^2+1}+1) \cdot 1}{x^2}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}+1}$$

$$\times \frac{\frac{x^2}{\sqrt{x^2+1}} - (\sqrt{x^2+1}+1)}{x^2}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}+1}$$

$$\times \frac{x^2 - (\sqrt{x^2+1})(\sqrt{x^2+1}+1)}{x^2(\sqrt{x^2+1})}$$

$$= \frac{x}{\sqrt{x^2+1}} - \left[\frac{x^2 - x^2 - 1 - \sqrt{x^2+1}}{x(\sqrt{x^2+1}+1)(\sqrt{x^2+1})} \right]$$

(1½)

$$= \frac{x}{\sqrt{x^2+1}} - \frac{(-\sqrt{x^2+1}-1)}{x(\sqrt{x^2+1}+1)(\sqrt{x^2+1})}$$

$$= \frac{1}{\sqrt{x^2+1}} \left[x + \frac{(\sqrt{x^2+1}+1)}{x(\sqrt{x^2+1}+1)} \right]$$

$$= \frac{1}{\sqrt{x^2+1}} \left(x + \frac{1}{x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} \cdot \frac{(x^2+1)}{x}$$

Hence, $\frac{dy}{dx} = \frac{\sqrt{x^2+1}}{x}$ (1)

76. Differentiate the following function w.r.t. x .

$$\tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right)$$

Delhi 2008



Reduce the given function into simplest form by putting $x = \cos \theta$ and by using the half angle formulae

$$1 + \cos \theta = 2 \cos^2 \frac{\theta}{2} \text{ and } 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$$

Then, find its derivative with respect to x .

Given function is

$$y = \tan^{-1} \left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right]$$

On putting $x = \cos \theta$

$\Rightarrow \theta = \cos^{-1} x$, we get

$$\therefore y = \tan^{-1} \left[\frac{\sqrt{1+\cos \theta} - \sqrt{1-\cos \theta}}{\sqrt{1+\cos \theta} + \sqrt{1-\cos \theta}} \right] \dots(i) \quad (1)$$

We know that,

$$1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}$$

$$\text{and } 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$$

On putting the above values in Eq. (i), we get

$$y = \tan^{-1} \left[\frac{\sqrt{2 \cos^2 \frac{\theta}{2}} - \sqrt{2 \sin^2 \frac{\theta}{2}}}{\sqrt{2 \cos^2 \frac{\theta}{2}} + \sqrt{2 \sin^2 \frac{\theta}{2}}} \right]$$

$$\Rightarrow y = \tan^{-1} \left[\frac{\sqrt{2} \cos \frac{\theta}{2} - \sqrt{2} \sin \frac{\theta}{2}}{\sqrt{2} \cos \frac{\theta}{2} + \sqrt{2} \sin \frac{\theta}{2}} \right]$$

$$\Rightarrow y = \tan^{-1} \left(\frac{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}} \right) \quad (1)$$

On dividing numerator and denominator by $\cos \frac{\theta}{2}$, we get

$$y = \tan^{-1} \left(\frac{1 - \tan \frac{\theta}{2}}{1 + \tan \frac{\theta}{2}} \right)$$

$$\Rightarrow y = \tan^{-1} \left(\frac{\tan \frac{\pi}{4} - \tan \frac{\theta}{2}}{1 + \tan \frac{\pi}{4} \tan \frac{\theta}{2}} \right)$$

$$\left[\begin{array}{l} \because 1 = \tan \frac{\pi}{4} \\ \text{and } \tan \frac{\theta}{2} = 1 \times \tan \frac{\theta}{2} = \tan \frac{\pi}{4} \tan \frac{\theta}{2} \end{array} \right]$$

$$\Rightarrow y = \tan^{-1} \tan \left(\frac{\pi}{4} - \frac{\theta}{2} \right)$$

$$\left[\because \frac{\tan A - \tan B}{1 + \tan A \tan B} = \tan (A - B) \right]$$

$$\Rightarrow y = \frac{\pi}{4} - \frac{\theta}{2} \quad [\because \tan^{-1} \tan x = x]$$

$$\Rightarrow y = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x \quad [\because \theta = \cos^{-1} x] \quad (1)$$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = 0 - \frac{1}{2} \times \frac{-1}{\sqrt{1-x^2}} \left[\because \frac{d}{dx} (\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \right]$$

$$\text{Hence, } \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}} \quad (1)$$