

## Application of Derivatives

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- For a quantity  $y$  varying with another quantity  $x$ , satisfying the rule  $y = f(x)$ , the rate of change of  $y$  with respect to  $x$  is given by  $\frac{dy}{dx}$  or  $f'(x)$

The rate of change of  $y$  with respect to  $x$  at the point  $x = x_0$  is given by  $\left. \frac{dy}{dx} \right|_{x=x_0}$  or  $f'(x_0)$ .

- If the variables  $x$  and  $y$  are expressed in form of  $x = f(t)$  and  $y = g(t)$ , then the rate of change of  $y$  with respect to  $x$  is given by  $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$  provided  $f'(t) \neq 0$

a. A function  $f: (a, b) \rightarrow \mathbf{R}$  is said to be

- increasing on  $(a, b)$ , if  $x_1 < x_2$  in  $(a, b)$
- decreasing on  $(a, b)$ , if  $x_1 < x_2$  in  $(a, b)$

**OR**

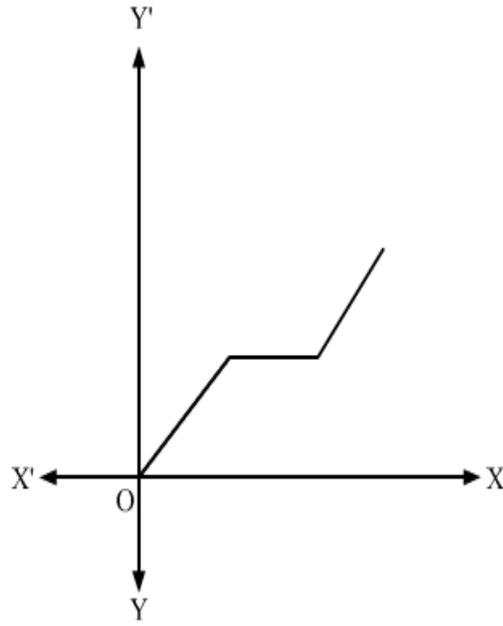
If a function  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then

- $f$  is increasing in  $[a, b]$ , if  $f'(x) > 0$  for each  $x \in (a, b)$
- $f$  is decreasing in  $[a, b]$ , if  $f'(x) < 0$  for each  $x \in (a, b)$
- $f$  is constant function in  $[a, b]$ , if  $f'(x) = 0$  for each  $x \in (a, b)$

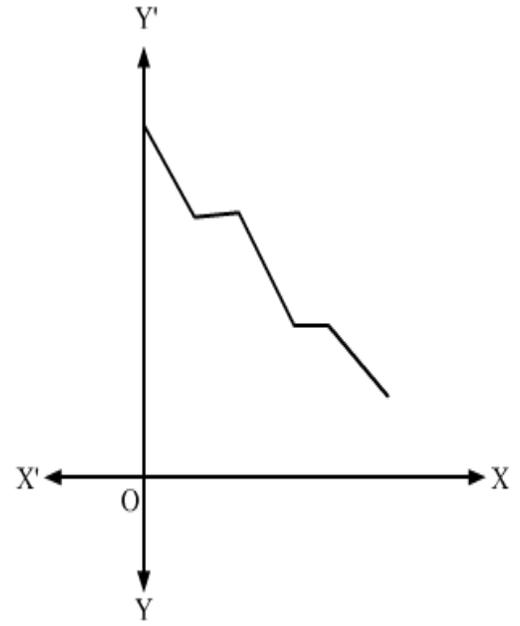
a. A function  $f: (a, b) \rightarrow \mathbf{R}$  is said to be

- strictly increasing on  $(a, b)$ , if  $x_1 < x_2$  in  $(a, b) \Rightarrow f(x_1) < f(x_2) \Rightarrow x_1, x_2 \in (a, b)$
- strictly decreasing on  $(a, b)$ , if  $x_1 < x_2$  in  $(a, b) \Rightarrow f(x_1) > f(x_2) \Rightarrow x_1, x_2 \in (a, b)$

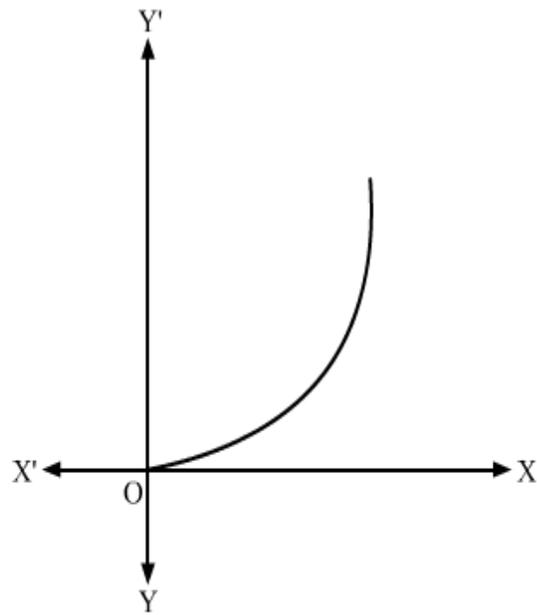
c. The graphs of various types of functions can be shown as follows:



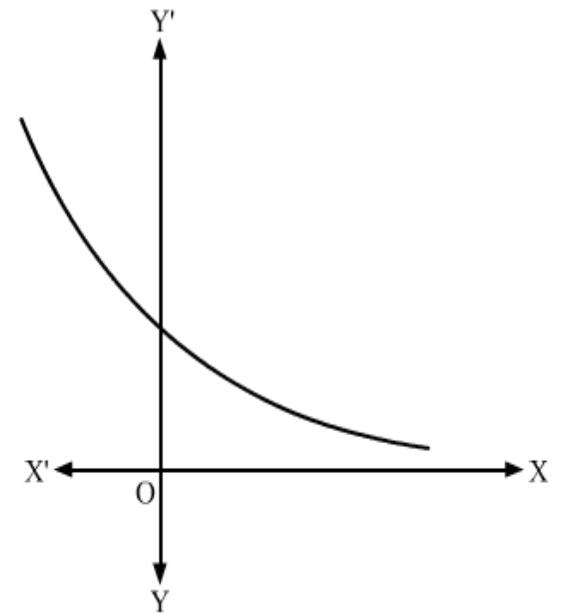
Increasing Function



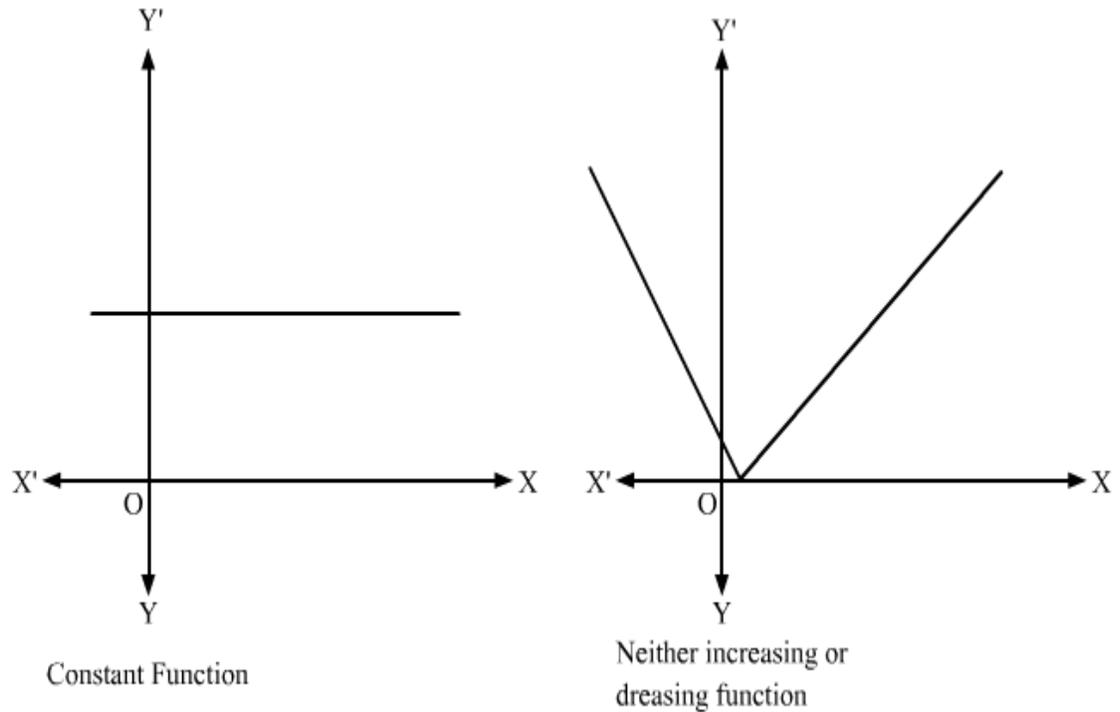
Decreasing Function



Strictly Increasing Function



Strictly Decreasing Function



**Example 1:** Find the intervals in which the function  $f$  given by  $f(x) = \sqrt{3} \sin x - \cos x, x \in [0, 2\pi]$  is strictly increasing or decreasing.

**Solution:**

$$f(x) = \sqrt{3} \sin x - \cos x$$

$$\therefore f'(x) = \sqrt{3} \cos x + \sin x$$

$$f'(x) = 0 \text{ gives } \tan x = -\sqrt{3}$$

$$\therefore x = \frac{2\pi}{3}, \frac{5\pi}{3}$$

The points  $x = \frac{2\pi}{3}$  and  $x = \frac{5\pi}{3}$  divide the interval  $[0, 2\pi]$  into three disjoint

intervals,  $\left[0, \frac{2\pi}{3}\right), \left(\frac{2\pi}{3}, \frac{5\pi}{3}\right), \left(\frac{5\pi}{3}, 2\pi\right]$ .

Now,  $f'(x) > 0$ , if  $x \in \left[0, \frac{2\pi}{3}\right) \cup \left(\frac{5\pi}{3}, 2\pi\right]$

$f$  is strictly increasing in the intervals  $\left[0, \frac{2\pi}{3}\right)$  and  $\left(\frac{5\pi}{3}, 2\pi\right]$ .

Also,  $f'(x) < 0$ , if  $x \in \left(\frac{2\pi}{3}, \frac{5\pi}{3}\right)$

$f$  is strictly decreasing in the interval  $\left(\frac{2\pi}{3}, \frac{5\pi}{3}\right)$ .

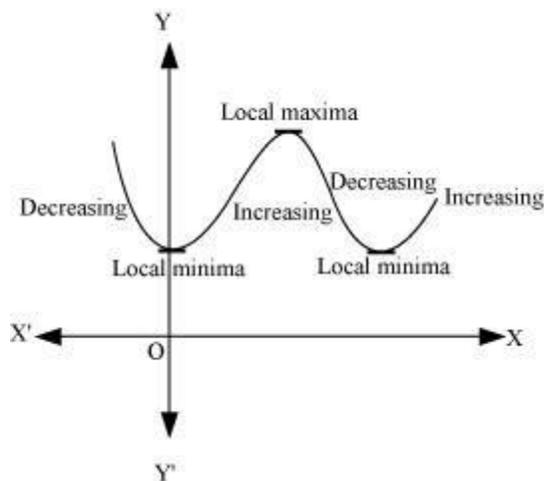
- For the curve  $y = f(x)$ , the slope of tangent at the point  $(x_0, y_0)$  is given by  $\left.\frac{dy}{dx}\right|_{(x_0, y_0)}$  or  $f'(x_0)$ .
- For the curve  $y = f(x)$ , the slope of normal at the point  $(x_0, y_0)$  is given by  $\frac{-1}{\left.\frac{dy}{dx}\right|_{(x_0, y_0)}}$  or  $\frac{-1}{f'(x_0)}$ .
- The equation of tangent to the curve  $y = f(x)$  at the point  $(x_0, y_0)$  is given by,  $y - y_0 = f'(x_0) \times (x - x_0)$
- If  $f'(x_0)$  does not exist, then the tangent to the curve  $y = f(x)$  at the point  $(x_0, y_0)$  is parallel to the  $y$ -axis and its equation is given by  $x = x_0$ .
- The equation of normal to the curve  $y = f(x)$  at the point  $(x_0, y_0)$  is given by,  $y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$
- If  $f'(x_0)$  does not exist, then the normal to the curve  $y = f(x)$  at the point  $(x_0, y_0)$  is parallel to the  $x$ -axis and its equation is given by  $y = y_0$ .
- If  $f'(x_0) = 0$ , then the respective equations of the tangent and normal to the curve  $y = f(x)$  at the point  $(x_0, y_0)$  are  $y = y_0$  and  $x = x_0$ .
- Let  $y = f(x)$  and let  $\Delta x$  be a small increment in  $x$  and  $\Delta y$  be the increment in  $y$  corresponding to the increment in  $x$  i.e.,  $\Delta y = f(x + \Delta x) - f(x)$

Then,  $dy = f'(x)dx$  or  $dy = \left(\frac{dy}{dx}\right)\Delta x$  is a good approximation of  $\Delta y$ , when  $dx = \Delta x$  is relatively small and we denote it by  $dy \approx \Delta y$ .

- **Maxima and Minima:** Let a function  $f$  be defined on an interval  $I$ . Then,  $f$  is said to have maximum value in  $I$ , if there exists  $c \in I$  such that  $f(c) > f(x), \forall x \in I$  [In this case,  $c$  is called the point of maxima]
- minimum value in  $I$ , if there exists  $c \in I$  such that  $f(c) < f(x), \forall x \in I$  [In this case,  $c$  is called the point of minima]
- an extreme value in  $I$ , if there exists  $c \in I$  such that  $c$  is either point of maxima or point of minima [In this case,  $c$  is called an extreme point]

**Note:** Every continuous function on a closed interval has a maximum and a minimum value.

- **Local maxima and local minima:** Let  $f$  be a real-valued function and  $c$  be an interior point in the domain of  $f$ . Then,  $c$  is called a point of
  - local maxima, if there exists  $h > 0$  such that  $f(c) > f(x), \forall x \in (c - h, c + h)$  [In this case,  $f(c)$  is called the local maximum value of  $f$ ]
  - local minima, if there exists  $h > 0$  such that  $f(c) < f(x), \forall x \in (c - h, c + h)$  [In this case,  $f(c)$  is called the local minimum value of  $f$ ]



- A point  $c$  in the domain of a function  $f$  at which either  $f'(c) = 0$  or  $f$  is not differentiable is called a critical point of  $f$ .
- **First derivative test:** Let  $f$  be a function defined on an open interval  $I$ . Let  $f$  be continuous at a critical point  $c$  in  $I$ . Then:
  - If  $f'(x)$  changes sign from positive to negative as  $x$  increases through  $c$ , i.e. if  $f'(x) > 0$  at every point sufficiently close to and to the left of  $c$ , and  $f'(x) < 0$  at every point sufficiently close to and to the right of  $c$ , then  $c$  is a point of local maxima.
  - If  $f'(x)$  changes sign from negative to positive as  $x$  increases through  $c$ , i.e. if  $f'(x) < 0$  at every point sufficiently close to and to the left of  $c$ , and  $f'(x) > 0$  at every point sufficiently close to and to the right of  $c$ , then  $c$  is a point of local minima.
  - If  $f'(x)$  does not change sign as  $x$  increases through  $c$ , then  $c$  is neither a point of local maxima nor a point of local minima. Such a point  $c$  is called point of inflection.
- **Second derivative test:** Let  $f$  be a function defined on an open interval  $I$  and  $c \in I$ . Let  $f$  be twice differentiable at  $c$  and  $f'(c) = 0$ . Then:
  - If  $f''(c) < 0$ , then  $c$  is a point of local maxima. In this situation,  $f(c)$  is local maximum value of  $f$ .
  - If  $f''(c) > 0$ , then  $c$  is a point of local minima. In this situation,  $f(c)$  is local minimum value of  $f$ .

- If  $f'(c) = 0$ , then the test fails. In this situation, we follow first derivative test and find whether  $c$  is a point of maxima or minima or a point of inflection.

**Example 1:** Find all the points of local maxima or local minima of the function  $f$  given by  $f(x) = x^3 - 12x^2 + 36x - 4$ .

**Solution:**

We have,

$$f(x) = x^3 - 12x^2 + 36x - 4$$

$$\therefore f'(x) = 3x^2 - 24x + 36 = 3(x^2 - 8x + 12)$$

$$\text{and } f''(x) = 3(2x - 8) = 6(x - 4)$$

$$\text{Now, } f'(x) = 0 \text{ gives } x^2 - 8x + 12 = 0$$

$$\Rightarrow (x - 2)(x - 6) = 0$$

$$\Rightarrow x = 2 \text{ or } x = 6$$

$$\text{However, } f''(2) = -12 \text{ and } f''(6) = 12$$

Therefore, the point of local maxima and local minima are at the points  $x = 2$  and  $x = 6$  respectively.

The local maximum value is  $f(2) = 28$

The local minimum value is  $f(6) = -4$