4 Complex Functions

4.1 Introduction

Many engineering problems may be treated and solved by methods involving complex numbers and complex functions. There are two kinds of such problems. The first of them consists of "elementary problems" for which some acquaintance with complex numbers is sufficient. This includes many applications to electric circuits or mechanical vibrating systems.

The second kind consists of more advanced problems for which we must be familiar with the theory of complex analytic functions— **"complex function theory"** or **"complex analysis,"** for short— and with its powerful and elegant methods. Interesting problems in heat conduction, fluid flow, and electrostatics belong to this category.

We shall see that the importance of complex analytic functions in engineering mathematics has the following two main roots.

1. The real and imaginary parts of an analytic function are solutions of Laplace's equation in two independent variables. Consequently, two-dimensional potential problems can be treated by methods developed for analytic functions.
2. Most higher functions is engineering mathematics are analytic functions, and their study for complex values of the independent variable leads to a much deeper understanding of their properties. Furthermore, complex integration can help evaluating complicated complex and real integrals of practical interest.

4.2 Complex Functions

If for each value of the complex variable $z(= x + xy)$ in a given region R, we have one or more values of $w(= u + iv)$, then w is said to be a complex function of z and we write $w = u(x, y) + iv(x, y) = f(z)$ where u, v are real functions of x and y.

If to each value of z, there corresponds one and only one value of w, then w is said to be a single-valued function of z otherwise a multi-valued function. For example $w = 1/z$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function of z. The former is defined at all points of the z-plane except at $z = 0$ and the latter assumes two values for each value of z except at $z = 0$.

4.2.1 Exponential Function of a Complex Variable

When x is real, we are already familiar with the exponential function

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots\infty$$

Similarly, we define the exponential function of the complex variable $z = x + iy$, as

$$e^z \text{ or } \exp(z) = 1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots+\frac{z^n}{n!}+\ldots\infty \qquad \ldots \text{(i)}$$

Putting $x = 0$ in (i), we get, $z = iy$ and

$$e^{iy} = 1+\frac{iy}{1!}+\frac{(iy)^2}{2!}+\frac{(iy)^3}{3!}+\frac{(iy)^4}{4!}+\ldots\infty$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right)$$

$$= \cos y + i \sin y$$

Thus $e^z = e^x . e^{iy} = e^x (\cos y + i \sin y)$

Also $x + iy = r(\cos\theta + i \sin\theta) = re^{i\theta}$

$\therefore$ Exponential form of $z (= x + iy) = re^{i\theta}$.

4.2.2 Circular Function of a Complex Variable

Since, $e^{iy} = \cos y + i \sin y$

and $e^{-iy} = \cos y - i \sin y$

$\therefore$ The circular functions of real angles can be written as

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and so on.}$$

It is, therefore, natural to define the circular functions of the complex variable z by the equations:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}, \tan z = \frac{\sin z}{\cos z}$$

with cosec z, sec z and cot z as their respective reciprocals.

Cor. 1. Euler's Theorem. By definition

$$\cos z + i \sin z = \frac{e^{iz} - e^{-iz}}{2i} + i\frac{e^{iz} - e^{-iz}}{2i} = e^{iz} \qquad \text{where } z = x + iy$$

Also we have shown that $e^{iy} = \cos y + i \sin y$, where y is real.

Thus $e^{i\theta} = \cos\theta + i \sin\theta$, where θ is real or complex. This is called the Euler's theorem.*

Cor. 2. De Moivre's theorem for complex numbers.

Whether θ is real or complex, we have

$$(\cos\theta + i \sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

Thus De Moivre's theorem is true for all θ (real or complex).

4.2.3 Hyperbolic Functions

1. **Def.** If x be real or complex,

   (a) $\frac{e^x - e^{-x}}{2}$ is defined as hyperbolic sine of x and is written as $\sinh x$.

   (b) $\frac{e^x + e^{-x}}{2}$ is defined as hyperbolic cosine of x and is written as $\cosh x$.

   Thus, $$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}$$

   Also we define, $$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

   $$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}; \text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

   Cor. $\sinh 0 = 0$, $\cosh 0 = 1$ and $\tanh 0 = 0$.

2. Relations between hyperbolic and circular functions.

   Since for all values of θ, $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$\therefore$ Putting $\theta = ix$, we have

$$\sin ix = i^2 \frac{e^{-x} - e^{x}}{2i} = -\left[\frac{e^{x} - e^{-x}}{2i}\right] \qquad [\because e^{i\theta} = e^{i \cdot ix} = e^{-x}]$$

$$= i^2 \frac{e^{x} - e^{-x}}{2i} = i \cdot \frac{e^{x} - e^{-x}}{2} = i \sinh x$$

and
$$\cos ix = \frac{e^{-x} + e^{x}}{2} = \cosh x$$

Thus,
$$\sin ix = i \sinh x \quad \text{... (i)}$$
$$\cos ix = \cosh x \quad \text{... (ii)}$$

and $\therefore$
$$\tan ix = i \tanh x \quad \text{... (iii)}$$

Cor.
$$\sinh ix = i \sin x \quad \text{... (iv)}$$
$$\cosh ix = \cos x \quad \text{... (v)}$$
$$\tanh ix = i \tan x \quad \text{... (vi)}$$

4.2.4 Inverse Hyperbolic Functions

Def. If $\sinh u = z$, then u is called the hyperbolic sine inverse of z and is written as $u = \sinh^{-1} z$. Similarly we define $\cosh^{-1} z$, $\tanh^{-1} z$, etc.

The inverse hyperbolic functions like other inverse functions are many-valued, but we shall consider only their principal values.

4.2.5 Logarithmic Function of a Complex Variable

1. **Def.** If $z(= x + iy)$ and $w(= u + iv)$ be so related that $e^w = z$, then w is said to be a logarithm of z to the base e and is written as $w = \log_e z$.... (i)

   Also
   $$e^{w + 2in\pi} = e^{w} \cdot e^{2in\pi} = z \qquad [\because e^{2in\pi} = 1]$$

   $\therefore$
   $$\log z = w + 2in\pi \quad \text{... (ii)}$$

   i.e. the logarithm of a complex number has an infinite number of values and is, therefore, a multi-valued function. The general value of the logarithm of z is written as Log z (beginning with capital L) so as to distinguish it from its principal value which is written as log z. This principal value is obtained by taking $n = 0$ in Log z.

   Thus from (i) and (ii), $\quad \text{Log}(x + iy) = 2in\pi + \log(x + iy)$.

   Obs.

   (a) If $y = 0$, then $\text{Log}\, x = 2in\pi + \log x$.
   This shows that the logarithm of a real quantity is also multi-valued. Its principal value is real while all other values are imaginary.

   (b) We know that the logarithm of a negative quantity has no real value. But we can now evaluate this.

   e.g.
   $$\log_e(-2) = \log_e 2(-1)$$
   $$= \log_e 2 + \log_e(-1)$$
   $$= \log_e 2 + i\pi \qquad [\because -1 = \cos\pi + i\sin\pi = e^{i\pi}]$$
   $$= 0.6931 + i(3.1416)$$

2. Real and imaginary parts of $\text{Log}(x + iy)$.

$$\text{Log}(x + iy) = 2in\pi + \log(x + iy) \qquad \text{Put, } x = r\cos\theta,\ y = r\sin\theta$$

$$= 2in\pi + \log[r(\cos\theta + i\sin\theta)] \qquad \text{so that } r = \sqrt{(x^2 + y^2)}$$

$$= 2in\pi + \log(re^{i\theta}) \qquad \text{and } \theta = \tan^{-1}(y/x)$$

$$= 2in\pi + \log r + i\theta$$

$$= \log\sqrt{(x^2 + y^2)} + i[2n\pi + \tan^{-1}(y/x)]$$

3. Real and imaginary parts of $(a + i\beta)^{x+iy}$

$$(a + i\beta)^{x+iy} = e^{(x+iy)\,\text{Log}(\alpha + i\beta)} \qquad \begin{cases} \text{Put } \alpha = r\cos\theta, \beta = r\sin\theta \text{ so that} \\ r = \sqrt{(\alpha^2 + \beta^2)} \text{ and } \theta = \tan^{-1}\beta/\alpha \end{cases}$$

$$= e^{(x+iy)\,[2in\pi + \log(\alpha + i\beta)]}$$

$$= e^{(x+iy)\,[2in\pi + \log r\, e^{i\theta}]}$$

$$= e^{(x+iy)\,[\log r + i\,(2n\pi + \theta)]}$$

$$= e^{A + iB}$$

$$= e^{A}(\cos B + i\sin B)$$

where $A = x\log r - y(2n\pi + \theta)$ and $B = y\log r + x(2n\pi + \theta)$

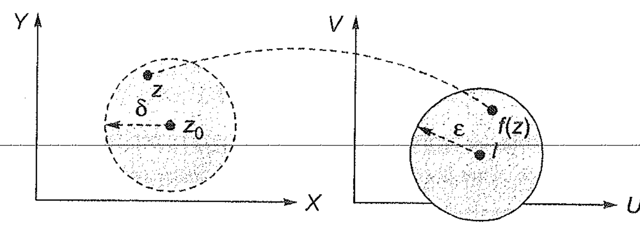
4.3 Limit of a Complex Function

A function $w = f(z)$ is said to tend to limit l as z approaches a point z_0, if for real $\in$, we can find a positive real δ such that

$$|f(z) - l| < \in \text{ for } |z - z_0| < \delta$$

i.e. for every $z \neq z_0$ in the δ–disc (dotted) of z-plane, $f(z)$ has a value lying in the $\in$-disc of w-plane (see figure below). In symbols, we write $\underset{z \to z_0}{Lt} f(z) = l$.

This definition of limit though similar to that in ordinary calculus, is quite different, for in real calculus x approaches x_0 only along the line whereas here z approaches z_0 from any direction in the z-plane.



Continuity of $f(z)$: A function $w = f(z)$ is said to be **continuous** at $z = z_0$, if

$$\underset{z \to z_0}{Lt} f(z) = f(z_0)$$

Further $f(z)$ is said to be continuous in any region R of the z-plane, if it is continuous at every point of that region.

Also if $w = f(z) = u(x, y) + iv(x, y)$ is continuous at $z = z_0$, then $u(x, y)$ and $v(x, y)$ are also continuous at $z = z_0$, i.e. at $x = x_0$ and $y = y_0$. Conversely if $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0), then $f(z)$ will be continuous at $z = z_0$.

4.4 Singularity

A point at which a function $f(z)$ is not analytic is singular point or singularity point. Example the function $\dfrac{1}{z-2}$ has a singular point at $z - 2 = 0$ or at $z = 2$.

4.4.1 Isolated Singular Point

If $z = a$ is a singularity of $f(z)$ and there is no other singularity with in a small circle surrounding the point $z = a$, then $z = a$ is said to be an isolated singularity of the function $f(z)$; otherwise it is called non-isolated.

Example the function $\frac{1}{(z-1)(z-3)}$, has two isolated singular points at $z = 1$, $z = 3$.

The function $\frac{1}{\sin \pi/z} = 0$ i.e., $\frac{\pi}{z} = n\pi$ or $z = \frac{1}{n}$ $[n = 1, 2, \ldots]$.

Here $z = 0$ is non-isolated singularity.

4.4.2 Essential Singularity

If the function $f(z)$ has pole $z = a$ is poles of order m. If

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 \ldots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} \ldots$$

the negative power in expansion are infinite then $z = a$ is called an essential singularity.

4.4.3 Removable Singularity

If

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

$\Rightarrow$

$$f(z) = a_0 + a_1(z-a) \ldots a_n(z-a)^n$$

Here the coefficient of negative power are zero. Then $z = a$ is called removable singularity i.e., $f(z)$ can be made analytic by redefining $f(a)$ suitably i.e., if $\lim_{x \to \infty} f(z)$ exists.

Example, $f(z) = \frac{\sin(z-a)}{(z-a)}$ has removable singularity at $z = a$.

4.4.4 Steps to Find Singularity

Step-1: If $\lim_{z \to a} f(z)$ exists and is finite then $z = a$ is a removable singular point.

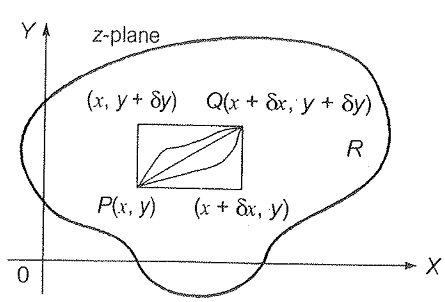
Step-2: If $\lim_{z \to a} f(z)$ does not exist then $z = a$ is an essential singular point.

Step-3: If $\lim_{z \to a} f(z)$ exists and is finite then $f(z)$ has a pole at $z = a$. The order of the pole is same as the number of negative power terms in the series expansion of $f(z)$.

4.5 Derivative of $f(z)$

Let $w = f(z)$ be a single-valued function of the variable $z = x + iy$. Then the derivative of $w = f(z)$ is defined to be

$$\frac{dw}{dz} = f'(z) = \underset{\delta z \to 0}{Lt} \frac{f(z+\delta z) - f(z)}{\delta z},$$



provided the limit exists and has the same value for all the different ways in which δz approaches zero.

Suppose $P(z)$ is fixed and $Q(z + \delta z)$ is a neighbouring point (Figure above). The point Q may approach P along any straight or curved path in the given region, i.e. δz may tend to zero in any manner and dw/dz may not exist. It, therefore, becomes a fundamental problem to determine the necessary and sufficient conditions for dw/dz to exist. The fact is settled by the following theorem.

Theorem. The necessary and sufficient conditions for the derivative of the function $w = u(x, y) + iv(x, y) = f(z)$ to exist for all values of z in a region R, are

1. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in R;

2. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

The relations in (ii) are known as Cauchy-Riemann equations or briefly C-R equations.

4.6 Analytic Functions

4.6.1 Analytic Functions

A function $f(z)$ which is single-valued and possesses a unique derivative with respect to z at all points of a region R, is called an **analytic** or a **regular function** of z in that region.

A point at which an analytic function ceases to possess a derivative is called a **singular point** of the function.

Thus if u and v are real single-valued functions of x and y such that $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ are continuous throughout a region R, then the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{... (i)}$$

are both necessary and sufficient conditions for the function $f(z) = u + iv$ to be analytic in R. The derivative of $f(z)$ is then, given by

$$f'(z) = \underset{\delta x \to 0}{Lt}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + iv_x \quad \text{... (ii)}$$

or

$$f'(z) = \underset{\delta y \to 0}{Lt}\left(\frac{\partial u}{i\delta y} + i\frac{\partial v}{i\delta y}\right)$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = v_y - iu_y \quad \text{... (iii)}$$

The real and imaginary parts of an analytic function are called **conjugate functions**. The relation between two conjugate functions is given by the C-R equations (i) above.

C-R equations in Polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial \theta}$$

Example 1.

Is $f(z) = z^3$ analytic?

Solution:

$$z = x + iy$$

$$\Rightarrow \quad z^2 = (x+iy)(x+iy) = x^2 - y^2 + 2ixy$$

$$\Rightarrow \quad z^3 = (x^2 - y^2 + 2ixy)(x+iy)$$

$$= (x^3 - 3xy^2) + (3x^2y - y^3)i$$

here $u = x^3 - 3xy^2$

$$v = 3x^2y - y^3$$

$$u_x = 3x^2 - 3y^2, \quad v_y = 3x^2 - 3y^2$$

$$u_y = -6xy, \quad v_x = 6xy$$

So $u_x = v_y$ and $u_y = -v_x$

So C–R equations are satisfied and also the partial derivatives are continuous at all points. Hence z^3 is analytic for every z.

Example 2.

If $w = \log z$, find dw/dz and determine where w is non-analytic.

Solution:

We have $w = u + iv = \log(x+iy)$

$$= \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1} y/x$$

so that $u = \frac{1}{2}\log(x^2+y^2)$, $v = \tan^{-1} y/x$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} = -\frac{\partial v}{\partial x}$$

Since the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous except at (0, 0). Hence w is analytic everywhere except at $z = 0$.

$$\therefore \quad \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x - iy}{(x+iy)(x-iy)}$$

$$= \frac{1}{x+iy} = \frac{1}{z} (z \neq 0).$$

Obs. The definition of the derivative of a function of complex variable is identical in form to that of the rivative of a function of real variable. Hence the rules of differentiation for complex functions are the same as ›se of real calculus. Thus if, a complex function is once known to be analytic, it can be differentiated just the ordinary way.

›.2 Harmonic Functions

Any function which satisfies Laplace equation is known as harmonic function.

If $f(z) = u + iv$ is analytic, then u and v are both harmonic functions.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Differentiating w.r.t. x

$$\frac{\partial^2 u}{\partial^2 x} = \frac{\partial^2 v}{\partial x \partial y}$$

Differentiating w.r.t. y

$$\frac{\partial^2 u}{\partial y} = -\frac{\partial^2 v}{\partial x \partial y}$$

adding these, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Therefore both u and v are harmonic functions.

4.6.3 Orthogonal Curves

Two curves are said to be orthogonal to each other, when they intersect at right angle at each of their point of intersection.

At the point of intersection, tangents at both the curves are also perpendicular.

The analytic function $f(z) = u(x, y) + iv(x, y)$ consists of two families of curves, $u(x, y) = c_1$ and $v(x, y) = c_2$ which form an orthogonal pair.

$$u(x, y) = c_1$$

$$\frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy = 0$$

$$\frac{dy}{dx} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = m_1 \qquad \text{(say)}$$

$$v(x, y) = c_2$$

$$\frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy = 0$$

$$\frac{dy}{dx} = -\frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y} = m_2 \qquad \text{(say)}$$

for orthogonality

$$m_1 m_2 = \left(-\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y}\right) \times \left(-\frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y}\right)$$

we know that, for an analytic function

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

we get,

$$m_1 m_2 = -1$$

$\Rightarrow$ The curves $u(x, y) = c_1$ and $v(x, y) = c_2$, are orthogonal.

4.7 Complex Integration

4.7.1 Line integral in the complex plane

As in calculus we distinguish between definite integrals and indefinite integrals or antiderivatives. An **indefinite integral** is a function whose derivative equals a given analytic function in a region. By known differentiation formulas we may find many types of indefinite integrals.

Complex definite integrals are called (complex) **line integrals.** They are written as

$$\int_C f(z)dz$$

Here the **integrand** $f(z)$ is integrated over a given curve C in the complex plane, called the **path of integration.** We may represent such a curve C by a parametric representation.

(1) $$x(t) = x(t) + iy(t) \qquad (a \le t \le b).$$

The sense of increasing t is called the **positive sense** on C, and we say that in this way, (1) **orients** C.

We assume C to be a **smooth curve,** that is, C has a continuous and nonzero derivative $\dot{z} = dz/dt$ at each point. Geometrically this means that C has a unique and continuously turning tangent.

4.7.2 Definition of the Complex Line Integral

This is similar to the method in calculus. Let C be a smooth curve in the complex plane given by (1), and let $f(z)$ be a continuous function given (at least) at each point of C. We now subdivide (we "**partition**") the interval $a \le t \le b$ in (1) by points

$$t_0 (=a),\ t_1,\ \ldots,\ t_{n-1},\ t_n (=b)$$

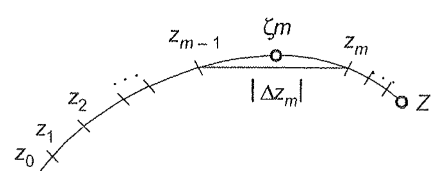
where $t_0 < t_1 \cdots < t_n$. To this subdivision there corresponds a subdivision of C by points

$$z_0,\ z_1, \ldots,\ z_{n-1},\ z_n (= Z)$$

where $z_j = z(t_j)$. On each portion of subdivision of C we choose an arbitrary point, say, a point ζ_1 between z_0 and z_1 (that is, $\zeta_1 = z(t)$) where t satisfies $t_0 \le t \le t_1$), a point ζ_2 between z_1 and z_2, etc. Then we form the sum

(2) $$S_n = \sum_{m=1}^{n} f(\zeta_m)\Delta z_m \qquad \text{where } \Delta z_m = z_m - z_{m-1}.$$

We do this for each $n = 2, 3, \cdots$ in a completely independent manner, but so that the greatest $|\Delta t_m| = |t_m - t_{m-1}|$ approaches zero as $n \to \infty$. This implies that the greatest $|\Delta z_m|$ also approaches zero because it cannot exceed the length of the arc of C from z_{m-1} to z_m and the latter goes to zero since the arc length of the smooth curve C is a continuous function of t. The limit of the sequence of complex numbers $S_2, S_3, \cdots$ thus obtained is called the **line integral** (or simply the integral) of $f(z)$ over the oriented curve C. This



Complex line integral

curve C is called **path of integration.** The line integral is denoted by

(3) $\int_C f(z)dz$, or by $\oint_C f(z)dz$

if C is a **closed path** (one whose terminal point Z coincides with its initial point z_0, as for a circle or an 8-shaped curve).

General Assumption. All paths of integration for complex line integrals are assumed to be **piecewise smooth,** that is, they consist of finitely many smooth curves joined end to end.

4.7.3 First Method: Indefinite Integration and Substitution of Limits

This method is simpler than the next one, but is less general. It is restricted to analytic functions. Its formula is the analog of the familiar formula from calculus

$$\int_a^b f(x)dx = F(b) - F(a) \qquad [F'(x) = f(x)].$$

Theorem 1: (Indefinite integration of analytic functions)

Let $f(z)$ be analytic in a simply connected domain D. A domain D is called **simply connected** if every simple closed curve (closed curve without self-intersections in D encloses only points of D). Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have

$$\text{(4)} \qquad \int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)].$$

(Note that we can write z_0 and z_1 instead of C, since we get the same value for all those C from z_0 to z_1). This theorem will be proved in the next section.

Simple connectedness is quite essential in Theorem 1, as we shall see in Example 5. Since analytic functions are our main concern, and since differentiation formulas will often help in finding $F(z)$ for a given $f(z) = F'(z)$, the present method is of great practical interest.

If $f(z)$ is entire, we can take for D the complex plane (which is certainly simply connected).

Example 1. $$\int_0^{1+i} z^2 dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = \frac{2}{3} + \frac{2}{3}i$$

Example 2. $$\int_{-\pi i}^{\pi i} \cos z\, dz = \sin z\Big|_{-\pi i}^{\pi i} = 2\sin \pi i = 2i \sinh \pi = 23.097i$$

Example 3. $$\int_{8+\pi i}^{8-3\pi i} e^{z/2} dz = 2e^{z/2}\Big|_{8+\pi i}^{8-3\pi i} = 2(e^{4-3\pi i/2} - e^{4+\pi i/2}) = 0$$

Since e^z is periodic with period $2\pi i$.

4.7.4 Second Method: Use of a Representation of the Path

This method is not restricted to analytic functions but applies to any continuous complex function.

Theorem 2: (Integration by the use of the path)

Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \le t \le b$. Let $f(z)$ be a continuous function on C. Then

$$\text{(5)} \qquad \int_C f(z)dz = \int_a^b f[z(t)]\, \dot{z}(t)dt \qquad \left(\dot{z} = \frac{dz}{dt}\right)$$

Proof: The left side of (5) is given in terms of real line integrals as $\int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)$. We now show that the right side of (5) also equals the same.

We have $z = x + iy$, hence $\dot{z} = \dot{x} + i\dot{y}$. We simply write u for $u[x(t), y(t)]$ and v for $v[x(t), y(t)]$. We also have $dx = \dot{x}\,dt$ and $dy = \dot{y}\,dt$.

Consequently, in (5)

$$\int_a^b f[z(t)\dot{z}(t)]dt = \int_a^b (u + iv)(\dot{x} + i\dot{y})dt = \int_C [u\,dx - v\,dy + i(u\,dy + v\,dx)]$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)$$

Steps in applying Theorem 2

1. Represent the path C in the form $z(t)$ $(a \le t \le b)$.
2. Calculate the derivative $\dot{z}(t) = dz/dt$.
3. Substitute $z(t)$ for every z in $f(z)$ (hence $x(t)$ for x and $y(t)$ for y).
4. Integrate $f[z(t)]\dot{z}(t)$ over t from a to b.

Example 1: A basic result: Integral of $1/z$ around the unit circle

We show that by integrating $1/z$ counterclockwise around the unit circle (the circle of radius 1 and center 0), ve obtain

(6) $$\oint_C \frac{dz}{z} = 2\pi i \qquad (C \text{ the unit circle, counterclockwise}).$$

This is a very important result that we shall need quite often.

Solution: We may represent the unit circle C in the form

$$z(t) = \cos t + i \sin t = e^{it} \qquad (0 \le t \le 2\pi),$$

so that the counterclockwise integration corresponds to an increase of t from 0 to 2π. By differentiation, $\dot{z}(t) = ie^{it}$ (chain rule) and with $f(z(t)) = 1/z(t) = e^{-it}$ we get from (10) the result

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it} ie^{it} dt = i\int_0^{2\pi} dt = 2\pi i$$

Check this result by using $z(t) = \cos t + i \sin t$.

Simple connectedness is essential in Theorem 1. Equation (4) in Theorem 1 gives 0 for any closed path ecause then $z_1 = z_0$, so that $F(z_1) - F(z_0) = 0$. Now $1/z$ is not analytic at $z = 0$. But any simply connected domain ontaining the unit circle must contain $z = 0$, so that Theorem 1 does not apply—it is not enough that $1/z$ is analytic ı an annulus, say $\frac{1}{2} < |z| < \frac{3}{2}$, because an annulus is not simply connected!

Example 2: Integral of integer powers

Let $f(z) = (z - z_0)^m$ where m is an integer and z_0 a constant. Integrate counterclockwise around the circle C f radius ρ with center at z_0 (Fig. below)

Solution: We may represent C in the form

$$z(t) = z_0 + \rho(\cos t + i \sin t) = z_0 + pe^{it} \qquad (0 \le i \le 2\pi).$$

Then we have

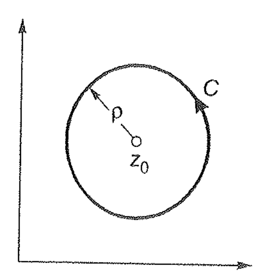
$$(z - z_0)^m = p^m e^{imt}, \qquad dz = i\rho e^{it}\, dt$$

and obtain

$$\oint_C (z - z_0)^m dz = \int_0^{2\pi} \rho^m e^{imt} i\rho e^{it} dt = i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t} dt$$

By the Euler formula, the right side equals

$$i\rho^{m+1} \left[\int_0^{2\pi} \cos(m+1)t\, dt + i \int_0^{2\pi} \sin(m+1)t\, dt \right].$$



Path in Example 2

If $m = -1$, we have $\rho^{m+1} = 1$, $\cos 0 = 1$, $\sin 0 = 0$. We thus obtain $2\pi i$. For integer $m \ne 1$ each of the two tegrals is zero because we integrate over an interval of length 2π, equal to a period of sine cosine. Hence the sult is

(7) $$\oint_C (z - z_0)^m dz = \begin{cases} 2\pi i & (m = -1), \\ 0 & (m \neq -1 \text{ and integer}) \end{cases}$$

Dependence on path. Now comes a very important fact. If we integrate a given function $f(z)$ from a point z_0 to a point z_1 along different paths, the integrals will in general have different values. In other words, **a complex line integral depends not only on the endpoints of the path but in general also on the path itself.** See the next example.

Example 3: Integral of a non-analytic function. Dependence on path

Integrate $f(z) = \text{Re } z = x$ from 0 to $1 + 2i$

(a) along C^* in Fig. below,

(b) along C consisting of C_1 and C_2.

Solution:

(a) C^* can be represented by $z(t) = t + 2it$ $(0 \le t \le 1)$. Hence $\dot{z}(t) = 1 + 2i$ and $f[f(t)] = x(t) = t$ on C^*. We now calculate

$$\int_{C^*} \text{Re } z\, dz = \int_0^1 t(1+2i)dt = \frac{1}{2}(1+2i) = \frac{1}{2} + i$$

(b) We now have

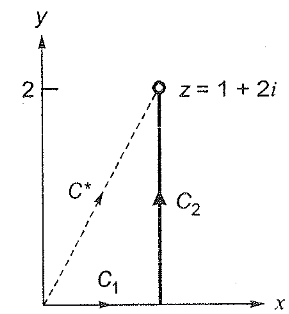
C_1: $z(t) = t$, $\quad \dot{z}(t) = 1$, $\quad f(z(t)) = x(t) = t \quad (0 \le t \le 1)$

C_2: $z(t) = 1 + it$, $\quad \dot{z}(t) = i$, $\quad f(z(t)) = x(t) = t \quad (0 \le t \le 2)$

We calculate by partitioning the path C into two paths C_1 and C_2 as shown below

$$\int_C \text{Re } z\, dz = \int_{C_1} \text{Re} z\, dz + \int_{C_2} \text{Re } z\, dz = \int_0^1 t\, dt + \int_0^2 1 \cdot i\, dt = \frac{1}{2} + 2i$$

Note that this result differs from the result in (a).



Paths in Example 3

4.8 Cauchy's Theorem

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a closed curve C, then $\int_C f(z)dz = 0$.

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that $dz = dx + idy$

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

Since $f'(z)$ is continuous, therefore, $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are also continuous in the region D enclosed by

Hence the Green's theorem can be applied to (i), giving

$$\int_C f(z)dz = -\iint_C \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] dx\,dy + i\iint_D \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dx\,dy \quad \text{... (ii)}$$

Now $f(z)$ being analytic, u and v necessarily satisfy the Cauchy-Riemann equations and thus the integrands of the two double integrals in (ii) vanish identically.

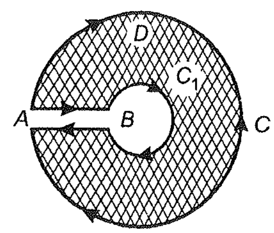
Hence, $$\iint_C f(z)dz = 0.$$

Obs. 1 The Cauchy-Riemann equations are precisely the conditions for the two real integrals in (1) to be independent of the path. Hence the line integral of a function $f(z)$ which is analytic in the region D, is independent of the path joining any two points of D.

Obs. 2 Extension of Cauchy's theorem. If $f(z)$ is analytic in the region D between two simple closed curves C and C_1, then $\int_C f(z)dz = \int_{C_1} f(z)dz$.

To prove this, we need to introduce the cross-cut AB. Then $\int f(z)dz = 0$ where the path is as indicated by arrows in Figure below i.e. along AB—along C_1 in clockwise sense and along BA—along C in anti-clockwise sense

i.e. $\int_{AB} f(z)dz + \int_{C_1} f(z)dz + \int_{BA} f(z)dz + \int_C f(z)dz = 0.$



But, since the integral along AB and along BA cancel, it follows that

$$\int_C f(z)dx + \int_{C_1} f(z)dz = 0.$$

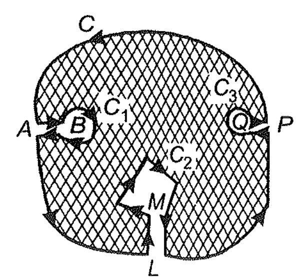
Reversing the direction of the integral around C_1 and transposing, we get

$$\int_C f(z)dz = \int_{C_1} f(z)dz$$

each integration being taken in the anti-clockwise sense.

If $C_1, C_2, C_3, \ldots$, be any number of closed curves within C (Figure below), then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \ldots$$



4.9 Cauchy's Integral Formula

If $f(z)$ is analytic within and on a closed curve and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a}.$$

Consider the function $f(z)/(z-a)$ which is analytic at all points within C except at $z = a$. With the point a as centre and radius r, draw a small circle C_1 lying entirely within C.

Now $f(z)/(z-a)$ being analytic in the region enclosed by C and C_1, we have by Cauchy's theorem,

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz \qquad \left\{\begin{array}{l}\text{For any point on } C_1, \\ z-a = re^{i\theta} \text{ and } dz = ire^{i\theta}d\theta\end{array}\right.$$

$$= \int_{C_1} \frac{f(a+re^{i\theta})}{re^{i\theta}} \cdot ire^{i\theta}d\theta = i\int_{C_1} f(a+re^{i\theta})d\theta. \quad \text{... (i)}$$

In the limiting form, as the circle C_1 shrinks to the point a, i.e. as $r \to 0$, the integral **(i)** will approach to

$$i\int_{C_1} f(a)d\theta = if(a)\int_0^{2\pi} d\theta = 2\pi i f(a). \text{ Thus } \int_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

i.e.

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \quad \text{... (ii)}$$

which is the desired Cauchy's integral formula.

Cor. Differentiating both sides of (2) w.r.t. a,

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right]dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz. \quad \text{... (iii)}$$

Similarly,

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}dz \quad \text{... (iv)}$$

and in general,

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz \quad \text{... (v)}$$

Thus it follows from the results (2) to (5) that if a function $f(z)$ is known to be analytic on the simple closed curve C then the values of the function and all its derivatives can be found at any point of C. Incidently, we have established a remarkable fact that **an analytic function possesses derivatives of all orders and these are themselves all analytic.**

4.10 Series of Complex Terms

1. **Taylor's series:** If $f(z)$ is analytic inside a circle C with centre at a, then for z inside C,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots + \frac{f^n(a)}{n!}(z-a)^n + \ldots \quad \text{... (i)}$$

2. **Laurent's series:** If $f(z)$ is analytic in the ring-shaped region R bounded by two concentric circles C and C_1 of radii r and r_1 $(r > r_1)$ and with centre at a, then for all z in R

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a\_1(z-a)^{-1} + a_{-2}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots$$

$$a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{(t-a)^{n+1}}dt$$

Γ being any curve in R, encircling C_1.

Obs. 1. As $f(z)$ is analytic inside, G, then $a_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(t)}{(t-a)^{n+1}}dt \neq \frac{f^n(a)}{n!}$

However, if f(z) is analytic inside G, then $a_{-n} = 0$; $a_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(t)}{(t-a)^{n+1}}dt = \frac{f^n(a)}{n!}$

and Laurent's series reduces to Taylor's series.

Obs. 2. To obtain Taylor's or Laurent's series, simply expand $f(z)$ by binomial theorem, instead of finding a_n by complex integration which is quite complicated.

Obs. 3. Laurent series of a given analytic function $f(z)$ in its annulus of convergence is unique. There may be different Laurent series of $f(z)$ in two annuli with the same centre.

1.11 Zeros and Singularities or Poles of an Analytic Function

1.11.1 Zeros of an Analytic Function

Definition: A zero of an analytic function $f(z)$ is that value of z for which $f(z) = 0$

If $f(z)$ is analytic in the neighbourhood of a point $z = a$, then by Taylor's theorem

$$f(z) = a_0 + a_1 (z-a) + a_2(z-a)^2 + \ldots + a_n (z-a)^n + \ldots \text{ where } a_n = \frac{f^n(a)}{n!}$$

If $a_0 = a_1 = a_2 = \ldots = a_{m-1} = 0$ but $a_m \neq 0$, then $f(z)$ is said to have a zero of order m at $z = a$.

When $m = 1$, the zero is said to be simple. In the neighbourhood of zero ($z = a$) of order m.

$$f(z) = a_m (z-a)^m + a_{m+1} (z-a)^{m+1} + \ldots\infty$$

$$= (z-a)^m \phi(z)$$

where,

$$\phi(z) = a_m + a_{m+1} (z-a) + \ldots$$

Then $\phi(z)$ is analytic and non-zero in the neighbourhood of $z = a$.

Example 1.

Poles and Essential singularities

The function

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

has a simple pole at $z = 0$ and a pole of fifth order at $z = 2$. Examples of functions having an isolated essential singularity at $z = 0$ are

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

and

$$\sin\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!z^{2n+1}}$$

$$= \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - + \cdots$$

Note: The classification of singularities into poles and essential singularities is not merely a formal matter, because the behaviour of an analytic function in a neighborhood of an essential singularity is entirely from that in the neighborhood of a pole.

Example 2.

Find the nature of singularities of following functions

(a) $f(z) = \dfrac{1}{z(z-2)^5} + \dfrac{3}{(z-2)^2}$ (b) $e^{\frac{1}{z}}$ (c) $\sin\dfrac{1}{z}$

Example 3.

Find the nature and location of singularities of the following functions:

(a) $\dfrac{z - \sin z}{z^2}$ (b) $(z+1)\sin\dfrac{1}{z-2}$ (c) $\dfrac{1}{\cos z - \sin z}$

Solution:

(a) Here $z = 0$ is a singularity.

Also
$$\frac{z-\sin z}{z^2} = \frac{1}{z^2}\left\{z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\ldots\right)\right\}$$
$$= \frac{z}{3!}-\frac{z^3}{5!}+\frac{z^5}{7!}-\ldots$$

Since there are no negative powers of z in the expansion, $z = 0$ is a removable singularity.

(b)
$$(z+1)\sin\frac{1}{z-2} = (t+2+1)\sin\frac{1}{t}, \text{ where } t = z-2$$
$$= (t+3)\left\{\frac{1}{t}-\frac{1}{3!\,t^3}+\frac{1}{5!\,t^5}-\ldots\right\}$$
$$= \left(1-\frac{1}{3!\,t^2}+\frac{1}{5!\,t^4}-\ldots\right)+\left(\frac{3}{t}-\frac{1}{2t^3}+\frac{3}{5!\,t^5}-\ldots\right)$$
$$= 1+\frac{3}{t}-\frac{1}{6t^2}-\frac{1}{2t^3}+\frac{1}{120t^4}-\ldots$$
$$= 1+\frac{3}{z-2}-\frac{1}{6(z-2)^2}-\frac{1}{2(z-2)^3}+\ldots$$

Since there are infinite number of terms in the negative powers of $(z-2)$, $z = 2$ is an essential singularity.

(c) Poles of $f(z) = \dfrac{1}{\cos z - \sin z}$ are given by equating the denominator to zero, i.e. by $\cos z - \sin z = 0$ or $\tan z = 1$ or $z = \pi/4$. Clearly $z = \pi/4$ is a simple pole of $f(z)$.

4.12 Residues

The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity is called the residue of $f(z)$ at that point. Thus is the Laurent's series expansion of $f(z)$ around $z = a$ i.e. $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots$, the residue of $f(z)$ at $z = a$ is a_{-1}.

Since,
$$a_n = \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$\therefore$
$$a_{-1} = \text{Res } f(a) = \frac{1}{2\pi i}\int_C f(z)dz$$

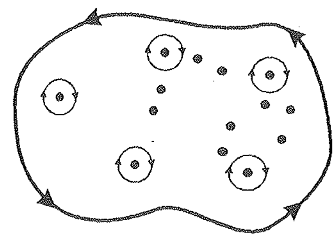
$\therefore$
$$\int_C f(z)dz = 2\pi i \text{ Res } f(a) \quad \ldots \text{(i)}$$

4.12.1 Residue Theorem

If $f(z)$ is analytic in a closed curve C except at a finite number of singular points within C, then

$$\int_C f(z)dz = 2\pi i \times (\text{sum of the residues at the singular points within } C)$$

Let us surround each of the singular points $a_1, a_2, \ldots, a_n$ by a small circle such that it encloses no other singular point. Then these circles $C_1, C_2, \ldots, C_n$ together with C, form a multiply connected region in which $f(z)$ is analytic.



∴ Applying Cauchy's theorem, we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots \int_{C_n} f(z)dz \qquad \text{[by (i)]}$$

$$= 2\pi i\,[\text{Res } f(a_1) + \text{Res } f(a_2) + \ldots + \text{Res } f(a_n)]$$

which is the desired result.

4.12.2 Calculation of Residues

1. If $f(z)$ has a simple pole at $z = a$, then

$$\text{Res } f(a) = \underset{z \to a}{\text{Lt}}\,[(z-a)f(z)] \qquad \ldots \text{(i)}$$

Laurent's series in this case is

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 \ldots + c_{-1}(z-a)^{-1}$$

Multiplying throughout by $z - a$, we have

$$(z-a)\,f(z) = c_0(z-a) + c_1(z-a)^2 + \ldots + c_{-1}$$

Taking limits as $z \to a$, we get

$$\underset{z \to a}{\text{Lt}}\,[(z-a)f(z)] = c_{-1} = \text{Res } f(a)$$

2. Another formula for Res $f(a)$:

Let $\quad f(z) = \phi(z)/\psi(z)$, where $\psi(z) = (z-a)\,F(z)$, $F(a) \neq 0$.

Then
$$\underset{z \to a}{\text{Lt}}\,[(z-a)\phi(z)/\psi(z)] = \underset{z \to a}{\text{Lt}}\,\frac{(z-a)[\phi(a) + (z-a)\phi'(a) + \ldots..]}{\psi(a) + (z-a)\psi'(a) + \ldots...}$$

$$= \underset{z \to a}{\text{Lt}}\,\frac{\phi(a) + (z-a)\phi'(a) + \ldots..}{\psi'(a) + (z-a)\psi''(a) + \ldots.}, \text{ since } \gamma(a) = 0$$

Thus,
$$\text{Res } f(a) = \frac{\phi(a)}{\psi'(a)}$$

3. If $f(z)$ has a pole of order n at $z = a$, then

$$\text{Res } f(a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$$

Obs. In many cases, the residue of a pole ($z = a$) can be found , by putting $z = a+t$ in $f(z)$ and expanding it in powers of t where $|\,t\,|$ is quite small.

Previous GATE and ESE Questions

Q.1 Consider likely applicability of Cauchy's Integral Theorem to evaluate the following integral counter clockwise around the unit circle c.

$$I = \oint_c \sec z\,dz,$$

z being a complex variable. The value of I will be

(a) $I = 0$: singularities set = ϕ

(b) $I = 0$: singularities set= $\left\{\pm \frac{2n+1}{2}\pi;\ n = 0, 1, 2...\right\}$

(c) $I = \pi/2$: singularities set = $\{\pm n\pi;\ n = 0, 1, 2...\}$

(d) None of above

[CE, GATE-2005, 2 marks]

Q.2 Using Cauchy's integral theorem, the value of the integral (integration being taken in counterclockwise direction) $\oint_c \frac{z^3 - 6}{3z - i}dz$ is

(a) $\frac{2\pi}{81} - 4\pi i$ (b) $\frac{\pi}{8} - 6\pi i$

(c) $\frac{4\pi}{81} - 6\pi i$ (d) 1

[CE, GATE-2006, 2 marks]

Q.3 The value of the contour integral $\oint_{|z-i|=2} \frac{1}{z^2 + 4}dz$ in positive sense is

(a) $i\pi/2$ (b) $-\pi/2$

(c) $-i\pi/2$ (d) $\pi/2$

[EC, GATE-2006, 2 marks]

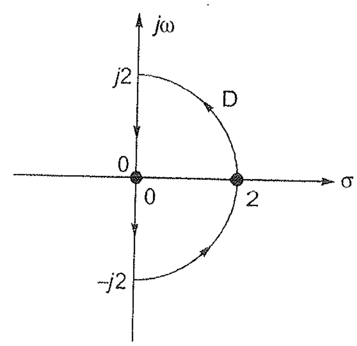
Q.4 The value of $\oint_C \frac{dz}{(1+z^2)}$ where C is the contour $|z - i/2| = 1$ is

(a) $2\pi i$ (b) π

(c) $\tan^{-1}z$ (d) $\pi i \tan^{-1}z$

[EE, GATE-2007, 2 marks]

Q.5 If the semi-circular contour D of radius 2 is as shown in the figure, then the value of the integral $\oint_D \frac{1}{(s^2 - 1)}ds$ is



(a) $j\pi$ (b) $-j\pi$

(c) $-\pi$ (d) π

[EC, GATE-2007, 2 marks]

Q.6 The integral $\oint f(z)dz$ evaluated around the unit circle on the complex plane for $f(z) = \frac{\cos z}{z}$ is

(a) $2\pi i$ (b) $4\pi i$

(c) $-2\pi i$ (d) 0

[ME, GATE-2008, 2 marks]

Q.7 The residue of the function $f(z) = \frac{1}{(z+2)^2(z-2)^2}$ at $z = 2$ is

(a) $-\frac{1}{32}$ (b) $-\frac{1}{16}$

(c) $\frac{1}{16}$ (d) $\frac{1}{32}$

[EC, GATE-2008, 2 marks]

Q.8 An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x, y) + i\,v(x, y)$ where $i = \sqrt{-1}$. If $u = xy$, the expression for v should be

(a) $\frac{(x+y)^2}{2} + k$ (b) $\frac{x^2 - y^2}{2} + k$

(c) $\frac{y^2 - x^2}{2} + k$ (d) $\frac{(x-y)^2}{2} + k$

[ME, GATE-2009, 2 marks]

.9 The value of the integral $\int_C \frac{\cos(2\pi z)}{(2z-1)(z-3)} dz$ (where C is a closed curve given by $|z| = 1$) is

(a) $-\pi i$ (b) $\frac{\pi i}{5}$

(c) $\frac{2\pi i}{5}$ (d) πi

[CE, GATE-2009, 2 marks]

.10 The analytic function $f(z) = \frac{z-1}{z^2+1}$ has singularities at

(a) 1 and – 1 (b) 1 and i

(c) 1 and $-i$ (d) i and $-i$

[CE, GATE-2009, 1 mark]

11 If $f(z) = C_0 + C_1 z^{-1}$, then $\oint_{\text{unit circle}} \frac{1+f(z)}{z} dz$ is given by

(a) $2\pi C_1$ (b) $2\pi(1 + C_0)$

(c) $2\pi j C_1$ (d) $2\pi j(1 + C_0)$

[EC, GATE-2009, 1 mark]

12 The modulus of the complex number $\left(\frac{3+4i}{1-2i}\right)$ is

(a) 5 (b) $\sqrt{5}$

(c) $1/\sqrt{5}$ (d) 1/5

[ME, GATE-2010, 1 mark]

13 The residues of a complex function $X(z) = \frac{1-2z}{z(z-1)(z-2)}$ at its poles are

(a) $\frac{1}{2}, -\frac{1}{2}$ and 1 (b) $\frac{1}{2}, \frac{1}{2}$ and -1

(c) $\frac{1}{2}$, 1 and $-\frac{3}{2}$ (d) $\frac{1}{2}$, -1 and $\frac{3}{2}$

[EC, GATE-2010, 2 marks]

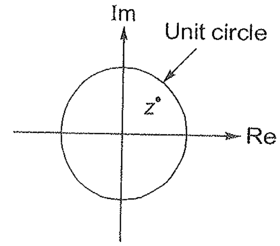
14 For an analytic function, $f(x + iy) = u(i, y) + iv(i, y)$, u is given by $u = 3x^2 - 3y^2$. The expression for v, considering K to be a constant is

(a) $3y^2 - 3x^2 + K$ (b) $6x - 6y + K$

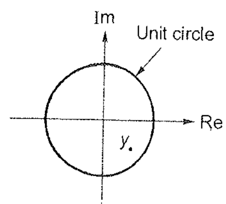
(c) $6y - 6x + K$ (d) $6xy + K$

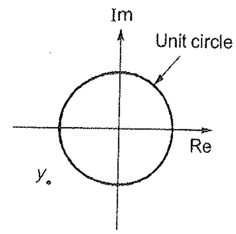
[CE, GATE-2011, 2 mark]

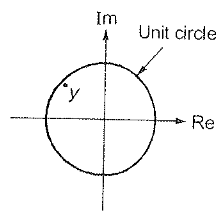
Q.15 A point z has been plotted in the complex plane, as shown in figure below.

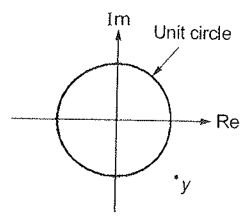


The plot for point $\frac{1}{z}$ is

(a) 

(b) 

(c) 

(d) 

[EE, GATE-2011, 1 marks]

Q.16 The value of the integral $\oint_c \frac{-3z+4}{(z^2+4z+5)} dz$ where c is the circle $|z| = 1$ is given by

(a) 0 (b) 1/10

(c) 4/5 (d) 1

[EC GATE-2011, 1 mark]

Q.17 If $x = \sqrt{-1}$, then the value of x^x is

(a) $e^{-\pi/2}$ (b) $e^{\pi/2}$

(c) x (d) 1

[EC, EE, IN, GATE-2012, 1 mark]

Q.18 Given $f(z) = \frac{1}{z+1} - \frac{2}{z+3}$. If C is a counter clockwise path in the z-plane such that $|z+1| = 1$, the value of $\frac{1}{2\pi j}\oint_C f(z)dz$ is

(a) – 2 (b) – 1

(c) 1 (d) 2

[EC, EE, IN, GATE-2012, 1 mark]

Q.19 Square roots of $-i$, where $i = \sqrt{-1}$, are

(a) $i, -i$

(b) $\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

(c) $\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$

(d) $\cos\left(\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right), \cos\left(-\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

[EE, GATE-2013, 1 Mark]

Q.20 The complex function tan $h(s)$ is analytic over a region of the imaginary axis of the complex s-plane if the following is TRUE everywhere in the region for all integers n

(a) $\text{Re}(s) = 0$ (b) $\text{Im}(s) \neq n\pi$

(c) $\text{Im}(s) \neq \frac{n\pi}{3}$ (d) $\text{Im}(s) \neq \frac{(2n+1)\pi}{2}$

[IN, GATE-2013 : 1 mark]

Q.21 $\oint \frac{z^2-4}{z^2+4} dz$ evaluated anticlockwise around the circle $|z - i| = 2$, where $i = \sqrt{-1}$, is

(a) -4π (b) 0

(c) $2 + \pi$ (d) $2 + 2i$

[EE, GATE-2013, 2 Marks]

Q.22 $z = \frac{2-3i}{-5+i}$ can be expressed as

(a) $-0.5 - 0.5i$ (b) $-0.5 + 0.5i$

(c) $0.5 - 0.5i$ (d) $0.5 + 0.5i$

[CE, GATE-2014 : 1 Mark]

Q.23 An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x, y) + i\,v(x, y)$, where $i = \sqrt{-1}$. If $u(x, y) = 2xy$, then $v(x, y)$ must be

(a) $x^2 + y^2$ + constant

(b) $x^2 - y^2$ + constant

(c) $-x^2 + y^2$ + constant

(d) $-x^2 - y^2$ + constant

[ME, GATE-2014 : 2 Marks]

Q.24 An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x, y) + iv(x, y)$, where $i = \sqrt{-1}$. If $u(x, y) = x^2 - y^2$, then expression for $v(x, y)$ in terms of x, y and a general constant c would be

(a) $xy + c$ (b) $\frac{x^2+y^2}{2} + c$

(c) $2xy + c$ (d) $\frac{(x-y)^2}{2} + c$

[ME, GATE-2014 : 2 Marks]

Q.25 The argument of the complex number $\frac{1+i}{1-i}$, where $i = \sqrt{-1}$, is

(a) $-\pi$ (b) $-\frac{\pi}{2}$

(c) $\frac{\pi}{2}$ (d) π

[ME, GATE-2014 : 1 Mark]

Q.26 Let S be the set of points in the complex plane corresponding to the unit circle. (That is, $S = \{z : |z| = 1\}$. Consider the function $f(z) = zz^*$ where z^* denotes the complex conjugate of z. The $f(z)$ maps S to which one of the following in the complex plane

(a) unit circle

(b) horizontal axis line segment from origin to (1, 0)

(c) the point (1, 0)

(d) the entire horizontal axis

[EE, GATE-2014 : 1 Mark]

Q.27 All the values of the multi-valued complex function 1^i, where $i = \sqrt{-1}$, are

(a) purely imaginary

(b) real and non-negative

(c) on the unit circle

(d) equal in real and imaginary parts

[EE, GATE-2014 : 1 Mark]

Q.28 The real part of an analytic function $f(z)$ where $z = x + jy$ is given by $e^{-y}\cos(x)$. The imaginary part of $f(z)$ is

(a) $e^{y}\cos(x)$ (b) $e^{-y}\sin(x)$

(c) $-e^{y}\sin(x)$ (d) $-e^{-y}\sin(x)$

[EC, GATE-2014 : 2 Marks]

Q.29 If z is a complex variable, the value of $\int_5^{3i} \frac{dz}{z}$ is

(a) $-0.511 - 1.57i$ (b) $-0.511 + 1.57i$

(c) $0.511 - 1.57i$ (d) $0.511 + 1.57i$

[ME, GATE-2014 : 2 Marks]

Q.30 Integration of the complex function $f(z) = \frac{z^2}{z^2 - 1}$, in the counterclockwise direction, around $|z - 1| = 1$, is

(a) $-\pi i$ (b) 0

(c) πi (d) $2\pi i$

[EE, GATE-2014 : 2 Marks]

Q.31 The Taylor series expansion of $3 \sin x + 2 \cos x$ is ____.

(a) $2+3x-x^2-\frac{x^3}{2}+\cdots$

(b) $2-3x+x^2-\frac{x^3}{2}+\cdots$

(c) $2+3x+x^2+\frac{x^3}{2}+\cdots$

(d) $2-3x-x^2+\frac{x^3}{2}+\cdots$

[EC, GATE-2014 : 2 Marks]

Q.32 The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges to

(a) $2 \ln 2$ (b) $\sqrt{2}$

(c) 2 (d) e

[EC, GATE-2014 : 1 Mark]

Q.33 Given two complex numbers $z_1 = 5 + (5\sqrt{3})i$ and $z_2 = \frac{2}{\sqrt{3}} + 2i$ the argument of $\frac{z_1}{z_2}$ in degree is

(a) 0 (b) 30

(c) 60 (d) 90

[ME, GATE-2015 : 1 Mark]

Q.34 Given $f(z) = g(z) + h(z)$, where f, g, h are complex valued functions of a complex variable z. Which one of the following statements is TRUE?

(a) If $f(z)$ is differentiable at z_0, then $g(z)$ and $h(z)$ are also differentiable at z_0.

(b) If $g(z)$ and $h(z)$ are differentiable at z_0, then $f(z)$ is also differentiable at z_0.

(c) If $f(z)$ is continuous at z_0, then it is differentiable at z_0.

(d) If $f(z)$ is differentiable at z_0, then so are its real and imaginary parts.

[EE, GATE-2015 : 1 Mark]

Q.35 Let $z = x + iy$ be a complex variable. Consider that contour integration is performed along the unit circle in anticlockwise direction. Which one of the following statements is NOT TRUE?

(a) The residue of $\frac{z}{z^2 - 1}$ at $z = 1$ is 1/2

(b) $\oint_C z^2 dz = 0$

(c) $\frac{1}{2\pi i} \oint_C \frac{1}{z} dz = 1$

(d) $\bar{z}$ (complex conjugate of z) is analytical function

[EC, GATE-2015 : 1 Mark]

Q.36 Let $f(z) = \frac{az + b}{cz + d}$. If $f(z_1) = f(z_2)$ for all $z_1 \neq z_2$, $a = 2$, $b = 4$ and $c = 5$, then d should be equal to ________.

[EC, GATE-2015 : 1 Mark]

Q.37 The value of $\oint \frac{1}{z^2} dz$, where the contour is the unit circle traversed clockwise, is

(a) $-2\pi i$ (b) 0

(c) $2\pi i$ (d) $4\pi i$

[IN, GATE-2015 : 1 Mark]

Q.38 If C denotes the counterclockwise unit circle, the value of the contour integral $\frac{1}{2\pi j} \oint_C \text{Re}\{z\} dz$ is_____.

[EC, GATE-2015 : 2 Marks]

Q.39 If C is a circle of radius r with centre z_0, in the complex z-plane and if n is a non-zero integer, then $\oint \frac{dz}{(z - z_0)^{n+1}}$ equals

(a) $2\pi nj$ (b) 0

(c) $\frac{\pi j}{2\pi}$ (d) $2\pi n$

[EC, GATE-2015 : 1 Mark]

Q.40 Consider the following complex function

$$f(z) = \frac{9}{(z - 1)(z + 2)^2}$$

Which of the following is one of the residues of the above function?

(a) −1 (b) $\frac{9}{16}$

(c) 2 (d) 9

[CE, GATE-2015 : 2 Marks]

Q.41 In the neighborhood of $z = 1$, the function $f(z)$ has a power series expansion of the form

$f(z) = 1 + (1 - z) + (1 - z)^2 + \ldots$

Then $f(z)$ is

(a) $\frac{1}{z}$ (b) $\frac{-1}{z-2}$

(c) $\frac{z-1}{z+2}$ (d) $\frac{1}{2z-1}$

[IN, GATE-2016 : 1 Mark]

Q.42 Consider the complex valued function $f(z) = 2z^3 + b|z|^3$ where z is a complex variable. The value of b for which the function $f(z)$ is analytic is ______.

[EC, GATE-2016 : 1 Mark]

Q.43 $f(z) = u(x, y) + iv(x, y)$ is an analytic function or complex variable $z = x + iy$ where $i = \sqrt{-1}$ $u(x, y) = 2xy$, then $v(x, y)$ may be expressed as

(a) $-x^2 + y^2$ + constant

(b) $x^2 - y^2$ + constant

(c) $x^2 + y^2$ + constant

(d) $-(x^2 + y^2)$ + constant

[ME, GATE-2016 : 1 Mark]

Q.44 A function f of the complex variable $z = x + iy$, is given as $f(x, y) = u(x, y) + iv(x, y)$, where $u(x, y) = 2kxy$ and $v(x, y) = x^2 - y^2$. The value of k, for which the function is analytic, is _____.

[ME, GATE-2016 : 1 Mark]

Q.45 Consider the function $f(z) = z + z^*$ where z is a complex variable and z^* denotes its complex conjugate. Which one of the following is TRUE?

(a) $f(z)$ is both continuous and analytic

(b) $f(z)$ is continuous but not analytic

(c) $f(z)$ is not continuous but is analytic

(d) $f(z)$ is neither continuous nor analytic

[EE, GATE-2016 : 1 Mark]

Q.46 The value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 2x + 2} dx$$

evaluated using contour integration and the residue theorem is

(a) $\frac{-\pi \sin(1)}{e}$ (b) $\frac{-\pi \cos(1)}{e}$

(c) $\frac{\sin(1)}{e}$ (d) $\frac{\cos(1)}{e}$

[ME, GATE-2016 : 2 Marks]

Q.47 The value of the integral

$$\oint \frac{2z+5}{\left(z-\frac{1}{2}\right)(z^2-4z+5)} dz$$

over the contour $|z| = 1$, taken in the anti-clockwise direction, would be

(a) $\frac{24\pi i}{13}$ (b) $\frac{48\pi i}{13}$

(c) $\frac{24}{13}$ (d) $\frac{12}{13}$

[EE, GATE-2016 : 1 Mark]

Q.48 The value of the integral $\frac{1}{2\pi j}\int_C \frac{z^2+1}{z^2-1} dz$ where z is a complex number and C is a unit circle with center at $1 + 0j$ in the complex plane is_______.

[IN, GATE-2016 : 2 Marks]

Q.49 In the following integral, the contour C encloses the points $2\pi j$ and $-2\pi j$ $-\frac{1}{2\pi}\oint_C \frac{\sin z}{(z-2\pi j)^3} dz$. The value of the integral is ________.

[EC, GATE-2016 : 2 Marks]

Q.50 The values of the integral $\frac{1}{2\pi j}\oint_c \frac{e^z}{z-2} dz$ along a closed contour c in anti-clockwise direction for

(*i*) the point $z_0 = 2$ inside the contour c, and

(*ii*) the point $z_0 = 2$ outside the contour c, respectively, are

(a) (*i*) 2.72, (*ii*) 0 (b) (*i*) 7.39, (*ii*) 0

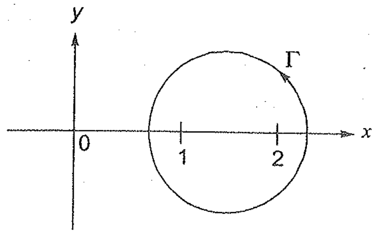
(c) (*i*) 0, (*ii*) 2.72 (d) (*i*) 0, (*ii*) 7.39

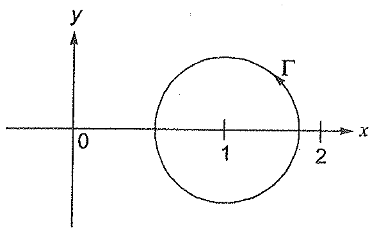
[EC, GATE-2016 : 2 Marks]

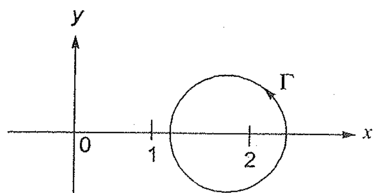
Q.51 For $f(z) = \frac{\sin(z)}{z^2}$, the residue of the pole at $z = 0$ is ________.

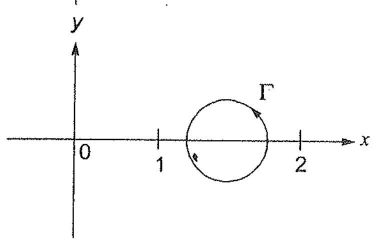
[EC, GATE-2016 : 1 Mark]

Q.52 The value of $\oint_\Gamma \frac{3z-5}{(z-1)(z-2)} dz$ along a closed path Γ is is equal to $(4\pi i)$, where $z = x + iy$ and $i = \sqrt{-1}$. The correct path Γ is

(a) 

(b) 

(c) 

(d) 

[ME, GATE-2016 : 2 Marks]

Q.53 If $f(z) = (x^2 + ay^2) + ibxy$ is a complex analytic function of $z = x + iy$, where $i = \sqrt{-1}$, then

(a) $a = -1, b = -1$ (b) $a = -1, b = 2$
(c) $a = 1, b = 2$ (d) $a = 2, b = 2$

[ME, GATE-2017 : 2 Marks]

Q.54 Let $z = x + jy$ where $j = \sqrt{-1}$. Then $\overline{\cos z} =$

(a) $\cos z$ (b) $\cos \bar{z}$
(c) $\sin z$ (d) $\sin \bar{z}$

[IN, GATE-2017 : 1 Mark]

Q.55 The value of the contour integral in the complex-plane

$$\oint \frac{z^3 - 2z + 3}{z - 2} dz$$

along the contour $|z| = 3$, taken counter-clockwise is

(a) $-18\pi i$ (b) 0
(c) $14\pi i$ (d) $48\pi i$

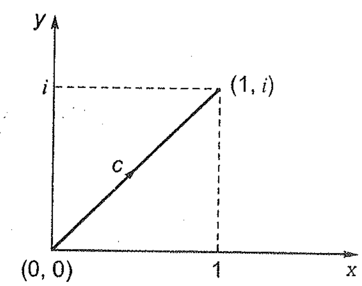
[EE, GATE-2017 : 2 Marks]

Q.56 For a complex number z, $\lim_{z \to i} \frac{z^2 + 1}{z^3 + 2z - i(z^2 + 2)}$ is

(a) $-2i$ (b) $-i$
(c) i (d) $2i$

[EE, GATE-2017 : 1 Mark]

Q.57 Consider the line integral $I = \int_c (x^2 + iy^2) dz$, where $z = x + iy$. The line c is shown in the figure below.



The value of I is

(a) $\frac{1}{2}i$ (b) $\frac{2}{3}i$

(c) $\frac{3}{4}i$ (d) $\frac{4}{5}i$

[EE, GATE-2017 : 2 Marks]

Q.58 The residues of a function

$$f(z) = \frac{1}{(z-4)(z+1)^3}$$

are

(a) $\frac{-1}{27}$ and $\frac{-1}{125}$ (b) $\frac{1}{125}$ and $\frac{-1}{125}$

(c) $\frac{-1}{27}$ and $\frac{1}{5}$ (d) $\frac{1}{125}$ and $\frac{-1}{5}$

[EC, GATE-2017 : 1 Mark]

Q.59 An integral I over a counter-clockwise circle C is given by

$$I = \oint_C \frac{z^2 - 1}{z^2 + 1} e^z \, dz.$$

If C is defined as $|z| = 3$, then the value of I is

(a) $-\pi i \sin(1)$ (b) $-2\pi i \sin(1)$
(c) $-3\pi i \sin(1)$ (d) $-4\pi i \sin(1)$

[EC, GATE-2017 : 2 Marks]

Q.60 If $W = \phi + i\psi$ represents the complex potential for an electric field.

Given $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$, then the function ϕ is

(a) $-2xy + \frac{y}{x^2+y^2} + C$

(b) $2xy + \frac{y}{x^2+y^2} + C$

(c) $-2xy + \frac{x}{x^2+y^2} + C$

(d) $2xy + \frac{x}{x^2+y^2} + C$

[ESE Prelims-2017]

Q.61 The residue of $f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}$ at $z = 3$ is

(a) –8 (b) $\frac{101}{16}$

(c) 0 (d) $\frac{27}{16}$

[ESE Prelims-2017]

Q.62 Evaluate: $\int_c \frac{dz}{z\sin z}$, where c is $x^2 + y^2 = 1$.

(a) 1 (b) 2

(c) 0 (d) –1

[EE, ESE-2017]

Q.63 The sum of residues of $f(z) = \frac{2z}{(z-1)^2(z-2)}$ at its singular point is

(a) –8 (b) –4

(c) 0 (d) 4

[EE, ESE-2017]

Q.64 $F(z)$ is a function of the complex variable $z = x + iy$ given by

$F(z) = iz + k\,Re(z) + i\,Im(z)$

For what value of k will F(z) satisfy the Cauchy-riemann equations?

(a) 0 (b) 1

(c) –1 (d) y

[ME, GATE-2018 : 1 Mark]

Q.65 Let z be a complex variable. For a counter-clockwise integration around a unit circle C, centered at origin.

$$\oint_C \frac{1}{5z-4}dz = A\pi i$$

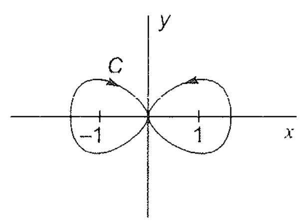
the value of A is

(a) $\frac{2}{5}$ (b) $\frac{1}{2}$

(c) 2 (d) $\frac{4}{5}$

[ME, GATE-2018 : 2 Marks]

Q.66 The contour C given below is on the complex plane $z = x + jy$, where $j = \sqrt{-1}$.



The value of the integral $\frac{1}{\pi j}\oint_C \frac{dz}{z^2-1}$ is ______.

[EC, GATE-2018 : 2 Marks]

Q.67 The value of the integral $\oint \frac{z+1}{z^2-4}dz$ in counter clockwise direction around a circle C of radius 1 with center at the point $z = -2$ is

(a) $\frac{\pi i}{2}$ (b) $2\pi i$

(c) $-\frac{\pi i}{2}$ (d) $-2\pi i$

[EE, GATE-2018 : 1 Mark]

Q.68 If C is a circle $|z| = 4$ and $f(z) = \frac{z^2}{(z^2-3z+2)^2}$,

then $\oint_C f(z)\,dz$ is

(a) 1 (b) 0

(c) –1 (d) –2

[EE, GATE-2018 : 2 Marks]

Q.69 Let $f_1(z) = z^2$ and $f_2(z) = \bar{z}$ be two complex variable function. Here $\bar{z}$ is the complex conjugate of z. Choose the correct answer.

(a) Both $f_1(z)$ and $f_2(z)$ are analytic
(b) Only $f_1(z)$ is analytic
(c) Only $f_2(z)$ is analytic
(d) Both $f_1(z)$ and $f_2(z)$ are not analytic

[IN, GATE-2018 : 1 Mark]

Q.70 In the Laurent expansion of $f(z) = \dfrac{1}{(z-1)(z-2)}$ valid in the region $1 < |z| < 2$, the coefficient of $\dfrac{1}{z^2}$ is

(a) 0 (b) $\dfrac{1}{2}$
(c) 1 (d) −1

[ESE Prelims-2018]

Q.71 What is the residue of the function $\dfrac{1-e^{2x}}{z^4}$ at its pole?

(a) $\dfrac{4}{3}$ (b) $-\dfrac{4}{3}$
(c) $-\dfrac{2}{3}$ (d) $\dfrac{2}{3}$

[ESE Prelims-2018]

Q.72 If $Z = e^{ax+by} F(ax - by)$; the value of $b \cdot \dfrac{\partial Z}{\partial x} + a \cdot \dfrac{\partial Z}{\partial y}$ is

(a) $2Z$ (b) $2a$
(c) $2b$ (d) $2abZ$

[EE, ESE-2018]

Q.73 Evaluate $\oint_c \dfrac{1}{(z-1)^3.(z-3)} dz$ where c is the rectangular region defined by $x = 0, x = 4, y = -1$ and $y = 1$

(a) 1 (b) 0
(c) $\dfrac{\pi}{2} i$ (d) $\pi(3 + 2i)$

[EE, ESE-2018]

■■■■

Answers Complex Functions

1. (a)	2. (a)	3. (d)	4. (b)	5. (a)	6. (a)	7. (a)	8. (c)	9. (c)
10. (d)	11. (d)	12. (b)	13. (c)	14. (d)	15. (d)	16. (a)	17. (a)	18. (c)
19. (b)	20. (d)	21. (a)	22. (b)	23. (c)	24. (c)	25. (c)	26. (c)	27. (b)
28. (b)	29. (b)	30. (c)	31. (a)	32. (d)	33. (a)	34. (b)	35. (d)	36. (10)
37. (b)	38. (0)	39. (b)	40. (a)	41. (a)	42. (0)	43. (a)	44. (–1)	45. (b)
46. (a)	47. (b)	48. (1)	49. (–134)	50. (b)	51. (1)	52. (b)	53. (b)	54. (b)
55. (c)	56. (d)	57. (b)	58. (b)	59. (d)	60. (a)	61. (d)	62. (c)	63. (c)
64. (b)	65. (a)	66. (2)	67. (a)	68. (b)	69. (b)	70. (d)	71. (b)	72. (d)
73. (b)								

Explanations Complex Functions

1. (a)

$$\int \sec z\,dz = \int \frac{1}{\cos z}dz$$

The poles are at

$z_0 = (n+1/2)\pi = \ldots -3\pi/2, -\pi/2, \pi/2, +3\pi/2\ldots$

None of these poles lie inside the unit circle $|z| = 1$.

Hence, sum of residues at poles = 0

$\therefore$ Singularities set = ϕ and

$I = 2\pi i$ [sum of residues of $f(z)$ at the poles]

$= 2\pi i \times 0 = 0$

2. (a)

Cauchy's integral theorem is

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

i.e. $\oint_C \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$

Now, $\oint_C \frac{z^3-6}{3z-i}dz = \frac{1}{3}\oint_C \frac{z^3-6}{\left(z-\frac{i}{3}\right)}$

Applying Cauchy's integral theorem, using

$f(z) = z^3 - 6,$

$$= \frac{1}{3}\left(2\pi i\, f\left(\frac{i}{3}\right)\right) = \frac{1}{3}\left(2\pi i\left[\left(\frac{i}{3}\right)^3 - 6\right]\right)$$

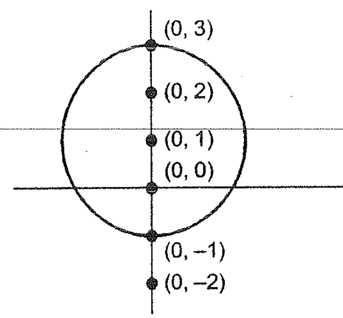
$$= \frac{1}{3}\left[2\pi i\left(\frac{i^3}{27} - 6\right)\right]\frac{2\pi}{81}i^4 - 4\pi i$$

$$= \frac{2\pi}{81} - 4\pi i$$

3. (d)

$$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$$

Pole (0, 2) lies inside the circle $|z-i| = 2$ while pole (0, –2) is outside the circle $|z-i| = 2$ as can be seen from figure below:



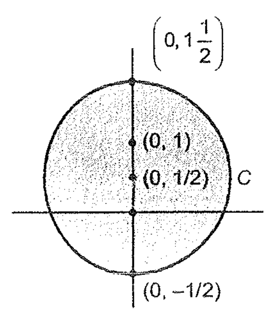
$\int_C f(z)\,dz = 2\pi i$ [Residue at those poles which are inside C]

$$= 2\pi i \text{ Res } f(2i) = 2\pi i \frac{1}{(2i+2i)} = \frac{\pi}{2}$$

4. (b)

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$$

Poles at i and $-i$, i.e. (0, 1) and (0, –1)



$$\left|z - \frac{i}{2}\right| = 1$$

From figure of $|z - i/2| = 1$ below we see that pole (0, 1) i.e. i is inside C, while pole (0, –1) i.e. $-i$ is outside C.

So, $I = 2\pi i$ Res $f(i) = 2\pi i \cdot \frac{1}{(i-i)(i+i)} = \pi$

5. (a)

$$I = \oint \frac{1}{(s^2-1)} ds = \oint \frac{1}{(s+1)(s-1)} ds$$

$$= 2\pi j \times \text{(Sum of residues)}$$

pole $s = -1$ is not inside the contour D, but $s = 1$ is inside D

residue at pole $s = 1$ is

$$z = \lim_{s \to 1} \frac{(s-1)}{(s-1)(s+1)} = \frac{1}{2}$$

$$\Rightarrow \oint \frac{1}{(s^2-1)} ds = 2\pi j \times \frac{1}{2} = j\pi$$

6. (a)

$$f(z) = \frac{\cos z}{z}$$

has simple pole at $z = 0$ and $z = 0$ is inside unit circle on complex plane

∴Residue of $f(z)$ at $z = 0$

$$\underset{z\to 0}{\text{Lt}}\, f(z)\cdot z \cdot = \underset{z\to 0}{\text{Lt}} \cos z = 1$$

$$\int_C f(z)dz = 2\pi i \text{ (Residue at } z = 0)$$

$$= 2\pi i.1 = 2\pi i$$

7. (a)

Since $\underset{z\to 2}{\text{Lt}} [(z-2)^2 f(z)]$ is finite and non-zero,

$f(z)$ has a pole of order two at $z = 2$.

The residue at $z = a$ is given for a pole of order n as

$$\text{Res } f(a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=2}$$

Here $n = 2$ (pole of order 2) and $a = 2$

$$\therefore \quad \text{Res } f(2) = \frac{1}{1!}\left\{\frac{d}{dz}\left[(z-2)^2 f(z)\right]\right\}_{z=2}$$

$$= \left\{\frac{d}{dz}\left[(z-2)^2 \frac{1}{(z+2)^2 (z-2)^2}\right]\right\}_{z=2}$$

$$= \left\{\frac{d}{dz}\left[\frac{1}{(z+2)^2}\right]\right\}_{z=2} = \left[-2(z+2)^{-3}\right]_{z=2}$$

$$= \frac{-2}{(2+2)^3} = -\frac{1}{32}$$

8. (c)

$f(z) = u + iv$ is analytic (given)

∴ it must satisfy the Cauchy-Reimann equations

$u_x = v_y$ ... (i)

and $v_x = -u_y$ ... (ii)

Here since, $u = xy$ (given)

$\Rightarrow$ $u_x = y$ and $u_y = x$

Now substituting u_x and u_y (i) and (ii) we get

$v_y = y$ ... (iii)

and $v_x = -x$ ... (iv)

Integrating (iii) and (iv) we can now get v as follows:

$$v_y = y$$

$$\Rightarrow \quad \frac{\partial v}{\partial y} = y$$

$$\Rightarrow \quad \int \partial v = \int y\, \partial y$$

$$\Rightarrow \quad v = \frac{y^2}{2} + f(x) \quad \text{... (v)}$$

from (v) we have,

$v_x = f'(x)$ ... (vi)

Since from (iv) we have,

$$v_x = -x$$

Substituting this is (vi) we get,

$$f'(x) = -x$$

$$\Rightarrow \quad \frac{df}{dx} = -x$$

$$\Rightarrow \quad \int df = \int -x\, dx$$

$$\Rightarrow \quad f = \frac{-x^2}{2} + k$$

Now substitute this is (v) we get,

$$v = \frac{y^2}{2} - \frac{x^2}{2} + k; \; v = \frac{y^2 - x^2}{2} + k$$

9. (c)

Here, $\quad I = \int_C \frac{\cos(2\pi z)}{(2z-1)(z-3)} dz$

$$= \frac{1}{2}\int_C \frac{\left[\frac{\cos(2\pi z)}{(z-3)}\right]}{\left[z-\frac{1}{2}\right]}$$

Since, $z = 1/2$ is a point within $|z| = 1$ (the closed curve C) we can use Cauchy's integral theorem and say that

$$I = \frac{1}{2} f\left(\frac{1}{2}\right)$$

where $f(z) = \frac{\cos(2\pi z)}{(z-3)}$

[Notice that $f(z)$ is analytic on all pts inside $|z| = 1$]

$$\therefore \quad I = \frac{1}{2} \frac{\cos\left(2\pi \times \frac{1}{2}\right)}{\left(\frac{1}{2}-3\right)} = \frac{2\pi i}{5}$$

10. (d)

$$f(z) = \frac{z-1}{z^2+1} = \frac{z-1}{z^2-i^2} = \frac{z-1}{(z-i)(z+i)}$$

$\therefore$ The singularities arc at $z = i$ and $-i$

11. (d)

$$f(z) = c_0 + c_1 z^{-1}$$

$$\oint \frac{1+f(z)}{z} dz = ?$$

It has one pole at origin, which is inside unit circle

So, $\oint \frac{\{1+f(z)\}}{z} dz = 2\pi j$[Residue of $f(z)$ at $z = 0$]

$$= 2\pi j [1 + f(0)]$$

Since, $f(z) = C_0 + C_1 z^{-1} \Rightarrow f(0) = C_0$

$\therefore$ Answer $= 2\pi j(1 + C_0)$

12. (b)

$$Z = \frac{3+4i}{1-2i} = \frac{(3+4i)(1+2i)}{(1-2i)(1+2i)}$$

$$= \frac{-5+10i}{5} = -1+2i$$

$$|Z| = \sqrt{(-1)^2+(2)^2} = \sqrt{5}$$

13. (c)

$$x(z) = \frac{1-2z}{z(z-1)(z-2)}$$

poles are $z = 0$, $z = 1$ and $z = 2$

Residue at $z = 0$

residue = value of $\frac{1-2z}{(z-1)(z-2)}$ at $z = 0$

$$= \frac{1-2\times 0}{(0-1)(0-2)} = \frac{1}{2}$$

Residue at $z = 1$

residue = value of $\frac{1-2z}{z(z-2)}$ at $z = 1$

$$= \frac{1-2\times 1}{1(1-2)} = 1$$

Residue at $z = 2$

residue = value of $\frac{1-2z}{z(z-1)}$ at $z = 2$

$$= \frac{1-2\times 2}{2(2-1)} = -\frac{3}{2}$$

$\therefore$ The residues at its poles are $\frac{1}{2}$, 1 and $-\frac{3}{2}$.

14. (d)

$$f = u + iv$$

$$u = 3x^2 - 3y^2$$

for f to be analysis, we have Cauchy-Riemann conditions,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{...}(i)$$

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \quad \text{...}(ii)$$

From (i) we have

$$6x = \frac{\partial v}{\partial y}$$

$$\Rightarrow \quad \int \partial v = \int 6x \partial y$$

$$v = 6xy + f(x)$$

i.e. $\quad v = 6xy + f(x) \quad \text{...}(iii)$

Now applying equation (ii) we get

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

$$\Rightarrow \quad -6y = -\left[6x + \frac{df}{dx}\right]$$

$$\Rightarrow \quad 6x + \frac{df}{dx} = 6y$$

$$\frac{df}{dx} = 6y - 6x$$

By integrating,

$$f(x) = 6yx - 3x^2 + K$$

Substitute in equation (*iii*)

$$v = 3x^2 + 6yx - 3x^2 + K$$

$$\Rightarrow \quad v = 6yx + K$$

15. **(d)**

Let $z = a + bi$

Since z is shown inside the unit circle in I quadrant, a and b are both +ve and

$$0 < \sqrt{a^2 + b^2} < 1$$

Now $\frac{1}{z} = \frac{1}{a + bi}$

$$\frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

Since $a, b > 0$,

$$\frac{a}{\sqrt{a^2 + b^2}} > 0$$

$$\frac{-b}{a^2 + b^2} < 0$$

So $\frac{1}{z}$ is in IV quadrant.

$$\left|\frac{1}{z}\right| = \sqrt{\left(\frac{a}{a^2 + b^2}\right)^2 + \left(\frac{-b}{a^2 + b^2}\right)^2}$$

$$= \sqrt{\frac{1}{a^2 + b^2}} = \frac{1}{\sqrt{a^2 + b^2}}$$

Since $0 < \sqrt{a^2 + b^2} < 1$

$$\frac{1}{\sqrt{a^2 + b^2}} > 1$$

So $\frac{1}{z}$ is outside the unit circle is IV quadrant.

16. **(a)**

$$I = \oint_C \frac{-3z + 4}{(z^2 + 4z + 5)}\, dz$$

$$= 2\pi i \text{ (sum of residues)}$$

Poles of $\frac{-3z + 4}{(z^2 + 4z + 5)}$ are given by

$$z^2 + 4z + 5 = 0$$

$$z = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i$$

Since the poles lie outside the circle $|z| = 1$.

So $f(z)$ is analytic inside the circle $|z| = 1$.

Hence $\oint_C f(z)\, dz = 2\pi i\,(0) = 0$

17. **(a)**

$x = i$, then in polar coordinates,

$$x = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{\frac{\pi}{2}i}$$

Now, $x^x = i^i = \left(e^{\pi i/2}\right)^i = e^{i^2 \pi/2} = e^{-\pi/2}$

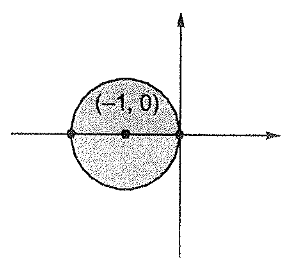
18. **(c)**

Given, $f(z) = \frac{1}{z+1} - \frac{2}{z+3} = \frac{(z+3) - 2(z+1)}{(z+1)(z+3)}$

$$= \frac{-z+1}{(z+1)(z+3)}$$

Poles are at –1 and –3 i.e. (–1, 0) and (–3, 0).

From figure below of $|z + 1| = 1$,



we see that (–1, 0) is inside the circle and (–3, 0) is outside the circle.

Residue theorem says,

$\frac{1}{2\pi j}\oint_C f(z)\, dz$ = Residue of those poles which are inside C.

So the required integral $\frac{1}{2\pi j}\oint_C f(z)dz$ is given by the residue of function at pole (–1, 0) (which is inside the circle).

This residue is $= \frac{-(-1)+1}{(-1+3)} = \frac{2}{2} = 1$

19. (b)

$$-i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$$
$$= \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)$$
$$(-i)^{1/2} = \left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]^{1/2}$$
$$= \cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right)$$
$$= \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$$

20. (d)

$$\tan h\, s = \frac{e^s - e^{-s}}{e^s + e^{-s}}$$

it is analytic if $e^s + e^{-s} \neq 0$

$\therefore \quad e^s \neq e^{-s}$

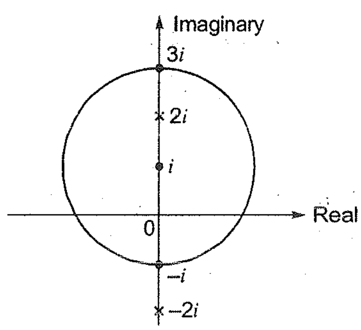
$e^{2s} \neq -1$

$$s \neq \frac{i(2n+1)}{2}\pi$$

$$\therefore \quad \text{Im}(s) \neq \frac{(2n+1)}{2}\pi$$

21. (a)

$$\frac{z^2-4}{z^2+4} = \frac{z^2-4}{(z+2i)(z-2i)}$$



Poles at $2i$ and $-2i$ i.e. $(0, 2i)$ and $(0, -2i)$

From figure of $|Z - i| = 2$, we see that pole, is incide C,

While pole, $-2i$ is outside C.

$$\therefore \oint \frac{z^2-4}{z^2+4}\,dz = 2\pi i \times \text{Res. } F(z)$$
$$= 2\pi i \cdot \left.\frac{(z-2i)(z^2-4)}{(z+2i)(z-2i)}\right|_{z=2i}$$
$$= 2\pi i \frac{\left[(2i)^2 - 4\right]}{(2i+2i)} = -4\pi$$

22. (b)

$$\frac{(2-3i)}{(-5+i)} = \frac{(2-3i)}{(-5+i)} \times \frac{(-5-i)}{(-5-i)}$$
$$= \frac{-10-2i+15i-3}{25+1} = \frac{-13+13i}{26}$$
$$= -0.5 + 0.5i$$

23. (c)

As per Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = 2y$$

and $$\frac{\partial u}{\partial y} = 2x$$

$$\frac{\partial v}{\partial y} = 2y$$

$\Rightarrow \quad v = y^2 + f(x)$

$$\frac{\partial v}{\partial x} = 0 + f'(x) = -2x$$

$\therefore \quad f(x) = -x^2 + \text{constant}$

$\therefore \quad v = y^2 - x^2 + \text{constant}$

24. (c)

As per Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$$u = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$$

$$\therefore \quad \frac{\partial v}{\partial x} = 2x$$

$\Rightarrow \quad v = 2xy + f(x)$

$$\frac{\partial v}{\partial x} = 2y + f'(x)$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = 2y + f'(x)$$

$\Rightarrow \quad f'(x) = 0 \quad$ i.e. $f(x) = C$

$\therefore \quad v = 2xy + C$

25. (c)

Let $z = \dfrac{1+i}{1-i}$

or $z = \dfrac{(1+i)(1+i)}{(1-i)(1+i)} = \dfrac{1+i^2+2i}{1-i^2}$

$= \dfrac{2i}{2} = i$

$z = x + iy = i$

so, $x = 0$

$y = 1$

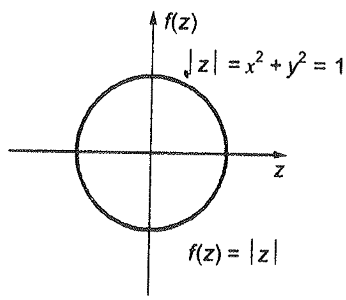
$\text{Arg}(z) = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{1}{0}\right) = \tan^{-1}\infty$

$= \dfrac{\pi}{2}$

26. (c)

zz^*

$\Rightarrow$ $z = x + xy$



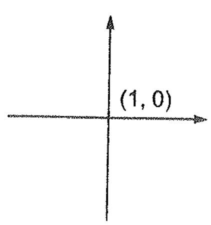
$z^* = x - iy$

$zz^* = (x + iy)(x - iy) = x^2 + y^2$

which is equal to (1) always as given

$|z| = 1$

$zz^* = x^2 + y^2$



7. (b)

Let $z = 1^i = 1^{e^{i(4n+1)\frac{\pi}{2}}}$ $n \in \mathrm{I}$

$z = 1$ which is purely real and non negative.

8. (b)

$Z = x + iy$

$f(z) = u + iv$

$u = e^{-y}\cos(x)$

$\dfrac{\partial u}{\partial x} = -e^{-y}\sin(x)$

For analytical function

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

$\Rightarrow$ $\dfrac{\partial v}{\partial y} = -e^{-y}\sin(x)$

Integrating w.r.t. y

$v = e^{-y}\sin(x)$

30. (c)

$f(z) = \int \dfrac{z^2}{z^2-1} = \int f(z)$

Given circle

$|z-1| = 1$

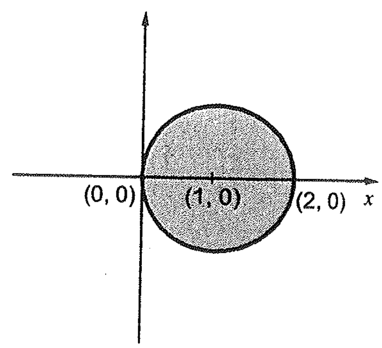
$\Rightarrow |(x+iy)-1| = 1$

$(x-1)^2 + y^2 = 1$

$x = 1, y = 0, r = 1$

Poles of $f(z)$

$z^2 - 1 = 0$



$[z = +1, 1]$

So, $-1 \rightarrow$ Outside circle

$+1 \rightarrow$ Inside circle

$$\int \frac{z^2}{(z-1)(z+1)} = 2\pi i\left[\frac{z^2}{z+1}\right]_{z=+1} = 2\pi i\left[\frac{1}{2}\right] = \pi i$$

For pole $(z = -1) = \int \dfrac{z^2}{(z-1)(z+1)} = 0$

as it lies outside from counter.

31. (a)

The Taylor's series expansion for

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad -\infty < x < \infty$$

and $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} \cdots \quad -\infty < x < \infty$

$$\therefore 3\sin x + 2\cos x = 2 + 3x - \frac{2x^2}{2!} - \frac{3x^3}{3!} + \frac{2x^4}{4!} + \frac{3x^5}{5!} \cdots$$

$$= 2 + 3x - x^2 - \frac{x^3}{2} + \cdots$$

32. (d)

Given

Let $x(n) = \sum_{n=0}^{\infty} \frac{1}{n!}$

$$= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

Also we know that expression of e^x

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$

Put $x = 1$ in above expression

$$e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

33. (a)

$$z_1 = 5 + (5\sqrt{3})i \; ; \; z_2 = \frac{2}{\sqrt{3}} + 2i$$

$$\arg(z_1) = \theta_1 = \tan^{-1}\left(\frac{5\sqrt{3}}{5}\right) \; ; \quad \theta_1 = 60°$$

$$\arg(z_2) = \theta_2 = \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) \; ; \quad \theta_2 = 60°$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = 60° - 60° = 0°$$

35. (d)

$$f(z) = \bar{z} = x - iy$$

$u = x \qquad v = -y$

$\Rightarrow \quad u_x = 1 \qquad v_x = 0$

$u_y = 0 \qquad v_y = -1$

$u_x \neq v_y$ i.e. $C - R$ not satisfied

$\Rightarrow \bar{z}$ is not analytic function.

36. Sol.

$$f(z_1) = \frac{az_1 + b}{cz_1 + d}$$

$$f(z_2) = \frac{az_2 + b}{cz_2 + d}$$

$$\frac{az_1 + b}{cz_1 + d} = \frac{az_2 + b}{cz_2 + d}$$

$$acz_1z_2 + bcz_2 + adz_1 + bd = acz_1z_2 + bcz_1 + adz_2 + bd$$

$$bc(z_2 - z_1) = ad(z_2 - z_1)$$

$$z_2 \neq z_1$$

$\Rightarrow \quad bc = ad$

$$d = \frac{bc}{a} = \frac{4 \times 5}{2} = 10$$

37. (b)

Given, $\oint \frac{1}{z^2} dz$ where C is the unit circle. By Cauchy's residue theorem

$$\oint \frac{1}{z^2} dz = 2\pi i \text{ [sum of residues]}$$

$\because \frac{1}{z^2}$ is NOT analytical at $z = 0$

So, $z = 0$ is the pole of order 2.

So, residue at $z = 0 = \frac{1}{(2-1)!}\left[\frac{d}{dz} z^2 . \frac{1}{z^2}\right]_{z=0} = 0$

So, $\oint \frac{1}{z^2} dz = 2\pi i [0] = 0$

38. Sol.

$$\text{Re}\{z\} = \frac{z + \bar{z}}{2}$$

Since there is no pole inside unit circle, so Residue at poles is zero

$$\Rightarrow \frac{1}{2\pi i} \oint_C \text{Re}\{z\} dz = 0$$

39. (b)

By Cauchy integral formula

$$\oint \frac{f(z)}{(z - z_o)^{n+1}} dz = \frac{2\pi i \, f^*(z_o)}{n!}$$

$$\oint \frac{dz}{(z - z_o)^{n+1}} = \frac{2\pi i}{n!} \cdot 0 = 0$$

40. (a)

$f(z)$ has poles at $z = 1, -2$

Residue of $f(z)$ at $(z = 1)$

$$= \lim_{z \to 1} (z - 1) f(z) = \lim_{z \to 1} \frac{9}{(z + 2)^2}$$

Residue of $f(z)$ at $(z = -2)$

$$= \lim_{z \to -2} \frac{d}{dz^{2-1}}\left[(z+2)^2 f(z)\right]$$

$$= \lim_{z \to -2} \frac{d}{dz}\left(\frac{9}{z-1}\right)$$

$$= \lim_{z \to -2} \frac{-9}{(z-1)^2} = -1$$

41. (a)

$$f(z) = 1+(1-z)+(1-z)^2+---$$

$$= \frac{1}{1-(1-z)} = \frac{1}{1-1+z} = \frac{1}{z}$$

42. Sol.

$$f(z) = 2z^3 + b_1|z|^3$$

Given that $f(z)$ is analytic.

which is possible only when $b = 0$

since $|z^3|$ is differentiable at the origin but not analytic.

$2z^3$ is analytic everywhere

$\therefore$ $f(z) = 2z^3 + b|z^3|$ is analytic

only when $b = 0$

43. (a)

$$u = 2xy$$

$$u_x = 2y \qquad u_y = 2x$$

In option (a)

$$V_x = -2x \quad u_y = -V_x$$

$$V_y = 2y$$

($-R$ equation are satisfied only in option a)

44. Sol.

Given that $f(z) = u + iv$ is analytic

$$u(x, y) = 2kxy \qquad v = x^2 - y^2$$

$$u_x = 2ky \qquad v_y = -2y$$

$$u_x = v_y$$

$$k = -1$$

$$u_y = 2kx \qquad v_x = 2x$$

$$u_y = -v_x$$

$$2kx = -2x$$

$$k = -1$$

45. (b)

$$f(z) = z + z^*$$

$f(z) = 2x$ is continuous (polynomial)

$$u = 2x \qquad v = 0$$

$$u_x = 2 \qquad u_y = 0$$

$$v_x = 0 \qquad v_y = 0$$

C.R. equation not satisfied.

$\therefore$ No where analytic.

46. (a)

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2}\,dx$$

$$I = \int_{-\infty}^{\infty} \frac{\sin z}{z^2+2z+2}\,dz$$

$\sin z$ = imaginary part of e^{iz}

$$= \int_{-\infty}^{\infty} \frac{\text{I.P of } e^{iz}}{z^2+2z+2}\,dz$$

Poles are $z^2 + 2z + 2 = 0$

$$z = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$z = -1 - i$

↓

Outside upper half

↓

Residue is 0

$-1 + i$

↓

inside upperhalf

$\operatorname{Res}_{z=-1+i} \phi(z)$

$$= \underset{z \to -1+i}{Lt}\; z-(-1+i)\frac{e^{iz}}{(z-(-1+i))(z-(-1-i))}$$

$$= \frac{e^{i(-1+i)}}{(-1+i)-(-1-i)} = \frac{e^{-i-1}}{-1+i+1+i} = \frac{e^{-i-1}}{2i}$$

$$I = \text{I.P. of } 2\pi i\left(\frac{e^{-i-1}}{2i}\right) = \text{I.P of } \pi\left(e^{-i}\cdot e^{-1}\right)$$

$$= \text{I.P of } \pi e^{-1}(\cos 1 - i\sin 1) = \frac{-\pi \sin 1}{e}$$

47. (b)

Singularities, $z = \frac{1}{2}, 2 \pm i$

only, $z = \frac{1}{2}$ lies inside C

By residue theorem,

$$\oint_C = 2\pi i(R) = \frac{48\pi i}{13}$$

Residue at $\frac{1}{2} = R_{1/2}$

$$= \lim_{z \to 1/2}\left[\left(z-\frac{1}{2}\right)\cdot\frac{2z+5}{\left(z-\frac{1}{2}\right)(z^2+4z+5)}\right] = \frac{24}{13}$$

48. Sol.

$$\frac{1}{2\pi j}\int_C \frac{z^2+1}{(z^2-1)}dz = \frac{1}{2\pi j}\int_C \frac{z^2+1}{(z-1)(z+1)}dz$$

Poles are at $z = 1, -1$

Given circle is $|z-1| = 1$

pole $z = 1$ lies inside C

pole $z = -1$ lies outside C

Res $f(z)$ at $z = 1$ is

$$= \underset{z\to 1}{\text{Lt}}(z-1)\frac{z^2+1}{(z-1)(z+1)} = \frac{2}{2} = 1$$

Res $f(z)$ at $z = -1$ is $= 0$

By Cauchy's residue theorem

$$\frac{1}{2\pi j}\int_C \frac{z^2+1}{z^2-1}dz = \frac{1}{2\pi j}\times 2\pi j(1+0) = 1$$

49. Sol.

$$I = -\frac{1}{2\pi}\int_c \frac{\sin z}{(z-2\pi j)^3}dz$$

$$= -\frac{1}{2\pi}\times\frac{2\pi j f''(2\pi j)}{2!}$$

$f(z) = \sin z$

$f'(z) = \cos z$

$f''(z) = -\sin z$

$$I = -\frac{1}{2\pi}\times 2\pi j\frac{-\sin(2\pi j)}{2}$$

$$= -\frac{1}{2}\sinh 2\pi = -133.87$$

50. (b)

(*i*) $Z_0 = 2$ -lies inside C,

so $\text{Res } f(z) = \lim_{z\to 2}(z-2)\cdot\frac{e^z}{z-2}$

$$= e^2 = 7.39$$

$$\frac{1}{2\pi i}\int_c \frac{e^z}{z-2}dz = 2\pi i\cdot\frac{1}{2\pi i}(7.39) = 7.39$$

(*ii*) $Z_0 = -2$ lies out side C then

Res $f(z) = 0$

$$\text{so}\int_c \frac{e^z}{z-2}dz = 2\pi i\frac{1}{2\pi i}(0) = 0\{\ \}$$

51. Sol.

Residue of $\frac{\sin z}{z^2}$ = Coefficient of $\frac{1}{z}$ in

$$\left\{\frac{z-\frac{z^3}{3!}+\frac{z^5}{5!}---}{z^2}\right\}$$

$$= \text{Coefficient of } \frac{1}{z} \text{ in } \left\{\frac{1}{z}-\frac{z}{3!}+\frac{z^3}{5!}----\right\} = 1$$

52. (b)

$$\int_\Gamma \frac{3z-5}{(z-1)(z-2)}dz = 4\pi i$$

$$\int_\Gamma \frac{3z-5}{(z-1)(z-2)}dz = 2\pi i\,(2)$$

Sum of residues must be equal to 2.

$$\underset{z=1}{\text{Res}f(z)} = \underset{z\to 1}{u}(z-1)\frac{3z-5}{(z-1)(z-2)} = \frac{-2}{-1} = 2$$

$$\underset{z=2}{\text{Res}f(z)} = \underset{z\to 2}{u}(z-2)\frac{3z-5}{(z-1)(z-2)} = \frac{6-5}{2-1} = 1$$

Therefore $z = 1$ must lies inside C

$z = 2$ lies outside C

then only we will get the given integral values is equal to $4\pi i$.

53. (b)

Given that the analytic function

$f(z) = (x^2 + ay^2) + i\ bxy$

$u + i\ v = (x^2 + ay^2) + i\ (bxy)$

$u = x^2 + ay^2$

$v = bxy$

$u_x = 2x;\quad u_y = 2ay$

$v_x = by;\quad v_y = bx$

$u_x = v_y;\quad u_y = -v_x$

$2x = bx;\quad 2ay = -by$

$b = 2$

$2a = -b$ since $b = 2$

$2a = -2$

$a = -1$

54. (b)

$$\overline{\cos z} = \overline{\cos(x+iy)}$$

$$= \overline{\cos x\cos iy - \sin x\sin iy}$$

$$= \overline{\cos x\cosh y - i\sin x\sinh y}$$

$$= \cos x\cosh y + i\sin x\sinh y$$

$$= \cos x\cos iy + \sin x\sin iy$$

$$= \cos(x-iy) = \cos\overline{z}$$

55. (c)

Pole, $z = 2$ lies inside $|z| = 3$

$$\text{Res } f(z) = \lim_{z\to 2}(z-2)\frac{z^2-2z+3}{z-2}$$

$z = 2,$ $= 8 - 4 + 3 = 7$

By Cauche residue theorem

$$I = 2\pi i(7) = 14\pi i$$

56. (d)

$$\lim_{Z\to i} \frac{z^2+1}{z^3+2z-i(z^2+2)} \qquad \left(\frac{0}{0} \text{ from}\right)$$

$$\lim_{Z\to i} \frac{2z}{3z^2+2-i(2z)}$$

$$= \frac{2i}{3i^2+2-i(2i)} = \frac{2i}{-3+2+2} = \frac{2i}{-3+4} = 2i$$

57. (b)

From the diagram C is $y = x$

$$I = \int_C (x^2+iy^2)\,dz$$

$$= \int_C (x^2+iy^2)(dx+idy)$$

$$= \int_C (x^2+ix^2)(dx+idx)$$

$$= \int x^2dx + ix^2dx + ix^2dx - x^2dx$$

$$= 2i\int_0^1 x^2dx = 2i\left(\frac{x^3}{3}\right)\Big|_0^1 = \frac{2i}{3}$$

58. (b)

Residue at $z = 4$ is

$$= \underset{z\to 4}{\text{Lt}}(z-4)\frac{1}{(z-4)(z+1)^3} = \frac{1}{(4+1)^3} = \frac{1}{125}$$

Residue at $z = -1$ is

$$= \underset{z\to -1}{\text{Lt}} \frac{1}{2!}\frac{d^2}{dz^2}\left((z+1)^3\frac{1}{(z-4)(z+1)^3}\right)$$

$$= \underset{z\to -1}{\text{Lt}} \frac{1}{2!}\left(\frac{2}{(z-4)^3}\right) = \frac{1}{(-1-4)^3} = \frac{-1}{125}$$

59. (d)

Poles are

$$z^2 + 1 = 0$$

$$z = \pm i$$

$z = i$ lies inside $|z| = 3$

$z = -i$ lies inside $|z| = 3$

Residue at $z = i$ is

$$= \underset{z\to i}{\text{Lt}}(z-i)\frac{z^2-1}{(z-i)(z+i)}e^z = \frac{-1-1}{2i}e^i = ie^i$$

Residue at $z = -i$ is

$$= \underset{z\to -i}{\text{Lt}}(z+i)\frac{z^2-1}{(z-i)(z+i)}e^z$$

$$= \frac{-1-1}{-2i}e^{-i} = \frac{1}{i}e^{-i} = -ie^{-i}$$

By Residues theorem

$$I = 2\pi i(ie^i - ie^{-i})$$

$$= -2\pi(e^i - e^{-i})$$

$$= -2\pi(\cos 1 + i\sin 1 - \cos 1 + i\sin 1)$$

$$= -2\pi(2i\sin 1) = -4\pi i\sin(1)$$

60. (a)

$$W = \phi + i\Psi$$

$$\Psi = x^2 - y^2 + \frac{x}{x^2+y^2}$$

$$\Psi_x = 2x + \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = -\phi_y$$

CR equation,

$$\Psi_y = -2y - \frac{x}{(x^2+y^2)}(2y) = \phi_x$$

$$\phi = -2xy + \frac{y}{(x^2+y^2)}C$$

$$\phi_x = -2y + \frac{y}{(x^2+y^2)}(2x)$$

$$\phi_y = -2x + \frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2}$$

$$= -2x + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= -\left[2x - \frac{y^2-x^2}{(x^2+y^2)^2}\right]$$

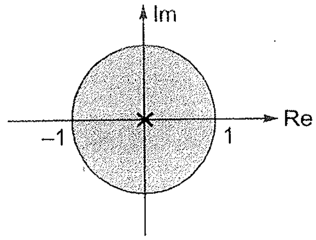
61. (d)

Residue at $z = 3$ is

$$= \lim_{z\to 3}\left[(z-3)\frac{z^3}{(z-1)^4(z-2)(z-3)}\right]$$

$$= \frac{3^3}{(3-1)^4(3-2)} = \frac{27}{16}$$

62. (c)



The value of $\int_c \frac{dz}{z \sin z}$; where c is $x^2 + y^2 = 1$

Singular points are,

$$z \sin z = 0$$
$$z = 0, \pm n\pi$$
$$z = 0 \text{ lies inside } c$$
$$z = \pm n\pi \text{ lies outside } c$$

Res $f(z)$ = the coefficient of $1/z$ in series $z = 0$

Expansion at $z = 0$,

$$\frac{1}{z \sin z} = \frac{1}{z\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...\right)}$$

$$= \frac{1}{z^2\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + ...\right)}$$

$$= \frac{1}{z^2}\left(1 + \frac{z^2}{3!} - \frac{z^4}{5!}...\right)$$

$$= \frac{1}{z^2} + \frac{1}{3!} - \frac{z^2}{5!} + ...$$

Res $f(z)$ = coefficient of $1/z$ in the above expansion is 0.

$$\therefore \quad \int_c \frac{1}{z \sin z} dz = 2\pi i\,(0) = 0$$

63. (c)

$$f(z) = \frac{2z}{(z-1)^2(z-2)}$$

$z = 1$ is pole of order 2

$z = 2$ pole of order 1

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} \frac{1}{1!}\frac{d}{dz}\left((z-1)^2 \cdot \frac{2z}{(z-1)^2(z-2)}\right)$$

$$= \lim_{z \to 1}\left(\frac{(z-2)(2) - 2z(1)}{(z-2)^2}\right)$$

$$= \frac{-2-2}{(1-2)^2} = -4$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z \to 2} (z-2) \cdot \frac{2z}{(z-1)^2(z-2)}$$

$$= \frac{4}{(2-1)^2} = 4$$

Sum of residues

$$= -4 + 4 = 0$$

64. (b)

$$F(z) = iz + k\,Re(z) + i\,Im(z)$$
$$u + iv = i(x + iy) + kx + iy$$
$$u + iv = kx - y + i(x + y)$$
$$u = kx - y,\; v = x + y$$
$$u_x = k,\; u_y = -1$$
$$V = x + y$$
$$v_x = 1$$
$$v_y = 1$$
$$u_x = v_y$$
$$K = 1$$

65. (a)

$$5z - 4 = 0$$

$z = \frac{4}{5}$ lies inside circle,

$$|z| = 1$$

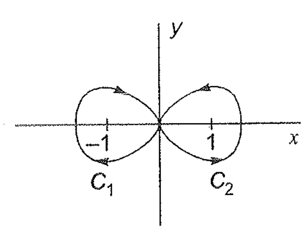
$$\int \frac{1}{(5z-4)} dz = A\pi i$$

$$\frac{1}{5}\int \frac{1}{\left(Z - \frac{4}{5}\right)} dz = A\pi i$$

$$\int \frac{\left(\frac{1}{5}\right)}{\left(Z - \frac{4}{5}\right)} dz = 2\pi i \cdot f\left(\frac{4}{5}\right) = 2\pi i \times \left(\frac{1}{5}\right) = \frac{2}{5}\pi i$$

$$A = \frac{2}{5} = 0.4$$

66. Sol.



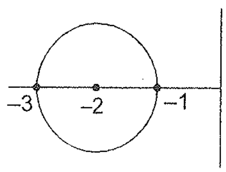
$$\frac{1}{\pi i}\oint_C \frac{dz}{z^2 - 1}$$

$$= 2\left[\frac{1}{2\pi i}\oint_{C_1}\frac{dz}{(z+1)(z-1)} + \frac{1}{2\pi i}\oint_{C_2}\frac{dz}{(z+1)(z-1)}\right]$$

$$= 2\left[-\left(\frac{1}{z-1}\right)\Big|_{z=-1} + \left(\frac{1}{z+1}\right)\Big|_{z=1}\right]$$

$$= 2\left[-\left(-\frac{1}{2}\right) + \left(\frac{1}{2}\right)\right] = 2$$

67. (a)



$$\int \frac{z+1}{z^2-4}\,dz$$

$$\int \frac{z+1}{(z-2)(z+2)}\,dz$$

$$\int \frac{\left(\frac{z+1}{z-2}\right)}{(z+2)}\,dz$$

where, $f(z) = \dfrac{z+1}{z-2}$

$$= 2\pi i\, f(-2)$$

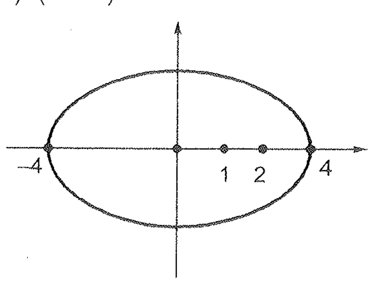
$$= 2\pi i\left(\frac{-2+1}{-2-2}\right)$$

$$= 2\pi i\left(\frac{-1}{-4}\right) = \frac{\pi i}{2}$$

68. (b)

$$\int \frac{z^2}{(z^2-3z+2)}\,dz$$

$$\int \frac{z^2}{(z-1)^2(z-2)^2}\,dz$$



$$\underset{z=1}{\text{Res.}} f(z) = \lim_{z\to 1}\frac{1}{1!}\frac{d}{dz}\left((z-1)^2\cdot\frac{z^2}{(z-1)^2(z-2)^2}\right)$$

$$= \lim_{z\to 1}\left(\frac{2z(z-2)^2 - 2z^2(z-2)}{(z-2)^4}\right)$$

$$= \lim_{z\to 1}\left(\frac{2z(z-2)-2z^2}{(z-2)^3}\right)$$

$$= \frac{-4}{-1} = 4$$

$$\underset{z=2}{\text{Res.}} f(z) = \lim_{z\to 2}\frac{1}{1!}\frac{d}{dz}\left((z-2)^2\cdot\frac{z^2}{(z-1)^2(z-2)^2}\right)$$

$$= \lim_{z\to 2}\left(\frac{(z-1)^2.2z - z^2 2(z-1)}{(z-1)^4}\right)$$

$$= \lim_{z\to 2}\left(\frac{2z(z-1)-2z^2}{(z-1)^3}\right)$$

$$= \frac{4-8}{1} = -4$$

By residue theorem,

$$I = 2\pi i\,(4-4) = 0$$

69. (b)

Only $f_1(z) = z^2$ is analytic

$f_2(z) = \bar{z}$ is not analytic

70. (d)

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$= \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= \frac{-1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{-1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right) - \frac{1}{2}\left(1+\frac{z}{2}+\frac{z}{2}\right)+\left(\frac{z}{2}\right)$$

$$= -\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}+\ldots+\frac{1}{2}-\frac{z}{4}-\frac{z^2}{8}$$

Coefficient of $\dfrac{1}{z^2} = -1$

71. (b)

$$\frac{1-e^{2Z}}{Z^4} = \frac{1-\left[1+2Z+\frac{(2Z)^2}{2!}+\frac{(2Z)^3}{3!}+\frac{(2Z)^4}{4!}+\ldots\right]}{Z^4}$$

$$= -\frac{2}{Z^3}-\frac{4}{2Z^2}-\frac{8}{3!Z}-\ldots$$

Residue at $Z = 0$

$$= \frac{-8}{3!} = \frac{-8}{6} = \frac{-4}{3}$$

72. (d)

$$z = e^{ax+by} f(ax - by)$$

$$z = e^{ax+by} f(ax - by) \quad ...(i)$$

$$\frac{\partial z}{\partial x} = e^{ax+by}(a)\cdot f + e^{ax+by}\cdot f'(a) \quad ...(ii)$$

From equation (i),

$$f = \frac{z}{e^{ax+by}} \text{ (or) } fe^{ax+by} = z \quad ...(iii)$$

Substituting in (ii),

$$\frac{\partial z}{\partial x} = az + ae^{ax+by}f' \quad ...(iv)$$

$$\frac{\partial z}{\partial y} = e^{ax+by}\cdot b\cdot f + e^{ax+by}f'(b)$$

$$= bz - be^{ax+by}\cdot f'$$

(Using equation iii)

$$\frac{\partial z}{\partial y} = bz - be^{ax+by}f'$$

$$f'(e^{ax+by}) = \frac{bz-q}{b} \quad ...(v)$$

Substituting (v) in (iv),

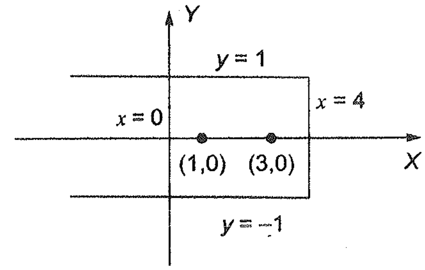
$$P = az + a\left(\frac{bz-q}{b}\right)$$

$$bP + aq = 2abZ$$

73. (b)

$$\int \frac{1}{(Z-1)^3(Z-3)}dz$$

Where,



$z = 1$ is a pole of order 3 (inside C)

$$\text{Res at } z = 1 = \lim_{z\to 1}\frac{1}{2!}\frac{d^2}{dz^2}\left((z-1)^3\cdot\frac{1}{(z-1)^3(z-3)}\right)$$

$$= \lim_{z\to 1}\frac{d}{dz}\left(\frac{-1}{(z-3)^2}\right)$$

$$= \lim_{z\to 1}\frac{1}{2!}\left(-1\cdot\left(\frac{-2}{(z-3)^3}\right)\right)$$

$$= \frac{2}{2(1-3)^3} = -\frac{1}{8}$$

$z = 3$ is a simple pole (inside)

$$\text{Res } f(z) = \lim_{z\to 3}(z-3)\frac{1}{(z-1)^3(z-3)} = \frac{1}{8}$$

$$I = 2\pi i\left(-\frac{1}{8}+\frac{1}{8}\right) = 0$$

$$I = 0$$