Long Answer Questions–II

[6 Mark]

Q.1. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$.

Ans.

For B^{-1}

$$|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} = 1(3-0) - 2(-1-0) - 2(2-0) = 3 + 2 - 4 = 1 \neq 0$$

i.e., B is invertible matrix $\Rightarrow$ B^{-1} exists and have unique solution.

Now
$$C_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} = 3 - 0 = 3; \quad C_{12} = (-1)^{1+2}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -(-1-0) = 1$$
$$C_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2 - 0 = 2; \quad C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = -(2-4) = 2$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1; \quad C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = -(-2-0) = 2$$
$$C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = 0 + 6 = 6; \quad C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} = -(0-2) = 2$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 3 + 2 = 5$$

$$\therefore \quad \text{adj } B = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$\Rightarrow \qquad B^{-1} = \frac{1}{|B|}(\text{ adj } B) = \frac{1}{1}\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$\text{Now} \quad (\text{ AB })^{-1} = B^{-1}.A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-30+30 & -3+12-12 & 3-10+12 \\ 3-15+10 & -1+6-4 & 1-5+4 \\ 6-30+25 & -2+12-10 & 2-10+10 \end{bmatrix} = \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Q.2. If $A = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$**, find adj *A* and verify that *A*(adj *A*) = (adj *A*)*A* =|*A*|*I*$_3$.**

Ans.

$$\text{Given} \quad A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For adj A

Let A_{ij} is co-factor of a_{ij}.

$$\therefore \quad A_{11} = \begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} = \cos\alpha; \quad A_{12} = -\begin{vmatrix} \sin\alpha & 0 \\ 0 & 1 \end{vmatrix} = -\sin\alpha; \; A_{13} = 0$$

$$A_{21} = -\begin{vmatrix} -\sin\alpha & 0 \\ 0 & 1 \end{vmatrix} = \sin\alpha; \; A_{22} = \begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} = \cos\alpha; \quad A_{23} = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{31} = \begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & 0 \end{vmatrix} = 0; \quad A_{32} = -\begin{vmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \end{vmatrix} = 0; \quad A_{33} = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = 1$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now $A.\text{ adj } A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha.\sin\alpha - \sin\alpha.\cos\alpha & 0 \\ \sin\alpha.\cos\alpha - \sin\alpha.\cos\alpha & \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A.\text{ adj } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A.\text{ adj } A = |A| I_3 \ldots (i) \quad \left[\because |A| = \begin{vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = 1\right]$$

Again $adj\ A.A = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\sin\alpha.\cos\alpha + \sin\alpha.\cos\alpha & 0 \\ -\sin\alpha.\cos\alpha + \sin\alpha.\cos\alpha & \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad adj\ A.A = |A|\ I_3 \ldots (ii)$

From (i) and (ii)

$A.\text{adj } A = \text{adj } A.A = |A| I_3.$

Q.3. Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ to solve the system of equations:

***x* – *y* + 2*z* = 1, 2*y* – 3*z* = 1, 3*x* – 2*y* + 4*z* = 2**

Ans.

Given system of equations are

$x - y + 2z = 1, \quad 2y - 3z = 1, \quad 3x - 2y + 4z = 2$

Above system of equations can be written in matrix form

as $\qquad AX = B \qquad \Rightarrow \qquad X = A^{-1} B$

where $\qquad A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Let $\qquad C = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

Now

$$\mathbf{AC} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \qquad AC = I$

$\Rightarrow \qquad A^{-1}(AC) = A^{-1} I \quad$ [Pre-multiply by A^{-1}]

$\Rightarrow \qquad (A^{-1} A) C = A^{-1} \quad$ [By Associativity]

$\Rightarrow \qquad IC = A^{-1} \Rightarrow A^{-1} = C$

$\Rightarrow \qquad A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

Putting X, A^{-1} and B in $X = A^{-1}B$, we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix} \Rightarrow x = 0, y = 5 \text{ and } z = 3$$

Q.4. Use product $\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$ **to solve the system of equations**

***x* + 3*z* = 9, –*x* + 2*y* –2*z* = 4, 2*x* – 3*y* + 4*z* = –3.**

Ans.

$$\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix} = \begin{bmatrix}-2-9+12 & 0-2+2 & 1+3-4\\ 0+18-18 & 0+4-3 & 0-6+6\\ -6-18+24 & 0-4+4 & 3+6-8\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = I$$

$$\Rightarrow \quad \begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}^{-1} = \begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}1 & 0 & 3\\ -1 & 2 & -2\\ 2 & -3 & 4\end{bmatrix}^{-1} = \begin{bmatrix}-2 & 9 & 6\\ 0 & 2 & 1\\ 1 & -3 & -2\end{bmatrix} \qquad [\because\ (A^T)^{-1} = (A^{-1})^T]$$

Now given system of equations can be written in matrix form as

$$AX = B, \text{ where },\ A = \begin{bmatrix}1 & 0 & 3\\ -1 & 2 & -2\\ 2 & -3 & 4\end{bmatrix}, X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}.\ B = \begin{bmatrix}9\\ 4\\ -3\end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1} B$$

$$\Rightarrow \quad \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-2 & 9 & 6\\ 0 & 2 & 1\\ 1 & -3 & -2\end{bmatrix}\begin{bmatrix}9\\ 4\\ -3\end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-18+36-16\\ 0+8-3\\ 9-12+6\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 5\\ 3\end{bmatrix}$$

$\Rightarrow x = 0,\ y = 5$ and $z = 3$

Q.5. Using matrices, solve the following system of equations:

4x + 3y + 3z = 60; x + 2y + 3z = 45; 6x + 2y + 3z = 70

Ans.

The system can be written as $AX = B \quad \Rightarrow X = A^{-1}B \quad \ldots(i)$

where $A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$

$$|A| = 4(6-6) - 3(3-18) + 2(2-12) = 0 + 45 - 20 \neq 0$$

For *adj A*

$A_{11} = 6 - 6 = 0 \qquad A_{21} = -(9-4) = -5 \qquad A_{31} = (9-4) = 5$

$A_{12} = -(3-18) = 15 \qquad A_{22} = (12-12) = 0 \qquad A_{32} = -(12-2) = -10$

$A_{13} = (2-12) = -10 \qquad A_{23} = -(8-18) = 10 \qquad A_{33} = (8-3) = 5$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 0 & 15 & -10 \\ -5 & 0 & 10 \\ 5 & -10 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{25}\begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}$$

$$= \frac{5}{25}\begin{bmatrix} 0 & -1 & 1 \\ 3 & 0 & -2 \\ -2 & 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 & -1 & 1 \\ 3 & 0 & -2 \\ -2 & 2 & 1 \end{bmatrix}$$

Now putting values in (i), we get

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 & -1 & 1 \\ 3 & 0 & -2 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 - 45 + 70 \\ 180 + 0 - 140 \\ -120 + 90 + 70 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 25 \\ 40 \\ 40 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

Hence, $x = 5$, $y = 8$, $z = 8$.

Q.6. Determine the product $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ **and use it to solve the system of equations**

***x* – *y* + *z* = 4, *x* – 2*y* – 2*z* = 9, 2*x* + *y* + 3*z* = 1.**

Ans.

$$\begin{bmatrix}-4 & 4 & 4\\-7 & 1 & 3\\5 & -3 & -1\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\1 & -2 & -2\\2 & 1 & 3\end{bmatrix} = \begin{bmatrix}-4+4+8 & 4-8+4 & 4-8+12\\-7+1+6 & 7-2+3 & -7-2+9\\5-3-2 & -5+6-1 & 5+6-3\end{bmatrix}$$

$$= \begin{bmatrix}8 & 0 & 0\\0 & 8 & 0\\0 & 0 & 8\end{bmatrix} = 8\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = 8I$$

$$\Rightarrow \quad \frac{1}{8}\begin{bmatrix}-4 & 4 & 4\\-7 & 1 & 3\\5 & -3 & -1\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\1 & -2 & -2\\2 & 1 & 3\end{bmatrix} = I$$

$$\Rightarrow \quad \begin{bmatrix}1 & -1 & 1\\1 & -2 & -2\\2 & 1 & 3\end{bmatrix}^{-1} = \frac{1}{8}\begin{bmatrix}-4 & 4 & 4\\-7 & 1 & 3\\5 & -3 & -1\end{bmatrix}$$

Given system of equation can be written in matrix form as

$AX = B \Rightarrow X = A^{-1} B,$ ...(*i*)

Where $A = \begin{bmatrix}1 & -1 & 1\\1 & -2 & -2\\2 & 1 & 3\end{bmatrix}, X = \begin{bmatrix}x\\y\\z\end{bmatrix}, B = \begin{bmatrix}4\\9\\1\end{bmatrix}$

We have $A^{-1} = \frac{1}{8}\begin{bmatrix}-4 & 4 & 4\\-7 & 1 & 3\\5 & -3 & -1\end{bmatrix}$

Now from (*i*)

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{8}\begin{bmatrix}-4 & 4 & 4\\-7 & 1 & 3\\5 & -3 & -1\end{bmatrix}\begin{bmatrix}4\\9\\1\end{bmatrix} = \frac{1}{8}\begin{bmatrix}-16+36+4\\-28+9+3\\20-27-1\end{bmatrix} = \frac{1}{8}\begin{bmatrix}24\\-16\\-8\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\-2\\-1\end{bmatrix}$$

$\Rightarrow \quad x = 3, y = -2, z = -1$

Q.7. A shopkeeper has 3 varieties of pens '*A*', '*B*' and '*C*'. Meenu purchased 1 pen of each variety for a total of ₹ 21. Jeevan purchased 4 pens of '*A*' variety, 3 pens of 'B' variety and 2 pens of '*C*' variety for ₹ 60. While Shikha purchased 6 pens of

'*A*' variety, 2 pens of '*B*' variety and 3 pens of '*C*' variety for ₹ 70. Using matrix method, find cost of each variety of pen.

Ans.

Let the cost of varieties of pens *A*, *B* and *C* be *x*, *y*, and *z* respectively.

From question

$$x + y + z = 21$$

$$4x + 3y + 2z = 60$$

$$6x + 2y + 3z = 70$$

The given system of linear equation in matrix equation is as follows

$AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$$

$\because \quad AX = B \quad \Rightarrow \quad X = A^{-1}B \quad \ldots(i)$

Now $$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{vmatrix} = 1(9-4) - 1(12-12) + 1(8-18) = 5 - 0 - 10 = -5 \neq 0$$

$A_{11} = (9 - 4) = 5 \qquad A_{21} = -(3 - 2) = -1 \qquad A_{31} = (2 - 3) = -1$

$A_{12} = -(12 - 12) = 0 \qquad A_{22} = (3 - 6) = -3 \qquad A_{32} = -(2 - 4) = 2$

$A_{13} = (8 - 18) = -10 \qquad A_{23} = -(2 - 6) = 4 \qquad A_{33} = (3 - 4) = -1$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} 5 & 0 & -10 \\ -1 & -3 & 4 \\ -1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}.\text{ adj } A = -\frac{1}{5} \begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}$$

Now from (i) $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}\begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -25 \\ -40 \\ -40 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix} \quad \Rightarrow \quad x = 5, y = 8, z = 8$$

$\Rightarrow$ Cost of pen A = ₹ 5

Cost of pen B = ₹ 8

Cost of pen C = ₹ 8

Q.8. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ **are two square matrices, find *AB* and hence solve the following system of linear equations:**

***x* – *y* = 3, 2*x* + 3*y* + 4*z* = 17 and *y* + 2*z* = 7**

Ans.

We have $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$

Now, $AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 2+4+0 & 2-2+0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2-2 & 0-4+10 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad AB = 6I \quad \Rightarrow \quad A^{-1} = \frac{1}{6}B$

$\Rightarrow \quad A^{-1} = \frac{1}{6} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$

The given system of linear equations can be written in matrix form as

$AX = B$, where $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$

$\Rightarrow \quad X = A^{-1} B$

$\Rightarrow \quad X = \frac{1}{6} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} \quad \Rightarrow \quad X = \frac{1}{6} \begin{bmatrix} 2 \times 3 + 2 \times 17 - 4 \times 7 \\ -4 \times 3 + 2 \times 17 - 4 \times 7 \\ 2 \times 3 - 1 \times 17 + 5 \times 7 \end{bmatrix}$

$\Rightarrow \quad X = \frac{1}{6} \begin{bmatrix} 6+34-28 \\ -12+34-28 \\ 6-17+35 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \quad \Rightarrow \quad x = 2, y = -1$ and $z = 4$

Q.9. Using matrices, solve the following system of equations:

3*x* – 2*y* + 3*z* = –1;   2*x* + *y* – *z* = 6;   4*x* – 3*y* + 2*z* = 5

Ans.

$$3x - 2y + 3z = -1$$

$$2x + y - z = 6$$

$$4x - 3y + 2z = 5$$

Now the matrix equation form of above three equations is

$$\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 5 \end{bmatrix}$$

i.e., $\quad AX = B \quad \Rightarrow \quad X = A^{-1}B$

we know that $\quad A^{-1} = \frac{1}{|A|} \text{adj } A$

$$|A| = \begin{vmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix} = 3\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix}$$

$= -3 + 16 - 30 = -17 \neq 0$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}' = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{-17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B = -\frac{1}{17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17} \begin{bmatrix} -34 \\ 17 \\ 51 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$$

By comparing both sides, we get

$x = 2, \ y = -1, \ z = -3$

Q.10. A mixture is to be made of three foods *A, B, C*. The three foods *A, B, C* contain nutrients *P, Q, R* as shown below:

[*HOTS*]

Food	**Grams per kg of nutrient**		
	P	*Q*	*R*
A	1	2	5
B	3	1	1
C	4	2	1

How to form a mixture which will have 8 grams of *P*, 5 grams of *Q* and 7 grams of *R*?

Ans.

Let food needed be x kg of *A*, y kg of *B* and z kg of *C*. Therefore x kg of *A* contains 1 gram of nutrient *P*. So, x kg of *A* will contain x grams of nutrient *P*. Similarly, the amount of nutrient *P* in y kg of food *B* and z kg of food *C* are $3y$ and $4z$ grams respectively. So, total quantity of nutrient *P* in x kg of food *A*, y kg of food *B* and z kg of food *C* is $x + 3y + 4z$ grams.

$x + 3y + 4z = 8$

Similarly, $2x + y + 2z = 5$ [For Q]

and $5x + y + z = 7$ [For R]

The above system of simultaneous linear equations can be written in matrix form as $AX = B$.

$$\text{or,}\quad \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

$$\text{Now,}\quad |A| = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{vmatrix} = 1(1-2) - 3(2-10) + 4(2-5) = -1 + 24 - 12 = 11 \neq 0$$

So, A^{-1} exists and system have unique solution.

Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$C_{11} = -1$; $C_{12} = 8$; $C_{13} = -3$

$C_{21} = 1$; $C_{22} = -19$; $C_{23} = 14$

$C_{31} = 2$; $C_{32} = 6$; $C_{33} = -5$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -1 & 8 & -3 \\ 1 & -19 & 14 \\ 2 & 6 & -5 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{11}\begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

Putting value of X, A^{-1} and B in $X = A^{-1}B$, we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -8+5+14 \\ 64-95+42 \\ -24+70-35 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 11 \\ 11 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow x = 1$, $y = 1$ and $z = 1$.

Thus, the mixture is formed by mixing 1 kg of each of the food A, B and C.