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Continuity and Differentiability

TOPIC 1 Continuity

1. Let $f(x) = x.\left[\frac{x}{2}\right]$, for $-10 < x < 10$, where $[t]$ denotes the greatest integer function. Then the number of points of discontinuity of f is equal to ______. **[NA Sep. 05, 2020 (I)]**

2. If a function $f(x)$ defined by

$$f(x) = \begin{cases} ae^x + be^{-x}, & -1 \le x < 1 \\ cx^2, & 1 \le x \le 3 \\ ax^2 + 2cx, & 3 < x \le 4 \end{cases}$$

be continuous for some $a, b, c \in \mathbf{R}$ and $f'(0) + f'(2) = e$, then the value of a is : **[Sep. 02, 2020 (I)]**

(a) $\frac{1}{e^2 - 3e + 13}$ (b) $\frac{e}{e^2 - 3e - 13}$

(c) $\frac{e}{e^2 + 3e + 13}$ (d) $\frac{e}{e^2 - 3e + 13}$

3. Let $[t]$ denote the greatest integer $\le t$ and $\lim\limits_{x\to 0} x\left[\frac{4}{x}\right] = A$. Then the function, $f(x) = [x^2]\sin(\pi x)$ is discontinuous, when x is equal to : **[Jan. 9, 2020 (II)]**

(a) $\sqrt{A+1}$ (b) $\sqrt{A+5}$

(c) $\sqrt{A+21}$ (d) $\sqrt{A}$

4. If the function f defined on $\left(-\frac{1}{3}, \frac{1}{3}\right)$ by

$$f(x) = \begin{cases} \frac{1}{x}\log_e\left(\frac{1+3x}{1-2x}\right), & \text{when } x \ne 0 \\ k, & \text{when } x = 0 \end{cases}$$

is continuous, then k is equal to __________. **[NA Jan. 7, 2020 (II)]**

5. If the function f defined on $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ by

$$f(x) = \begin{cases} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}, & x \ne \frac{\pi}{4} \\ k, & x = \frac{\pi}{4} \end{cases}$$

is continuous, then k is equal to: **[April 09, 2019 (I)]**

(a) 2 (b) $\frac{1}{2}$ (c) 1 (d) $\frac{1}{\sqrt{2}}$

6. If $f(x) = [x] - \left[\frac{x}{4}\right], x \in R$, where $[x]$ denotes the greatest integer function, then: **[April 09, 2019 (II)]**

(a) f is continuous at $x = 4$.

(b) $\lim\limits_{x\to 4+} f(x)$ exists but $\lim\limits_{x\to 4-} f(x)$ does not exist.

(c) Both $\lim\limits_{x\to 4-} f(x)$ and $\lim\limits_{x\to 4+} f(x)$ exist but are not equal.

(d) $\lim\limits_{x\to 4-} f(x)$ exists but $\lim\limits_{x\to 4+} f(x)$ does not exist.

7. If the function

$$f(x) = \begin{cases} a|\pi - x| + 1, & x \le 5 \\ b|x - \pi| + 3, & x > 5 \end{cases}$$

is continuous at $x = 5$, then the value of $a - b$ is: **[April 09, 2019 (II)]**

(a) $\frac{2}{\pi+5}$ (b) $\frac{-2}{\pi+5}$ (c) $\frac{2}{\pi-5}$ (d) $\frac{2}{5-\pi}$

8. Let $f: [-1, 3] \to R$ be defined as

$$f(x) = \begin{cases} |x| + [x], & -1 \le x < 1 \\ x + |x|, & 1 \le x < 2 \\ x + [x], & 2 \le x \le 3, \end{cases}$$

where $[t]$ denotes the greatest integer less than or equal to t. Then, f is discontinuous at : **[April 08, 2019 (II)]**

(a) only one point (b) only two points

(c) only three points (d) four or more points

9. Let $f: \mathbf{R} \to \mathbf{R}$ be a function defined as

$$f(x)=\begin{cases} 5, & \text{if } x \le 1 \\ a+bx, & \text{if } 1<x<3 \\ b+5x, & \text{if } 3 \le x<5 \\ 30, & \text{if } x \ge 5 \end{cases}$$

Then, f is : **[Jan 09, 2019 (I)]**

(a) continuous if a = 5 and b = 5
(b) continuous if a = – 5 and b = 10
(c) continous if a = 0 and b = 5
(d) not continuous for any values of a and b

10. If the function f defined as

$$f(x)=\frac{1}{x}-\frac{k-1}{e^{2x}-1}$$

$x \neq 0$, is continuous at $x = 0$,
then the ordered pair $(k, f(0))$ is equal to? **[Online April 16, 2018]**

(a) (3, 1) (b) (3, 2) (c) $\left(\frac{1}{3}, 2\right)$ (d) (2, 1)

11. Let $f(x)=\begin{cases} (x-1)^{\frac{1}{2-x}}, & x>1, x \neq 2 \\ k, & x=2 \end{cases}$

The value of k for which f is continuous at $x = 2$ is **[Online April 15, 2018]**

(a) e^{-2} (b) e (c) e^{-1} (d) 1

12. The value of k for which the function

$$f(x)=\begin{cases} \left(\frac{4}{5}\right)^{\frac{\tan 4x}{\tan 5x}}, & 0<x<\frac{\pi}{2} \\ k+\frac{2}{5}, & x=\frac{\pi}{2} \end{cases}$$ is continuous at $x=\frac{\pi}{2}$, is :

[Online April 9, 2017]

(a) $\frac{17}{20}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) $-\frac{2}{5}$

13. Let a, b $\in R, (a \neq 0)$. if the function f defined as

$$f(x)=\begin{cases} \frac{2x^2}{a}, & 0 \le x<1 \\ a, & 1 \le x<\sqrt{2} \\ \frac{2b^2-4b}{x^3}, & \sqrt{2} \le x<\infty \end{cases}$$

is continuous in the interval $[0, \infty)$, then an ordered pair (a, b) is : **[Online April 10, 2016]**

(a) $(-\sqrt{2}, 1-\sqrt{3})$ (b) $(\sqrt{2}, -1+\sqrt{3})$
(c) $(\sqrt{2}, 1-\sqrt{3})$ (d) $(-\sqrt{2}, 1+\sqrt{3})$

14. Let k be a non–zero real number. **[Online April 11, 2015]**

If $f(x)=\begin{cases} \frac{(e^x-1)}{\sin\left(\frac{x}{k}\right)\log\left(1+\frac{x}{4}\right)}, & x \neq 0 \\ 12, & x=0 \end{cases}$

is a continuous function then the value of k is:

(a) 4 (b) 1 (c) 3 (d) 2

15. If the function

$$f(x)=\begin{cases} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}, & x \neq \pi \\ k, & x=\pi \end{cases}$$

is continuous at $x = \pi$, then k equals: **[Online April 19, 2014]**

(a) 0 (b) $\frac{1}{2}$ (c) 2 (d) $\frac{1}{4}$

16. If f(x) is continuous and $f\left(\frac{9}{2}\right)=\frac{2}{9}$, then $\lim_{x \to 0} f\left(\frac{1-\cos 3x}{x^2}\right)$ is equal to: **[Online April 9, 2014]**

(a) $\frac{9}{2}$ (b) $\frac{2}{9}$ (c) 0 (d) $\frac{8}{9}$

17. Consider the function :
$f(x) = [x] + |1 - x|,\ -1 \le x \le 3$ where $[x]$ is the greatest integer function.
Statement 1 : f is not continuous at $x = 0, 1, 2$ and 3.

Statement 2 : $f(x)=\begin{cases} -x, & -1 \le x<0 \\ 1-x, & 0 \le x<1 \\ 1+x, & 1 \le x<2 \\ 2+x, & 2 \le x \le 3 \end{cases}$

[Online April 25, 2013]

(a) Statement 1 is true ; Statement 2 is false,
(b) Statement 1 is true; Statement 2 is true; Statement 2 is not correct explanation for Statement 1.
(c) Statement 1 is true; Statement 2 is true; Statement It is a correct explanation for Statement 1.
(d) Statement 1 is false; Statement 2 is true.

18. Let f be a composite function of x defined by

$$f(u)=\frac{1}{u^2+u-2}, u(x)=\frac{1}{x-1}.$$

Then the number of points x where f is discontinuous is : **[Online April 23, 2013]**

(a) 4 (b) 3 (c) 2 (d) 1

19. Let $f(x) = -1 + |x-2|$, and $g(x) = 1 - |x|$; then the set of all points where fog is discontinuous is : **[Online April 22, 2013]**

(a) $\{0, 2\}$ (b) $\{0, 1, 2\}$
(c) $\{0\}$ (d) an empty set

20. If $f : R \to R$ is a function defined by $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$, where $[x]$ denotes the greatest integer function, then f is . **[2012]**

(a) continuous for every real x.
(b) discontinuous only at $x = 0$
(c) discontinuous only at non-zero integral values of x.
(d) continuous only at $x = 0$.

21. Let $f : [1, 3] \to R$ be a function satisfying $\frac{x}{[x]} \le f(x) \le \sqrt{6-x}$, for all $x \ne 2$ and $f(2) = 1$, where R is the set of all real numbers and $[x]$ denotes the largest integer less than or equal to x.

Statement 1: $\lim_{x \to 2^-} f(x)$ exists. **[Online May 19, 2012]**

Statement 2: f is continuous at $x = 2$.

(a) Statement 1 is true, Statement 2 is true, Statement 2 is a correct explanation for Statement 1.
(b) Statement 1 is false, Statement 2 is true.
(c) Statement 1 is true, Statement 2 is true, Statement 2 is not a correct explanation for Statement 1.
(d) Statement 1 is true, Statement 2 is false.

22. **Statement 1:** A function $f : R \to R$ is continuous at x_0 if and only if $\lim_{x \to x_0} f(x)$ exists and $\lim_{x \to x_0} f(x) = f(x_0)$

Statement 2: A function $f : R \to R$ is discontinuous at x_0 if and only if, $\lim_{x \to x_0} f(x)$ exists and $\lim_{x \to x_0} f(x) \ne f(x_0)$. **[Online May 12, 2012]**

(a) Statement 1 is true, Statement 2 is true, Statement 2 is not a correct explanation of Statement 1.
(b) Statement 1 is false, Statement 2 is true.
(c) Statement 1 is true, Statement 2 is true, Statement 2 is a correct explanation of Statement 1.
(d) Statement 1 is true, Statement 2 is false.

23. Define $f(x)$ as the product of two real functions **[2011RS]**

$$f_1(x) = x, x \in R, \text{and } f_2(x) = \begin{cases} \sin\frac{1}{x}, & if\ x \ne 0 \\ 0, & if\ x = 0 \end{cases}$$

as follows :

$$f(x) = \begin{cases} f_1(x).f_2(x), & if\ x = 0 \\ 0 & if\ x = 0 \end{cases}$$

Statement - 1 : $f(x)$ is continuous on R.

Statement - 2 : $f_1(x)$ and $f_2(x)$ are continuous on R.

(a) Statement -1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is true; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

24. The values of p and q for which the function **[2011]**

$$f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x} & , x < 0 \\ q & , x = 0 \\ \frac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}} & , x > 0 \end{cases}$$

is continuous for all x in R, are

(a) $p = \frac{5}{2}, q = \frac{1}{2}$ (b) $p = -\frac{3}{2}, q = \frac{1}{2}$
(c) $p = \frac{1}{2}, q = \frac{3}{2}$ (d) $p = \frac{1}{2}, q = -\frac{3}{2}$

25. The function $f : R/\{0\} \to R$ given by **[2007]**

$$f(x) = \frac{1}{x} - \frac{2}{e^{2x} - 1}$$

can be made continuous at $x = 0$ by defining $f(0)$ as

(a) 0 (b) 1
(c) 2 (d) -1

26. Let $f(x) = \frac{1 - \tan x}{4x - \pi}, x \ne \frac{\pi}{4}, x \in \left[0, \frac{\pi}{2}\right]$.

If $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$, then $f\left(\frac{\pi}{4}\right)$ is **[2004]**

(a) -1 (b) $\frac{1}{2}$
(c) $-\frac{1}{2}$ (d) 1

27. f is defined in [-5, 5] as **[2002]**

$f(x) = x$ if x is rational
$= -x$ if x is irrational. Then

(a) $f(x)$ is continuous at every x, except $x = 0$
(b) $f(x)$ is discontinuous at every x, except $x = 0$
(c) $f(x)$ is continuous everywhere
(d) $f(x)$ is discontinuous everywhere

TOPIC 2 Differentiability

28. Let $f: R \to R$ be a function defined by $f(x) = \max\{x, x^2\}$. Let S denote the set of all points in R, where f is not differentiable. Then: **[Sep. 06, 2020 (II)]**
(a) $\{0, 1\}$ (b) $\{0\}$
(c) ϕ (an empty set) (d) $\{1\}$

29. If the function $f(x) = \begin{cases} k_1(x-\pi)^2 - 1, & x \le \pi \\ k_2 \cos x, & x > \pi \end{cases}$ is twice differentiable, then the ordered pair (k_1, k_2) is equal to: **[Sep. 05, 2020 (I)]**
(a) $\left(\frac{1}{2}, 1\right)$ (b) $(1, 0)$
(c) $\left(\frac{1}{2}, -1\right)$ (d) $(1, 1)$

30. Let f be a twice differentiable function on $(1, 6)$. If $f(2) = 8$, $f'(2) = 5, f'(x) \ge 1$ and $f''(x) \ge 4$, for all $x \in (1, 6)$, then: **[Sep. 04, 2020 (I)]**
(a) $f(5) + f'(5) \le 26$ (b) $f(5) + f'(5) \ge 28$
(c) $f'(5) + f''(5) \le 20$ (d) $f(5) \le 10$

31. Suppose a differentiable function $f(x)$ satisfies the identity $f(x+y) = f(x) + f(y) + xy^2 + x^2y$, for all real x and y. If $\lim_{x \to 0} \frac{f(x)}{x} = 1$, then $f'(3)$ is equal to ________. **[NA Sep. 04, 2020 (I)]**

32. The function $f(x) = \begin{cases} \frac{\pi}{4} + \tan^{-1} x, & |x| \le 1 \\ \frac{1}{2}(|x| - 1), & |x| > 1 \end{cases}$
is: **[Sep. 04, 2020 (II)]**
(a) continuous on $\mathbf{R} - \{1\}$ and differentiable on $\mathbf{R} - \{-1, 1\}$.
(b) both continuous and differentiable on $\mathbf{R} - \{1\}$.
(c) continuous on $\mathbf{R} - \{-1\}$ and differentiable on $\mathbf{R} - \{-1, 1\}$.
(d) both continuous and differentiable on $\mathbf{R} - \{-1\}$.

33. If $f(x) = \begin{cases} \frac{\sin(a+2)x + \sin x}{x} & ; \ x < 0 \\ b & ; \ x = 0 \\ \frac{(x+3x^2)^{1/3} - x^{1/3}}{x^{4/3}} & ; \ x > 0 \end{cases}$
is continuous at $x = 0$, then $a + 2b$ is equal to: **[Jan. 9, 2020 (I)]**
(a) 1 (b) –1 (c) 0 (d) –2

34. Let f and g be differentiable functions on $\mathbf{R}$ such that fog is the identity function. If for some $a, b \in \mathbf{R}$, $g'(a) = 5$ and $g(a) = b$, then $f'(b)$ is equal to: **[Jan. 9, 2020 (II)]**
(a) $\frac{1}{5}$ (b) 1 (c) 5 (d) $\frac{2}{5}$

35. Let S be the set of all functions $f: [0,1] \to R$, which are continuous on $[0, 1]$ and differentiable on $(0,1)$. Then for every f in S, there exists a $c \in (0,1)$, depending on f, such that: **[Jan. 8, 2020 (II)]**
(a) $|f(c) - f(1)| < (1-c)|f'(c)|$
(b) $\frac{f(1) - f(c)}{1-c} = f'(c)$
(c) $|f(c) + f(1)| < (1+c)|f'(c)|$
(d) $|f(c) - f(1)| < |f'(c)|$

36. Let the function, $f: [-7, 0] \to R$ be continuous on $[-7, 0]$ and differentiable on $(-7, 0)$. If $f(-7) = -3$ and $f'(x)$ d" 2, for all $x \in (-7, 0)$, then for all such functions f, $f'(-1) + f(0)$ lies in the interval: **[Jan. 7, 2020 (I)]**
(a) $(-\infty, 20]$ (b) $[-3, 11]$
(c) $(-\infty, 11]$ (d) $[-6, 20]$

37. Let S be the set of points where the function, $f(x) = |2 - |x - 3||, x \in \mathbf{R}$, is not differentiable.
Then $\sum_{x \in S} f(f(x))$ is equal to———. **[NA Jan. 7, 2020 (I)]**

38. If $f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x}, & x < 0 \\ q, & x = 0 \\ \frac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}}, & x > 0 \end{cases}$
is continuous at $x = 0$, then the ordered pair (p, q) is equal to: **[April 10, 2019 (I)]**
(a) $\left(-\frac{3}{2}, -\frac{1}{2}\right)$ (b) $\left(-\frac{1}{2}, \frac{3}{2}\right)$
(c) $\left(-\frac{3}{2}, \frac{1}{2}\right)$ (d) $\left(\frac{5}{2}, \frac{1}{2}\right)$

39. Let $f(x) = \log_e(\sin x)$, $(0 < x < \pi)$ and $g(x) = \sin^{-1}(e^{-x})$, $(x \ge 0)$. If α is a positive real number such that $a = (fog)'(\alpha)$ and $b = (fog)(\alpha)$, then: **[April 10, 2019 (II)]**
(a) $a\alpha^2 + b\alpha + a = 0$ (b) $a\alpha^2 - b\alpha - a = 1$
(c) $a\alpha^2 - b\alpha - a = 0$ (d) $a\alpha^2 + b\alpha - a = -2a^2$

40. Let f : $\mathbf{R} \to \mathbf{R}$ be differentiable at $c \in \mathbf{R}$ and f (c) = 0. If $g(x) = |f(x)|$, then at x = c, g is : **[April 10, 2019 (I)]**

(a) not differentiable if f'(c) = 0

(b) differentiable if f"(c) $\neq 0$

(c) differentiable if f' (c) = 0

(d) not differentiable

41. Let f(x) = 15 – |x – 10|; $x \in R$. Then the set of all values of x, at which the function, g(x) = f (f(x)) is not differentiable, is: **[April 09, 2019 (I)]**

(a) {5, 10, 15} (b) {10, 15}

(c) {5, 10, 15, 20} (d) {10}

42. If $f(1) = 1, f'(1) = 3$, then the derivative of $f(f(f(x))) + (f(x))^2$ at $x = 1$ is : **[April 08, 2019 (II)]**

(a) 33 (b) 12 (c) 15 (d) 9

43. Let f be a differentiable function such that $f(1) = 2$ and $f'(x) = f(x)$ for all $x \in R$. If $h(x) = f(f(x))$, then $h'(1)$ is equal to : **[Jan. 12, 2019 (II)]**

(a) $2e^2$ (b) 4e (c) 2e (d) $4e^2$

44. Let $f(x) = \begin{cases} -1, & -2 \le x < 0 \\ x^2 - 1, & 0 \le x \le 2 \end{cases}$ and $g(x) = |f(x)| + f(|x|)$. Then, in the interval (–2, 2), g is : **[Jan. 11, 2019 (I)]**

(a) differentiable at all points

(b) not continuous

(c) not differentiable at two points

(d) not differentiable at one point

45. If $x\log_e(\log_e x) - x^2 + y^2 = 4 (y > 0)$, then $\frac{dy}{dx}$ at x = e is equal to : **[Jan. 11, 2019 (I)]**

(a) $\frac{(1+2e)}{2\sqrt{4+e^2}}$ (b) $\frac{(2e-1)}{2\sqrt{4+e^2}}$

(c) $\frac{(1+2e)}{\sqrt{4+e^2}}$ (d) $\frac{e}{\sqrt{4+e^2}}$

46. Let K be the set of all real values of x where the function $f(x) = \sin|x| - |x| + 2(x-\pi)\cos|x|$ is not differentiable. Then the set K is equal to : **[Jan. 11, 2019 (II)]**

(a) ϕ (an empty set) (b) $\{\pi\}$

(c) {0} (d) $\{0, \pi\}$

47. Let $f(x) = \begin{cases} \max\{|x|, x^2\} & |x| \le 2 \\ 8 - 2|x|, & 2 < |x| \le 4 \end{cases}$

Let S be the set of points in the interval (– 4, 4) at which f is not differentiable. Then S: **[Jan 10, 2019 (I)]**

(a) is an empty set

(b) equals {– 2, – 1, 0, 1, 2}

(c) equals {– 2, – 1, 1, 2}

(d) equals {– 2, 2}

48. Let f : (– 1, 1) $\to$ **R** be a function defined by $f(x) = \max\left\{-|x|, -\sqrt{1-x^2}\right\}$. If K be the set of all points at which f is not differentiable, then K has exactly: **[Jan. 10, 2019 (II)]**

(a) five elements (b) one element

(c) three elements (d) two elements

49. Let S = $\{t \in R : f(x) = |x-\pi|(e^{|x|}-1)\sin|x|$ is not differentiable at t$\}$. Then the set S is equal to : **[2018]**

(a) {0} (b) $\{\pi\}$

(c) $\{0, \pi\}$ (d) ϕ (an empty set)

50. Let $S = \{(\lambda, \mu) \in R \times R : f(t) = (|\lambda|e^{|t|} - \mu).\sin(2|t|), t \in R$, is a differentiable function$\}$. Then S is a subest of? **[Online April 15, 2018]**

(a) $R \times [0, \infty)$ (b) $(-\infty, 0) \times R$

(c) $[0, \infty) \times R$ (d) $R \times (-\infty, 0)$

51. If the function

$$f(x) = \begin{cases} -x, & x < 1 \\ a + \cos^{-1}(x+b), & 1 \le x \le 2 \end{cases}$$ is differentiable at

x = 1, then $\frac{a}{b}$ is equal to : **[Online April 9, 2016]**

(a) $\frac{\pi+2}{2}$ (b) $\frac{\pi-2}{2}$

(c) $\frac{-\pi-2}{2}$ (d) $-1 - \cos^{-1}(2)$

52. If the function.

$$g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx + 2, & 3 < x \le 5 \end{cases}$$ is differentiable, then the

value of k + m is : **[2015]**

(a) $\frac{10}{3}$ (b) 4

(c) 2 (d) $\frac{16}{5}$

53. Let f: R $\to$ R be a function such that $|f(x)| \le x^2$, for all $x \in R$. Then, at $x = 0$, f is: **[Online April 19, 2014]**

(a) continuous but not differentiable.

(b) continuous as well as differentiable.

(c) neither continuous nor differentiable.

(d) differentiable but not continuous.

54. Let f, g: R → R be two functions defined by

$$f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0 \end{cases}, \text{ and } g(x)=x\,f(x)$$

Statement I: f is a continuous function at $x = 0$.
Statement II: g is a differentiable function at $x = 0$. **[Online April 12, 2014]**

(a) Both statement I and II are false.
(b) Both statement I and II are true.
(c) Statement I is true, statement II is false.
(d) Statement I is false, statement II is true.

55. Consider the function, $f(x)=|x-2|+|x-5|, x\in R$.
Statement-1 : $f'(4)=0$
Statement-2 : f is continuous in [2,5], differentiable in (2,5) and $f(2)=f(5)$. **[2012]**

(a) Statement-1 is false, Statement-2 is true.
(b) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.
(c) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for Statement-1.
(d) Statement-1 is true, statement-2 is false.

56. If $f(x) = a|\sin x| + be^{|x|} + c|x|^3$, where $a, b, c \in R$, is differentiable at $x = 0$, then **[Online May 26, 2012]**

(a) $a = 0$, b and c are any real numbers
(b) $c = 0, a = 0$, b is any real number
(c) $b = 0, c = 0$, a is any real number
(d) $a = 0, b = 0$, c is any real number

57. If $x + |y| = 2y$, then y as a function of x, at $x = 0$ is **[Online May 7, 2012]**

(a) differentiable but not continuous
(b) continuous but not differentiable
(c) continuous as well as differentiable
(d) neither continuous nor differentiable

58. If function $f(x)$ is differentiable at $x = a$,

then $\lim_{x\to a}\dfrac{x^2 f(a)-a^2 f(x)}{x-a}$ is : **[2011RS]**

(a) $-a^2 f'(a)$ (b) $a\,f(a)-a^2 f'(a)$
(c) $2af(a)-a^2 f'(a)$ (d) $2af(a)+a^2 f'(a)$

59. Let $f(x)=\begin{cases}(x-1)\sin\dfrac{1}{x-1} & \text{if } x\neq 1\\ 0 & \text{if } x=1\end{cases}$ **[2008]**

Then which one of the following is true?
(a) f is neither differentiable at $x = 0$ nor at $x = 1$
(b) f is differentiable at $x = 0$ and at $x = 1$
(c) f is differentiable at $x = 0$ but not at $x = 1$
(d) f is differentiable at $x = 1$ but not at $x = 0$

60. Let $f: R\to R$ be a function defined by

$f(x) = \min\{x+1, |x|+1\}$, Then which of the following is true?

(a) $f(x)$ is differentiable everywhere **[2007]**
(b) $f(x)$ is not differentiable at $x = 0$
(c) $f(x) \geq 1$ for all $x \in R$
(d) $f(x)$ is not differentiable at $x = 1$

61. The set of points where $f(x)=\dfrac{x}{1+|x|}$ is differentiable is **[2006]**

(a) $(-\infty,0)\cup(0,\infty)$ (b) $(-\infty,-1)\cup(-1,\infty)$
(c) $(-\infty,\infty)$ (d) $(0,\infty)$

62. If f is a real valued differentiable function satisfying $|f(x)-f(y)| \leq (x-y)^2, x, y \in R$ and $f(0) = 0$, then $f(1)$ equals **[2005]**

(a) -1 (b) 0
(c) 2 (d) 1

63. Suppose $f(x)$ is differentiable at $x = 1$ and $\lim_{h\to 0}\frac{1}{h}f(1+h)=5$, then $f'(1)$ equals **[2005]**

(a) 3 (b) 4 (c) 5 (d) 6

64. If $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\neq 0\\ 0, & x=0\end{cases}$ then f(x) is

(a) discontinuous every where **[2003]**
(b) continuous as well as differentiable for all x
(c) continuous for all x but not differentiable at $x = 0$
(d) neither differentiable nor continuous at $x = 0$

TOPIC 3 Chain Rule of Differentiation, Differentiation of Explicit & Implicit Functions, Parametric & Composite Functions, Logarithmic & Exponential Functions, Inverse Functions, Differentiation by Trigonometric Substitution

65. The derivative of $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\dfrac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x=\dfrac{1}{2}$ is : **[Sep. 05, 2020 (II)]**

(a) $\dfrac{2\sqrt{3}}{5}$ (b) $\dfrac{\sqrt{3}}{12}$ (c) $\dfrac{2\sqrt{3}}{3}$ (d) $\dfrac{\sqrt{3}}{10}$

66. If $(a+\sqrt{2}b\cos x)(a-\sqrt{2}b\cos y)=a^2-b^2$, where $a>b>0$, then $\frac{dx}{dy}$ at $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$ is: **[Sep. 04, 2020 (I)]**

(a) $\frac{a-2b}{a+2b}$ (b) $\frac{a-b}{a+b}$ (c) $\frac{a+b}{a-b}$ (d) $\frac{2a+b}{2a-b}$

67. If $y=\sum_{k=1}^{6} k\cos^{-1}\left\{\frac{3}{5}\cos kx-\frac{4}{5}\sin kx\right\}$, then $\frac{dy}{dx}$ at $x=0$ is ____________. **[NA Sep. 02, 2020 (II)]**

68. If $x=2\sin\theta-\sin 2\theta$ and $y=2\cos\theta-\cos 2\theta$, $\theta\in[0, 2\pi]$, then $\frac{d^2y}{dx^2}$ at $\theta=\pi$ is : **[Jan. 9, 2020 (II)]**

(a) $\frac{3}{4}$ (b) $-\frac{3}{8}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{4}$

69. If $y(\alpha)=\sqrt{2\left(\frac{\tan\alpha+\cot\alpha}{1+\tan^2\alpha}\right)+\frac{1}{\sin^2\alpha}}, \alpha\in\left(\frac{3\pi}{4},\pi\right)$, then $\frac{dy}{d\alpha}$ at $\alpha=\frac{5\pi}{6}$ is: **[Jan. 7, 2020 (I)]**

(a) 4 (b) $\frac{4}{3}$ (c) –4 (d) $-\frac{1}{4}$

70. Let $y=y(x)$ be a function of x satisfying $y\sqrt{1-x^2}=k-x\sqrt{1-y^2}$ where k is a constant and $y\left(\frac{1}{2}\right)=-\frac{1}{4}$. Then $\frac{dy}{dx}$ at $x=\frac{1}{2}$, is equal to: **[Jan. 7, 2020 (II)]**

(a) $-\frac{\sqrt{5}}{4}$ (b) $-\frac{\sqrt{5}}{2}$ (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{\sqrt{5}}{2}$

71. If $e^y+xy=e$, the ordered pair $\left(\frac{dy}{dx},\frac{d^2y}{dx^2}\right)$ at $x=0$ is equal to : **[April 12, 2019 (I)]**

(a) $\left(\frac{1}{e},-\frac{1}{e^2}\right)$ (b) $\left(-\frac{1}{e},\frac{1}{e^2}\right)$

(c) $\left(\frac{1}{e},\frac{1}{e^2}\right)$ (d) $\left(-\frac{1}{e},-\frac{1}{e^2}\right)$

72. The derivative of $\tan^{-1}\left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)$, with respect to $\frac{x}{2}$, where $\left(x\in\left(0,\frac{\pi}{2}\right)\right)$ is : **[April 12, 2019 (II)]**

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) 2

73. If $2y=\left(\cot^{-1}\left(\frac{\sqrt{3}\cos x+\sin x}{\cos x-\sqrt{3}\sin x}\right)\right)^2$, $x\in\left(0,\frac{\pi}{2}\right)$ then $\frac{dy}{dx}$ is equal to : **[April 08, 2019 (I)]**

(a) $\frac{\pi}{6}-x$ (b) $x-\frac{\pi}{6}$ (c) $\frac{\pi}{3}-x$ (d) $2x-\frac{\pi}{3}$

74. Let S be the set of all points in $(-\pi, \pi)$ at which the function $f(x) = \min\{\sin x, \cos x\}$ is not differentiable. Then S is a subset of which of the following? **[Jan. 12, 2019 (I)]**

(a) $\left\{-\frac{\pi}{4},0,\frac{\pi}{4}\right\}$ (b) $\left\{-\frac{3\pi}{4},-\frac{\pi}{4},\frac{3\pi}{4},\frac{\pi}{4}\right\}$

(c) $\left\{-\frac{\pi}{2},-\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2}\right\}$ (d) $\left\{-\frac{3\pi}{4},-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{4}\right\}$

75. For $x > 1$, if $(2x)^{2y}=4e^{2x-2y}$, then $(1+\log_e 2x)^2\frac{dy}{dx}$ is equal to : **[Jan. 12, 2019 (I)]**

(a) $\frac{x\log_e 2x-\log_e 2}{x}$ (b) $\log_e 2x$

(c) $\frac{x\log_e 2x+\log_e 2}{x}$ (d) $x\log_e 2x$

76. Let $f:\mathbf{R}\to\mathbf{R}$ be a function such that $f(x)=x^3+x^2f'(1)+xf''(2)+f'''(3), x\in\mathbf{R}$. Then $f(2)$ equals: **[Jan 10, 2019 (I)]**

(a) –4 (b) 30 (c) –2 (d) 8

77. If $x = 3\tan t$ and $y = 3\sec t$, then the value of $\frac{d^2y}{dx^2}$ at $t=\frac{\pi}{4}$, is: **[Jan. 09, 2019 (II)]**

(a) $\frac{1}{3\sqrt{2}}$ (b) $\frac{1}{6\sqrt{2}}$ (c) $\frac{3}{2\sqrt{2}}$ (d) $\frac{1}{6}$

78. If $x=\sqrt{2^{\operatorname{cosec}^{-1}t}}$ and $y=\sqrt{2^{\sec^{-1}t}}$ $(|t|\ge 1)$, then $\frac{dy}{dx}$ is equal to. **[Online April 16, 2018]**

(a) $\frac{y}{x}$ (b) $-\frac{y}{x}$ (c) $-\frac{x}{y}$ (d) $\frac{x}{y}$

79. If $f(x)=\begin{vmatrix}\cos x & x & 1\\ 2\sin x & x^2 & 2x\\ \tan x & x & 1\end{vmatrix}$, then $\lim_{x\to 0}\frac{f'(x)}{x}$ **[Online April 15, 2018]**

(a) Exists and is equal to – 2
(b) Does not exist
(c) Exist and is equal to 0
(d) Exists and is equal to 2

80. If $f(x) = \sin^{-1}\left(\frac{2\times 3^x}{1+9^x}\right)$, then $f'\left(-\frac{1}{2}\right)$ equals. **[Online April 15, 2018]**

(a) $\sqrt{3}\log_e\sqrt{3}$ (b) $-\sqrt{3}\log_e\sqrt{3}$
(c) $-\sqrt{3}\log_e 3$ (d) $\sqrt{3}\log_e 3$

81. If $x^2 + y^2 + \sin y = 4$, then the value of $\frac{d^2y}{dx^2}$ at the point $(-2, 0)$ is **[Online April 15, 2018]**

(a) -34 (b) -32 (c) -2 (d) 4

82. If for $x \in \left(0, \frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x}.g(x)$, then g(x) equals : **[2017]**

(a) $\frac{3}{1+9x^3}$ (b) $\frac{9}{1+9x^3}$
(c) $\frac{3x\sqrt{x}}{1-9x^3}$ (d) $\frac{3x}{1-9x^3}$

83. For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then : **[2016]**

(a) $g'(0) = -\cos(\log 2)$
(b) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$
(c) g is not differentiable at $x = 0$
(d) $g'(0) = \cos(\log 2)$

84. If $f(x) = x^2 - x + 5$, $x > \frac{1}{2}$, and g(x) is its inverse function, then $g'(7)$ equals: **[Online April 12, 2014]**

(a) $-\frac{1}{3}$ (b) $\frac{1}{13}$ (c) $\frac{1}{3}$ (d) $-\frac{1}{13}$

85. If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to : **[2013]**

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) 1 (d) $\sqrt{2}$

86. If the curves $\frac{x^2}{\alpha} + \frac{y^2}{4} = 1$ and $y^3 = 16x$ intersect at right angles, then a value of α is : **[Online April 23, 2013]**

(a) 2 (b) $\frac{4}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

87. For $a > 0$, $t \in \left(0, \frac{\pi}{2}\right)$, let $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$,

Then, $1 + \left(\frac{dy}{dx}\right)^2$ equals : **[Online April 22, 2013]**

(a) $\frac{x^2}{y^2}$ (b) $\frac{y^2}{x^2}$ (c) $\frac{x^2+y^2}{y^2}$ (d) $\frac{x^2+y^2}{x^2}$

88. Let $f(x) = \frac{x^2 - x}{x^2 + 2x}$, $x \neq 0, -2$. Then $\frac{d}{dx}[f^{-1}(x)]$ (wherever it is defined) is equal to : **[Online April 9, 2013]**

(a) $\frac{-1}{(1-x)^2}$ (b) $\frac{3}{(1-x)^2}$
(c) $\frac{1}{(1-x)^2}$ (d) $\frac{-3}{(1-x)^2}$

89. If $f'(x) = \sin(\log x)$ and $y = f\left(\frac{2x+3}{3-2x}\right)$, then $\frac{dy}{dx}$ equals **[Online May 12, 2012]**

(a) $\sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$
(b) $\frac{12}{(3-2x)^2}$
(c) $\frac{12}{(3-2x)^2}\sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$
(d) $\frac{12}{(3-2x)^2}\cos\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$

90. Let $f: (-1, 1) \rightarrow R$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x) + 2)]^2$. Then $g'(0) =$ **[2010]**

(a) -4 (b) 0 (c) -2 (d) 4

91. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals **[2009]**

(a) 1 (b) log 2 (c) $-\log 2$ (d) -1

92. If $x^m . y^n = (x+y)^{m+n}$, then $\frac{dy}{dx}$ is **[2006]**

(a) $\frac{y}{x}$ (b) $\frac{x+y}{xy}$ (c) xy (d) $\frac{x}{y}$

93. If $x = e^{y+e^{y+\cdots\text{to}\infty}}$, $x > 0$, then $\frac{dy}{dx}$ is **[2004]**

(a) $\frac{1+x}{x}$ (b) $\frac{1}{x}$ (c) $\frac{1-x}{x}$ (d) $\frac{x}{1+x}$

94. Let $f(x)$ be a polynomial function of second degree. If $f(1) = f(-1)$ and a, b, c are in A. P, then $f'(a), f'(b), f'(c)$ are in **[2003]**

(a) Arithmetic -Geometric Progression
(b) A.P
(c) G.P
(d) H.P.

95. If $f(x+y) = f(x).f(y) \forall x.y$ and $f(5) = 2$, $f'(0) = 3$, then $f'(5)$ is **[2002]**

(a) 0 (b) 1 (c) 6 (d) 2

TOPIC 4 Differentiation of Infinite Series, Successive Differentiation, nth Derivative of Some Standard Functions, Leibnitz's Theorem, Rolle's Theorem, Lagrange's Mean Value Theorem

96. For all twice differentiable functios f : R→R, with $f(0)=f(1)=f'(0)=0$ **[Sep. 06, 2020 (II)]**

(a) $f''(x) \neq 0$ at every point $x \in (0,1)$

(b) $f''(x)=0$, for some $x \in (0,1)$

(c) $f''(0)=0$

(d) $f''(x)=0$, at every point $x \in (0,1)$

97. If $y^2 + \log_e(\cos^2 x) = y,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then : **[Sep. 03, 2020 (I)]**

(a) $y''(0)=0$ (b) $|y'(0)|+|y''(0)|=1$

(c) $|y''(0)|=2$ (d) $|y'(0)|+|y''(0)|=3$

98. If c is a point at which Rolle's theorem holds for the function, $f(x)=\log_e\left(\frac{x^2+a}{7x}\right)$ in the interval [3, 4], where $\alpha \in R$, then $f''(c)$ is equal to: **[Jan. 8, 2020 (I)]**

(a) $-\frac{1}{12}$ (b) $\frac{1}{12}$ (c) $-\frac{1}{24}$ (d) $\frac{\sqrt{3}}{7}$

99. Let $x^k + y^k = a^k$, $(a, k > 0)$ and $\frac{dy}{dx}+\left(\frac{y}{x}\right)^{\frac{1}{3}}=0$, then k is: **[Jan. 7, 2020 (I)]**

(a) $\frac{3}{2}$ (b) $\frac{4}{3}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

100. The value of c in the Lagrange's mean value theorem for the function $f(x) = x^3 - 4x^2 + 8x + 11$, when $x \in [0,1]$ is: **[Jan. 7, 2020 (II)]**

(a) $\frac{4-\sqrt{5}}{3}$ (b) $\frac{4-\sqrt{7}}{3}$

(c) $\frac{2}{3}$ (d) $\frac{\sqrt{7}-2}{3}$

101. If $2x = y^{\frac{1}{5}} + y^{-\frac{1}{5}}$ = and $(x^2-1)\frac{d^2y}{dx^2}+\lambda x\frac{dy}{dx}+ky=0$, then $\lambda + k$ is equal to : **[Online April 9, 2017]**

(a) –23 (b) –24 (c) 26 (d) –26

102. Let f be a polynomial function such that $f(3x) = f'(x) \cdot f''(x)$, for all $x \in \mathbf{R}$. Then : **[Online April 9, 2017]**

(a) $f(b) + f'(b) = 28$

(b) $f''(b) - f'(b) = 0$

(c) $f''(b) - f'(b) = 4$

(d) $f(b) - f'(b) + f''(b) = 10$

103. If $y = \left[x+\sqrt{x^2-1}\right]^{15} + \left[x-\sqrt{x^2-1}\right]^{15}$, then $(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}$ is equal to **[Online April 8, 2017]**

(a) $12y$ (b) $224y^2$ (c) $225y^2$ (d) $225y$

104. If Rolle's theorem holds for the function $f(x)\ 2x^3 + bx^2 + cx$, $x \in [-1, 1]$, at the point $x = \frac{1}{2}$, then $2b + c$ equals : **[Online April 10, 2015]**

(a) –3 (b) –1 (c) 2 (d) 1

105. If f and g are differentiable functions in [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0,1[$ **[2014]**

(a) $f'(c) = g'(c)$ (b) $f'(c) = 2g'(c)$

(c) $2f'(c) = g'(c)$ (d) $2f'(c) = 3g'(c)$

106. Let $f(x) = x|x|$, $g(x) = \sin x$ and $h(x) = (gof)(x)$. Then **[Online April 11, 2014]**

(a) h(x) is not differentiable at x = 0.

(b) h(x) is differentiable at x = 0, but h′(x) is not continuous at x = 0

(c) h′(x) is continuous at x = 0 but it is not differentiable at x = 0

(d) h′(x) is differentiable at x = 0

107. Let for i = 1, 2, 3, $p_i(x)$ be a polynomial of degree 2 in x, $p'_i(x)$ and $p''_i(x)$ be the first and second order derivatives of $p_i(x)$ respectively. Let,

$$A(x)=\begin{bmatrix} p_1(x) & p_1'(x) & p_1''(x) \\ p_2(x) & p_2'(x) & p_2''(x) \\ p_3(x) & p_3'(x) & p_3''(x) \end{bmatrix}$$

and $B(x) = [A(x)]^T A(x)$. Then determinant of B(x): **[Online April 11, 2014]**

(a) is a polynomial of degree 6 in x.

(b) is a polynomial of degree 3 in x.

(c) is a polynomial of degree 2 in x.

(d) does not depend on x.

108. If the Rolle's theorem holds for the function $f(x) = 2x^3 + ax^2 + bx$ in the interval [– 1, 1] for the point $c = \frac{1}{2}$, then the value of 2a + b is: **[Online April 9, 2014]**

(a) 1 (b) – 1 (c) 2 (d) – 2

109. If $f(x)=\sin(\sin x)$ and $f''(x)+\tan x\, f'(x)+g(x)=0$, then $g(x)$ is : **[Online April 23, 2013]**

(a) $\cos^2 x\cos(\sin x)$ (b) $\sin^2 x\cos(\cos x)$

(c) $\sin^2 x\sin(\cos x)$ (d) $\cos^2 x\sin(\sin x)$

110. Consider a quadratic equation $ax^2+bx+c=0$, where $2a+3b+6c=0$ and let $g(x)=a\frac{x^3}{3}+b\frac{x^2}{2}+cx$. **[Online May 19, 2012]**

Statement 1: The quadratic equation has at least one root in the interval (0, 1).

Statement 2: The Rolle's theorem is applicable to function $g(x)$ on the interval [0, 1].

(a) Statement 1 is false, Statement 2 is true.

(b) Statement 1 is true, Statement 2 is false.

(c) Statement 1 is true, Statement 2 is true, Statement 2 is not a correct explanation for Statement 1.

(d) Statement 1 is true, Statement 2 is true, , Statement 2 is a correct explanation for Statement 1.

111. $\frac{d^2x}{dy^2}$ equals : **[2011]**

(a) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (b) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$

(c) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ (d) $\left(\frac{d^2y}{dx^2}\right)^{-1}$

112. Let $f(x)=x|x|$ and $g(x)=\sin x$.

Statement-1 : gof is differentiable at $x=0$ and its derivative is continuous at that point.

Statement-2 : gof is twice differentiable at $x=0$. **[2009]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is false.

(c) Statement-1 is false, Statement-2 is true.

(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

113. A value of c for which conclusion of Mean Value Theorem holds for the function $f(x)=\log_e x$ on the interval [1, 3] is **[2007]**

(a) $\log_3 e$ (b) $\log_e 3$ (c) $2\log_3 e$ (d) $\frac{1}{2}\log_3 e$

114. Let f be differentiable for all x. If $f(1)=-2$ and $f'(x)\ge 2$ for $x\in[1,6]$, then **[2005]**

(a) $f(6)\ge 8$ (b) $f(6)<8$ (c) $f(6)<5$ (d) $f(6)=5$

115. If the equation $a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots\ldots + a_1 x = 0$ $a_1\neq 0, n\ge 2$, has a positive root $x=\alpha$, then the equation $na_n x^{n-1}+(n-1)a_{n-1}x^{n-2}+\ldots\ldots + a_1=0$ has a positive root, which is **[2005]**

(a) greater than α

(b) smaller than α

(c) greater than or equal to α

(d) equal to α

116. If $2a+3b+6c=0$, then at least one root of the equation $ax^2+bx+c=0$ lies in the interval **[2004]**

(a) $(1,3)$ (b) $(1,2)$ (c) $(2,3)$ (d) $(0,1)$

117. If $f(x)=x^n$, then the value of **[2003]**

$f(1)-\frac{f'(1)}{1!}+\frac{f''(1)}{2!}-\frac{f'''(1)}{3!}+\ldots\ldots\ldots\frac{(-1)^n f^n(1)}{n!}$ is

(a) 1 (b) 2^n (c) 2^n-1 (d) 0.

118. Let $f(a)=g(a)=k$ and their nth derivatives $f^n(a), g^n(a)$ exist and are not equal for some n. Further if

$$\lim_{x\to a}\frac{f(a)g(x)-f(a)-g(a)f(x)+f(a)}{g(x)-f(x)}=4$$

then the value of k is **[2003]**

(a) 0 (b) 4 (c) 2 (d) 1

119. If $y=(x+\sqrt{1+x^2})^n$, then $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}$ is **[2002]**

(a) n^2y (b) $-n^2y$ (c) $-y$ (d) $2x^2y$

120. If $2a+3b+6c=0$, $(a,b,c\in R)$ then the quadratic equation $ax^2+bx+c=0$ has **[2002]**

(a) at least one root in [0, 1]

(b) at least one root in [2, 3]

(c) at least one root in [4, 5]

(d) None of these

Hints & Solutions

1. **(8)** We know $[x]$ discontinuous for $x \in Z$

$f(x) = x\left[\frac{x}{2}\right]$ may be discontinuous where $\frac{x}{2}$ is an integer.

So, points of discontinuity are,

$x = \pm 2, \pm 4, \pm 6, \pm 8$ and 0

but at $x = 0$

$\lim_{x\to 0^+} f(x) = 0 = f(0) = \lim_{x\to 0^-} f(x)$

So, $f(x)$ will be discontinuous at $x = \pm 2, \pm 4, \pm 6$ and ± 8.

2. **(d)** Since, function $f(x)$ is continuous at $x = 1, 3$

$\therefore f(1) = f(1^+)$

$\Rightarrow ae + be^{-1} = c$ ...(i)

$f(3) = f(3^+)$

$\Rightarrow 9c = 9a + 6c \Rightarrow c = 3a$ ...(ii)

From (i) and (ii),

$b = ae(3-e)$ ...(iii)

$$f'(x) = \begin{cases} ae^x - be^{-x} & -1 < x < 1 \\ 2cx & 1 < x < 3 \\ 2ax + 2c & 3 < x < 4 \end{cases}$$

$f'(0) = a - b,\ f'(2) = 4c$

Given, $f'(0) + f'(2) = e$

$a - b + 4c = e$ ...(iv)

From eqs. (i), (ii), (iii) and (iv),

$a - 3ae + ae^2 + 12a = e$

$\Rightarrow 13a - 3ae + ae^2 = e$

$\Rightarrow a = \frac{e}{e^2 - 3e + 13}$

3. **(a)** $\lim_{x\to 0} x\left[\frac{4}{x}\right] = A \Rightarrow \lim_{x\to 0} x\left[\frac{4}{x} - \left\{\frac{4}{x}\right\}\right] = A$

$\Rightarrow \lim_{x\to 0} 4 - x\left\{\frac{4}{x}\right\} = A \Rightarrow 4 - 0 = \text{A}$

As, $f(x) = [x^2]\sin(\pi x)$ will be discontinuous at non-integers

And, when $x = \sqrt{A+1} \Rightarrow x = \sqrt{5}$,

which is not an integer.

Hence, $f(x)$ is discontinuous when x is equal to $\sqrt{A+1}$

4. **(5)** $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{1}{x}\ln\left(\frac{1+3x}{1-2x}\right)\right)$

$= \lim_{x\to 0}\left(\frac{\ln(1+3x)}{x} - \frac{\ln(1-2x)}{x}\right)$

$= \lim_{x\to 0}\left(\frac{3\ln(1+3x)}{3x} - \frac{2\ln(1-2x)}{-2x}\right)$

$= 3 + 2 = 5$

$\because$ $f(x)$ will be continuous

$\therefore$ k $= f(0) = \lim_{x\to 0} f(x) = 5$

5. **(b)** Since, f(x) is continuous, then

$\lim_{x\to \frac{\pi}{4}} f(x) = f\left(\frac{\pi}{4}\right)$

$\lim_{x\to \frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\cot x - 1} = k$

Now by L- hospital's rule

$\lim_{x\to \frac{\pi}{4}} \frac{\sqrt{2}\sin x}{\operatorname{cosec}^2 x} = k \Rightarrow \frac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)}{\left(\sqrt{2}\right)^2} = k \quad \Rightarrow k = \frac{1}{2}$

6. **(a)** L.H.L. $\lim_{x\to 4^-}\left([x] - \left[\frac{x}{4}\right]\right) = 3 - 0 = 3$

R.H.L. $\lim_{x\to 4^+}[x] - \left[\frac{x}{4}\right] = 4 - 1 = 3$

$f(4) = [4] - \left[\frac{4}{4}\right] = 4 - 1 = 3$

$\because$ LHL $= f(4) =$ RHL

$\therefore f(x)$ is continuous at $x = 4$

7. **(d)** R.H.L. $\lim_{x\to 5^+} b|(x-\pi)| + 3 = (5-\pi)b + 3$

$f(5) =$ L.H.L. $\lim_{x\to 5^-} a|(\pi - x)| + 1 = a(5-\pi) + 1$

$\because$ function is continuous at $x = 5$

$\therefore$ LHL = RHL

$(5-\pi)\,b + 3 = (5-\pi)\,a + 1$

$\Rightarrow 2 = (a-b)(5-\pi) \Rightarrow a - b = \frac{2}{5-\pi}$

8. **(c)** Given function is,

$$f(x)=\begin{cases}|x|+[x], & -1\le x<1\\ x+|x|, & 1\le x<2\\ x+[x], & 2\le x\le 3\end{cases}$$

$$=\begin{cases}-x-1, & -1\le x<0\\ x, & 0\le x<1\\ 2x, & 1\le x<2\\ x+2, & 2\le x<3\\ 6, & x=3\end{cases}$$

$\Rightarrow f(-1)=0, f(-1^+)=0;$

$f(0^-)=-1, f(0)=0, f(0^+)=0;$

$f(1^-)=1, f(1)=2, f(1^+)=2;$

$f(2^-)=4, f(2)=4, f(2^+)=4;$

$f(3^-)=5, f(3)=6$

$f(x)$ is discontinuous at $x=\{0, 1, 3\}$

Hence, $f(x)$ is discontinuous at only three points.

9. **(d)** Let $f(x)$ is continuous at $x=1$, then

$f(1^-)=f(1)=f(1^+)$

$\Rightarrow \quad 5=a+b \qquad ...(1)$

Let $f(x)$ is continuous at $x=3$, then

$f(3^-)=f(3)=f(3^+)$

$\Rightarrow \quad a+3b=b+15 \quad ...(2)$

Let $f(x)$ is continuous at $x=5$, then

$f(5^-)=f(5)=f(5^+)$

$\Rightarrow \quad b+25=30$

$\Rightarrow \quad b=30-25=5$

From (1), $a=0$

But $a=0$, $b=5$ do not satisfy equation (2)

Hence, $f(x)$ is not continuous for any values of a and b

10. **(a)** If the function is continuous at $x=0$, then

$\lim\limits_{x\to 0} f(x)$ will exist and $f(0)=\lim\limits_{x\to 0} f(x)$

Now, $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{k-1}{e^{2x}-1}\right)$

$$=\lim_{x\to 0}\left(\frac{e^{2x}-1-kx+x}{(x)(e^{2x}-1)}\right)$$

$$=\lim_{x\to 0}\left[\frac{\left(1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+...\right)-1-kx+x}{(x)\left(\left(1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+...\right)-1\right)}\right]$$

$$=\lim_{x\to 0}\left[\frac{(3-k)x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+...}{\left(2x^2+\frac{4x^3}{2!}+\frac{8x^3}{3!}+...\right)}\right]$$

For the limit to exist, power of x in the numerator should be greater than or equal to the power of x in the denominator. Therefore, coefficient of x in numerator is equal to zero

$\Rightarrow 3-k=0$

$\Rightarrow k=3$

So the limit reduces to

$$\lim_{x\to 0}\frac{(x^2)\left(\frac{4}{2!}+\frac{8x}{3!}+...\right)}{(x^2)\left(2+\frac{4x}{2!}+\frac{8x^2}{3!}+...\right)}$$

$$=\lim_{x\to 0}\frac{\frac{4}{2!}+\frac{8x}{3!}+...}{2+\frac{4x}{2!}+\frac{8x^2}{3!}+...}=1$$

Hence, $f(0)=1$

11. **(c)** Since $f(x)$ is continuous at $x=2$.

$\therefore \lim\limits_{x\to 2} f(x)=f(2)$

$\Rightarrow \lim\limits_{x\to 2}(x-1)^{\frac{1}{2-x}}=k \qquad (1^\infty \text{ form})$

$\therefore e^l=k$

where $l=\lim\limits_{x\to 2}(x-1-1)\times\dfrac{1}{2-x}=\lim\limits_{x\to 2}\dfrac{x-2}{2-x}$

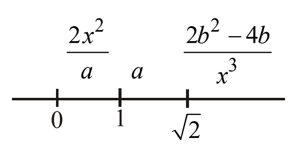
$=\lim\limits_{x\to 2}\left(\dfrac{x-2}{x-2}\right)$

$\Rightarrow k=e^{-1}$

12. **(c)** $\lim\limits_{x\to\pi/2} f(x)=f(\pi/2)$

$\Rightarrow \quad k+2/5=1 \Rightarrow k=1-\dfrac{2}{5} \quad \Rightarrow \quad k=\dfrac{3}{5}$

13. **(c)**



Continuity at $x=1$

$\dfrac{2}{a}=a \Rightarrow a=\pm\sqrt{2}$

Continuity at $x=\sqrt{2}$ $a=\sqrt{2}$

$a=\dfrac{2b^2-4b}{2\sqrt{2}}$

Put $a=\sqrt{2}$

$2=b^2-2b \quad \Rightarrow b^2-2b-2=0$

$b=\dfrac{2\pm\sqrt{4+4.2}}{2}=1\pm\sqrt{3}$

So, $(a,b)=(\sqrt{2},1-\sqrt{3})$

14. (c) Since f(x) is a continuous function therefore limit of f(x) at $x \to 0$ = value of f(x) at 0.

$$\therefore \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(e^x-1)^2}{\sin\left(\frac{x}{k}\right)\log\left(1+\frac{x}{4}\right)}$$

$$= \lim_{x\to 0} \frac{x^2\left(\frac{e^x-1}{x}\right)^2}{\frac{x}{R}\left[\frac{\sin\left(\frac{x}{R}\right)}{\frac{x}{R}}\right]\cdot\frac{\log\left(1+\frac{x}{4}\right)}{\left(\frac{x}{4}\right)}} \times \left(\frac{x}{4}\right)$$

$$= \lim_{x\to 0} \frac{x^2\left(\frac{e^x-1}{x}\right)^2 4k}{\frac{\sin\frac{x}{k}}{\frac{x}{k}}\cdot\frac{\log\left(1+\frac{x}{4}\right)}{\frac{x}{4}}}$$

on applying limit we get

$4k = 12 \Rightarrow k = 3$

15. (d) Since $f(x) = \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}$ is

Continuous at $x = \pi$

$\therefore$ L.H.L = R.H.L = f (π)

Let $(\pi - x) = \theta$, $\theta \to 0$ when $x \to \pi$

$$\therefore \lim_{\theta\to 0} \frac{\sqrt{2-\cos\theta}-1}{\theta^2}$$

$$= \lim_{\theta\to 0} \frac{(2-\cos\theta)-1}{\theta^2} \times \frac{1}{\sqrt{2-\cos\theta}+1}$$

$$= \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2}\cdot\frac{1}{2} \quad (\because \cos 0 = 1)$$

$$= \frac{1}{2}\lim_{\theta\to 0} \frac{2\sin^2\theta/2}{\theta^2} = \frac{2}{2}\lim_{\theta\to 0} \frac{\sin^2\theta/2}{\frac{\theta^2}{4}.4}$$

$$= \frac{1}{4}\left(\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

16. (b) Given that $f\left(\frac{9}{2}\right) = \frac{2}{9}$

$$\lim_{x\to 0} f\left(\frac{1-\cos 3x}{x^2}\right) = \lim_{x\to 0}\left(\frac{x^2}{1-\cos 3x}\right)$$

$$= \lim_{x\to 0}\left(\frac{x^2}{2\sin^2\frac{3x}{2}}\right) = \frac{1}{2}\lim_{x\to 0}\left(\frac{\frac{9}{4}.x^2.\frac{4}{9}}{\sin^2\frac{3x}{2}}\right)$$

$$= \frac{4}{9\times 2}\lim_{x\to 0}\left(\frac{1}{\frac{\sin^2\frac{3x}{2}}{\left(\frac{3x}{2}\right)^2}}\right)$$

$$= \frac{2}{9}\left[\frac{\lim_{x\to 0} 1}{\lim_{x\to 0}\frac{\sin^2\frac{3x}{2}}{\left(\frac{3x}{2}\right)^2}}\right] \quad \left\{\lim_{x\to 0}\frac{\sin x}{x} = 1\right\}$$

$$= \frac{2}{9}\cdot\left[\frac{1}{1}\right] = \frac{2}{9}$$

17. (a) Let $f(x) = [x] + |1-x|, \ -1 \le x \le 3$

where $[x]$ = greatest integer function.

f is not continuous at $x = 0, 1, 2, 3$

But in statement-2 $f(x)$ is continuous at $x = 3$.

Hence, statement-1 is true and 2 is false.

18. (b) $\mu(x) = \frac{1}{x-1}$, which is discontinous at $x = 1$

$$f(u) = \frac{1}{u^2+u-2} = \frac{1}{(u+2)(u-1)},$$

which is discontinous at $u = -2, 1$

when $u = -2$, then $\frac{1}{x-1} = -2 \Rightarrow x = \frac{1}{2}$

when $u = 1$, then $\frac{1}{x-1} = 1 \Rightarrow x = 2$

Hence given composite function is discontinous at three points, $x = 1, \frac{1}{2}$ and 2.

19. (d) $fog = f(g(x)) = f(1-|x|)$

$= -1 + \left|1-|x|-2\right|$

$= -1 + \left|-|x|-1\right| = -1 + \left||x|+1\right|$

Let $fog = y$

$\therefore y = -1 + \left||x|+1\right|$

$$\Rightarrow y = \begin{cases} -1+x+1, & x \ge 0 \\ -1-x+1, & x < 0 \end{cases}$$

$$\Rightarrow y = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$$

LHL at $(x = 0) = \lim_{x\to 0}(-x) = 0$

RHL at $(x = 0) = \lim_{x\to 0}(x) = 0$

When $x = 0$, then $y = 0$

Hence, LHL at $(x = 0)$ = RHL at $(x = 0)$
= value of y at $(x = 0)$

Hence y is continuous at $x = 0$.

Clearly at all other point y continuous. Therefore, the set of all points where fog is discontinuous is an empty set.

20. **(a)** Let $f(x)=[x]\cos\left(\frac{2x-1}{2}\right)$

We know that [x] is discontinuous at all integral points and cos x is continuous at $x\in R$.

So, check at $x=n, n\in I$

$$\text{L.H.L} = \lim_{x\to n^-}[x]\cos\left(\frac{2x-1}{2}\right)\pi$$

$$=(n-1)\cos\left(\frac{2n-1}{2}\right)\pi=0$$

($\because$ [x] is the greatest integer function)

$$\text{R.H.L} = \lim_{x\to n^+}[x]\cos\left(\frac{2x-1}{2}\right)\pi$$

$$=n\cos\left(\frac{2n-1}{2}\right)\pi=0$$

Now, value of the function at $x=n$ is

$f(n)=0$

Since, L.H.L = R.H.L. $=f(n)$

$\therefore f(x)=[x]\cos\left(\frac{2x-1}{2}\right)$ is continuous for every real x.

21. **(d)** Consider $\frac{x}{[x]}\le f(x)\le\sqrt{6-x}$

$$\Rightarrow \lim_{x\to 2^-}\frac{x}{[x]} = \frac{2}{1} = 2$$

$$\Rightarrow \lim_{x\to 2^-}\sqrt{6-x} = 2$$

$\therefore \lim_{x\to 2^-} f(x) = 2$ [By Sandwich theorem]

Now $\lim_{x\to 2^+}\frac{x}{[x]}=1,\ \lim_{x\to 2^+}\sqrt{6-x}=2$

Hence by Sandwich theorem $\lim_{x\to 2^+} f(x)$ does not exists.

Therefore f is not continuous at x = 2. Thus statement-1 is true but statement-2 is not true

22. **(d)** Statement - 1 is true.

It is the definition of continuity.

Statement - 2 is false.

23. **(c)** Given that $f(x)=\begin{cases}x\sin(1/x), & x\ne 0\\ 0, & x=0\end{cases}$

At $x=0$

$$\text{LHL} = \lim_{h\to 0^-}\left\{-h\sin\left(-\frac{1}{h}\right)\right\}$$

= 0 × a finite quantity between – 1 and 1 = 0

$$\text{RHL} = \lim_{h\to 0^+} h\sin\frac{1}{h}=0$$

Also, $f(0)=0$

Thus LHL = RHL = $f(0)$

$\therefore$ $f(x)$ is continuous on R.

but $f_2(x)$ is not continuous at $x=0$

24. **(b)** $\underset{(at\,x=0)}{L.H.L} = \lim_{x\to 0^-} f(x)$

$$=\lim_{h\to 0}\frac{\sin\{(p+1)(-h)\}-\sinh}{-h}=p+1+1=p+2$$

$\underset{(at\,x=0)}{R.H.L} = \lim_{x\to 0^+} f(x)$

$$=\lim_{h\to 0}\frac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}}\times\frac{\sqrt{x+x^2}+\sqrt{x}}{\sqrt{x+x^2}+\sqrt{x}} = \frac{1}{1+1}=\frac{1}{2}$$

$f(0)=2$

Given that f(x) is continuous at $x=0$

$\therefore p+2=q=\frac{1}{2}$

$\Rightarrow p=-\frac{3}{2}, q=\frac{1}{2}$

25. **(b)** Given, $f(x)=\frac{1}{x}-\frac{2}{e^{2x}-1}$ is continuous at $x=0$

$$\Rightarrow f(0)=\lim_{x\to 0}\frac{1}{x}-\frac{2}{e^{2x}-1}$$

$$=\lim_{x\to 0}\frac{(e^{2x}-1)-2x}{x(e^{2x}-1)};\ \left[\frac{0}{0}\text{ form}\right]$$

$\therefore$ Applying, L'Hospital rule

Differentiate two times, we get

$$f(0)=\lim_{x\to 0}\frac{4e^{2x}}{2(xe^{2x}2+e^{2x}.1)+e^{2x}.2}$$

$$=\lim_{x\to 0}\frac{4e^{2x}}{4xe^{2x}+2e^{2x}+2e^{2x}}\quad\left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to 0}\frac{4e^{2x}}{4(xe^{2x}+e^{2x})}=\frac{4.e^0}{4(0+e^0)}=1$$

26. **(c)** Given that $f(x)=\frac{1-\tan x}{4x-\pi}$ is continuous in $\left[0,\frac{\pi}{2}\right]$

$$\therefore f\left(\frac{\pi}{4}\right)=\lim_{x\to\frac{\pi}{4}^-}f(x)=\lim_{x\to\frac{\pi}{4}^+}f(x)$$

$$\lim_{x\to\frac{\pi}{4}^-}f(x)=\lim_{h\to 0}f\left(\frac{\pi}{4}+h\right)$$

$$=\lim_{h\to 0}\frac{1-\tan\left(\frac{\pi}{4}+h\right)}{4\left(\frac{\pi}{4}+h\right)-\pi}, h>0 = \lim_{h\to 0}\frac{1-\frac{1+\tan h}{1-\tan h}}{4h}$$

$$=\lim_{h\to 0}\frac{-2}{1-\tan h}.\frac{\tan h}{4h}=\frac{-2}{4}=-\frac{1}{2}\quad\left[\because \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1\right]$$

27. **(b)** Let a is a rational number other than 0, in [–5, 5], then $f(a)=a$ and $\lim_{x\to a}f(x)=-a$

$\therefore x\to a^-$ and $x\to a^+$ is tends to irrational number

$\therefore f(x)$ is discontinuous at any rational number

If a is irrational number, then

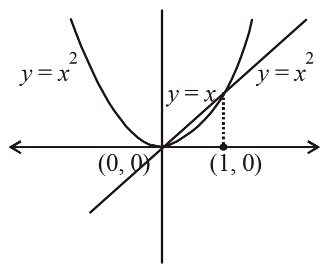
$f(a) = -a$ and $\lim_{x \to a} f(x) = a$

$\therefore f(x)$ is not continuous at any irrational number. For $x = 0$,

$\lim_{x \to 0} f(x) = f(0) = 0$

$\therefore f(x)$ is continuous at $x = 0$

28. **(a)**



$f(x) = \max.\{x, x^2\}$

$$\Rightarrow f(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \le x < 1 \\ x^2, & x \ge 1 \end{cases}$$

$\therefore f(x)$ is not differentiable at $x = 0, 1$

29. **(a)** $f(x)$ is differentiable then, $f(x)$ is also continuous.

$\therefore \lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^-} f(x) = f(\pi)$

$\Rightarrow -1 = -K_2 \Rightarrow K_2 = 1$

$$\therefore f'(x) = \begin{cases} 2K_1(x - \pi) & : \; x \le \pi \\ -K_2 \sin x & \;\; x > \pi \end{cases}$$

Then, $\lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^-} f(x) = 0$

$$f''(x) = \begin{cases} 2K_1 & ; \; x \le \pi \\ -K_2 \cos x & ; \; x > \pi \end{cases}$$

Then, $\lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^-} f(x)$

$\Rightarrow 2K_1 = K_2 \Rightarrow K_1 = \frac{1}{2}$

So, $(K_1, K_2) = \left(\frac{1}{2}, 1\right)$

30. **(b)** Let f be twice differentiable function

$\because f'(x) \ge 1$

$\Rightarrow \frac{f(5) - f(2)}{3} \ge 1$

$\Rightarrow f(5) \ge 3 + f(2)$

$\Rightarrow f(5) \ge 3 + 8 \Rightarrow f(5) \ge 11$

and also $f''(x) \ge 4$

$\Rightarrow \frac{f'(5) - f'(2)}{5 - 2} \ge 4 \Rightarrow f'(5) \ge 12 + f'(2)$

$\Rightarrow f'(5) \ge 17$

Hence, $f(5) + f'(5) \ge 28$

31. **(10.00)**

$f(x + y) = f(x) + f(y) + xy^2 + x^2y$

Differentiate w.r.t. x:

$f'(x + y) = f'(x) + 0 + y^2 + 2xy$

Put $y = -x$

$f'(0) = f'(x) + x^2 - 2x^2$ ...(i)

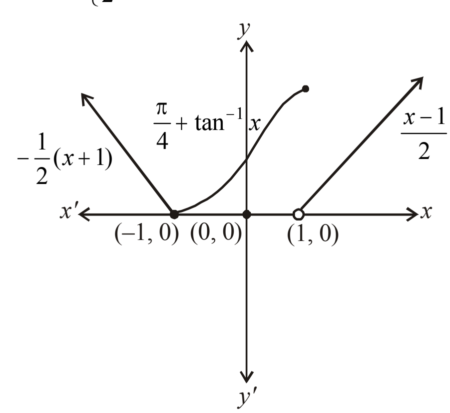
$\because \lim_{x \to 0} \frac{f(x)}{x} = 1 \Rightarrow f(0) = 0$

$\therefore f'(0) = 1$ ...(ii)

From equations (i) and (ii),

$f'(x) = (x^2 + 1) \Rightarrow f'(3) = 10.$

32. **(a)** $$f(x) = \begin{cases} \frac{-x - 1}{2}, & x < -1 \\ \frac{\pi}{4} + \tan^{-1} x, & -1 \le x \le 1 \\ \frac{1}{2}(x - 1), & x > 1 \end{cases}$$



It is clear from above graph that,
$f(x)$ is discontinuous at $x = 1$.
i.e. continuous on $R - \{1\}$
$f(x)$ is non-differentiable at $x = -1, 1$
i.e. differentiable on $R - \{-1, 1\}$.

33. **(c)** LHL $= \lim_{x \to 0} \frac{\sin(a + 2)x + \sin x}{x}$

$= \lim_{x \to 0} \left(\frac{\sin(a + 2)x}{(a + 2)x}\right)(a + 2) + \lim_{x \to 0} \frac{\sin x}{x} = a + 3$

$f(0)=b$

$$\text{RHL} = \lim_{h\to 0}\left(\frac{(1+3h)^{\frac{1}{3}}-1}{h}\right)=1$$

$\because$ Function $f(x)$ is continuous

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\therefore\ a+3=1 \Rightarrow a=-2$

and $b=1$

Hence, $a+2b=0$

34. **(a)** It is given that functions f and g are differentiable and fog is identity function.

$\therefore\ (fog)(x)=x \Rightarrow f(g(x))=x$

Differentiating both sides, we get

$f'(g(x))\cdot g'(x)=1$

Now, put $x=a$, then

$f'(g(a))\cdot g'(a)=1$

$f'(b)\cdot 5=1$

$f'(b)=\frac{1}{5}$

35. **(Bonus)** For a constant function $f(x)$, option (1), (3) and (4) doesn't hold and by LMVT theorem, option (2) is incorrect.

36. **(a)** From, LMVT for $x\in[-7,-1]$

$$\frac{f(-1)-f(-7)}{(-1+7)}\le 2 \Rightarrow \frac{f(-1)+3}{6}\le 2 \Rightarrow f(-1)\le 9$$

From, LMVT for $x\in[-7,0]$

$$\frac{f(0)-f(-7)}{(0+7)}\le 2$$

$$\frac{f(0)+3}{7}\le 2 \Rightarrow f(0)\le 11$$

$\therefore\ f(0)+f(-1)\le 20$

37. **(c)** $\because$ $f(x)$ is non differentiable at $x=1,3,5$

$[\because |x-3|$ is not differentiable at $x=3]$

$\Sigma f(f(x)) = f(f(1))+f(f(3))+f(f(5))$

$=1+1+1=3$

38. **(c)** $$f(x)=\begin{cases}\dfrac{\sin(p+1)x+\sin x}{x} & x<0\\ q & x=0\\ \dfrac{\sqrt{x^2+x}-\sqrt{x}}{x^{\frac{3}{2}}} & x>0\end{cases}$$ is continuous at $x=0$

Therefore, $f(0^-)=f(0)=f(0^+)$ ...(1)

$$f(0^-)=\lim_{h\to 0} f(0-h) = \lim_{h\to 0}\frac{\sin(p+1)(-h)+\sin(-h)}{-h}$$

$$=\lim_{h\to 0}\left[\frac{-\sin(p+1)h}{-h}+\frac{\sin h}{h}\right]$$

$$=\lim_{h\to 0}\frac{\sin(p+1)h}{h(p+1)}\times(p+1)+\lim_{h\to 0}\frac{\sin h}{h}$$

$=(p+1)+1=p+2$ ...(2)

And $$f(0^+)=\lim_{h\to 0} f(0+h) = \frac{\sqrt{h^2+h}-\sqrt{h}}{h^{3/2}}$$

$$=\lim_{h\to 0}\frac{(h)^{\frac{1}{2}}\left[\sqrt{h+1}-1\right]}{h\left(h^{\frac{1}{2}}\right)}$$

$$=\lim_{h\to 0}\frac{\sqrt{h+1}-1}{h}\times\frac{\sqrt{h+1}+1}{\sqrt{h+1}+1}=\lim_{h\to 0}\frac{h+1-1}{h(\sqrt{h+1}+1)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{h+1}+1}=\frac{1}{1+1}=\frac{1}{2}$$ ...(3)

Now, from equation (1),

$f(0^-)=f(0)=f(0^+)\Rightarrow p+2=q=\frac{1}{2}$

$\Rightarrow q=\frac{1}{2}$ and $p=\frac{-3}{2}$ $\quad\therefore (p,q)=\left(-\frac{3}{2},\frac{1}{2}\right)$

39. **(b)** $f(x)=\ln(\sin x)$, $g(x)=\sin^{-1}(e^{-x})$

$\Rightarrow f(g(x))=\ln(\sin(\sin^{-1}e^{-x}))=-x$

$\Rightarrow f(g(x))=-\alpha$

But given that $(fog)(\alpha)=b$

$\therefore\ -\alpha=b$ and $f'(g(\alpha))=a$, i.e., $a=-1$

$\therefore\ a\alpha^2-b\alpha-a=-\alpha^2+\alpha^2-(-1)$

$\Rightarrow a\alpha^2-b\alpha-a=1$.

40. **(c)** $$g'(c)=\lim_{x\to c}\frac{g(x)-g(c)}{x-c}$$

$$\Rightarrow g'(c)=\lim_{x\to c}\frac{|f(x)|-|f(c)|}{x-c}$$

Since, $f(c)=0$

Then, $$g'(c)=\lim_{x\to c}\frac{|f(x)|}{x-c}$$

$$\Rightarrow g'(c)=\lim_{x\to c}\frac{f(x)}{x-c};\ \text{if } f(x)>0$$

and $$g'(c)=\lim_{x\to c}\frac{-f(x)}{x-c};\ \text{if } f(x)<0$$

$\Rightarrow g'(c)=f'(c)=-f'(c)$

$\Rightarrow 2f'(c)=0\Rightarrow f'(c)=0$

Hence, $g(x)$ is differentiable if $f'(c)=0$

41. **(a)** Since, $f(x) = 15 - |(10 - x)|$

$\therefore g(x) = f(f(x)) = 15 - |10 - [15 - |10 - x|]|$

$= 15 - ||10 - x| - 5|$

$\therefore$ Then, the points where function g(x) is Non-differentiable are

$10 - x = 0$ and $|10 - x| = 5$

$\Rightarrow x = 10$ and $x - 10 = \pm 5$

$\Rightarrow x = 10$ and $x = 15, 5$

42. **(a)** Let $g(x) = f(f(f(x))) + (f(x))^2$

Differentiating both sides w.r.t. x, we get

$g'(x) = f'(f(f(x)))f'(f(x))f'(x) + 2f(x)f'(x)$

$g'(1) = f'(f(f(1)))f'(f(1))f'(1) + 2f(1)f'(1)$

$= f'(f(1))f'(1)f'(1) + 2f(1)f'(1)$

$= 3 \times 3 \times 3 + 2 \times 1 \times 3 = 27 + 6 = 33$

43. **(b)** Since, $f'(x) = f(x)$

Then, $\dfrac{f'(x)}{f(x)} = 1$

$\Rightarrow \dfrac{f'(x)}{f(x)} = dx \Rightarrow \dfrac{f'(x)}{f(x)}\, dx = \int dx$

$\Rightarrow \ln|f(x)| = x + c$

$f(x) = \pm e^{x+c}$ ...(1)

Since, the given condition

$f(1) = 2$

From eq^n (1) $f(x) = e^{x+c} = e^c e^x$

Then, $f(1) = e^c \cdot e^1$

$\Rightarrow 2 = e^c \cdot e$

$\Rightarrow \dfrac{2}{e} = e^c$

Then, from eq^n (1)

$f(x) = \dfrac{2}{e}e^x$

$\Rightarrow f'(x) = \dfrac{2}{e}e^x$

Now $h(x) = f(f(x))$

$\Rightarrow h'(x) = f'(f(x)) \cdot f'(x)$

$h'(1) = f'(2) \cdot f'(1) = \dfrac{2}{e}e^2 \cdot \dfrac{2}{e} \cdot e = 4e$

44. **(d)** $f(x) = \begin{cases} -1, & -2 \le x < 0 \\ x^2 - 1, & 0 \le x \le 2 \end{cases}$

Then, $f(|x|) = \begin{cases} -1, & -2 \le |x| < 0 \\ |x|^2 - 1, & 0 \le |x| \le 2 \end{cases}$

$\Rightarrow f(|x|) = x^2 - 1, -2 \le x \le 2$

$\Rightarrow g(x) = \begin{cases} 1 + x^2 - 1, & -2 \le x < 0 \\ (x^2 - 1) + |x^2 - 1|, & 0 \le x \le 2 \end{cases}$

$= \begin{cases} x^2, & -2 \le x < 0 \\ 0, & 0 \le x < 1 \\ 2(x^2 - 1), & 1 \le x \le 2 \end{cases}$

$g'(0^-) = 0, g'(0^+) = 0, g'(1^-) = 0, g'(1^+) = 4$

$\Rightarrow$ $g(x)$ is non-differentiable at $x = 1$

$\Rightarrow$ $g(x)$ is not differentiable at one point.

45. **(b)** Consider the equation,

$x \log_e(\log_e x) - x^2 + y^2 = 4$

Differentiate both sides w.r.t. x,

$\log_e(\log_e x) + x \cdot \dfrac{1}{x \cdot \log_e x} - 2x + 2y\dfrac{dy}{dx} = 0$

$\log_e(\log_e x) + \dfrac{1}{\log_e x} - 2x + 2y\dfrac{dy}{dx} = 0$ ...(1)

When $x = e, y = \sqrt{4 + e^2}$. Put these values in (1),

$0 + 1 - 2e + 2\sqrt{4 + e^2}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{2e - 1}{2\sqrt{4 + e^2}}$.

46. **(a)** $f(x) = \sin|x| - |x| + 2(x - \pi)\cos|x|$

There are two cases,

Case (1), $x > 0$

$f(x) = \sin x - x + 2(x - \pi)\cos x$

$f'(x) = \cos x - 1 + 2(1 - 0)\cos x - 2\sin(x - \pi)$

$f'(x) = 3\cos x - 2(x - \pi)\sin x - 1$

Then, function $f(x)$ is differentiable for all $x > 0$

Case (2) $x < 0$

$f(x) = -\sin x + x + 2(x - \pi)\cos x$

$f'(x) = -\cos x + 1 - 2(x - \pi)\sin x + 2\cos x$

$f'(x) = \cos x + 1 - 2(x - \pi)\sin x$

Then, function $f(x)$ is differentiable for all $x < 0$

Now check for $x = 0$

$f'(0^+)$ R.H.D. $= 3 - 1 = 2$

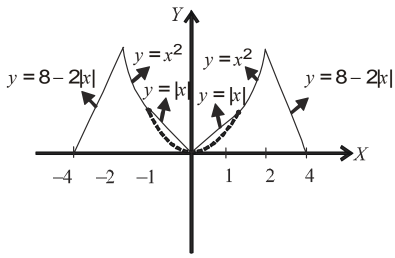
$f'(0^-)$ L.H.D. $= 1 + 1 = 2$

L.H.D. = R.H.D.

Then, function $f(x)$ is differentiable for $x = 0$. So it is differentiable everywhere

Hence, $k = \phi$

47. **(b)** Given $f(x)=\begin{cases}\max\{|x|,x^2\} & |x|\le 2\\ 8-2|x| & 2<|x|\le 4\end{cases}$



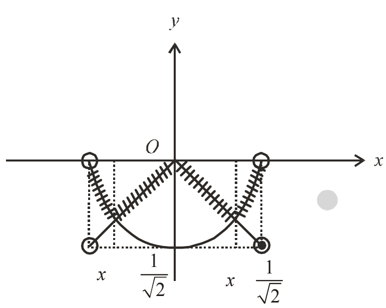
$\because f(x)$ is not differentiable at $-2, -1, 0, 1$ and 2.

$\therefore S=\{-2,-1,0,1,2\}$

48. **(c)** Consider the function

$f(x)=\max\left\{-|x|,-\sqrt{1-x^2}\right\}$

Now, the graph of the function



From the graph, it is clear that $f(x)$ is not differentiable at $x=0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}$

Then, $K=\left\{-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right\}$

Hence, K has exactly three elements.

49. **(d)** $f(x)=|x-\pi|(e^{|x|}-1)\sin|x|$

Check differentiability of f(x) at $x=\pi$ and $x=0$

at $x=\pi$:

$$\text{R.H.D.}=\lim_{h\to 0}\frac{|\pi+h-\pi|(e^{|x+h|}-1)\sin|\pi+h|-0}{h}$$

$$\text{L.H.D}=\lim_{h\to 0}\frac{|\pi-h-\pi|(e^{|\pi-h|}-1)\sin|\pi-h|-0}{-h}=0$$

$\because$ RHD = LHD

Therefore, function is differentiable at $x=\pi$

at $x=0$:

$$\text{R.H.D.}=\lim_{h\to 0}\frac{|h-\pi|(e^{|h|}-1)\sin|h|-0}{h}=0$$

$$\text{L.H.D.}=\lim_{h\to 0}\frac{|-h-\pi|(e^{|-h|}-1)\sin|-h|-0}{-h}=0$$

$\therefore$ RHD = LHD

Therefore, function is differentiable.

at $x=0$.

Since, the function f(x) is differentiable at all the points including π and 0.

i.e., f(x) is every where differentiable .

Therefore, there is no element in the set S.

$\Rightarrow S=\phi$ (an empty set)

50. **(a)** $S=\{(\lambda,\mu)\in R\times R: f(t)=(|\lambda|e^{|t|}-\mu)\sin(2|t|),\ t\in R$

$f(t)=(|\lambda|e^{|t|}-\mu)\sin(2|t|)$

$$=\begin{cases}(|\lambda|e^{t}-\mu)\sin 2t, & t>0\\ (|\lambda|e^{-t}-\mu)(-\sin 2t), & t<0\end{cases}$$

$$f'(t)=\begin{cases}(|\lambda|e^{t})\sin 2t+(|\lambda|e^{t}-\mu)(2\cos 2t), & t>0\\ |\lambda|e^{-t}\sin 2t+(|\lambda|e^{-t}-\mu)(-2\cos 2t), & t<0\end{cases}$$

As, $f(t)$ is differentiable

$\therefore$ LHD = RHD at $t=0$

$\Rightarrow\ |\lambda|.\sin 2(0)+(|\lambda|e^{0}-\mu)2\cos(0)$

$=|\lambda|e^{-0}.\sin 2(0)-2\cos(0)(|\lambda|e^{-0}-\mu)$

$\Rightarrow\ 0+(|\lambda|-\mu)2=0-2(|\lambda|-\mu)$

$\Rightarrow\ 4(|\lambda|-\mu)=0$

$\Rightarrow\ |\lambda|=\mu$

So, $S\equiv(\lambda,\mu)=\{\lambda\in R\ \&\ \mu\in[0,\infty)\}$

Therefore set S is subset of $R\times[0,\infty)$

51. **(a)** $f(x)=\begin{cases}-x & x<1\\ a+\cos^{-1}(x+b) & 1\le x\le 2\end{cases}$

$f(x)$ is continuous

$\Rightarrow\ \lim_{x\to 1^-}f(x)=\lim_{x\to 1^+}a+\cos^{-1}(x+b)=f(x)$

$\Rightarrow\ -1=a+\cos^{-1}(1+b)$

$\cos^{-1}(1+b)=-1-a$ .....(a)

$f(x)$ is differentiate

$\Rightarrow$ LHD = RHD

$$\Rightarrow\ -1=\frac{-1}{\sqrt{1-(1+b)^2}}$$

$\Rightarrow\ 1-(1+b)^2=1\ \Rightarrow\ b=-1$ .....(b)

From (a) $\Rightarrow\cos^{-1}(0)=-1-a$

$\therefore\ -1-a=\frac{\pi}{2}$

$a=-1-\frac{\pi}{2}\ \Rightarrow\ a=\frac{-\pi-2}{2}$ .....(c)

$\therefore\ \frac{a}{b}=\frac{\pi+2}{2}$

52. **(c)** Since g (x) is differentiable, it will be continuous at x = 3

$\therefore \lim_{x \to 3^-} g(x) = \lim_{x \to 3^+} g(x)$

$2k = 3m + 2$ ...(1)

Also g(x) is differentiable at x = 0

$\therefore \lim_{x \to 3^-} g'(x) = \lim_{x \to 3^+} g'(x)$

$$\frac{k}{2\sqrt{3+1}} = m$$

$k = 4m$ ...(2)

Solving (1) and (2), we get

$$m = \frac{2}{5},\ k = \frac{8}{5}$$

$k + m = 2$

53. **(b)** Let $|f(x)| \le x^2,\ \forall x \in R$

Now, at $x = 0$, $|f(0)| \le 0$

$\Rightarrow f(0) = 0$

$$\therefore f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h - 0} = \lim_{h \to 0} \frac{f(h)}{h} \quad ...(1)$$

Now, $\left|\frac{f(h)}{h}\right| \le |h| \quad (\because |f(x)| \le x^2)$

$$\Rightarrow -|h| \le \frac{f(h)}{h} \le |h|$$

$$\Rightarrow \lim_{h \to 0} \frac{f(h)}{h} \to 0 \quad ...(2)$$

(using sandwich Theorem)

$\therefore$ from (1) and (2), we get $f'(0) = 0$,

i.e. $-f(x)$ is differentiable, at $x = 0$

Since, differentiability $\Rightarrow$ Continutity

$\therefore |f(x)| \le x^2$, for all $x \in R$ is continuous as well as differentiable at $x = 0$.

54. **(b)** $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$

and $g(x) = x\, f(x)$

For $f(x)$

$$\text{LHL} = \lim_{h \to 0^-} \left\{-h\sin\left(-\frac{1}{h}\right)\right\}$$

$= 0 \times$ a finite quantity between -1 and $1 = 0$

$$\text{RHL} = \lim_{h \to 0^+} h\sin\frac{1}{h} = 0$$

Also, $f(0) = 0$

Thus LHL = RHL = $f(0)$

$\therefore f(x)$ is continuous at $x = 0$

$$g(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

For $g(x)$

$$\text{LHL} = \lim_{h \to 0^-} \left\{-h^2\sin\left(\frac{1}{h}\right)\right\}$$

$= 0^2 \times$ a finite quantity between -1 and $1 = 0$

$$\text{RHL} = \lim_{h \to 0^+} h^2\sin\left(\frac{1}{h}\right) = 0$$

Also $g(0) = 0$

$\therefore g(x)$ is continuous at $x = 0$

55. **(c)** $f(x) = |x-2| = \begin{cases} x-2, & x-2 \ge 0 \\ 2-x, & x-2 \le 0 \end{cases}$

$$= \begin{cases} x-2, & x \ge 2 \\ 2-x, & x \le 2 \end{cases}$$

Similarly,

$$f(x) = |x-5| = \begin{cases} x-5, & x \ge 5 \\ 5-x, & x \le 5 \end{cases}$$

$\therefore \quad f(x) = |x-2| + |x-5|$

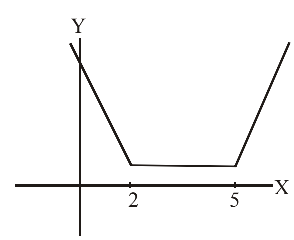
$$= \{x - 2 + 5 - x = 3,\ 2 \le x \le 5\}$$

Thus $f(x) = 3,\ 2 \le x \le 5$

$f'(x) = 0,\ 2 < x < 5$

$f'(4) = 0$

$\because$ Statement-1 is true



Since $f(x) = 3,\ 2 \le x \le 5$ is constant function.

So, it continuous in 2, 5 and differentiable in (2, 5)

$\because f(2) = 0 + |2-5| = 3$

and $f(5) = |5-2| + 0 = 3$ statement-2 is also true.

56. **(d)** $|\sin x|$ and $e^{|x|}$ are not differentiable at $x = 0$ and $|x|^3$ is differentiable at $x = 0$.

$\therefore$ for $f(x)$ to be differentiable at $x = 0$, we must have $a = 0$, $b = 0$ and c is any real number.

57. **(b)** Given $x + |y| = 2y$

$\Rightarrow \quad x + y = 2y$ or $x - y = 2y$

$\Rightarrow \quad x = y$ or $x = 3y$

This represent a straight line which passes through origin.
Hence, $x+|y|=2y$ is continuous at $x=0$.
Now, we check differentiability at $x=0$

$x+|y|=2y \Rightarrow x+y=2y, y\geq 0$

$x-y=2y,\ \ y<0$

Thus, $f(x)=\begin{cases} x, & y<0 \\ x/3, & y\geq 0 \end{cases}$

Now, L.H.D. $= \lim_{h\to 0^-} \dfrac{f(x+h)-f(x)}{-h}$

$= \lim_{h\to 0^-} \dfrac{x+h-x}{-h} = -1$

R.H.D $= \lim_{h\to 0^+} \dfrac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0^+} \dfrac{\frac{x+h}{3}-\frac{x}{3}}{h} = \lim_{h\to 0^+} \dfrac{1}{3} = \dfrac{1}{3}$

Since, L.H.D $\neq$ R.H.D. at $x=0$

$\therefore$ given function is not differentiable at $x=0$

58. **(c)** $\lim_{x\to a} \dfrac{x^2 f(a)-a^2 f(x)}{x-a}$

Applying L-Hospital rule

$= \lim_{x\to a} \dfrac{2xf(a)-a^2 f'(x)}{1} = 2af(a)-a^2 f'(a)$

59. **(c)** Given that, $f(x)=\begin{cases} (x-1)\sin\left(\dfrac{1}{x-1}\right), & \text{if } x\neq 1 \\ 0 & \text{, if } x=1 \end{cases}$

At $x=1$

R.H.D. $= \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0} \dfrac{h\sin\frac{1}{h}-0}{h} = \lim_{h\to 0} \sin\dfrac{1}{h}$ = a finite number

Let this finite number be l

L.H.D. $= \lim_{h\to 0} \dfrac{f(1-h)-f(1)}{-h}$

$= \lim_{h\to 0} \dfrac{-h\sin\left(\frac{1}{-h}\right)}{-h} = \lim_{h\to 0} \sin\left(\dfrac{1}{-h}\right)$

$= -\lim_{h\to 0} \sin\left(\dfrac{1}{h}\right) = -(\text{a finite number}) = -l$

Thus R.H.D $\neq$ L.H.D

$\therefore$ f is not differentiable at $x=1$

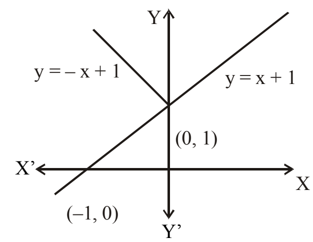
At $x=0$ $\quad f'(0)=\left.\sin\dfrac{1}{(x-1)}-\dfrac{x-1}{(x-1)^2}\cos\left(\dfrac{1}{x-1}\right)\right]_{x=0}$

$=-\sin 1+\cos 1$

$\therefore$ f is differentiable at $x=0$

60. **(a)** $f(x)=\min\{x+1, |x|+1\}$

$\Rightarrow f(x)=x+1\ \forall\ x\in R$



Since $f(x)=x+1$ is polynomial function
Hence, $f(x)$ is differentiable everywhere for all $x\in R$.

61. **(c)** $f(x)=\begin{cases} \dfrac{x}{1-x}, & x<0 \\ \dfrac{x}{1+x}, & x\geq 0 \end{cases}$

$f(x)=\dfrac{x}{1-x}$ is not define at $x\neq 1$ but here $x<0$ and $f(x)$ $=\dfrac{x}{1+x}$ is not define at $x=-1$ but here $x>0$. So, $f(x)$ is continuous for $x\in R$.

and $f'(x)=\begin{cases} \dfrac{x}{(1-x)^2}, & x<0 \\ \dfrac{x}{(1+x)^2}, & x\geq 0 \end{cases}$

$\therefore$ $f'(x)$ exist at everywhere.

62. **(b)** Given that $|f(x)-f(y)|\leq (x-y)^2$, $x, y\in R$ ...(i) and $f(0)=0$

$f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$|f'(x)| = \lim_{h\to 0}\left|\dfrac{f(x+h)-f(x)}{h}\right| \leq \lim_{h\to 0}\left|\dfrac{(h)^2}{h}\right|$

$\Rightarrow |f'(x)|\leq 0 \Rightarrow f'(x)=0$

$\Rightarrow f(x)=\text{constant}$

As $f(0)=0$

$\Rightarrow f(1)=0.$

63. **(c)** $f'(1) = \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$;

Given that function is differentiable so it is continuous also and $\lim_{h\to 0}\frac{f(1+h)}{h}=5$ and hence $f(1)=0$

Hence, $f'(1)=\lim_{h\to 0}\frac{f(1+h)}{h}=5$

64. **(c)** Given that $f(0)=0$; $f(x)=xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$

R.H.L. $=\lim_{h\to 0}(0+h)e^{-2/h}=\lim_{h\to 0}\frac{h}{e^{2/h}}=0$

L.H.L. $=\lim_{h\to 0}(0-h)e^{-\left(\frac{1}{h}-\frac{1}{h}\right)}=0$

therefore, $f(x)$ is continuous at $x=0$.

Now, R.H.D $=\lim_{h\to 0}\frac{(0+h)e^{-\left(\frac{1}{h}+\frac{1}{h}\right)}-0}{h}=0$

L.H.D. $=\lim_{h\to 0}\frac{(0-h)e^{-\left(\frac{1}{h}-\frac{1}{h}\right)}-0}{-h}=1$

therefore, L.H.D. ≠ R.H.D.

$f(x)$ is not differentiable at $x=0$.

65. **(d)** Let $u=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x=\tan\theta\Rightarrow\theta=\tan^{-1}x$

$\therefore u=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x$

$\therefore \frac{du}{dx}=\frac{1}{2}\times\frac{1}{(1+x^2)}$

Let $v=\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$

Put $x=\sin\phi\Rightarrow\phi=\sin^{-1}x$

$v=\tan^{-1}\left(\frac{2\sin\phi\cos\phi}{\cos 2\phi}\right)=\tan^{-1}(\tan 2\phi)$

$=2\phi=2\sin^{-1}x$

$\frac{dv}{dx}=2\frac{1}{\sqrt{1-x^2}}$

$\frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{\sqrt{1-x^2}}{4(1+x^2)}$

$\therefore\left(\frac{du}{dv}\right)_{\left(x=\frac{1}{2}\right)}=\frac{\sqrt{3}}{10}$

66. **(c)** $(a+\sqrt{2}b\cos x)(a-\sqrt{2}b\cos y)=a^2-b^2$

Differentiating both sides,

$(-\sqrt{2}b\sin x)(a-\sqrt{2}b\cos y)+(a+\sqrt{2}b\cos x)$

$(\sqrt{2}b\sin y)\frac{dy}{dx}=0$

$\Rightarrow\frac{dy}{dx}=\frac{(\sqrt{2}b\sin x)(a-\sqrt{2}b\cos y)}{(a+\sqrt{2}b\cos x)(\sqrt{2}b\sin y)}$

$\therefore\left[\frac{dy}{dx}\right]_{\left(\frac{\pi}{4},\frac{\pi}{4}\right)}=\frac{a-b}{a+b}\Rightarrow\frac{dx}{dy}=\frac{a+b}{a-b}$

67. **(91)** $y=\sum_{k=1}^{6}k\cos^{-1}\left\{\frac{3}{5}\cos kx-\frac{4}{5}\sin kx\right\}$

Let $\cos a=\frac{3}{5}$ and $\sin a=\frac{4}{5}$

$\therefore y=\sum_{k=1}^{6}k\cos^{-1}\{\cos a\cos kx-\sin a\sin kx\}$

$=\sum_{k=1}^{6}k\cos^{-1}(\cos(kx+a))$

$=\sum_{k=1}^{6}k(kx+a)=\sum_{k=1}^{6}(k^2x+ak)$

$\therefore\frac{dy}{dx}=\sum_{k=1}^{6}k^2=\frac{6(7)(13)}{6}=91.$

68. **(Bonus)** It is given that

$x=2\sin\theta-\sin 2\theta$ ...(i)

$y=2\cos\theta-\cos 2\theta$ ...(ii)

Differentiating (i) w.r.t. θ, we get

$\frac{dx}{d\theta}=2\cos\theta-2\cos 2\theta$

Differentiating (ii) w.r.t. θ; we get

$\frac{dy}{d\theta}=-2\sin\theta+2\sin 2\theta$

From (ii) ÷ (i), we get

$\therefore\quad\frac{dy}{dx}=\frac{\sin 2\theta-\sin\theta}{\cos\theta-\cos 2\theta}$

$=\frac{2\sin\frac{\theta}{2}.\cos\frac{3\theta}{2}}{2\sin\frac{\theta}{2}.\sin\frac{3\theta}{2}}=\cot\frac{3\theta}{2}$ ...(iii)

Again, differentiating eqn. (iii), we get

$\frac{d^2y}{dx^2}=\frac{-3}{2}\text{cosec}^2\frac{3\theta}{2}.\frac{d\theta}{dx}$

$$\frac{d^2y}{dx^2}=\frac{\frac{-3}{2}\operatorname{cosec}^2\frac{3\theta}{2}}{2(\cos\theta-\cos 2\theta)}$$

$$\frac{d^2y}{dx^2(\theta=\pi)}=-\frac{3}{4(-1-1)}=\frac{3}{8}$$

69. **(a)** $y(\alpha)=\sqrt{\frac{\frac{2\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}}{\sec^2\alpha}}=\sqrt{\frac{2\cos^2\alpha}{\sin\alpha\cos\alpha}+\frac{1}{\sin^2\alpha}}$

$=\sqrt{2\cot\alpha+\operatorname{cosec}^2\alpha}=\sqrt{2\cot\alpha+1+\cot^2\alpha}$

$=|1+\cot\alpha|=-1-\cot\alpha \qquad \left[\because \alpha\in\left(\frac{3\pi}{4},\pi\right)\right]$

$\frac{dy}{d\alpha}=\operatorname{cosec}^2\alpha \Rightarrow \left(\frac{dy}{d\alpha}\right)_{\alpha=\frac{5\pi}{6}}=4$

70. **(b)** Given, $x=\frac{1}{2}, y=\frac{-1}{4} \Rightarrow xy=\frac{-1}{8}$

$y.\frac{1.(-2x)}{2\sqrt{1-x^2}}+y'\sqrt{1-x^2}$

$=-\left\{1.\sqrt{1-y^2}+\frac{x.(-2y)}{2\sqrt{1-y^2}}y'\right\}$

$\Rightarrow -\frac{xy}{\sqrt{1-x^2}}+y'\sqrt{1-x^2}=-\sqrt{1-y^2}+\frac{xy.y'}{\sqrt{1-y^2}}$

$\Rightarrow y'\left(\sqrt{1-x^2}-\frac{xy}{\sqrt{1-y^2}}\right)=\frac{xy}{\sqrt{1-x^2}}-\sqrt{1-y^2}$

$\Rightarrow y'\left(\frac{\sqrt{3}}{2}+\frac{1}{8.\frac{\sqrt{15}}{4}}\right)=\frac{-1}{8.\sqrt{\frac{3}{2}}}-\frac{\sqrt{15}}{4}$

$\Rightarrow y'\left(\frac{\sqrt{45}+1}{2\sqrt{15}}\right)=-\frac{(1+\sqrt{45})}{4\sqrt{3}}$

$\therefore y'=-\frac{\sqrt{5}}{2}$

71. **(b)** Given, $e^y+xy=e$ ...(i)

Putting $x=0$ in (i), $\Rightarrow e^y=e\Rightarrow y=1$

On differentiating (i) w. r. to x

$e^y\frac{dy}{dx}+x\frac{dy}{dx}+y=0$ ...(ii)

Putting $y=1$ and $x=0$ in (ii),

$e\frac{dy}{dx}+0+1=0\Rightarrow\frac{dy}{dx}=-\frac{1}{e}$

On differentiating (ii) w. r. to x,

$e^y\frac{d^2y}{dx^2}+\frac{dy}{dx}.e^y.\frac{dy}{dx}+x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{dy}{dx}=0$ ...(iii)

Putting $y=1, x=0$ and $\frac{dy}{dx}=-\frac{1}{e}$ in (iii),

$e\frac{d^2y}{dx^2}+\frac{1}{e}-\frac{2}{e}=0\Rightarrow\frac{d^2y}{dx^2}=\frac{1}{e^2}$

Hence, $\left(\frac{dy}{dx},\frac{d^2y}{dx^2}\right)\equiv\left(-\frac{1}{e},\frac{1}{e^2}\right)$

72. **(d)** $f(x)=\tan^{-1}\left(\frac{\tan x-1}{\tan x+1}\right)$

$=-\tan^{-1}\left(\tan\left(\frac{\pi}{4}-x\right)\right) \qquad \left[\because \frac{\pi}{4}-x\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\right]$

So, $f(x)=-\left(\frac{\pi}{4}-x\right)=x-\frac{\pi}{4}$

Let $y=\Rightarrow f(y)=2y-\frac{\pi}{4}$

Now, differentiate w.r.t. y, $\frac{df(y)}{dy}=2.$

73. **(none)** $2y=\left[\cot^{-1}\left(\frac{\frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x}{\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x}\right)\right]^2$

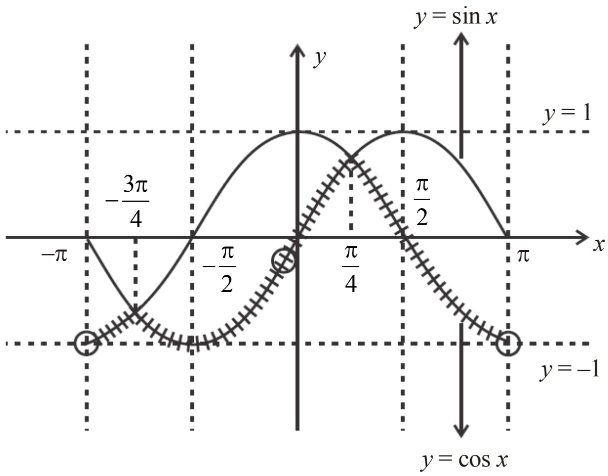
$\Rightarrow 2y=\left[\cot^{-1}\left(\frac{\cos\left(\frac{\pi}{6}-x\right)}{\sin\left(\frac{\pi}{6}-x\right)}\right)\right]^2$

$\Rightarrow 2y=\left[\cot^{-1}\left(\cot\left(\frac{\pi}{6}-x\right)\right)\right]^2 \because \frac{\pi}{6}-x\in\left(-\frac{\pi}{3},\frac{\pi}{6}\right)$

$$\Rightarrow 2y=\begin{cases}\left(\frac{7\pi}{6}-x\right)^2, & \text{if } \frac{\pi}{6}-x\in\left(\frac{-\pi}{3},0\right)\\ \left(\frac{\pi}{6}-x\right)^2, & \text{if } \frac{\pi}{6}-x\in\left(0,\frac{\pi}{0}\right)\end{cases}$$

$$\Rightarrow \frac{dy}{dx}=\begin{cases}x-\frac{7\pi}{6} & \text{if } x\in\left(\frac{\pi}{6},\frac{\pi}{2}\right)\\ x-\frac{\pi}{6} & \text{if } x\in\left(0,\frac{\pi}{6}\right)\end{cases}$$

74. **(b)** $f(x) = \min\{\sin x, \cos x\}$



$\because$ $f(x)$ is not differentiable at $x = -\frac{3\pi}{4}, \frac{\pi}{4}$

$\therefore \quad S = \left\{-\frac{3\pi}{4}, \frac{\pi}{4}\right\}$

$\Rightarrow \quad S \subseteq \left\{-\frac{3\pi}{4}, \frac{-\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}\right\}$

75. **(a)** Consider the equation,

$(2x)^{2y} = 4e^{2x-2y}$

Taking log on both sides

$2y \ln(2x) = \ln 4 + (2x - 2y) \qquad \ldots(1)$

Differentiating both sides w.r.t. x,

$$2y\frac{1}{2x}2 + 2\ln(2x)\frac{dy}{dx} = 0 + 2 - 2\frac{dy}{dx}$$

$$2\frac{dy}{dx}(1+\ln(2x)) = 2 - \frac{2y}{x} = \frac{2x-2y}{x} \qquad \ldots(2)$$

From (1) and (2),

$$\frac{dy}{dx}(1+\ln 2x) = 1 - \frac{1}{x}\left(\frac{\ln 2 + x}{1+\ln 2x}\right)$$

$$\Rightarrow \quad (1+\ln 2x)^2\frac{dy}{dx} = 1 + \ln(2x) - \left(\frac{x+\ln 2}{x}\right)$$

$$= \frac{x\ln(2x) - \ln 2}{x}$$

76. **(c)** Let $f(x) = x^3 + ax^2 + bx + c$

$f'(x) = 3x^2 + 2ax + b \Rightarrow f'(1) = 3 + 2a + b$

$f''(x) = 6x + 2a \Rightarrow f''(2) = 12 + 2a$

$f'''(x) = 6 \Rightarrow f'''(3) = 6$

$\because \quad f(x) = x^3 + f'(1)x^2 + f''(2)x + f'''(3)$

$\therefore \quad f'(1) = a \Rightarrow 3 + 2a + b = a \Rightarrow a + b = -3 \qquad \ldots(1)$

also $f''(2) = b \Rightarrow 12 + 2a = b \Rightarrow 2a - b = -12 \qquad \ldots(2)$

and $f'''(3) = c \Rightarrow c = 6$

Add (1) and (2)

$3a = -15 \Rightarrow a = -5 \Rightarrow b = 2$

$\Rightarrow \quad f(x) = x^3 - 5x^2 + 2x + 6$

$\Rightarrow \quad f(2) = 8 - 20 + 4 + 6 = -2$

77. **(b)** $\because \quad x = 3\tan t \Rightarrow \frac{dx}{dt} = 3\sec^2 t$

and $y = 3\sec t \Rightarrow \frac{dy}{dt} = 3\sec t \cdot \tan t$

$$\because \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \therefore \frac{dy}{dx} = \frac{\tan t}{\sec t} = \sin t$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dt}(\sin t)\cdot\frac{dt}{dx}$$

$$= \cos t \cdot \frac{1}{3\sec^2 t}$$

$$\therefore \quad \frac{d^2y}{dx^2}\left(\text{at } t = \frac{\pi}{4}\right) = \frac{1}{3}\cdot\left(\frac{1}{\sqrt{2}}\right)^3$$

$$= \frac{1}{6\sqrt{2}}$$

78. **(b)** Here, $\frac{dx}{dt} = \frac{1}{2\sqrt{2^{\text{cosec}^{-1}t}}}2^{\text{cosec}^{-1}t}\log 2 . \frac{-1}{x\sqrt{x^2-1}}$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{2^{\sec^{-1}t}}}2^{\sec^{-1}t}\log 2 . \frac{1}{x\sqrt{x^2-1}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sqrt{2^{\text{cosec}^{-1}t}}}{\sqrt{2^{\sec^{-1}t}}}\,\frac{2^{\sec^{-1}t}}{2^{\text{cosec}^{-1}t}}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\sqrt{\frac{2^{\sec^{-1}t}}{2^{\text{cosec}^{-1}t}}} = \frac{-y}{x}$$

79. **(a)** $f(x)=\begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$

$= \cos x\,(x^2-2x^2) - x\,(2\sin x - 2x\tan x) + 1(2x\sin x - x^2\tan x)$

$= -x^2\cos x - 2x\sin x + 2x^2\tan x + 2x\sin x - x^2\tan x$

$= x^2\tan x - x^2\cos x \quad = x^2(\tan x - \cos x)$

$\Rightarrow f'(x) = 2x(\tan x - \cos x) + x^2(\sec^2 x + \sin x)$

$\therefore \quad \lim_{x\to 0}\frac{f'(x)}{x} =$

$\lim_{x\to 0}\frac{2x(\tan x - \cos x) + x^2(\sec^2 x + \sin x)}{x}$

$= \lim_{x\to 0}(\tan x - \cos x) + x(\sec^2 x + \sin x)$

$= 2(0-1) + 0 = -2$

So, $\lim_{x\to 0}\frac{f'(x)}{x} = -2$

80. **(a)** Since $f(x) = \sin^{-1}\left(\frac{2\times 3^x}{1+9^x}\right)$

Suppose $3^x = \tan t$

$\Rightarrow f(x) = \sin^{-1}\left(\frac{2\tan t}{1+\tan^2 t}\right) = \sin^{-1}(\sin 2t) = 2t$

$= 2\tan^{-1}(3x)$

So, $f'(x) = \frac{2}{1+(3^x)^2}\times 3^x . \log_e 3$

$\therefore \quad f'\left(-\frac{1}{2}\right) = \frac{2}{1+\left(3^{-\frac{1}{2}}\right)^2}\times 3^{-\frac{1}{2}} . \log_e 3$

$= \frac{1}{2}\times\sqrt{3}\times\log_e 3 \qquad = \sqrt{3}\times\log_e\sqrt{3}$

81. **(a)** Given, $x^2+y^2+\sin y = 4$

After differentiating the above equation w. r. t. x we get

$$2x + 2y\frac{dy}{dx} + \cos y\frac{dy}{dx} = 0 \qquad ...(1)$$

$\Rightarrow 2x + (2y+\cos y)\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{2y+\cos y}$

At $(-2, 0)$, $\left(\frac{dy}{dx}\right)_{(-2,0)} = \frac{-2\times -2}{2\times 0+\cos 0}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(-2,0)} = \frac{4}{0+1}$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(-2,0)} = 4 \qquad ...(2)$$

Again differentiating equation (1) w. r. t to x, we get

$2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} - \sin y\left(\frac{dy}{dx}\right)^2 + \cos y\frac{d^2y}{dx^2} = 0$

$\Rightarrow 2 + (2-\sin y)\left(\frac{dy}{dx}\right)^2 + (2y+\cos y)\frac{d^2y}{dx^2} = 0$

$\Rightarrow (2y+\cos y)\frac{d^2y}{dx^2} = -2 - (2-\sin y)\left(\frac{dy}{dx}\right)^2$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2-(2-\sin y)\left(\frac{dy}{dx}\right)^2}{2y+\cos y}$

So, at $(-2, 0)$,

$\frac{d^2y}{dx^2} = \frac{-2-(2-0)\times 4^2}{2\times 0+1}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2-2\times 16}{1}$

$\Rightarrow \frac{d^2y}{dx^2} = -34$

82. **(b)** Let $F(x) = \tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ where $x \in \left(0, \frac{1}{4}\right)$.

$= \tan^{-1}\left(\frac{2.(3x^{3/2})}{1-(3x^{3/2})^2}\right) = 2\tan^{-1}(3x^{3/2})$

As $3x^{3/2} \in \left(0, \frac{3}{8}\right)$

$\left[\because 0 < x < \frac{1}{4} \Rightarrow 0 < x^{3/2} < \frac{1}{8} \Rightarrow 0 < 3x^{3/2} < \frac{3}{8}\right]$

So $\frac{dF(x)}{dx} = 2\times\frac{1}{1+9x^3}\times 3\times\frac{3}{2}\times x^{1/2} = \frac{9}{1+9x^3}\sqrt{x}$

On comparing

$\therefore g(x) = \frac{9}{1+9x^3}$

83. **(d)** $g(x) = f(f(x))$

In the neighbourhood of $x = 0$,

$f(x) = |\log 2 - \sin x| = (\log 2 - \sin x)$

$\therefore\ g(x) = |\log 2 - \sin|\log 2 - \sin x||$

$= (\log 2 - \sin(\log 2 - \sin x))$

$\therefore$ g (x) is differentiable

and $g'(x) = -\cos(\log 2 - \sin x)(-\cos x)$

$\Rightarrow g'(0) = \cos(\log 2)$

84. (c) $f(x) = y = x^2 - x + 5$

$x^2 - x + \frac{1}{4} - \frac{1}{4} + 5 = y$

$\left(x - \frac{1}{2}\right)^2 + \frac{19}{4} = y$

$\left(x - \frac{1}{2}\right)^2 = y - \frac{19}{4}$

$x - \frac{1}{2} = \pm\sqrt{y - \frac{19}{4}}$

$x = \frac{1}{2} \pm \sqrt{y - \frac{19}{4}}$

As $x > \frac{1}{2}$

$x = \frac{1}{2} + \sqrt{y - \frac{19}{4}}$

$g(x) = \frac{1}{2} + \sqrt{x - \frac{19}{4}}$

$g'(x) = \frac{1}{2\sqrt{x - \frac{19}{4}}}$

$g'(7) = \frac{1}{2\sqrt{7 - \frac{19}{4}}} = \frac{1}{2\frac{\sqrt{28-19}}{2}} = \frac{1}{3}$

85. (a) Let $y = \sec(\tan^{-1} x) = \sec\left(\sec^{-1}\sqrt{1+x^2}\right)$

$\Rightarrow \; y = \sqrt{1+x^2}$

$\Rightarrow \; \frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}} \cdot 2x$

At $x = 1$,

$\frac{dy}{dx} = \frac{1}{\sqrt{2}}$.

86. (b) $\frac{x^2}{\alpha} + \frac{y^2}{4} = 1 \Rightarrow \frac{2x}{\alpha} + \frac{2y}{4} \cdot \frac{dy}{dx} = 0$

$\Rightarrow \; \frac{dy}{dx} = \frac{-4x}{\alpha y}$ ...(i)

$y^3 = 16x \Rightarrow 3y^2 \cdot \frac{dy}{dx} = 16 \Rightarrow \frac{dy}{dx} = \frac{16}{3y^2}$ ...(ii)

Since curves intersects at right angles

$\therefore \; \frac{-4x}{\alpha y} \times \frac{16}{3y^2} = -1 \Rightarrow 3\alpha y^3 = 64x$

$\Rightarrow \; \alpha = \frac{64x}{3 \times 16x} = \frac{4}{3}$

87. (d) Let $x = \sqrt{a^{\sin^{-1} t}}$

$\Rightarrow \; x^2 = a^{\sin^{-1} t}$

$\Rightarrow \; 2 \log x = \sin^{-1} t \,.\, \log a$

$\Rightarrow \; \frac{2}{x} = \frac{\log a}{\sqrt{1-t^2}} \cdot \frac{dt}{dx}$

$\Rightarrow \; \frac{2\sqrt{1-t^2}}{x \log a} = \frac{dt}{dx}$ ...(1)

Now, let $y = \sqrt{a^{\cos^{-1} t}}$

$\Rightarrow \; 2 \log y = \cos^{-1} t . \log a$

$\Rightarrow \; \frac{2}{y} \cdot \frac{dy}{dx} = \frac{-\log a}{\sqrt{1-t^2}} \cdot \frac{dt}{dx}$

$\Rightarrow \; \frac{2}{y} \cdot \frac{dy}{dx} = \frac{-\log a}{\sqrt{1-t^2}} \times \frac{2\sqrt{1-t^2}}{x \log a}$ (from (1))

$\Rightarrow \; \frac{dy}{dx} = -\frac{y}{x}$

Hence, $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{-y}{x}\right)^2 = \frac{x^2 + y^2}{x^2}$

88. (b) Let $y = \frac{x^2 - x}{x^2 + 2x}$

$\Rightarrow \; (x^2 + 2x)\, y = x^2 - x$

$\Rightarrow \; x(x+2)y = x(x-1)$

$\Rightarrow \; x[(x+2)y - (x-1)] = 0$

$\because \; x \neq 0, \; \therefore \; (x+2)y - (x-1) = 0$

$\Rightarrow \; xy + 2y - x + 1 = 0$

$\Rightarrow \; x(y-1) = -(2y+1)$

$\therefore \; x = \frac{2y+1}{1-y} \Rightarrow f^{-1}(x) = \frac{2x+1}{1-x}$

$\frac{d}{dx}(f^{-1}(x)) = \frac{2(1-x) - (2x+1)(-1)}{(1-x)^2}$

$= \frac{2 - 2x + 2x + 1}{(1-x)^2} = \frac{3}{(1-x)^2}$

89. (c) Let $f'(x) = \sin[\log x]$ and $y = f\left(\frac{2x+3}{3-2x}\right)$

Now, $\frac{dy}{dx} = f'\left(\frac{2x+3}{3-2x}\right) \cdot \frac{d}{dx}\left(\frac{2x+3}{3-2x}\right)$

$= \sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right] \frac{\left[(6-4x) - (-4x-6)\right]}{(3-2x)^2}$

$= \frac{12}{(3-2x)^2} . \sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$

90. **(a)** Given that $g(x)=[f(2f(x))+2]^2$

$\therefore \quad g'(x)=2\left(f(2f(x)+2)\right)\left(\frac{d}{dx}\left(f(2f(x)+2)\right)\right)$

$=2f(2f(x)+2)f'(2f(x))+2).(2f'(x))$

$\Rightarrow \quad g'(0)=2f(2f(0)+2.f'(2f(0)+2)$

$.2f'(0)=4f(0)(f'(0))^2=4(-1)(1)^2=-4$

91. **(d)** $x^{2x}-2x^x\cot y-1=0$
$\Rightarrow 2\cot y=x^x-x^{-x}$
Let $u=x^x$

$\Rightarrow 2\cot y=u-\frac{1}{u}$

Differentiating both sides with respect to x, we get

$-2\text{cosec}^2 y\frac{dy}{dx}=\left(1+\frac{1}{u^2}\right)\frac{du}{dx}$

Now $u=x^x$ Taking log both sides

$\Rightarrow \log u=x\log x$

$\Rightarrow \frac{1}{u}\frac{du}{dx}=1+\log x$

$\Rightarrow \frac{du}{dx}=x^x(1+\log x)$

$\therefore$ We get

$-2\text{cosec}^2 y\frac{dy}{dx}=(1+x^{-2x}).x^x(1+\log x)$

$\Rightarrow \frac{dy}{dx}=\frac{\left(x^x+x^{-x}\right)(1+\log x)}{-2(1+\cot^2 y)}$ …(i)

Put $n=1$ in eqn. $x^{2x}-2x^x\cot y-1=0$, gives
$1-2\cot y-1=0$
$\Rightarrow \cot y=0$
$\therefore$ Putting $x=1$ and $\cot y=0$ in eqn. (i), we get

$y'(1)=\frac{(1+1)(1+0)}{-2(1+0)}=-1$

92. **(a)** $x^m.y^n=(x+y)^{m+n}$
taking log both sides
$\Rightarrow \quad m\,ln\,x+n\,ln\,y=(m+n)\,ln\,(x+y)$
Differentiating both sides, we get

$\therefore \quad \frac{m}{x}+\frac{n}{y}\frac{dy}{dx}=\frac{m+n}{x+y}\left(1+\frac{dy}{dx}\right)$

$\Rightarrow \left(\frac{m}{x}-\frac{m+n}{x+y}\right)=\left(\frac{m+n}{x+y}-\frac{n}{y}\right)\frac{dy}{dx}$

$\Rightarrow \frac{my-nx}{x(x+y)}=\left(\frac{my-nx}{y(x+y)}\right)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{y}{x}$

93. **(c)** Given that $x=e^{y+e^{y+\cdots\infty}} \Rightarrow x=e^{y+x}$.
Taking log both sides.

$\log x=y+x$ differentiating both side $\Rightarrow \frac{1}{x}=\frac{dy}{dx}+1$

$\therefore \frac{dy}{dx}=\frac{1}{x}-1=\frac{1-x}{x}$

94. **(b)** $f(x)=ax^2+bx+c$

$f(1)=f(-1)$

$\Rightarrow a+b+c=a-b+c \text{ or } b=0$

$\therefore f(x)=ax^2+c \text{ or } f'(x)=2ax$

Now $f'(a); f'(b)$ and $f'(c)$

are $2a(a); 2a(b); 2a(c)$

i.e. $2a^2, 2ab, 2ac$.
$\Rightarrow$ If a, b, c are in A.P. then $f'(a); f'(b)$ and $f'(c)$ are also in A.P.

95. **(c)** Given that $f(x+y)=f(x)\times f(y)$
Differentiate with respect to x, treating y as constant

$f'(x+y)=f'(x)f(y)$

Putting $x=0$ and $y=x$, we get $f'(x)=f'(0)f(x)$;

$\Rightarrow f'(5)=3f(5)=3\times 2=6$.

96. **(b)** Let $f:R\to R$, with $f(0)=f(1)=0$ and $f'(0)=0$

$\because f(x)$ is differentiable and continuous and $f(0)=f(1)=0$.

Then by Rolle's theorem, $f'(c)=0,\ c\in(0,1)$

Now again

$\because f'(c)=0,\ f'(0)=0$

Then, again by Rolle's theorem,

$f''(x)=0$ for some $x\in(0,1)$

97. **(c)** $y^2+2\log_e(\cos x)=y$ ...(i)

$\Rightarrow 2yy'-2\tan x=y'$ ...(ii)

From (i), $y(0)=0$ or 1

$\therefore y'(0)=0$

Again differentiating (ii) we get,

$2(y')^2+2yy''-2\sec^2 x=y''$

Put $x=0, y(0)=0, 1$ and $y'(0)=0$,
we get, $|y''(0)|=2$.

98. **(b)** Since, Rolle's theorem is applicable
$\therefore \quad f(a)=f(b)$
$f(3)=f(4) \Rightarrow \alpha=12$

$f'(x)=\frac{x^2-12}{x(x^2+12)}$

As $f'(c)=0$ (by Rolle's theorem)

$x=\pm\sqrt{12},\ \therefore\ c=\sqrt{12},\ \therefore\ f''(c)=\frac{1}{12}$

99. (c) $k.x^{k-1}+k.y^{k-1}\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=-\left(\frac{x}{y}\right)^{k-1}$

$\Rightarrow \frac{dy}{dx}+\left(\frac{x}{y}\right)^{k-1}=0$

$\Rightarrow k-1=-\frac{1}{3}$

$\Rightarrow k=1-\frac{1}{3}=\frac{2}{3}$

100. (b) Since, $f(x)$ is a polynomial function.

$\therefore$ It is continuous and differentiable in [0, 1]

Here, $f(0)=11, f(1)=1-4+8+11=16$

$f'(x)=3x^2-8x+8$

$\therefore f'(c)=\frac{f(1)-f(0)}{1-0}=\frac{16-11}{1}$

$=3c^2-8c+8$

$\Rightarrow 3c^2-8c+3=0$

$\Rightarrow c=\frac{8\pm 2\sqrt{7}}{6}=\frac{4\pm\sqrt{7}}{3}$

$\therefore c=\frac{4-\sqrt{7}}{3}\in(0,1)$

101. (b) $y^{1/5}+y^{-1/5}=2x$

$\Rightarrow \left(\frac{1}{5}y^{-4/5}-\frac{1}{5}y^{-6/5}\right)\frac{dy}{dx}=2$

$\Rightarrow y'\left(y^{1/5}-y^{-1/5}\right)=10y$

$\Rightarrow y^{1/5}+y^{-1/5}=2x$

$\Rightarrow y^{1/5}-y^{-1/5}=\sqrt{4x^2-4}$

$\Rightarrow y'\left(2\sqrt{x^2-1}\right)=10y$

$\Rightarrow y''\left(2\sqrt{x^2-1}\right)+y'2\frac{2x}{2\sqrt{x^2-1}}=10y'$

$\Rightarrow y''(x^2-1)+xy'=5\sqrt{x^2-1}\,(y')$

$\Rightarrow \boxed{y''(x^2-1)+xy'-25y=0}$

$\lambda=1, k=-25$

102. (b) Let $f(x)=ax^3+bx^2+cx+d$

$\Rightarrow f(3x)=27ax^3+9bx^2+3cx+d$

$\Rightarrow f'(x)=3ax^2+2bx+c$

$\Rightarrow f''(x)=6ax+2b$

$\Rightarrow f(3x)=f'(x)f''(x)$

$\Rightarrow 27a=18a^2$

$\Rightarrow a=\frac{3}{2}, b=0, c=0, d=0$

$\Rightarrow f(x)=\frac{3}{2}x^3,$

$f'(x)=\frac{9}{2}x^2,\ f'(x)=9x$

$\Rightarrow f'(2)=18$

and $f''(2)=18$

$\Rightarrow f''(b)-f'(b)=0$

103. (d) $y=\left\{x+\sqrt{x^2-1}\right\}^{15}+\left\{x-\sqrt{x^2-1}\right\}^{15}$

Differentiate w.r.t. 'x'

$\frac{dy}{dx}=15\left(x+\sqrt{x^2-1}\right)^{14}\left[1+\frac{x}{\sqrt{x^2-1}}\right]$

$+15\left(x-\sqrt{x^2-1}\right)^{14}\left(1-\frac{x}{\sqrt{x^2-1}}\right)$

$\Rightarrow \frac{dy}{dx}=\frac{15}{\sqrt{x^2-1}}.y$ ....(i)

$\Rightarrow \sqrt{x^2-1}.\frac{dy}{dx}=15y$

Again differentiating both sides w.r.t. x

$\frac{x}{\sqrt{x^2-1}}.\frac{dy}{dx}+\sqrt{x^2-1}\frac{d^2y}{dx^2}=15\frac{dy}{dx}$

$\Rightarrow x\frac{dy}{dx}+(x^2-1)\frac{d^2y}{dx^2}$

$=15\sqrt{x^2-1}.\frac{15}{\sqrt{x^2-1}}.y=225y$

104. (b) Conduction for Rolls theorem

$f(1)=f(-1)$

and $f'\left(\frac{1}{2}\right)=0$

$c=-2$ and $b=\frac{1}{2}$

$2b+c=-1$

105. (b) Since, f and g both are continuous function on [0, 1] and differentiable on (0, 1) then $\exists c\in(0,1)$ such that

$f'(c)=\frac{f(1)-f(0)}{1}=\frac{6-2}{1}=4$

and $g'(c)=\dfrac{g(1)-g(0)}{1}=\dfrac{2-0}{1}=2$

Thus, we get $f'(c)=2g'(c)$

106. (c) Let $f(x) = x|x| = x|x|$, $g(x) = \sin x$ and $h(x) = gof(x) = g[f(x)]$

$$\therefore\ h(x) = \begin{cases} \sin x^2, & x \ge 0 \\ -\sin x^2, & x < 0 \end{cases}$$

$$\text{Now, } h'(x) = \begin{cases} 2x\cos x^2, & x \ge 0 \\ -2x\cos x^2, & x < 0 \end{cases}$$

Since, L.H.L and R.H.L at $x = 0$ of $h'(x)$ is equal to 0 therefore $h'(x)$ is continuous at $x = 0$

Now, suppose $h'(x)$ is differentiable

$$\therefore\ h''(x) = \begin{cases} 2(\cos x^2 + 2x^2(-\sin x^2), & x \ge 0 \\ 2(-\cos x^2 + 2x^2 \sin x^2), & x < 0 \end{cases}$$

Since, L.H.L and R.H.L at $x = 0$ of $h''(x)$ are different therefore $h''(x)$ is not continuous.

$\Rightarrow h''(x)$ is not differentiable

$\Rightarrow$ our assumption is wrong

Hence $h'(x)$ is not differentiable at $x = 0$.

107. (a) Let $p_1(x) = a_1x^2 + b_1x + c_1$

$p_2(x) = a_2x^2 + b_2x + c_2$

and $p_3(x) = a_3x^2 + b_3x + c_3$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are real numbers.

$$\therefore\quad A(x) = \begin{bmatrix} a_1x^2+b_1x+c_1 & 2a_1x+b_1 & 2a_1 \\ a_2x^2+b_2x+c_2 & 2a_2x+b_2 & 2a_2 \\ a_3x^2+b_3x+c_3 & 2a_3x+b_3 & 2a_3 \end{bmatrix}$$

$$B(x) = \begin{bmatrix} a_1x^2+b_1x+c_1 & a_2x^2+b_2x+c_2 & a_3x^2+b_3x+c_3 \\ 2a_1x+b_1 & 2a_2x+b_2 & 2a_3x+b_2 \\ 2a_1 & 2a_2 & 2a_3 \end{bmatrix}$$

$$\times \begin{bmatrix} a_1x^2+b_1x+c_1 & 2a_1x+b_1 & 2a_1 \\ a_2x^2+b_2x+c_2 & 2a_2x+b_2 & 2a_2 \\ a_3x^2+b_3x+c_3 & 2a_3x+b_3 & 2a_3 \end{bmatrix}$$

It is clear from the above multiplication, the degree of determinant of $B(x)$ can not be less than 4.

108. (b) $f(x) = 2x^3 + ax^2 + bx$

let, $a = -1$, $b = 1$

Given that $f(x)$ satisfy Rolle's theorem in interval $[-1, 1]$

$f(x)$ must satisfy two conditions.

(1) $f(a) = f(b)$

(2) $f'(c) = 0$ (c should be between a and b)

$f(a) = f(1) = 2(1)^3 + a(1)^2 + b(1) = 2 + a + b$

$f(b) = f(-1) = 2(-1)^3 + a(-1)^2 + b(-1)$

$= -2 + a - b$

$f(a) = f(b)$

$2 + a + b = -2 + a - b$

$2b = -4$

$b = -2$

(given that $c = \dfrac{1}{2}$)

$f'(x) = 6x^2 + 2ax + b$

at $x = \dfrac{1}{2}$, $f'(x) = 0$

$$0 = 6\left(\frac{1}{2}\right)^2 + 2a\left(\frac{1}{2}\right) + b$$

$$\frac{3}{2} + a + b = 0$$

$$\frac{3}{2} + a - 2 = 0$$

$$a = 2 - \frac{3}{2} = \frac{1}{2}$$

$$2a + b = 2 \times \frac{1}{2} - 2 = 1 - 2 = -1$$

109. (d) $f(x) = \sin(\sin x)$

$\Rightarrow f'(x) = \cos(\sin x) . \cos x$

$\Rightarrow f''(x) = -\sin(\sin x) . \cos^2 x + \cos(\sin x). (-\sin x)$

$= -\cos^2 x . \sin(\sin x) - \sin x . \cos(\sin x)$

Now $f''(x) + \tan x . f'(x) + g(x) = 0$

$\Rightarrow g(x) = \cos^2 x . \sin(\sin x) + \sin x . \cos(\sin x)$

$- \tan x . \cos x . \cos(\sin x)$

$\Rightarrow g(x) = \cos^2 x . \sin(\sin x)$.

110. (d) Let $g(x) = \dfrac{ax^3}{3} + b.\dfrac{x^2}{2} + cx$

$g'(x) = ax^2 + bx + c$

Given: $ax^2 + bx + c = 0$ and $2a + 3b + 6c = 0$

Statement-2:

(*i*) $g(0) = 0$ and $g(1) = \dfrac{a}{3} + \dfrac{b}{2} + c = \dfrac{2a+3b+6c}{6}$

$= \dfrac{0}{6} = 0$

$\Rightarrow g(0) = g(1)$

(*ii*) g is continuous on $[0, 1]$ and differentiable on $(0, 1)$

$\therefore$ By Rolle's theorem $\exists\, k \in (0,1)$ such that $g'(k) = 0$

This holds the statement 2. Also, from statement-2, we can say $ax^2 + bx + c = 0$ has at least one root in $(0, 1)$.

Thus statement-1 and 2 both are true and statement-2 is a correct explanation for statement-1.

111. (c) $\frac{d^2x}{dy^2}=\frac{d}{dy}\left(\frac{dx}{dy}\right)=\frac{d}{dx}\left(\frac{dx}{dy}\right)\frac{dx}{dy}=\frac{d}{dx}\left(\frac{1}{dy/dx}\right)\frac{dx}{dy}$

$$=-\frac{1}{\left(\frac{dy}{dx}\right)^2}\cdot\frac{d^2y}{dx^2}\cdot\frac{1}{\frac{dy}{dx}}\left[\because \frac{d\left(\frac{1}{x}\right)}{dx}=-\frac{1}{x^2}\right]$$

$$=-\frac{1}{\left(\frac{dy}{dx}\right)^3}\frac{d^2y}{dx^2}$$

112. (b) Given that $f(x)=x|x|$ and $g(x)=\sin x$

So that

$gof(x)=g(f(x))=g(x|x|)=\sin x|x|$

$$=\begin{cases}\sin(-x^2), & \text{if } x<0\\ \sin(x^2), & \text{if } x\ge 0\end{cases}=\begin{cases}-\sin x^2, & \text{if } x<0\\ \sin x^2, & \text{if } x\ge 0\end{cases}$$

$$\therefore (gof)'(x)=\begin{cases}-2x\cos x^2, & \text{if } x<0\\ 2x\cos x^2, & \text{if } x\ge 0\end{cases}$$

Here we observe

$L(gof)'(0)=0=R(gof)'(0)$

$\Rightarrow$ gof is differentiable at $x=0$

and $(gof)'$ is continuous at $x=0$

$$\text{Now } (gof)''(x)=\begin{cases}-2\cos x^2+4x^2\sin x^2, x<0\\ 2\cos x^2-4x^2\sin x^2, x\ge 0\end{cases}$$

Here

$L(gof)''(0)=-2$ and $R(gof)''(0)=2$

$\because$ $L(gof)''(0)\ne R(gof)''(0)$

$\Rightarrow$ $gof(x)$ is not twice differentiable at $x=0$.

$\therefore$ Statement - 1 is true but statement -2 is false.

113. (c) Using Lagrange's Mean Value Theorem

Let $f(x)$ be a function defined on $[a, b]$

then, $f'(c)=\frac{f(b)-f(a)}{b-a}$ ....(i)

$c\in[a, b]$

$\therefore$ Given $f(x)=\log_e x$ $\quad\therefore f'(x)=\frac{1}{x}$

$\therefore$ equation (i) become

$$\frac{1}{c}=\frac{f(3)-f(1)}{3-1}$$

$$\Rightarrow \frac{1}{c}=\frac{\log_e 3-\log_e 1}{2}=\frac{\log_e 3}{2}$$

$$\Rightarrow c=\frac{2}{\log_e 3}\Rightarrow c=2\log_3 e$$

114. (a) As $f(1)=-2$ & $f'(x)\ge 2\ \forall\ x\in[1,6]$

Applying Lagrange's mean value theorem

$$\frac{f(6)-f(1)}{5}=f'(c)\ge 2$$

$\Rightarrow$ $f(6)\ge 10+f(1)$

$\Rightarrow$ $f(6)\ge 10-2\Rightarrow$ $f(6)\ge 8$.

115. (b) Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots\ldots+a_1x=0$

The other given equation,

$na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\ldots+a_1=0=f'(x)$

Given $a_1\ne 0\Rightarrow f(0)=0$

Again $f(x)$ has root α, $\Rightarrow f(\alpha)=0$

$\therefore$ $f(0)=f(\alpha)$

$\therefore$ By Rolle's theorem $f'(x)=0$ has root between $(0, \alpha)$

Hence $f'(x)$ has a positive root smaller than α.

116. (d) Let us define a function

$$f(x)=\frac{ax^3}{3}+\frac{bx^2}{2}+cx$$

Being polynomial, it is continuous and differentiable, also,

$$f(0)=0 \text{ and } f(1)=\frac{a}{3}+\frac{b}{2}+c$$

$$\Rightarrow f(1)=\frac{2a+3b+6c}{6}=0 \text{ (given)}$$

$\therefore f(0)=f(1)$

$\therefore f(x)$ satisfies all conditions of Rolle's theorem therefore $f'(x)=0$ has a root in $(0, 1)$

i.e. $ax^2+bx+c=0$ has at lease one root in $(0, 1)$

117. (d) Given that $f(x)=x^n\Rightarrow f(1)=1$

$$f'(x)=nx^{n-1}\Rightarrow f'(1)=n$$

$$f''(x)=n(n-1)x^{n-2}\Rightarrow f''(1)=n(n-1)$$

....

$$f^n(x)=n!\Rightarrow f^n(1)=n!$$

$$f(1)-\frac{f'(1)}{1!}+\frac{f''(1)}{2!}-\frac{f'''(1)}{3!}+\ldots+\frac{(-1)^nf^n(1)}{n!}$$

$$=1-\frac{n}{1!}+\frac{n(n-1)}{2!}-\frac{n(n-1)(n-2)}{3!}+\ldots+(-1)^n\frac{n!}{n!}$$

$$={}^nC_0-{}^nC_1+{}^nC_2-{}^nC_3+\ldots+(-1)^n\,{}^nC_n$$

$$=(1-1)^n=0$$

118. (b) $\lim_{x\to a}\frac{f(a)g'(x)-g(a)f'(x)}{g'(x)-f'(x)}=4$

(By Applying L' Hospital rule)

$\lim_{x\to a}\frac{k\,g'(x)-k\,f'(x)}{g'(x)-f'(x)}=4 \quad \therefore k=4.$

119. (a) Given that $y=(x+\sqrt{1+x^2})^n$ ...(i)

Differentiating both sides w.r. to x

$$\frac{dy}{dx}=n(x+\sqrt{1+x^2})^{n-1}\left(1+\frac{1}{2}(1+x^2)^{-1/2}.\,2x\right)$$

$$\frac{dy}{dx}=n(x+\sqrt{1+x^2})^{n-1}\frac{(\sqrt{1+x^2}+x)}{\sqrt{1+x^2}}$$

$$=\frac{n(\sqrt{1+x^2}+x)^n}{\sqrt{1+x^2}}$$

or $\sqrt{1+x^2}\frac{dy}{dx}=ny$ [from (i)]

$\Rightarrow \sqrt{1+x^2}\,y_1=ny$ $(\because y_1=\frac{dy}{dx})$ Squaring both sides,

we get $(1+x^2)y_1^2=n^2y^2$

Differentiating it w.r. to x,

$(1+x^2)2y_1y_2+y_1^2.2x=n^2.2yy_1$

$\Rightarrow (1+x^2)y_2+xy_1=n^2y$

120. (a) Let $f(x)=\frac{ax^3}{3}+\frac{bx^2}{2}+cx$

$\Rightarrow f(0)=0$ and

$f(1)=\frac{a}{3}+\frac{b}{2}+c=\frac{2a+3b+6c}{6}=0$

Also $f(x)$ is continuous and differentiable in [0, 1] and [0, 1[. So by Rolle's theorem, $f'(x)=0$.

i.e $ax^2+bx+c=0$ has at least one root in [0, 1].