Chapter 4

Electric Fields in Matter

4.1 Polarization

4.1.1 Dielectrics

In this chapter we shall study electric fields in matter. Matter, of course, comes in many varieties—solids, liquids, gases, metals, woods, glasses—and these substances do not all respond in the same way to electrostatic fields. Nevertheless, *most* everyday objects belong (at least, in good approximation) to one of two large classes: **conductors** and **insulators** (or **dielectrics**). We have already talked about conductors; these are substances that contain an "unlimited" supply of charges that are free to move about through the material. In practice what this ordinarily means is that many of the electrons (one or two per atom in a typical metal) are not associated with any particular nucleus, but roam around at will. In dielectrics, by contrast, *all charges are attached to specific atoms or molecules*—they're on a tight leash, and all they can do is move a bit *within* the atom or molecule. Such microscopic displacements are not as dramatic as the wholesale rearrangement of charge in a conductor, but their cumulative effects account for the characteristic behavior of dielectric materials. There are actually two principal mechanisms by which electric fields can distort the charge distribution of a dielectric atom or molecule: *stretching* and *rotating*. In the next two sections I'll discuss these processes.

4.1.2 Induced Dipoles

What happens to a neutral atom when it is placed in an electric field $\mathbf{E}$? Your first guess might well be: "Absolutely nothing—since the atom is not charged, the field has no effect on it." But that is incorrect. Although the atom as a whole is electrically neutral, there *is* a positively charged core (the nucleus) and a negatively charged electron cloud surrounding it. These two regions of charge within the atom are influenced by the field: the nucleus is pushed in the direction of the field, and the electrons the opposite way. In principle, if the field is large enough, it can pull the atom apart completely, "ionizing" it (the substance

then becomes a conductor). With less extreme fields, however, an equilibrium is soon established, for if the center of the electron cloud does not coincide with the nucleus, these positive and negative charges attract one another, and this holds the atoms together. The two opposing forces—**E** pulling the electrons and nucleus apart, their mutual attraction drawing them together—reach a balance, leaving the atom **polarized,** with plus charge shifted slightly one way, and minus the other. The atom now has a tiny dipole moment **p**, which points in the *same direction as* **E**. Typically, this induced dipole moment is approximately proportional to the field (as long as the latter is not too strong):

$$\mathbf{p} = \alpha \mathbf{E}. \tag{4.1}$$

The constant of proportionality α is called **atomic polarizability.** Its value depends on the detailed structure of the atom in question. Table 4.1 lists some experimentally determined atomic polarizabilities.

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.76	0.396	24.1	1.64	43.4	59.6

Table 4.1 Atomic Polarizabilities ($\alpha/4\pi\epsilon_0$, in units of 10^{-30} m^3).
Source: Handbook of Chemistry and Physics, 78th ed.
(Boca Raton: CRC Press, Inc., 1997).

Example 4.1

A primitive model for an atom consists of a point nucleus ($+q$) surrounded by a uniformly charged spherical cloud ($-q$) of radius a (Fig. 4.1). Calculate the atomic polarizability of such an atom.

Solution: In the presence of an external field **E**, the nucleus will be shifted slightly to the right and the electron cloud to the left, as shown in Fig. 4.2. (Because the actual displacements

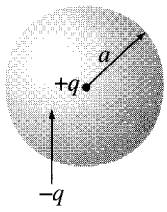


Figure 4.1

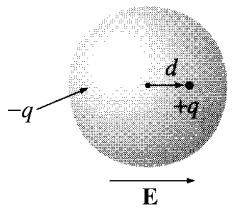


Figure 4.2

involved are extremely small, as you'll see in Prob. 4.1, it is reasonable to assume that the electron cloud retains its spherical shape.) Say that equilibrium occurs when the nucleus is displaced a distance d from the center of the sphere. At that point the external field pushing the nucleus to the right exactly balances the internal field pulling it to the left: $E = E_e$, where E_e is the field produced by the electron cloud. Now the field at a distance d from the center of a uniformly charged sphere is

$$E_e = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}$$

(Prob. 2.12). At equilibrium, then,

$$E = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}, \quad \text{or} \quad p = qd = (4\pi\epsilon_0 a^3)E.$$

The atomic polarizability is therefore

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v, \tag{4.2}$$

where v is the volume of the atom. Although this atomic model is extremely crude, the result (4.2) is not too bad—it's accurate to within a factor of four or so for many simple atoms.

For molecules the situation is not quite so simple, because frequently they polarize more readily in some directions than others. Carbon dioxide (Fig. 4.3), for instance, has a polarizability of 4.5×10^{-40} C$^2\cdot$m/N when you apply the field along the axis of the molecule, but only 2×10^{-40} for fields perpendicular to this direction. When the field is at some *angle* to the axis, you must resolve it into parallel and perpendicular components, and multiply each by the pertinent polarizability:

$$\mathbf{p} = \alpha_\perp \mathbf{E}_\perp + \alpha_\parallel \mathbf{E}_\parallel.$$

In this case the induced dipole moment may not even be in the same *direction* as $\mathbf{E}$. And CO_2 is relatively simple, as molecules go, since the atoms at least arrange themselves in a straight line; for a completely asymmetrical molecule Eq. 4.1 is replaced by the most general linear relation between $\mathbf{E}$ and $\mathbf{p}$:

$$\left.\begin{aligned} p_x &= \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z \\ p_y &= \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z \\ p_z &= \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z \end{aligned}\right\} \tag{4.3}$$

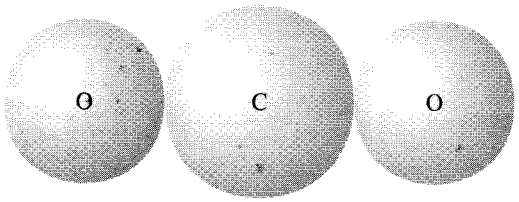


Figure 4.3

The set of nine constants α_{ij} constitute the **polarizability tensor** for the molecule. Their actual values depend on the orientation of the axes you chose, though it is always possible to choose "principal" axes such that all the off-diagonal terms (α_{xy}, α_{zx}, etc.) vanish, leaving just three nonzero polarizabilities: α_{xx}, α_{yy}, and α_{zz}.

Problem 4.1 A hydrogen atom (with the Bohr radius of half an angstrom) is situated between two metal plates 1 mm apart, which are connected to opposite terminals of a 500 V battery. What fraction of the atomic radius does the separation distance d amount to, roughly? Estimate the voltage you would need with this apparatus to ionize the atom. [Use the value of α in Table 4.1. *Moral:* The displacements we're talking about are *minute,* even on an atomic scale.]

Problem 4.2 According to quantum mechanics, the electron cloud for a hydrogen atom in the ground state has a charge density.

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a},$$

where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom. [*Hint:* First calculate the electric field of the electron cloud, $E_e(r)$; then expand the exponential, assuming $r \ll a$. For a more sophisticated approach, see W. A. Bowers, *Am. J. Phys.* **54,** 347 (1986).]

Problem 4.3 According to Eq. 4.1, the induced dipole moment of an atom is proportional to the external field. This is a "rule of thumb," not a fundamental law, and it is easy to concoct exceptions—in theory. Suppose, for example, the charge density of the electron cloud were proportional to the distance from the center, out to a radius R. To what power of E would p be proportional in that case? Find the condition on $\rho(r)$ such that Eq. 4.1 will hold in the weak-field limit.

Problem 4.4 A point charge q is situated a large distance r from a neutral atom of polarizability α. Find the force of attraction between them.

4.1.3 Alignment of Polar Molecules

The neutral atom discussed in Sect. 4.1.2 had no dipole moment to start with—**p** was *induced* by the applied field. Some molecules have built-in, permanent dipole moments. In the water molecule, for example, the electrons tend to cluster around the oxygen atom (Fig. 4.4), and since the molecule is bent at 105°, this leaves a negative charge at the vertex and a net positive charge at the opposite end. (The dipole moment of water is unusually large: 6.1×10^{-30} C·m; in fact, this is what accounts for its effectiveness as a solvent.) What happens when such molecules (called **polar molecules**) are placed in an electric field?

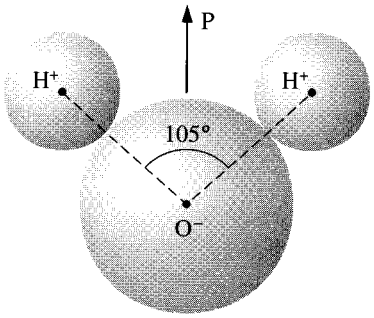


Figure 4.4

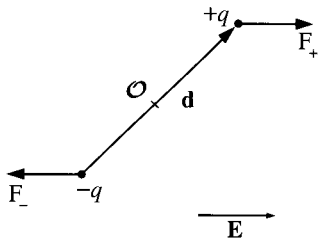


Figure 4.5

If the field is uniform, the *force* on the positive end, $\mathbf{F}_+ = q\mathbf{E}$, exactly cancels the force on the negative end, $\mathbf{F}_- = -q\mathbf{E}$ (Fig. 4.5). However, there will be a *torque:*

$$\begin{aligned} \mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) \\ &= [(\mathbf{d}/2) \times (q\mathbf{E})] + [(-\mathbf{d}/2) \times (-q\mathbf{E})] = q\mathbf{d} \times \mathbf{E}. \end{aligned}$$

Thus a dipole $\mathbf{p} = q\mathbf{d}$ in a uniform field $\mathbf{E}$ experiences a torque

$$\boxed{\mathbf{N} = \mathbf{p} \times \mathbf{E}.} \tag{4.4}$$

Notice that $\mathbf{N}$ is in such a direction as to line $\mathbf{p}$ up *parallel* to $\mathbf{E}$; a polar molecule that is free to rotate will swing around until it points in the direction of the applied field.

If the field is *non*uniform, so that $\mathbf{F}_+$ does not exactly balance $\mathbf{F}_-$, there will be a net *force* on the dipole, in addition to the torque. Of course, $\mathbf{E}$ must change rather abruptly for there to be significant variation in the space of one molecule, so this is not ordinarily a major consideration in discussing the behavior of dielectrics. Nevertheless, the formula for the force on a dipole in a nonuniform field is of some interest:

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q(\mathbf{E}_+ - \mathbf{E}_-) = q(\Delta\mathbf{E}),$$

where $\Delta\mathbf{E}$ represents the difference between the field at the plus end and the field at the minus end. Assuming the dipole is very short, we may use Eq. 1.35 to approximate the small change in E_x:

$$\Delta E_x \equiv (\nabla E_x) \cdot \mathbf{d},$$

with corresponding formulas for E_y and E_z. More compactly,

$$\Delta\mathbf{E} = (\mathbf{d} \cdot \nabla)\mathbf{E},$$

and therefore[1]

$$\boxed{\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}.} \tag{4.5}$$

For a "perfect" dipole of infinitesimal length, Eq. 4.4 gives the torque *about the center of the dipole* even in a *non*uniform field; about any *other* point $\mathbf{N} = (\mathbf{p} \times \mathbf{E}) + (\mathbf{r} \times \mathbf{F})$.

Problem 4.5 In Fig. 4.6, $\mathbf{p}_1$ and $\mathbf{p}_2$ are (perfect) dipoles a distance r apart. What is the torque on $\mathbf{p}_1$ due to $\mathbf{p}_2$? What is the torque on $\mathbf{p}_2$ due to $\mathbf{p}_1$? [In each case I want the torque on the dipole *about its own center*. If it bothers you that the answers are not equal and opposite, see Prob. 4.29.]

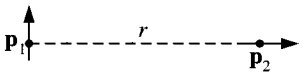


Figure 4.6

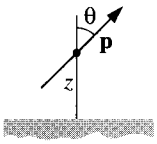


Figure 4.7

Problem 4.6 A (perfect) dipole $\mathbf{p}$ is situated a distance z above an infinite grounded conducting plane (Fig. 4.7). The dipole makes an angle θ with the perpendicular to the plane. Find the torque on $\mathbf{p}$. If the dipole is free to rotate, in what orientation will it come to rest?

Problem 4.7 Show that the energy of an ideal dipole $\mathbf{p}$ in an electric field $\mathbf{E}$ is given by

$$\boxed{U = -\mathbf{p} \cdot \mathbf{E}.} \tag{4.6}$$

Problem 4.8 Show that the interaction energy of two dipoles separated by a displacement $\mathbf{r}$ is

$$U = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\mathbf{p}_1 \cdot \hat{\mathbf{r}})(\mathbf{p}_2 \cdot \hat{\mathbf{r}})]. \tag{4.7}$$

[*Hint:* use Prob. 4.7 and Eq. 3.104.]

Problem 4.9 A dipole $\mathbf{p}$ is a distance r from a point charge q, and oriented so that $\mathbf{p}$ makes an angle θ with the vector $\mathbf{r}$ from q to $\mathbf{p}$.

(a) What is the force on $\mathbf{p}$?

(b) What is the force on q?

[1] In the present context Eq. 4.5 could be written more conveniently as $\mathbf{F} = \nabla(\mathbf{p} \cdot \mathbf{E})$. However, it is safer to stick with $(\mathbf{p} \cdot \nabla)\mathbf{E}$, because we will be applying the formula to materials in which the dipole moment (per unit volume) is itself a function of position and this second expression would imply (incorrectly) that $\mathbf{p}$ *too* is to be differentiated.

4.1.4 Polarization

In the previous two sections we have considered the effect of an external electric field on an individual atom or molecule. We are now in a position to answer (qualitatively) the original question: What happens to a piece of dielectric material when it is placed in an electric field? If the substance consists of neutral atoms (or nonpolar molecules), the field will induce in each a tiny dipole moment, pointing in the same direction as the field.[2] If the material is made up of polar molecules, each permanent dipole will experience a torque, tending to line it up along the field direction. (Random thermal motions compete with this process, so the alignment is never complete, especially at higher temperatures, and disappears almost at once when the field is removed.)

Notice that these two mechanisms produce the same basic result: *a lot of little dipoles pointing along the direction of the field*—the material becomes **polarized**. A convenient measure of this effect is

$$\mathbf{P} \equiv \textit{dipole moment per unit volume,}$$

which is called the **polarization**. From now on we shall not worry much about how the polarization *got* there. Actually, the two mechanisms I described are not as clear-cut as I tried to pretend. Even in polar molecules there will be some polarization by displacement (though generally it is a lot easier to rotate a molecule than to stretch it, so the second mechanism dominates). It's even possible in some materials to "freeze in" polarization, so that it persists after the field is removed. But let's forget for a moment about the *cause* of the polarization and study the field that a chunk of polarized material *itself* produces. Then, in Sect. 4.3, we'll put it all together: the original field, which was *responsible* for **P**, plus the new field, which is *due* to **P**.

4.2 The Field of a Polarized Object

4.2.1 Bound Charges

Suppose we have a piece of polarized material—that is, an object containing a lot of microscopic dipoles lined up. The dipole moment per unit volume **P** is given. *Question:* What is the field produced by this object (not the field that may have *caused* the polarization, but the field the polarization *itself* causes)? Well, we know what the field of an individual dipole looks like, so why not chop the material up into infinitesimal dipoles and integrate to get the total? As usual it's easier to work with the potential. For a single dipole **p** we have equation (Eq. 3.99),

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{\imath}}\cdot\mathbf{p}}{\imath^2}, \tag{4.8}$$

[2] In asymmetric molecules the induced dipole moment may not be parallel to the field, but if the molecules are randomly oriented, the perpendicular contributions will *average* to zero. Within a single crystal, the orientations are certainly *not* random, and we would have to treat this case separately.

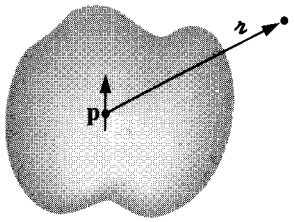


Figure 4.8

where $\mathcal{r}$ is the vector from the dipole to the point at which we are evaluating the potential (Fig. 4.8). In the present context we have a dipole moment $\mathbf{p} = \mathbf{P}\,d\tau'$ in each volume element $d\tau'$, so the total potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\hat{\mathcal{r}}\cdot\mathbf{P}(\mathbf{r}')}{\mathcal{r}^2}\,d\tau'. \tag{4.9}$$

That *does* it, in principle. But a little sleight-of-hand casts this integral into a much more illuminating form. Observing that

$$\boldsymbol{\nabla}'\left(\frac{1}{\mathcal{r}}\right) = \frac{\hat{\mathcal{r}}}{\mathcal{r}^2},$$

where (unlike Prob. 1.13) the differentiation is with respect to the *source* coordinates ($\mathbf{r}'$), we have

$$V = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \mathbf{P}\cdot\boldsymbol{\nabla}'\left(\frac{1}{\mathcal{r}}\right)\,d\tau'.$$

Integrating by parts, using product rule number 5, gives

$$V = \frac{1}{4\pi\epsilon_0}\left[\int_{\mathcal{V}} \boldsymbol{\nabla}'\cdot\left(\frac{\mathbf{P}}{\mathcal{r}}\right)\,d\tau' - \int_{\mathcal{V}} \frac{1}{\mathcal{r}}(\boldsymbol{\nabla}'\cdot\mathbf{P})\,d\tau'\right],$$

or, using the divergence theorem,

$$V = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{1}{\mathcal{r}}\mathbf{P}\cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{1}{\mathcal{r}}(\boldsymbol{\nabla}'\cdot\mathbf{P})\,d\tau'. \tag{4.10}$$

The first term looks like the potential of a surface charge

$$\boxed{\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}}} \tag{4.11}$$

(where $\hat{\mathbf{n}}$ is the normal unit vector), while the second term looks like the potential of a volume charge

$$\boxed{\rho_b \equiv -\nabla \cdot \mathbf{P}.} \tag{4.12}$$

With these definitions, Eq. 4.10 becomes

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{\sigma_b}{\imath} \, da' + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho_b}{\imath} \, d\tau'. \tag{4.13}$$

What this means is that the potential (and hence also the field) of a polarized object is the same as that produced by a volume charge density $\rho_b = -\nabla \cdot \mathbf{P}$ plus a surface charge density $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$. Instead of integrating the contributions of all the infinitesimal dipoles, as in Eq. 4.9, we just find those **bound charges**, and then calculate the fields *they* produce, in the same way we calculate the field of any other volume and surface charges (for example, using Gauss's law).

Example 4.2

Find the electric field produced by a uniformly polarized sphere of radius R.

Solution: We may as well choose the z axis to coincide with the direction of polarization (Fig. 4.9). The volume bound charge density ρ_b is zero, since $\mathbf{P}$ is uniform, but

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P \cos\theta,$$

where θ is the usual spherical coordinate. What we want, then, is the field produced by a charge density $P \cos\theta$ plastered over the surface of a sphere. But we have already computed the potential of such a configuration in Ex. 3.9:

$$V(r, \theta) = \begin{cases} \dfrac{P}{3\epsilon_0} r \cos\theta, & \text{for } r \leq R, \\ \dfrac{P}{3\epsilon_0} \dfrac{R^3}{r^2} \cos\theta, & \text{for } r \geq R. \end{cases}$$

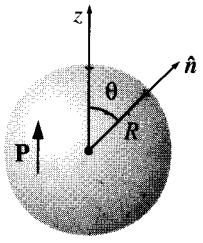


Figure 4.9

Since $r\cos\theta = z$, the *field* inside the sphere is uniform,

$$\mathbf{E} = -\nabla V = -\frac{P}{3\epsilon_0}\hat{\mathbf{z}} = -\frac{1}{3\epsilon_0}\mathbf{P}, \quad \text{for } r < R. \tag{4.14}$$

This remarkable result will be very useful in what follows. Outside the sphere the potential is identical to that of a perfect dipole at the origin,

$$V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, \quad \text{for } r \geq R, \tag{4.15}$$

whose dipole moment is, not surprisingly, equal to the total dipole moment of the sphere:

$$\mathbf{p} = \tfrac{4}{3}\pi R^3\mathbf{P}. \tag{4.16}$$

The field of the uniformly polarized sphere is shown in Fig. 4.10.

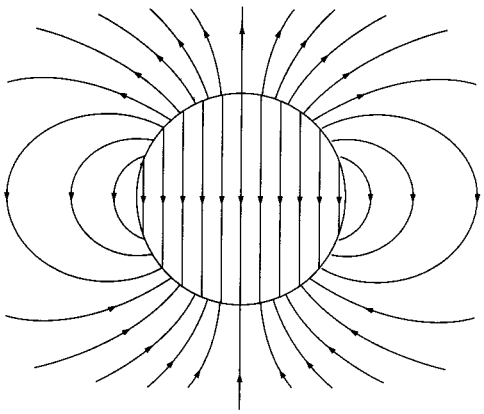


Figure 4.10

Problem 4.10 A sphere of radius R carries a polarization

$$\mathbf{P}(\mathbf{r}) = k\mathbf{r},$$

where k is a constant and $\mathbf{r}$ is the vector from the center.

(a) Calculate the bound charges σ_b and ρ_b.

(b) Find the field inside and outside the sphere.

Problem 4.11 A short cylinder, of radius a and length L, carries a "frozen-in" uniform polarization $\mathbf{P}$, parallel to its axis. Find the bound charge, and sketch the electric field (i) for $L \gg a$. (ii) for $L \ll a$, and (iii) for $L \approx a$. [This device is known as a **bar electret**; it is the electrical analog to a bar magnet. In practice, only very special materials—barium titanate is the most "familiar" example—will hold a permanent electric polarization. That's why you can't buy electrets at the toy store.]

Problem 4.12 Calculate the potential of a uniformly polarized sphere (Ex. 4.2) directly from Eq. 4.9.

4.2.2 Physical Interpretation of Bound Charges

In the last section we found that the field of a polarized object is identical to the field that would be produced by a certain distribution of "bound charges," σ_b and ρ_b. But this conclusion emerged in the course of abstract manipulations on the integral in Eq. 4.9, and left us with no clue as to the physical meaning of these bound charges. Indeed, some authors give you the impression that bound charges are in some sense "fictitious"—mere bookkeeping devices used to facilitate the calculation of fields. Nothing could be farther from the truth; ρ_b and σ_b represent *perfectly genuine accumulations of charge.* In this section I'll explain how polarization leads to such accumulations of charge.

The basic idea is very simple: Suppose we have a long string of dipoles, as shown in Fig. 4.11. Along the line, the head of one effectively cancels the tail of its neighbor, but at the ends there are two charges left over: plus at the right end and minus at the left. It is as if we had peeled off an electron at one end and carried it all the way down to the other end. though in fact no single electron made the whole trip—a lot of tiny displacements add up to one large one. We call the net charge at the ends *bound* charge to remind ourselves that it cannot be removed; in a dielectric every electron is attached to a specific atom or molecule. But apart from that, bound charge is no different from any other kind.

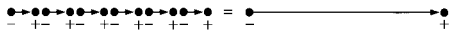


Figure 4.11

To calculate the actual *amount* of bound charge resulting from a given polarization. examine a "tube" of dielectric parallel to $\mathbf{P}$. The dipole moment of the tiny chunk shown in Fig. 4.12 is $P(Ad)$, where A is the cross-sectional area of the tube and d is the length of the chunk. In terms of the charge (q) at the end, this same dipole moment can be written qd. The bound charge that piles up at the right end of the tube is therefore

$$q = PA.$$

If the ends have been sliced off perpendicularly, the surface charge density is

$$\sigma_b = \frac{q}{A} = P.$$

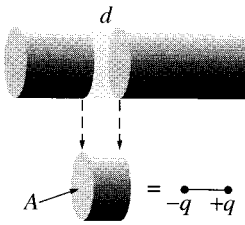


Figure 4.12

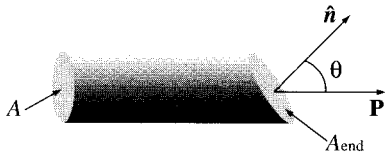


Figure 4.13

For an oblique cut (Fig. 4.13), the *charge* is still the same, but $A = A_{\text{end}} \cos\theta$, so

$$\sigma_b = \frac{q}{A_{\text{end}}} = P\cos\theta = \mathbf{P}\cdot\hat{\mathbf{n}}.$$

The effect of the polarization, then, is to paint a bound charge $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ over the surface of the material. This is exactly what we found by more rigorous means in Sect. 4.2.1. But now we know where the bound charge *comes* from.

If the polarization is nonuniform we get accumulations of bound charge *within* the material as well as on the surface. A glance at Fig. 4.14 suggests that a diverging $\mathbf{P}$ results in a pileup of negative charge. Indeed, the net bound charge $\int \rho_b\, d\tau$ in a given volume is equal and opposite to the amount that has been pushed out through the surface. The latter (by the same reasoning we used before) is $\mathbf{P}\cdot\hat{\mathbf{n}}$ per unit area, so

$$\int_{\mathcal{V}} \rho_b\, d\tau = -\oint_{\mathcal{S}} \mathbf{P}\cdot d\mathbf{a} = -\int_{\mathcal{V}} (\boldsymbol{\nabla}\cdot\mathbf{P})\, d\tau.$$

Since this is true for *any* volume, we have

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P},$$

confirming, again, the more rigorous conclusion of Sect. 4.2.1.

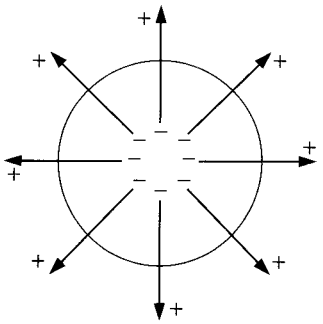


Figure 4.14

Example 4.3

There is another way of analyzing the uniformly polarized sphere (Ex. 4.2), which nicely illustrates the idea of a bound charge. What we have, really, is *two* spheres of charge: a positive sphere and a negative sphere. Without polarization the two are superimposed and cancel completely. But when the material is uniformly polarized, all the plus charges move slightly *upward* (the z direction), and all the minus charges move slightly *downward* (Fig. 4.15). The two spheres no longer overlap perfectly: at the top there's a "cap" of leftover positive charge and at the bottom a cap of negative charge. This "leftover" charge is precisely the bound surface charge σ_b.

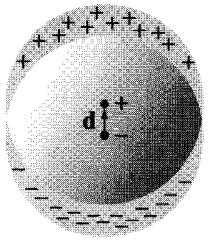


Figure 4.15

In Prob. 2.18 you calculated the field in the region of overlap between two uniformly charged spheres; the answer was

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{q\mathbf{d}}{R^3},$$

where q is the total charge of the positive sphere, $\mathbf{d}$ is the vector from the negative center the positive center, and R is the radius of the sphere. We can express this in terms of the polarization of the sphere, $\mathbf{p} = q\mathbf{d} = (\frac{4}{3}\pi R^3)\mathbf{P}$, as

$$\mathbf{E} = -\frac{1}{3\epsilon_0}\mathbf{P}.$$

Meanwhile, for points *outside,* it is as though all the charge on each sphere were concentrated at the respective center. We have, then, a dipole, with potential

$$V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}.$$

(Remember that $\mathbf{d}$ is some small fraction of an atomic radius; Fig. 4.15 is grossly exaggerated.) These answers agree, of course, with the results of Ex. 4.2.

Problem 4.13 A very long cylinder, of radius a, carries a uniform polarization $\mathbf{P}$ perpendicular to its axis. Find the electric field inside the cylinder. Show that the field *outside* the cylinder can be expressed in the form

$$\mathbf{E}(\mathbf{r}) = \frac{a^2}{2\epsilon_0 s^2}[2(\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}} - \mathbf{P}].$$

[*Careful:* I said "uniform," not "radial"!]

Problem 4.14 When you polarize a neutral dielectric, charge moves a bit, but the *total* remains zero. This fact should be reflected in the bound charges σ_b and ρ_b. Prove from Eqs. 4.11 and 4.12 that the total bound charge vanishes.

4.2.3 The Field Inside a Dielectric

I have been sloppy about the distinction between "pure" dipoles and "physical" dipoles. In developing the theory of bound charges, I assumed we were working with the pure kind—indeed, I started with Eq. 4.8, the formula for the potential of a pure dipole. And yet, an actual polarized dielectric consists of *physical* dipoles, albeit extremely tiny ones. What is more, I presumed to represent discrete molecular dipoles by a continuous density function $\mathbf{P}$. How can I justify this method? *Outside* the dielectric there is no real problem: here we are far away from the molecules (r is many times greater than the separation distance between plus and minus charges), so the dipole potential dominates overwhelmingly and the detailed "graininess" of the source is blurred by distance. *Inside* the dielectric, however, we can hardly pretend to be far from all the dipoles, and the procedure I used in Sect. 4.2.1 is open to serious challenge.

In fact, when you stop to think about it, the electric field inside matter must be fantastically complicated, on the microscopic level. If you happen to be very near an electron, the field is gigantic, whereas a short distance away it may be small or point in a totally different direction. Moreover, an instant later, as the atoms move about, the field will have altered entirely. This true **microscopic** field would be utterly impossible to calculate, nor would it be of much interest if you could. Just as, for macroscopic purposes, we regard water as a continuous fluid, ignoring its molecular structure, so also we can ignore the microscopic bumps and wrinkles in the electric field inside matter, and concentrate on the **macroscopic** field. This is defined as the *average* field over regions large enough to contain many thousands of atoms (so that the uninteresting microscopic fluctuations are smoothed over), and yet small enough to ensure that we do not wash out any significant large-scale variations in the field. (In practice, this means we must average over regions much smaller than the dimensions of the object itself.) Ordinarily, the macroscopic field is what people *mean* when they speak of "the" field inside matter.[3]

[3] In case the introduction of the macroscopic field sounds suspicious to you, let me point out that you do *exactly* the same averaging whenever you speak of the *density* of a material.

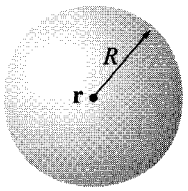


Figure 4.16

It remains to show that the macroscopic field is what we actually obtain when we use the methods of Sect. 4.2.1. The argument is subtle, so hang on. Suppose I want to calculate the macroscopic field at some point $\mathbf{r}$ within a dielectric (Fig. 4.16). I know I must average the true (microscopic) field over an appropriate volume, so let me draw a small sphere about $\mathbf{r}$, of radius, say, a thousand times the size of a molecule. The macroscopic field at $\mathbf{r}$, then. consists of two parts: the average field over the sphere due to all charges *outside*, plus the average due to all charges *inside:*

$$\mathbf{E} = \mathbf{E}_{\text{out}} + \mathbf{E}_{\text{in}}.$$

Now you proved in Prob. 3.41(d) that the average field (over a sphere), produced by charges *outside*, is equal to the field they produce at the center, so $\mathbf{E}_{\text{out}}$ is the field at $\mathbf{r}$ due to the dipoles exterior to the sphere. These are far enough away that we can safely use Eq. 4.9:

$$V_{\text{out}} = \frac{1}{4\pi\epsilon_0} \int_{\text{outside}} \frac{\hat{\boldsymbol{\imath}} \cdot \mathbf{P}(\mathbf{r}')}{\imath^2} \, d\tau'. \tag{4.17}$$

The dipoles *inside* the sphere are too close to treat in this fashion. But fortunately all we need is their *average* field, and that, according to Eq. 3.105, is

$$\mathbf{E}_{\text{in}} = -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3},$$

regardless of the details of the charge distribution within the sphere. The only relevant quantity is the total dipole moment, $\mathbf{p} = (\frac{4}{3}\pi R^3)\,\mathbf{P}$:

$$\mathbf{E}_{\text{in}} = -\frac{1}{3\epsilon_0}\mathbf{P}. \tag{4.18}$$

Now, by assumption the sphere is small enough that $\mathbf{P}$ does not vary significantly over its volume, so the term *left out* of the integral in Eq. 4.17 corresponds to the field at the center of a *uniformly* polarized sphere, to wit: $-(1/3\epsilon_0)\mathbf{P}$ (Eq. 4.14). But this is precisely what $\mathbf{E}_{\text{in}}$ (Eq. 4.18) puts back in! The macroscopic field, then, is given by the potential

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\boldsymbol{\imath}} \cdot \mathbf{P}(\mathbf{r}')}{\imath^2} \, d\tau', \tag{4.19}$$

where the integral runs over the *entire* volume of the dielectric. This is, of course, what we used in Sect. 4.2.1; without realizing it, we were correctly calculating the averaged, macroscopic field, for points inside the dielectric.

You may have to reread the last couple of paragraphs for the argument to sink in. Notice that it all revolves around the curious fact that the average field over *any* sphere (due to the charge inside) is the same as the field at the center of a *uniformly polarized* sphere with the same total dipole moment. This means that no matter how crazy the actual microscopic charge configuration, we can replace it by a nice smooth distribution of perfect dipoles, if all we want is the macroscopic (average) field. Incidentally, while the argument ostensibly relies on the spherical shape I chose to average over, the macroscopic field is certainly independent of the geometry of the averaging region, and this is reflected in the final answer, Eq. 4.19. Presumably, one could reproduce the same argument for a cube or an ellipsoid or whatever—the calculation might be more difficult, but the conclusion would be the same.

4.3 The Electric Displacement

4.3.1 Gauss's Law in the Presence of Dielectrics

In Sect. 4.2 we found that the effect of polarization is to produce accumulations of bound charge, $\rho_b = -\nabla \cdot \mathbf{P}$ within the dielectric and $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ on the surface. The field due to polarization of the medium is just the field of this bound charge. We are now ready to put it all together: the field attributable to bound charge plus the field due to everything *else* (which, for want of a better term, we call **free charge**). The free charge might consist of electrons on a conductor or ions embedded in the dielectric material or whatever; any charge, in other words, that is *not* a result of polarization. Within the dielectric, then, the total charge density can be written:

$$\rho = \rho_b + \rho_f, \tag{4.20}$$

and Gauss's law reads

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f,$$

where $\mathbf{E}$ is now the *total* field, not just that portion generated by polarization.

It is convenient to combine the two divergence terms:

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f.$$

The expression in parentheses, designated by the letter $\mathbf{D}$,

$$\boxed{\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P},} \tag{4.21}$$

is known as the **electric displacement**. In terms of $\mathbf{D}$, Gauss's law reads

$$\boxed{\nabla \cdot \mathbf{D} = \rho_f,} \tag{4.22}$$

or, in integral form,

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}, \tag{4.23}$$

where $Q_{f_{\text{enc}}}$ denotes the total free charge enclosed in the volume. This is a particularly useful way to express Gauss's law, in the context of dielectrics, because *it makes reference only to free charges,* and free charge is the stuff we control. Bound charge comes along for the ride: when we put the free charge in place, a certain polarization automatically ensues, by the mechanisms of Sect. 4.1, and this polarization produces the bound charge. In a typical problem, therefore, we know ρ_f, but we do not (initially) know ρ_b; Eq. 4.23 lets us go right to work with the information at hand. In particular, whenever the requisite symmetry is present, we can immediately calculate **D** by the standard Gauss's law methods.

Example 4.4

A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a (Fig. 4.17). Find the electric displacement.

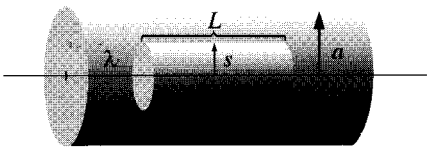


Figure 4.17

Solution: Drawing a cylindrical Gaussian surface, of radius s and length L, and applying Eq. 4.23, we find

$$D(2\pi s L) = \lambda L.$$

Therefore,

$$\mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}. \tag{4.24}$$

Notice that this formula holds both within the insulation and outside it. In the latter region, $\mathbf{P} = 0$, so

$$\mathbf{E} = \frac{1}{\epsilon_0}\mathbf{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}, \qquad \text{for } s > a.$$

Inside the rubber the electric field cannot be determined, since we do not know **P**.

It may have appeared to you that I left out the surface bound charge σ_b in deriving Eq. 4.22, and in a sense that is true. We cannot apply Gauss's law precisely *at* the surface of a dielectric, for here ρ_b blows up, taking the divergence of **E** with it. But everywhere *else* the logic is sound, and in fact if we picture the edge of the dielectric as having some finite thickness within which the polarization tapers off to zero (probably a more realistic model

than an abrupt cut-off anyway), then there *is* no surface bound charge; ρ_b varies rapidly but smoothly within this "skin," and Gauss's law can be safely applied *everywhere.* At any rate, the integral form (Eq. 4.23) is free from this "defect."

Problem 4.15 A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a "frozen-in" polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\hat{\mathbf{r}},$$

where k is a constant and r is the distance from the center (Fig. 4.18). (There is no *free* charge in the problem.) Find the electric field in all three regions by two different methods:

(a) Locate all the bound charge, and use Gauss's law (Eq. 2.13) to calculate the field it produces.

(b) Use Eq. 4.23 to find $\mathbf{D}$, and then get $\mathbf{E}$ from Eq. 4.21. [Notice that the second method is much faster, and avoids any explicit reference to the bound charges.]

Problem 4.16 Suppose the field inside a large piece of dielectric is $\mathbf{E}_0$, so that the electric displacement is $\mathbf{D}_0 = \epsilon_0\mathbf{E}_0 + \mathbf{P}$.

(a) Now a small spherical cavity (Fig. 4.19a) is hollowed out of the material. Find the field at the center of the cavity in terms of $\mathbf{E}_0$ and $\mathbf{P}$. Also find the displacement at the center of the cavity in terms of $\mathbf{D}_0$ and $\mathbf{P}$.

(b) Do the same for a long needle-shaped cavity running parallel to $\mathbf{P}$ (Fig. 4.19b).

(c) Do the same for a thin wafer-shaped cavity perpendicular to $\mathbf{P}$ (Fig. 4.19c).

[Assume the cavities are small enough that $\mathbf{P}$, $\mathbf{E}_0$, and $\mathbf{D}_0$ are essentially uniform. *Hint:* Carving out a cavity is the same as superimposing an object of the same shape but opposite polarization.]

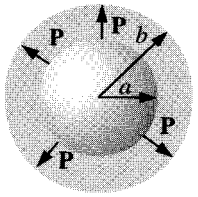


Figure 4.18

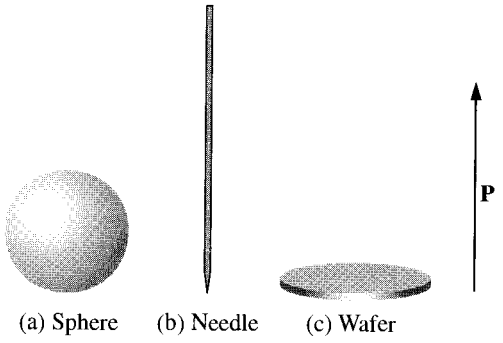


Figure 4.19

4.3.2 A Deceptive Parallel

Equation 4.22 looks just like Gauss's law, only the *total* charge density ρ is replaced by the *free* charge density ρ_f, and $\mathbf{D}$ is substituted for $\epsilon_0\mathbf{E}$. For this reason, you may be tempted to conclude that $\mathbf{D}$ is "just like" $\mathbf{E}$ (apart from the factor ϵ_0), except that its source is ρ_f instead of ρ: "To solve problems involving dielectrics, you just forget all about the bound charge—calculate the field as you ordinarily would, only call the answer $\mathbf{D}$ instead of $\mathbf{E}$." This reasoning is seductive, but the conclusion is false; in particular, there is no "Coulomb's law" for $\mathbf{D}$:

$$\mathbf{D}(\mathbf{r}) \neq \frac{1}{4\pi}\int \frac{\hat{\boldsymbol{\imath}}}{\imath^2}\rho_f(\mathbf{r}')\,d\tau'.$$

The parallel between $\mathbf{E}$ and $\mathbf{D}$ is more subtle than that.

For the divergence alone is insufficient to determine a vector field; you need to know the curl as well. One tends to forget this in the case of electrostatic fields because the curl of $\mathbf{E}$ is always zero. But the curl of $\mathbf{D}$ is *not* always zero.

$$\nabla \times \mathbf{D} = \epsilon_0(\nabla \times \mathbf{E}) + (\nabla \times \mathbf{P}) = \nabla \times \mathbf{P}, \tag{4.25}$$

and there is no reason, in general, to suppose that the curl of $\mathbf{P}$ vanishes. Sometimes it does, as in Ex. 4.4 and Prob. 4.15, but more often it does not. The bar electret of Prob. 4.11 is a case in point: here there is no free charge anywhere, so if you really believe that the only source of $\mathbf{D}$ is ρ_f, you will be forced to conclude that $\mathbf{D} = 0$ everywhere, and hence that $\mathbf{E} = (-1/\epsilon_0)\mathbf{P}$ inside and $\mathbf{E} = 0$ outside the electret, which is obviously wrong. (I leave it for you to find the place where $\nabla \times \mathbf{P} \neq 0$ in this problem.) Because $\nabla \times \mathbf{D} \neq 0$, moreover, $\mathbf{D}$ cannot be expressed as the gradient of a scalar—there is no "potential" for $\mathbf{D}$.

Advice: When you are asked to compute the electric displacement, first look for symmetry. If the problem exhibits spherical, cylindrical, or plane symmetry, then you can get $\mathbf{D}$ directly from Eq. 4.23 by the usual Gauss's law methods. (Evidently in such cases $\nabla \times \mathbf{P}$ is automatically zero, but since symmetry alone dictates the answer you're not really obliged to worry about the curl.) If the requisite symmetry is absent, you'll have to think of another approach and, in particular, you must *not* assume that $\mathbf{D}$ is determined exclusively by the free charge.

4.3.3 Boundary Conditions

The electrostatic boundary conditions of Sect. 2.3.5 can be recast in terms of $\mathbf{D}$. Equation 4.23 tells us the discontinuity in the component perpendicular to an interface:

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f, \tag{4.26}$$

while Eq. 4.25 gives the discontinuity in parallel components:

$$\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}}. \tag{4.27}$$

In the presence of dielectrics these are sometimes more useful than the corresponding boundary conditions on **E** (Eqs. 2.31 and 2.23):

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma, \tag{4.28}$$

and

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = 0. \tag{4.29}$$

You might try applying them, for example, to Probs. 4.16 and 4.17.

Problem 4.17 For the bar electret of Prob. 4.11, make three careful sketches: one of **P**, one of **E**, and one of **D**. Assume L is about $2a$. [*Hint:* **E** lines terminate on charges; **D** lines terminate on *free* charges.]

4.4 Linear Dielectrics

4.4.1 Susceptibility, Permittivity, Dielectric Constant

In Sects. 4.2 and 4.3 we did not commit ourselves as to the *cause* of **P**; we dealt only with the *effects* of polarization. From the qualitative discussion of Sect. 4.1, though, we know that the polarization of a dielectric ordinarily results from an electric field, which lines up the atomic or molecular dipoles. For many substances, in fact, the polarization is *proportional* to the field, provided **E** is not too strong:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}. \tag{4.30}$$

The constant of proportionality, χ_e, is called the **electric susceptibility** of the medium (a factor of ϵ_0 has been extracted to make χ_e dimensionless). The value of χ_e depends on the microscopic structure of the substance in question (and also on external conditions such as temperature). I shall call materials that obey Eq. 4.30 **linear dielectrics.**[4]

Note that **E** in Eq. 4.30 is the *total* field; it may be due in part to free charges and in part to the polarization itself. If, for instance, we put a piece of dielectric into an external field $\mathbf{E}_0$, we cannot compute **P** directly from Eq. 4.30; the external field will polarize the material, and this polarization will produce its own field, which then contributes to the total field, and this in turn modifies the polarization, which ... Breaking out of this infinite regress is not always easy. You'll see some examples in a moment. The simplest approach is to begin with the *displacement,* at least in those cases where **D** can be deduced directly from the free charge distribution.

[4] In modern optical applications, especially, *non*linear materials have become increasingly important. For these there is a second term in the formula for **P** as a function of **E**—typically a *cubic* one. In general, Eq. 4.30 can be regarded as the first (nonzero) term in the Taylor expansion of **P** in powers of **E**.

In linear media we have

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0\mathbf{E} + \epsilon_0\chi_e\mathbf{E} = \epsilon_0(1 + \chi_e)\mathbf{E}, \tag{4.31}$$

so $\mathbf{D}$ is *also* proportional to $\mathbf{E}$:

$$\mathbf{D} = \epsilon\mathbf{E}, \tag{4.32}$$

where

$$\epsilon \equiv \epsilon_0(1 + \chi_e). \tag{4.33}$$

This new constant ϵ is called the **permittivity** of the material. (In vacuum, where there is no matter to polarize, the susceptibility is zero, and the permittivity is ϵ_0. That's why ϵ_0 is called the **permittivity of free space**. I dislike the term, for it suggests that the vacuum is just a special kind of linear dielectric, in which the permittivity happens to have the value $8.85 \times 10^{-12}\ \mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$.) If you remove a factor of ϵ_0, the remaining dimensionless quantity

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0} \tag{4.34}$$

is called the **relative permittivity**, or **dielectric constant**, of the material. Dielectric constants for some common substances are listed in Table 4.2. Of course, the permittivity and the dielectric constant do not convey any information that was not already available in the susceptibility, nor is there anything essentially new in Eq. 4.32; the *physics* of linear dielectrics is all contained in Eq. 4.30.[5]

Material	Dielectric Constant	Material	Dielectric Constant
Vacuum	1	Benzene	2.28
Helium	1.000065	Diamond	5.7
Neon	1.00013	Salt	5.9
Hydrogen	1.00025	Silicon	11.8
Argon	1.00052	Methanol	33.0
Air (dry)	1.00054	Water	80.1
Nitrogen	1.00055	Ice (-30° C)	99
Water vapor (100° C)	1.00587	$KTaNbO_3$ (0° C)	34,000

Table 4.2 Dielectric Constants (unless otherwise specified, values given are for 1 atm, 20° C). *Source: Handbook of Chemistry and Physics,* 78th ed. (Boca Raton: CRC Press, Inc., 1997).

[5] As long as we are engaged in this orgy of unnecessary terminology and notation, I might as well mention that formulas for $\mathbf{D}$ in terms of $\mathbf{E}$ (Eq. 4.32, in the case of linear dielectrics) are called **constitutive relations**.

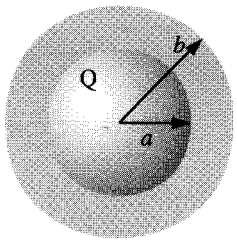


Figure 4.20

Example 4.5

A metal sphere of radius a carries a charge Q (Fig. 4.20). It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).

Solution: To compute V, we need to know $\mathbf{E}$; to find $\mathbf{E}$, we might first try to locate the bound charge; we could get the bound charge from $\mathbf{P}$, but we can't calculate $\mathbf{P}$ unless we already know $\mathbf{E}$ (Eq. 4.30). We seem to be in a bind. What we *do* know is the *free* charge Q, and fortunately the arrangement is spherically symmetric, so let's begin by calculating $\mathbf{D}$, using Eq. 4.23:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \qquad \text{for all points } r > a.$$

(Inside the metal sphere, of course, $\mathbf{E} = \mathbf{P} = \mathbf{D} = 0$.) Once we know $\mathbf{D}$, it is a trivial matter to obtain $\mathbf{E}$, using Eq. 4.32:

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } a < r < b, \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > b. \end{cases}$$

The potential at the center is therefore

$$\begin{aligned} V &= -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b \left(\frac{Q}{4\pi\epsilon_0 r^2}\right) dr - \int_b^a \left(\frac{Q}{4\pi\epsilon r^2}\right) dr - \int_a^0 (0)\, dr \\ &= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right). \end{aligned}$$

As it turns out, it was not necessary for us to compute the polarization or the bound charge explicitly, though this can easily be done:

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \frac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\hat{\mathbf{r}},$$

in the dielectric, and hence

$$\rho_b = -\nabla \cdot \mathbf{P} = 0,$$

while

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0 \chi_e Q}{4\pi\epsilon b^2}, & \text{at the outer surface,} \\ \dfrac{-\epsilon_0 \chi_e Q}{4\pi\epsilon a^2}, & \text{at the inner surface.} \end{cases}$$

Notice that the surface bound charge at a is *negative* ($\hat{\mathbf{n}}$ points outward *with respect to the dielectric,* which is $+\hat{\mathbf{r}}$ at b but $-\hat{\mathbf{r}}$ at a). This is natural, since the charge on the metal sphere attracts its opposite in all the dielectric molecules. It is this layer of negative charge that reduces the field, within the dielectric, from $1/4\pi\epsilon_0(Q/r^2)\hat{\mathbf{r}}$ to $1/4\pi\epsilon(Q/r^2)\hat{\mathbf{r}}$. In this respect a dielectric is rather like an imperfect conductor: on a *conducting* shell the induced surface charge would be such as to cancel the field of Q *completely* in the region $a < r < b$: the dielectric does the best it can, but the cancellation is only partial.

You might suppose that linear dielectrics would escape the defect in the parallel between **E** and **D**. Since **P** and **D** are now proportional to **E**, does it not follow that their curls, like **E**'s, must vanish? Unfortunately, it does *not,* for the line integral of **P** around a closed path that *straddles the boundary between one type of material and another* need not be zero, even though the integral of **E** around the same loop *must* be. The reason is that the proportionality factor $\epsilon_0\chi_e$ is different on the two sides. For instance, at the interface between a polarized dielectric and the vacuum (Fig. 4.21), **P** is zero on one side but not on the other. Around this loop $\oint \mathbf{P} \cdot d\mathbf{l} \neq 0$, and hence, by Stokes' theorem, the curl of **P** cannot vanish everywhere within the loop (in fact, it is *infinite* at the boundary).

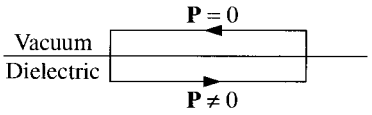


Figure 4.21

Of course, if the space is *entirely* filled with a homogeneous[6] linear dielectric, then this objection is void; in this rather special circumstance

$$\nabla \cdot \mathbf{D} = \rho_f \quad \text{and} \quad \nabla \times \mathbf{D} = 0,$$

so **D** can be found from the free charge just as though the dielectric were not there:

$$\mathbf{D} = \epsilon_0 \mathbf{E}_{\text{vac}},$$

where $\mathbf{E}_{\text{vac}}$ is the field the same free charge distribution would produce in the absence of any dielectric. According to Eqs. 4.32 and 4.34, therefore,

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \frac{1}{\epsilon_r}\mathbf{E}_{\text{vac}}. \tag{4.35}$$

[6] A **homogeneous** medium is one whose properties (in this case the susceptibility) do not vary with position.

Conclusion: When all space is filled with a homogeneous linear dielectric, the field everywhere is simply reduced by a factor of one over the dielectric constant. (Actually, it is not necessary for the dielectric to fill *all* space: in regions where the field is zero anyway, it can hardly matter whether the dielectric is present or not, since there's no polarization in any event.)

For example, if a free charge q is embedded in a large dielectric, the field it produces is

$$\mathbf{E} = \frac{1}{4\pi\epsilon}\frac{q}{r^2}\hat{\mathbf{r}} \tag{4.36}$$

(that's ϵ, not ϵ_0), and the force it exerts on nearby charges is reduced accordingly. But it's not that there is anything wrong with Coulomb's law; rather, the polarization of the medium partially "shields" the charge, by surrounding it with bound charge of the opposite sign (Fig. 4.22).[7]

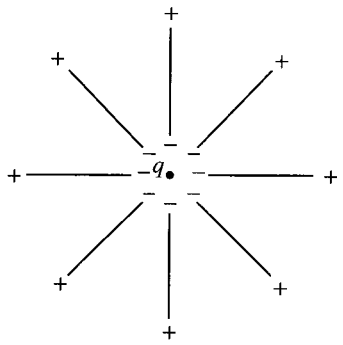


Figure 4.22

Example 4.6

A parallel-plate capacitor (Fig. 4.23) is filled with insulating material of dielectric constant ϵ_r. What effect does this have on its capacitance?

Solution: Since the field is confined to the space between the plates, the dielectric will reduce **E**, and hence also the potential difference V, by a factor $1/\epsilon_r$. Accordingly, the capacitance $C = Q/V$ is *increased by a factor of the dielectric constant,*

$$C = \epsilon_r C_{\text{vac}}. \tag{4.37}$$

This is, in fact, a common way to beef up a capacitor.

---

[7] In *quantum* electrodynamics the vacuum itself can be polarized, and this means that the effective (or "renormalized") charge of the electron, as you might measure it in the laboratory, is not its true ("bare") value, and in fact depends slightly on how far away you are!

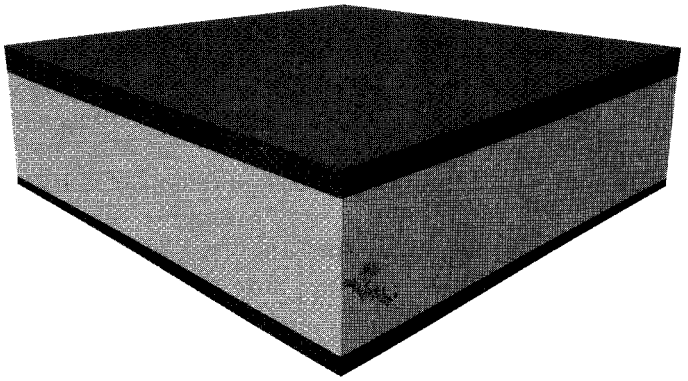


Figure 4.23

By the way, a *crystal* is generally easier to polarize in some directions than in others,[8] and in this case Eq. 4.30 is replaced by the general linear relation

$$\left.\begin{aligned} P_x &= \epsilon_0(\chi_{e_{xx}} E_x + \chi_{e_{xy}} E_y + \chi_{e_{xz}} E_z) \\ P_y &= \epsilon_0(\chi_{e_{yx}} E_x + \chi_{e_{yy}} E_y + \chi_{e_{yz}} E_z) \\ P_z &= \epsilon_0(\chi_{e_{zx}} E_x + \chi_{e_{zy}} E_y + \chi_{e_{zz}} E_z) \end{aligned}\right\}, \tag{4.38}$$

just as Eq. 4.1 was superseded by Eq. 4.3 for asymmetrical molecules. The nine coefficients, $\chi_{e_{xx}}, \chi_{e_{xy}}, \ldots$, constitute the **susceptibility tensor.**

Problem 4.18 The space between the plates of a parallel-plate capacitor (Fig. 4.24) is filled with two slabs of linear dielectric material. Each slab has thickness a, so the total distance between the plates is $2a$. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate $-\sigma$.

(a) Find the electric displacement **D** in each slab.

(b) Find the electric field **E** in each slab.

(c) Find the polarization **P** in each slab.

(d) Find the potential difference between the plates.

(e) Find the location and amount of all bound charge.

(f) Now that you know all the charge (free and bound), recalculate the field in each slab, and confirm your answer to (b).

---

[8] A medium is said to be **isotropic** if its properties (such as susceptibility) are the same in all directions. Thus Eq. 4.30 is the special case of Eq. 4.38 that holds for isotropic media. Physicists tend to be sloppy with their language, and unless otherwise indicated the term "linear dielectric" certainly means "isotropic linear dielectric," and probably means "homogeneous isotropic linear dielectric."

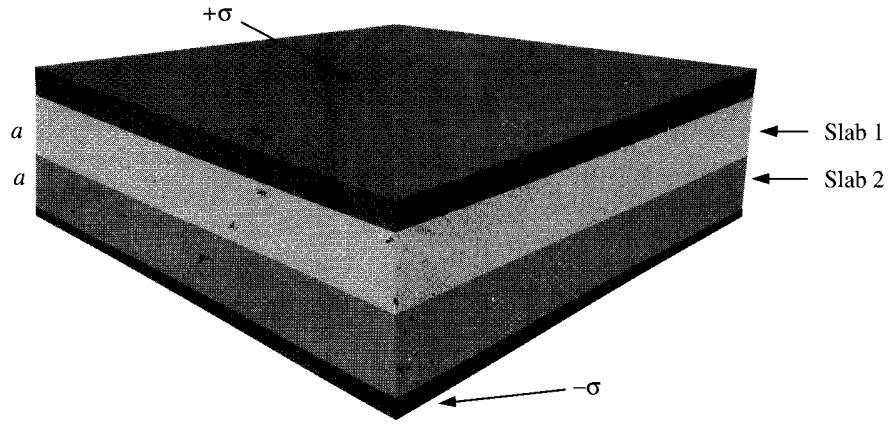
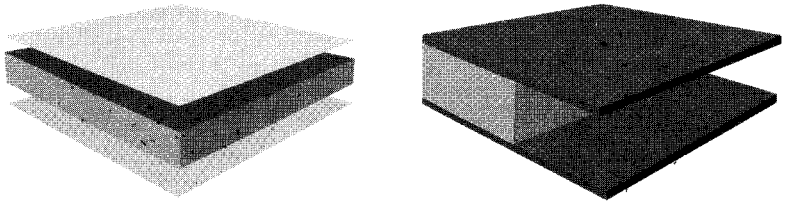


Figure 4.24

Problem 4.19 Suppose you have enough linear dielectric material, of dielectric constant ϵ_r, to *half*-fill a parallel-plate capacitor (Fig. 4.25). By what fraction is the capacitance increased when you distribute the material as in Fig. 4.25(a)? How about Fig. 4.25(b)? For a given potential difference V between the plates, find **E**, **D**, and **P**, in each region, and the free and bound charge on all surfaces, for both cases.



(a) (b)

Figure 4.25

Problem 4.20 A sphere of linear dielectric material has embedded in it a uniform free charge density ρ. Find the potential at the center of the sphere (relative to infinity), if its radius is R and its dielectric constant is ϵ_r.

Problem 4.21 A certain coaxial cable consists of a copper wire, radius a, surrounded by a concentric copper tube of inner radius c (Fig. 4.26). The space between is partially filled (from b out to c) with material of dielectric constant ϵ_r, as shown. Find the capacitance per unit length of this cable.

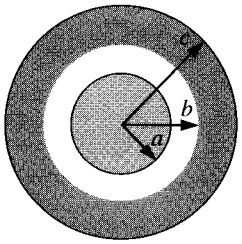


Figure 4.26

4.4.2 Boundary Value Problems with Linear Dielectrics

In a homogeneous linear dielectric the bound charge density (ρ_b) is proportional to the free charge density (ρ_f):[9]

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\epsilon_0 \frac{\chi_e}{\epsilon}\mathbf{D}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f. \tag{4.39}$$

In particular, unless free charge is actually embedded in the material, $\rho = 0$, and any net charge must reside at the surface. Within such a dielectric, then, the potential obeys Laplace's equation, and all the machinery of Chapter 3 carries over. It is convenient, however, to rewrite the boundary conditions in a way that makes reference only to the free charge. Equation 4.26 says

$$\epsilon_{\text{above}} E^{\perp}_{\text{above}} - \epsilon_{\text{below}} E^{\perp}_{\text{below}} = \sigma_f, \tag{4.40}$$

or (in terms of the potential),

$$\epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f, \tag{4.41}$$

whereas the potential itself is, of course, continuous (Eq. 2.34):

$$V_{\text{above}} = V_{\text{below}}. \tag{4.42}$$

Example 4.7

A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field $\mathbf{E}_0$ (Fig. 4.27). Find the electric field inside the sphere.

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[9] This does not apply to the surface charge (σ_b), because χ_e is not independent of position (obviously) at the boundary.

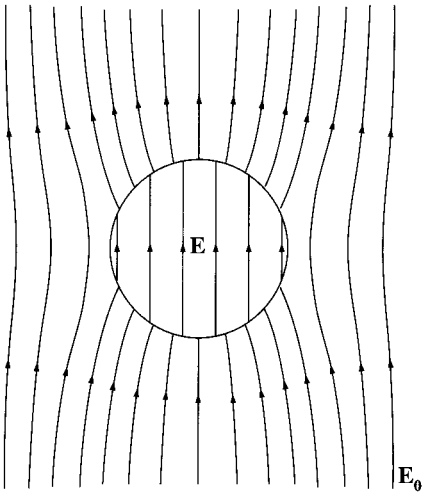

Figure 4.27

Solution: This is reminiscent of Ex. 3.8, in which an uncharged *conducting* sphere was introduced into a uniform field. In that case the field of the induced charge completely canceled $\mathbf{E}_0$ within the sphere; in a *dielectric,* the cancellation (from the bound charge) is only partial.

Our problem is to solve Laplace's equation, for $V_{\text{in}}(r, \theta)$ when $r \leq R$, and $V_{\text{out}}(r, \theta)$ when $r \geq R$, subject to the boundary conditions

$$\left.\begin{aligned} &\text{(i)} \quad V_{\text{in}} = V_{\text{out}}, \quad \text{at } r = R, \\ &\text{(ii)} \quad \epsilon \frac{\partial V_{\text{in}}}{\partial r} = \epsilon_0 \frac{\partial V_{\text{out}}}{\partial r}, \quad \text{at } r = R, \\ &\text{(iii)} \quad V_{\text{out}} \to -E_0 r \cos\theta, \quad \text{for } r \gg R. \end{aligned}\right\} \tag{4.43}$$

(The second of these follows from Eq. 4.41, since there is no free charge at the surface.) Inside the sphere Eq. 3.65 says

$$V_{\text{in}}(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta); \tag{4.44}$$

outside the sphere, in view of (iii), we have

$$V_{\text{out}}(r, \theta) = -E_0 r \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta). \tag{4.45}$$

Boundary condition (i) requires that

$$\sum_{l=0}^{\infty} A_l\, R^l\, P_l(\cos\theta) = -E_0 R\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta),$$

so[10]

$$\left.\begin{aligned} A_l R^l &= \frac{B_l}{R^{l+1}}, \quad \text{for } l \neq 1, \\ A_1 R &= -E_0 R + \frac{B_1}{R^2}. \end{aligned}\right\} \tag{4.46}$$

Meanwhile, condition (ii) yields

$$\epsilon_r \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -E_0\cos\theta - \sum_{l=0}^{\infty} \frac{(l+1)B_l}{R^{l+2}} P_l(\cos\theta),$$

so

$$\left.\begin{aligned} \epsilon_r l A_l R^{l-1} &= -\frac{(l+1)B_l}{R^{l+2}}, \quad \text{for } l \neq 1, \\ \epsilon_r A_1 &= -E_0 - \frac{2B_1}{R^3}. \end{aligned}\right\} \tag{4.47}$$

It follows that

$$\left.\begin{aligned} &A_l = B_l = 0, \qquad \text{for } l \neq 1, \\ &A_1 = -\tfrac{3}{\epsilon_r+2} E_0 \quad B_1 = \tfrac{\epsilon_r-1}{\epsilon_r+2} R^3 E_0. \end{aligned}\right\} \tag{4.48}$$

Evidently

$$V_{\text{in}}(r,\theta) = -\frac{3E_0}{\epsilon_r+2}\, r\cos\theta = -\frac{3E_0}{\epsilon_r+2}\, z,$$

and hence the field inside the sphere is (surprisingly) *uniform*:

$$\mathbf{E} = \frac{3}{\epsilon_r+2}\mathbf{E}_0. \tag{4.49}$$

Example 4.8

Suppose the entire region below the plane $z = 0$ in Fig. 4.28 is filled with uniform linear dielectric material of susceptibility χ_e. Calculate the force on a point charge q situated a distance d above the origin.

Solution: The surface bound charge on the xy plane is of opposite sign to q, so the force will be attractive. (In view of Eq. 4.39, there is no volume bound charge.) Let us first calculate σ_b. using Eqs. 4.11 and 4.30.

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = P_z = \epsilon_0\chi_e E_z,$$

[10]Remember, $P_1(\cos\theta) = \cos\theta$, and the coefficients must be equal for each l, as you could prove by multiplying by $P_{l'}(\cos\theta)\sin\theta$, integrating from 0 to π, and invoking the orthogonality of the Legendre polynomials (Eq. 3.68).

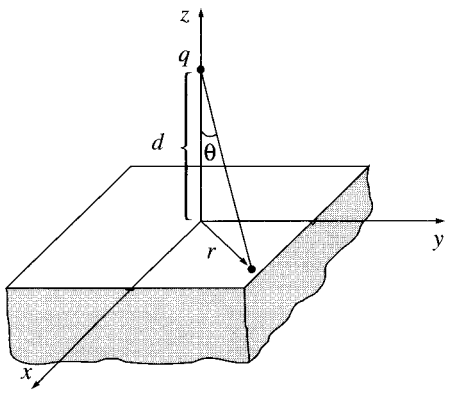

Figure 4.28

where E_z is the z-component of the total field just inside the dielectric, at $z = 0$. This field is due in part to q and in part to the bound charge itself. From Coulomb's law, the former contribution is

$$-\frac{1}{4\pi\epsilon_0}\frac{q}{(r^2+d^2)}\cos\theta = -\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}},$$

where $r = \sqrt{x^2+y^2}$ is the distance from the origin. The z component of the field of the bound charge, meanwhile, is $-\sigma_b/2\epsilon_0$ (see footnote 6, p. 89). Thus

$$\sigma_b = \epsilon_0\chi_e\left[-\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}} - \frac{\sigma_b}{2\epsilon_0}\right],$$

which we can solve for σ_b:

$$\sigma_b = -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{(r^2+d^2)^{3/2}}. \tag{4.50}$$

Apart from the factor $\chi_e/(\chi_e+2)$, this is exactly the same as the induced charge on an infinite *conducting* plane under similar circumstances (Eq. 3.10).[11] Evidently the *total* bound charge is

$$q_b = -\left(\frac{\chi_e}{\chi_e+2}\right)q. \tag{4.51}$$

We could, of course, obtain the field of σ_b by direct integration

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right)\sigma_b\, da.$$

[11] For some purposes a conductor can be regarded as the limiting case of a linear dielectric, with $\chi_e \to \infty$. This is often a useful check—try applying it to Exs. 4.5, 4.6, and 4.7.

But as in the case of the conducting plane, there is a nicer solution by the method of images. Indeed, if we replace the dielectric by a single point charge q_b at the image position $(0, 0, -d)$, we have

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_b}{\sqrt{x^2+y^2+(z+d)^2}}\right], \tag{4.52}$$

in the region $z > 0$. Meanwhile, a charge $(q + q_b)$ at $(0, 0, d)$ yields the potential

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q+q_b}{\sqrt{x^2+y^2+(z-d)^2}}\right], \tag{4.53}$$

for the region $z < 0$. Taken together, Eqs. 4.52 and 4.53 constitute a function which satisfies Poisson's equation with a point charge q at $(0, 0, d)$, which goes to zero at infinity, which is continuous at the boundary $z = 0$, and whose normal derivative exhibits the discontinuity appropriate to a surface charge σ_b at $z = 0$:

$$-\epsilon_0\left(\left.\frac{\partial V}{\partial z}\right|_{z=0^+} - \left.\frac{\partial V}{\partial z}\right|_{z=0^-}\right) = -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{(x^2+y^2+d^2)^{3/2}}.$$

Accordingly, this is the correct potential for our problem. In particular, the force on q is:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qq_b}{(2d)^2}\hat{\mathbf{z}} = -\frac{1}{4\pi\epsilon_0}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{q^2}{4d^2}\hat{\mathbf{z}}. \tag{4.54}$$

I do not claim to have provided a compelling *motivation* for Eqs. 4.52 and 4.53—like all image solutions, this one owes its justification to the fact that it *works:* it solves Poisson's equation, and it meets the boundary conditions. Still, discovering an image solution is not entirely a matter of guesswork. There are at least two "rules of the game": (1) You must never put an image charge into the region where you're computing the potential. (Thus Eq. 4.52 gives the potential for $z > 0$, but this image charge q_b is at $z = -d$; when we turn to the region $z < 0$ (Eq. 4.53), the image charge $(q + q_b)$ is at $z = +d$.) (2) The image charges must add up to the correct total in each region. (That's how I knew to use q_b to account for the charge in the region $z \leq 0$, and $(q + q_b)$ to cover the region $z \geq 0$.)

Problem 4.22 A very long cylinder of linear dielectric material is placed in an otherwise uniform electric field $\mathbf{E}_0$. Find the resulting field within the cylinder. (The radius is a, the susceptibility χ_e, and the axis is perpendicular to $\mathbf{E}_0$.)

Problem 4.23 Find the field inside a sphere of linear dielectric material in an otherwise uniform electric field $\mathbf{E}_0$ (Ex. 4.7) by the following method of successive approximations: First pretend the field inside is just $\mathbf{E}_0$, and use Eq. 4.30 to write down the resulting polarization $\mathbf{P}_0$. This polarization generates a field of its own, $\mathbf{E}_1$ (Ex. 4.2), which in turn modifies the polarization by an amount $\mathbf{P}_1$, which further changes the field by an amount $\mathbf{E}_2$, and so on. The resulting field is $\mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + \cdots$. Sum the series, and compare your answer with Eq. 4.49.

Problem 4.24 An uncharged conducting sphere of radius a is coated with a thick insulating shell (dielectric constant ϵ_r) out to radius b. This object is now placed in an otherwise uniform electric field $\mathbf{E}_0$. Find the electric field in the insulator.

! **Problem 4.25** Suppose the region *above* the xy plane in Ex. 4.8 is *also* filled with linear dielectric but of a different susceptibility χ_e'. Find the potential everywhere.

4.4.3 Energy in Dielectric Systems

It takes work to charge up a capacitor (Eq. 2.55):

$$W = \tfrac{1}{2}CV^2.$$

If the capacitor is filled with linear dielectric, its capacitance exceeds the vacuum value by a factor of the dielectric constant,

$$C = \epsilon_r C_{\text{vac}},$$

as we found in Ex. 4.6. Evidently the work necessary to charge a dielectric-filled capacitor is increased by the same factor. The reason is pretty clear: you have to pump on more (free) charge to achieve a given potential, because part of the field is canceled off by the bound charges.

In Chapter 2, I derived a general formula for the energy stored in any electrostatic system (Eq. 2.45):

$$W = \frac{\epsilon_0}{2}\int E^2\, d\tau. \tag{4.55}$$

The case of the dielectric-filled capacitor suggests that this should be changed to

$$W = \frac{\epsilon_0}{2}\int \epsilon_r E^2\, d\tau = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau,$$

in the presence of linear dielectrics. To *prove* it, suppose the dielectric material is fixed in position, and we bring in the free charge, a bit at a time. As ρ_f is increased by an amount $\Delta\rho_f$, the polarization will change and with it the bound charge distribution; but we're interested only in the work done on the incremental *free* charge:

$$\Delta W = \int (\Delta\rho_f) V\, d\tau. \tag{4.56}$$

Since $\nabla\cdot\mathbf{D} = \rho_f$, $\Delta\rho_f = \nabla\cdot(\Delta\mathbf{D})$, where $\Delta\mathbf{D}$ is the resulting change in $\mathbf{D}$, so

$$\Delta W = \int [\nabla\cdot(\Delta\mathbf{D})] V\, d\tau.$$

Now

$$\nabla\cdot[(\Delta\mathbf{D})V] = [\nabla\cdot(\Delta\mathbf{D})]V + \Delta\mathbf{D}\cdot(\nabla V),$$

and hence (integrating by parts):

$$\Delta W = \int \nabla\cdot[(\Delta\mathbf{D})V]\, d\tau + \int (\Delta\mathbf{D})\cdot\mathbf{E}\, d\tau.$$

The divergence theorem turns the first term into a surface integral, which vanishes if we integrate over all of space. Therefore, the work done is equal to

$$\Delta W = \int (\Delta \mathbf{D}) \cdot \mathbf{E}\, d\tau. \tag{4.57}$$

So far, this applies to *any* material. Now, if the medium is a linear dielectric, then $\mathbf{D} = \epsilon \mathbf{E}$, so

$$\tfrac{1}{2}\Delta(\mathbf{D} \cdot \mathbf{E}) = \tfrac{1}{2}\Delta(\epsilon E^2) = \epsilon(\Delta \mathbf{E}) \cdot \mathbf{E} = (\Delta \mathbf{D}) \cdot \mathbf{E}$$

(for infinitesimal increments). Thus

$$\Delta W = \Delta\left(\frac{1}{2}\int \mathbf{D} \cdot \mathbf{E}\, d\tau\right).$$

The total work done, then, as we build the free charge up from zero to the final configuration, is

$$W = \frac{1}{2}\int \mathbf{D} \cdot \mathbf{E}\, d\tau, \tag{4.58}$$

as anticipated.[12]

It may puzzle you that Eq. 4.55, which we derived quite generally in Chapter 2, does not seem to apply in the presence of dielectrics, where it is replaced by Eq. 4.58. The point is not that one or the other of these equations is *wrong*, but rather that they speak to somewhat different questions. The distinction is subtle, so let's go right back to the beginning: What do we *mean* by "the energy of a system"? *Answer:* It is the work required to assemble the system. Very well—but when dielectrics are involved there are two quite different ways one might construe this process: (1) We bring in all the charges (free *and* bound), one by one, with tweezers, and glue each one down in its proper final location. If *this* is what you mean by "assemble the system," the Eq. 4.55 is your formula for the energy stored. Notice, however, that this will *not* include the work involved in stretching and twisting the dielectric molecules (if we picture the positive and negative charges as held together by tiny springs, it does not include the spring energy, $\frac{1}{2}kx^2$, associated with polarizing each molecule).[13] (2) With the unpolarized dielectric in place, we bring in the *free* charges, one by one, allowing the dielectric to respond as it sees fit. If *this* is what you mean by "assemble the system" (and ordinarily it *is*, since free charge is what we actually push around), then Eq. 4.58 is the formula you want. In this case the "spring" energy *is* included, albeit indirectly, because the force you must apply to the *free* charge depends on the disposition of the *bound* charge; as you move the free charge you are automatically stretching those "springs." To put it another

[12] In case you are wondering why I did not do this more simply by the method of Sect. 2.4.3, starting with $W = \frac{1}{2}\int \rho_f V\, d\tau$, the reason is that *this* formula is untrue, in general. Study the derivation of Eq. 2.42 and you will see that it applies only to the *total* charge. For *linear* dielectrics it happens to hold for the free charge alone, but this is scarcely obvious a priori and, in fact, is most easily confirmed by working backward from Eq. 4.58.

[13] The "spring" itself may be electrical in nature, but it is still not included in Eq. 4.55, if $\mathbf{E}$ is taken to be the *macroscopic* field.

way, in method (2) the total energy of the system consists of three parts: the electrostatic energy of the free charge, the electrostatic energy of the bound charge, and the "spring" energy:

$$W_{\text{tot}} = W_{\text{free}} + W_{\text{bound}} + W_{\text{spring}}.$$

The last two are equal and opposite (in procedure (2) the bound charges are always in equilibrium, and hence the *net* work done on them is zero); thus method (2), in calculating W_{free}, actually delivers W_{tot}, whereas method (1), by calculating $W_{\text{free}}+W_{\text{bound}}$, leaves out W_{spring}.

Incidentally, it is sometimes alleged that Eq. 4.58 represents the energy even for *non*linear dielectrics, but this is false: To proceed beyond Eq. 4.57 one must assume linearity. In fact, for dissipative systems the whole notion of "stored energy" loses its meaning, because the work done depends not only on the final configuration but on *how it got there.* If the molecular "springs" are allowed to have some *friction,* for instance, then W_{spring} can be made as large as you like, by assembling the charges in such a way that the spring is obliged to expand and contract many times before reaching its final state. In particular, you get nonsensical results if you try to apply Eq. 4.58 to electrets, with frozen-in polarization (see Prob. 4.27).

Problem 4.26 A spherical conductor, of radius a, carries a charge Q (Fig. 4.29). It is surrounded by linear dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration (Eq. 4.58).

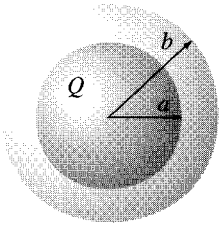

Figure 4.29

Problem 4.27 Calculate W, using both Eq. 4.55 and Eq. 4.58, for a sphere of radius R with frozen-in uniform polarization $\mathbf{P}$ (Ex. 4.2). Comment on the discrepancy. Which (if either) is the "true" energy of the system?

4.4.4 Forces on Dielectrics

Just as a conductor is attracted into an electric field (Eq. 2.51), so too is a dielectric—and for essentially the same reason: the bound charge tends to accumulate near the free charge of the opposite sign. But the calculation of forces on dielectrics can be surprisingly tricky.

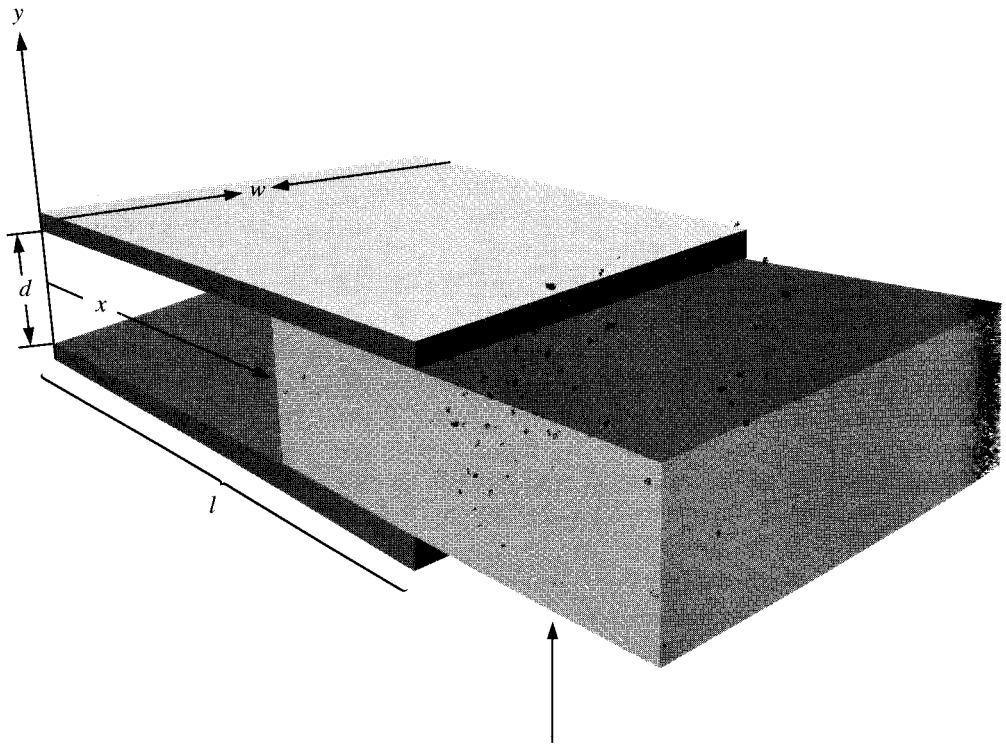

Figure 4.30

Consider, for example, the case of a slab of linear dielectric material, partially inserted between the plates of a parallel-plate capacitor (Fig. 4.30). We have always pretended that the field is uniform inside a parallel-plate capacitor, and zero outside. If this were literally true, there would be no net force on the dielectric at all, since the field everywhere would be perpendicular to the plates. However, there is in reality a **fringing field** around the edges, which for most purposes can be ignored but in this case is responsible for the whole effect. (Indeed, the field *could* not terminate abruptly at the edge of the capacitor, for if it did the line integral of $\mathbf{E}$ around the closed loop shown in Fig. 4.31 would not be zero.) It is this nonuniform fringing field that pulls the dielectric into the capacitor.

Fringing fields are notoriously difficult to calculate; luckily, we can avoid this altogether, by the following ingenious method. Let W be the energy of the system—it depends, of course, on the amount of overlap. If I pull the dielectric out an infinitesimal distance dx, the energy is changed by an amount equal to the work done:

$$dW = F_{\text{me}}\, dx, \tag{4.59}$$

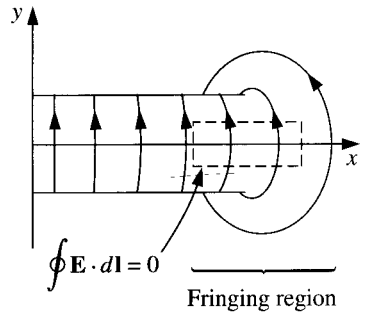

Figure 4.31

where F_{me} is the force I must exert, to counteract the electrical force F on the dielectric: $F_{\text{me}} = -F$. Thus the electrical force on the slab is

$$F = -\frac{dW}{dx}. \tag{4.60}$$

Now, the energy stored in the capacitor is

$$W = \tfrac{1}{2}CV^2, \tag{4.61}$$

and the capacitance in this case is

$$C = \frac{\epsilon_0 w}{d}(\epsilon_r l - \chi_e x), \tag{4.62}$$

where l is the length of the plates (Fig. 4.30). Let's assume that the total charge on the plates ($Q = CV$) is held constant, as the dielectric moves. In terms of Q,

$$W = \frac{1}{2}\frac{Q^2}{C}, \tag{4.63}$$

so

$$F = -\frac{dW}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}. \tag{4.64}$$

But

$$\frac{dC}{dx} = -\frac{\epsilon_0 \chi_e w}{d},$$

and hence

$$F = -\frac{\epsilon_0 \chi_e w}{2d}V^2. \tag{4.65}$$

(The minus sign indicates that the force is in the negative x direction; the dielectric is pulled *into* the capacitor.)

It is a common error to use Eq. 4.61 (with V constant), rather than Eq. 4.63 (with Q constant), in computing the force. One then obtains

$$F = -\frac{1}{2}V^2\frac{dC}{dx},$$

which is off by a sign. It is, of course, *possible* to maintain the capacitor at a fixed potential, by connecting it up to a battery. But in that case the *battery also does work* as the dielectric moves; instead of Eq. 4.59, we now have

$$dW = F_{\text{me}}\,dx + V\,dQ, \tag{4.66}$$

where $V\,dQ$ is the work done by the battery. It follows that

$$F = -\frac{dW}{dx} + V\frac{dQ}{dx} = -\frac{1}{2}V^2\frac{dC}{dx} + V^2\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}, \tag{4.67}$$

the same as before (Eq. 4.64), with the *correct* sign. (Please understand, the force on the dielectric cannot possibly depend on whether you plan to hold Q constant or V constant—it is determined entirely by the distribution of charge, free and bound. It's simpler to *calculate* the force assuming constant Q, because then you don't have to worry about work done by the battery; but if you insist, it can be done correctly either way.)

Notice that we were able to determine the force *without knowing anything about the fringing fields that are ultimately responsible for it!* Of course, it's built into the whole structure of electrostatics that $\nabla \times \mathbf{E} = 0$, and hence that the fringing fields must be present; we're not really getting something for nothing here—just cleverly exploiting the internal consistency of the theory. The energy stored in the fringing fields themselves (which was not accounted for in this derivation) stays constant, as the slab moves; what *does* change is the energy well *inside* the capacitor, where the field is nice and uniform.

Problem 4.28 Two long coaxial cylindrical metal tubes (inner radius a, outer radius b) stand vertically in a tank of dielectric oil (susceptibility χ_e, mass density ρ). The inner one is maintained at potential V, and the outer one is grounded (Fig. 4.32). To what height (h) does the oil rise in the space between the tubes?

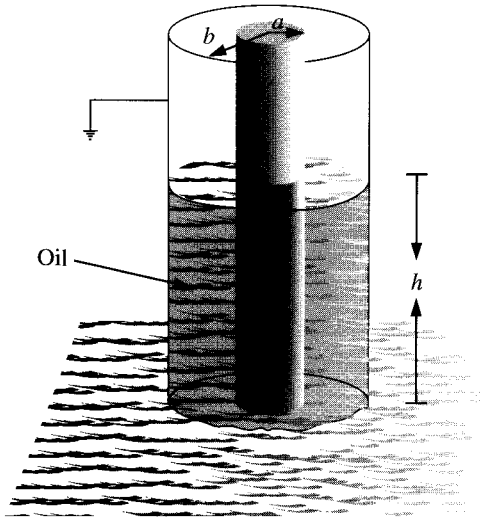

Figure 4.32

More Problems on Chapter 4

Problem 4.29

(a) For the configuration in Prob. 4.5, calculate the *force* on $\mathbf{p}_2$ due to $\mathbf{p}_1$, and the force on $\mathbf{p}_1$ due to $\mathbf{p}_2$. Are the answers consistent with Newton's third law?

(b) Find the total torque on $\mathbf{p}_2$ *with respect to the center of* $\mathbf{p}_1$, and compare it with the torque on $\mathbf{p}_1$ *about that same point.* [*Hint:* combine your answer to (a) with the result of Prob. 4.5.]

Problem 4.30 An electric dipole $\mathbf{p}$, pointing in the y direction, is placed midway between two large conducting plates, as shown in Fig. 4.33. Each plate makes a small angle θ with respect to the x axis, and they are maintained at potentials $\pm V$. What is the *direction* of the net force on $\mathbf{p}$? (There's nothing to *calculate*, here, but do explain your answer qualitatively.)

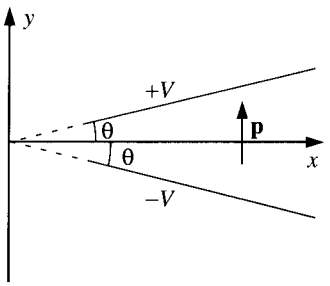

Figure 4.33

Problem 4.31 A dielectric cube of side a, centered at the origin, carries a "frozen-in" polarization $\mathbf{P} = k\mathbf{r}$, where k is a constant. Find all the bound charges, and check that they add up to zero.

Problem 4.32 A point charge q is imbedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius R). Find the electric field, the polarization, and the bound charge densities, ρ_b and σ_b. What is the total bound charge on the surface? Where is the compensating negative bound charge located?

Problem 4.33 At the interface between one linear dielectric and another the electric field lines bend (see Fig. 4.34). Show that

$$\tan\theta_2/\tan\theta_1 = \epsilon_2/\epsilon_1, \tag{4.68}$$

assuming there is no *free* charge at the boundary. [*Comment:* Eq. 4.68 is reminiscent of Snell's law in optics. Would a convex "lens" of dielectric material tend to "focus," or "defocus," the electric field?]

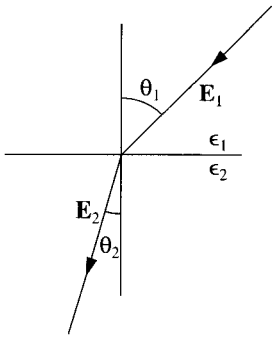

Figure 4.34

! **Problem 4.34** A point dipole $\mathbf{p}$ is imbedded at the center of a sphere of linear dielectric material (with radius R and dielectric constant ϵ_r). Find the electric potential inside and outside the sphere.

$$\left[\textit{Answer}: \ \frac{p\cos\theta}{4\pi\epsilon r^2}\left(1+2\frac{r^3}{R^3}\frac{(\epsilon_r-1)}{(\epsilon_r+2)}\right), \ (r\le R); \quad \frac{p\cos\theta}{4\pi\epsilon_0 r^2}\left(\frac{3}{\epsilon_r+2}\right), \ (r\ge R)\right]$$

Problem 4.35 Prove the following uniqueness theorem: A volume $\mathcal{V}$ contains a specified free charge distribution, and various pieces of linear dielectric material, with the susceptibility of each one given. If the potential is specified on the boundaries $\mathcal{S}$ of $\mathcal{V}$ ($V = 0$ at infinity would be suitable) then the potential throughout $\mathcal{V}$ is uniquely determined. [*Hint:* integrate $\boldsymbol{\nabla}\cdot(V_3\mathbf{D}_3)$ over $\mathcal{V}$.]

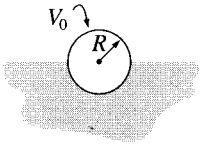

Figure 4.35

Problem 4.36 A conducting sphere at potential V_0 is half embedded in linear dielectric material of susceptibility χ_e, which occupies the region $z < 0$ (Fig. 4.35). *Claim:* the potential everywhere is exactly the same as it would have been in the absence of the dielectric! Check this claim, as follows:

(a) Write down the formula for the proposed potential $V(r)$, in terms of V_0, R, and r. Use it to determine the field, the polarization, the bound charge, and the free charge distribution on the sphere.

(b) Show that the total charge configuration would indeed produce the potential $V(r)$.

(c) Appeal to the uniqueness theorem in Prob. 4.35 to complete the argument.

(d) Could you solve the configurations in Fig. 4.36 with the same potential? If not, explain *why.*

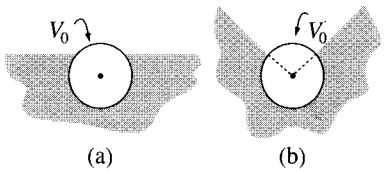

Figure 4.36

Problem 4.37 According to Eq. 4.5, the force on a single dipole is $(\mathbf{p} \cdot \nabla)\mathbf{E}$, so the *net* force on a dielectric object is

$$\mathbf{F} = \int (\mathbf{P} \cdot \nabla)\mathbf{E}_{\text{ext}}\, d\tau. \tag{4.69}$$

[Here $\mathbf{E}_{\text{ext}}$ is the field of everything *except* the dielectric. You might assume that it wouldn't matter if you used the *total* field; after all, the dielectric can't exert a force on *itself.* However, because the field of the dielectric is discontinuous at the location of any bound surface charge, the derivative introduces a spurious delta function, and you must either add a compensating surface term, or (better) stick with $\mathbf{E}_{\text{ext}}$, which suffers no such discontinuity.] Use Eq. 4.69 to determine the force on a tiny sphere of radius R, composed of linear dielectric material of susceptibility χ_e, which is situated a distance s from a fine wire carrying a uniform line charge λ.

! **Problem 4.38** In a linear dielectric, the polarization is proportional to the field: $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$. If the material consists of atoms (or nonpolar molecules), the induced dipole moment of each one is likewise proportional to the field $\mathbf{p} = \alpha \mathbf{E}$. *Question:* What is the relation between the atomic polarizability α and the susceptibility χ_e?

Since $\mathbf{P}$ (the dipole moment per unit volume) is $\mathbf{p}$ (the dipole moment per atom) times N (the number of atoms per unit volume), $\mathbf{P} = N\mathbf{p} = N\alpha\mathbf{E}$, one's first inclination is to say that

$$\chi_e = \frac{N\alpha}{\epsilon_0}. \tag{4.70}$$

And in fact this is not far off, if the density is low. But closer inspection reveals a subtle problem, for the field $\mathbf{E}$ in Eq. 4.30 is the *total macroscopic* field in the medium, whereas the field in Eq. 4.1 is due to everything *except* the particular atom under consideration (polarizability was defined for an isolated atom subject to a specified external field); call this field $\mathbf{E}_{\text{else}}$. Imagine that the space allotted to each atom is a sphere of radius R, and show that

$$\mathbf{E} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right)\mathbf{E}_{\text{else}}. \tag{4.71}$$

Use this to conclude that

$$\chi_e = \frac{N\alpha/\epsilon_0}{1 - N\alpha/3\epsilon_0},$$

or

$$\alpha = \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right). \tag{4.72}$$

Equation 4.72 is known as the **Clausius-Mossotti** formula, or, in its application to optics, the **Lorentz-Lorenz** equation.

Problem 4.39 Check the Clausius-Mossotti relation (Eq. 4.72) for the gases listed in Table 4.1. (Dielectric constants are given in Table 4.2.) (The densities here are so small that Eqs. 4.70 and 4.72 are indistinguishable. For experimental data that confirm the Clausius-Mossotti correction term see, for instance, the first edition of Purcell's *Electricity and Magnetism,* Problem 9.28.)[14]

! **Problem 4.40** The Clausius-Mossotti equation (Prob. 4.38) tells you how to calculate the susceptibility of a *nonpolar* substance, in terms of the atomic polarizability α. The Langevin equation tells you how to calculate the susceptibility of a *polar* substance, in terms of the permanent molecular dipole moment p. Here's how it goes:

(a) The energy of a dipole in an external field $\mathbf{E}$ is $u = -\mathbf{p} \cdot \mathbf{E}$ (Eq. 4.6); it ranges from $-pE$ to $+pE$, depending on the orientation. Statistical mechanics says that for a material in equilibrium at absolute temperature T, the probability of a given molecule having energy u is proportional to the Boltzmann factor,

$$\exp(-u/kT).$$

The average energy of the dipoles is therefore

$$<u> = \frac{\int u e^{-(u/kT)}\, du}{\int e^{-(u/kT)}\, du},$$

[14] E. M. Purcell, *Electricity and Magnetism* (Berkeley Physics Course, Vol. 2), (New York: McGraw-Hill, 1963).

where the integrals run from $-pE$ to $+pE$. Use this to show that the polarization of a substance containing N molecules per unit volume is

$$P = Np[\coth(pE/kT) - (kT/pE)]. \tag{4.73}$$

That's the **Langevin formula**. Sketch P/Np as a function of pE/kT.

(b) Notice that for large fields/low temperatures, virtually *all* the molecules are lined up, and the material is *non*linear. Ordinarily, however, kT is much greater than pE. Show that in this régime the material *is* linear, and calculate its susceptibility, in terms of N, p, T, and k. Compute the susceptibility of water at 20° C, and compare the experimental value in Table 4.2. (The dipole moment of water is 6.1×10^{-30} C·m.) This is rather far off, because we have again neglected the distinction between $\mathbf{E}$ and $\mathbf{E}_{\text{else}}$. The agreement is better in low-density gases, for which the difference between $\mathbf{E}$ and $\mathbf{E}_{\text{else}}$ is negligible. Try it for water vapor at 100° and 1 atm.