2

Damped Simple Harmonic Motion

Initially we discussed the case of ideal simple harmonic motion where the total energy remained constant and the displacement followed a sine curve, apparently for an infinite time. In practice some energy is always dissipated by a resistive or viscous process; for example, the amplitude of a freely swinging pendulum will always decay with time as energy is lost. The presence of resistance to motion means that another force is active, which is taken as being proportional to the velocity. The frictional force acts in the direction opposite to that of the velocity (see Figure 2.1) and so Newton's Second law becomes

$$m\ddot{x} = -sx - r\dot{x}$$

where r is the constant of proportionality and has the dimensions of force per unit of velocity. The presence of such a term will always result in energy loss.

The problem now is to find the behaviour of the displacement x from the equation

$$m\ddot{x} + r\dot{x} + sx = 0 \tag{2.1}$$

where the coefficients m, r and s are constant.

When these coefficients are constant a solution of the form $x = C\,e^{\alpha t}$ can always be found. Obviously, since an exponential term is always nondimensional, C has the dimensions of x (a length, say) and α has the dimensions of inverse time, T^{-1}. We shall see that there are three possible forms of this solution, each describing a different behaviour of the displacement x with time. In two of these solutions C appears explicitly as a constant length, but in the third case it takes the form

$$C = A + Bt^{*}$$

* The number of constants allowed in the general solution of a differential equation is always equal to the order (that is, the highest differential coefficient) of the equation. The two values A and B are allowed because equation (2.1) is second order. The values of the constants are adjusted to satisfy the initial conditions.

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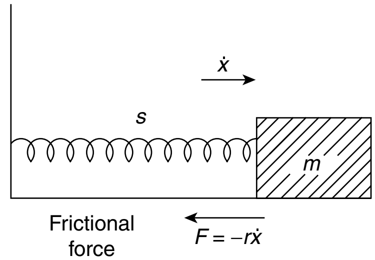


Figure 2.1 Simple harmonic motion system with a damping or frictional force $r\dot{x}$ acting against the direction of motion. The equation of motion is $m\ddot{x} + r\dot{x} + sx = 0$

where A is a length, B is a velocity and t is time, giving C the overall dimensions of a length, as we expect. From our point of view this case is not the most important.

Taking C as a constant length gives $\dot{x} = \alpha C\,\mathrm{e}^{\alpha t}$ and $\ddot{x} = \alpha^2 C\,\mathrm{e}^{\alpha t}$, so that equation (2.1) may be rewritten

$$C\,\mathrm{e}^{\alpha t}(m\alpha^2 + r\alpha + s) = 0$$

so that either

$$x = C\,\mathrm{e}^{\alpha t} = 0 \quad \text{(which is trivial)}$$

or

$$m\alpha^2 + r\alpha + s = 0$$

Solving the quadratic equation in α gives

$$\alpha = \frac{-r}{2m} \pm \sqrt{\frac{r^2}{4m^2} - \frac{s}{m}}$$

Note that $r/2m$ and $(s/m)^{1/2}$, and therefore, α, all have the dimensions of inverse time, T^{-1}, which we expect from the form of $\mathrm{e}^{\alpha t}$.

The displacement can now be expressed as

$$x_1 = C_1\,\mathrm{e}^{-rt/2m + (r^2/4m^2 - s/m)^{1/2}t}, \quad x_2 = C_2\,\mathrm{e}^{-rt/2m - (r^2/4m^2 - s/m)^{1/2}t}$$

or the sum of both these terms

$$x = x_1 + x_2 = C_1\,\mathrm{e}^{-rt/2m + (r^2/4m^2 - s/m)^{1/2}t} + C_2\,\mathrm{e}^{-rt/2m - (r^2/4m^2 - s/m)^{1/2}t}$$

The bracket $(r^2/4m^2 - s/m)$ can be positive, zero or negative depending on the relative magnitude of the two terms inside it. Each of these conditions gives one of the three possible solutions referred to earlier and each solution describes a particular kind of behaviour. We shall discuss these solutions in order of *increasing* significance from our point of view; the third solution is the one we shall concentrate upon throughout the rest of this book.

The conditions are:

(1) *Bracket positive* $(r^2/4m^2 > s/m)$. Here the damping resistance term $r^2/4m^2$ dominates the stiffness term s/m, and heavy damping results in a *dead beat* system.
(2) *Bracket zero* $(r^2/4m^2 = s/m)$. The balance between the two terms results in a *critically damped* system.

Neither (1) nor (2) gives oscillatory behaviour.

(3) *Bracket negative* $(r^2/4m^2 < s/m)$. The system is lightly damped and gives oscillatory *damped simple harmonic motion.*

Case 1. Heavy Damping

Writing $r/2m = p$ and $(r^2/4m^2 - s/m)^{1/2} = q$, we can replace

$$x = C_1\,\mathrm{e}^{-rt/2m+(r^2/4m^2-s/m)^{1/2}t} + C_2\,\mathrm{e}^{-rt/2m-(r^2/4m^2-s/m)^{1/2}t}$$

by

$$x = \mathrm{e}^{-pt}(C_1\,\mathrm{e}^{qt} + C_2\,\mathrm{e}^{-qt}),$$

where the C_1 and C_2 are arbitrary in value but have the same dimensions as C (note that two separate values of C are allowed because the differential equation (2.1) is second order).

If now $F = C_1 + C_2$ and $G = C_1 - C_2$, the displacement is given by

$$x = \mathrm{e}^{-pt}\left[\frac{F}{2}(\mathrm{e}^{qt} + \mathrm{e}^{-qt}) + \frac{G}{2}(\mathrm{e}^{qt} - \mathrm{e}^{-qt})\right]$$

or

$$x = \mathrm{e}^{-pt}(F\cosh qt + G\sinh qt)$$

This represents non-oscillatory behaviour, but the actual displacement will depend upon the initial (or boundary) conditions; that is, the value of x at time $t = 0$. If $x = 0$ at $t = 0$ then $F = 0$, and

$$x = G\,\mathrm{e}^{-rt/2m}\sinh\left(\frac{r^2}{4m^2} - \frac{s}{m}\right)^{1/2} t$$

Figure 2.2 illustrates such behaviour when a heavily damped system is disturbed from equilibrium by a sudden impulse (that is, given a velocity at $t = 0$). It will return to zero

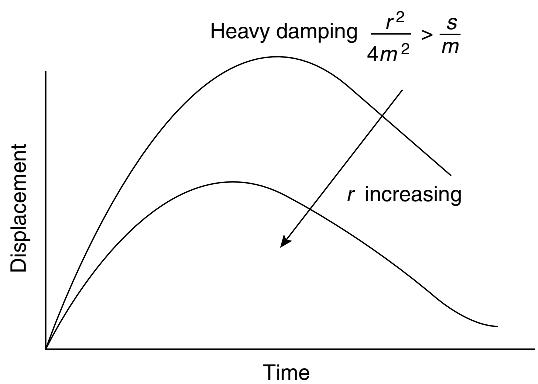


Figure 2.2 Non-oscillatory behaviour of damped simple harmonic system with heavy damping (where $r^2/4m^2 > s/m$) after the system has been given an impulse from a rest position $x = 0$

displacement quite slowly without oscillating about its equilibrium position. More advanced mathematics shows that the value of the velocity dx/dt vanishes only once so that there is only one value of maximum displacement.

(Problem 2.1)

Case 2. Critical Damping $(r^2/4m^2 = s/m)$

Using the notation of Case 1, we see that $q = 0$ and that $x = e^{-pt}(C_1 + C_2)$. This is, in fact, the limiting case of the behaviour of Case I as q changes from positive to negative. In this case the quadratic equation in α has equal roots, which, in a differential equation solution, demands that C must be written $C = A + Bt$, where A is a constant length and B a given velocity which depends on the boundary conditions. It is easily verified that the value

$$x = (A + Bt)e^{-rt/2m} = (A + Bt)e^{-pt}$$

satisfies $m\ddot{x} + r\dot{x} + sx = 0$ when $r^2/4m^2 = s/m$.

(Problem 2.2)

Application to a Damped Mechanical Oscillator

Critical damping is of practical importance in mechanical oscillators which experience sudden impulses and are required to return to zero displacement in the minimum time. Suppose such a system has zero displacement at $t = 0$ and receives an impulse which gives it an initial velocity V.

Then $x = 0$ (so that $A = 0$) and $\dot{x} = V$ at $t = 0$. However,

$$\dot{x} = B[(-pt)\mathrm{e}^{-pt} + \mathrm{e}^{-pt}] = B \text{ at } t = 0$$

so that $B = V$ and the complete solution is

$$x = Vt\,\mathrm{e}^{-pt}$$

The maximum displacement x occurs when the system comes to rest before returning to zero displacement. At maximum displacement

$$\dot{x} = V\,\mathrm{e}^{-pt}(1 - pt) = 0$$

thus giving $(1 - pt) = 0$, i.e. $t = 1/p$.

At this time the displacement is therefore

$$\begin{aligned} x &= Vt\,\mathrm{e}^{-pt} = \frac{V}{p}\,\mathrm{e}^{-1} \\ &= 0.368\frac{V}{p} = 0.368\frac{2mV}{r} \end{aligned}$$

The curve of displacement versus time is shown in Figure 2.3; the return to zero in a critically damped system is reached in *minimum* time.

Case 3. Damped Simple Harmonic Motion

When $r^2/4m^2 < s/m$ the damping is light, and this gives from the present point of view the most important kind of behaviour, *oscillatory damped simple harmonic motion.*

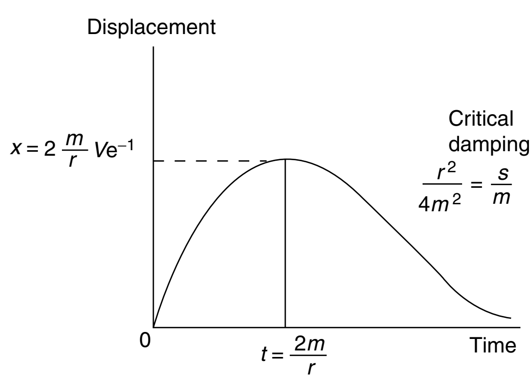


Figure 2.3 Limiting case of non-oscillatory behaviour of damped simple harmonic system where $r^2/4m^2 = s/m$ (critical damping)

The expression $(r^2/4m^2 - s/m)^{1/2}$ is an imaginary quantity, the square root of a negative number, which can be rewritten

$$\begin{aligned}\pm\left(\frac{r^2}{4m^2}-\frac{s}{m}\right)^{1/2} &= \pm\sqrt{-1}\left(\frac{s}{m}-\frac{r^2}{4m^2}\right)^{1/2}\\ &= \pm\mathrm{i}\left(\frac{s}{m}-\frac{r^2}{4m^2}\right)^{1/2}\text{(where } i=\sqrt{-1}\text{)}\end{aligned}$$

so the displacement

$$x = C_1\,\mathrm{e}^{-rt/2m}\,\mathrm{e}^{+\mathrm{i}(s/m-r^2/4m^2)^{1/2}t} + C_2\,\mathrm{e}^{-rt/2m}\,\mathrm{e}^{-\mathrm{i}(s/m-r^2/4m^2)^{1/2}t}$$

The bracket has the dimensions of inverse time; that is, of frequency, and can be written $(s/m - r^2/4m^2)^{1/2} = \omega'$, so that the second exponential becomes $\mathrm{e}^{\mathrm{i}\omega' t} = \cos\omega' t + \mathrm{i}\sin\omega' t$. This shows that the behaviour of the displacement x is oscillatory with a new frequency $\omega' < \omega = (s/m)^{1/2}$, the frequency of ideal simple harmonic motion. To compare the behaviour of the damped oscillator with the ideal case we should like to express the solution in a form similar to $x = A\,\sin(\omega' t + \phi)$ as in the ideal case, where ω has been replaced by ω'.

We can do this by writing

$$x = \mathrm{e}^{-rt/2m}(C_1\,\mathrm{e}^{\mathrm{i}\omega' t} + C_2\,\mathrm{e}^{-\mathrm{i}\omega' t})$$

If we now choose

$$C_1 = \frac{A}{2\mathrm{i}}\,\mathrm{e}^{\mathrm{i}\phi}$$

and

$$C_2 = -\frac{A}{2\mathrm{i}}\,\mathrm{e}^{-\mathrm{i}\phi}$$

where A and ϕ (and thus $\mathrm{e}^{\mathrm{i}\phi}$) are constants which depend on the motion at $t = 0$, we find after substitution

$$\begin{aligned}x &= A\,\mathrm{e}^{-rt/2m}\,\frac{[\mathrm{e}^{\mathrm{i}(\omega' t+\phi)} - \mathrm{e}^{-\mathrm{i}(\omega' t+\phi)}]}{2\mathrm{i}}\\ &= A\,\mathrm{e}^{-rt/2m}\sin(\omega' t + \phi)\end{aligned}$$

This procedure is equivalent to imposing the boundary condition $x = A\sin\phi$ at $t = 0$ upon the solution for x. The displacement therefore varies sinusoidally with time as in the case of simple harmonic motion, but now has a new frequency

$$\omega' = \left(\frac{s}{m}-\frac{r^2}{4m^2}\right)^{1/2}$$

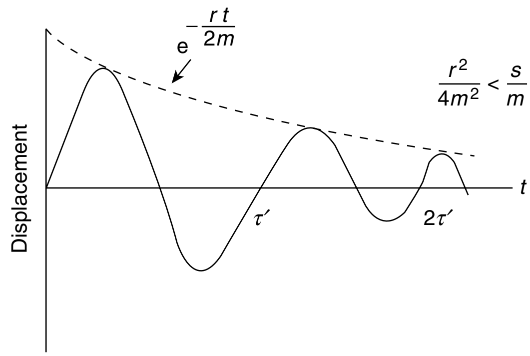


Figure 2.4 Damped oscillatory motion where $s/m > r^2/4m^2$. The amplitude decays with $e^{-rt/2m}$, and the reduced angular frequency is given by $\omega'^2 = s/m - r^2/4m^2$

and its amplitude A is modified by the exponential term $e^{-rt/2m}$, a term which decays with time.

If $x = 0$ at $t = 0$ then $\phi = 0$; Figure 2.4 shows the behaviour of x with time, its oscillations gradually decaying with the envelope of maximum amplitudes following the dotted curve $e^{-rt/2m}$. The constant A is obviously the value to which the amplitude would have risen at the first maximum if no damping were present.

The presence of the force term $r\dot{x}$ in the equation of motion therefore introduces a loss of energy which causes the amplitude of oscillation to decay with time as $e^{-rt/2m}$.

(Problem 2.3)

Methods of Describing the Damping of an Oscillator

Earlier in this chapter we saw that the energy of an oscillator is given by

$$E = \tfrac{1}{2}ma^2\omega^2 = \tfrac{1}{2}sa^2$$

that is, proportional to the square of its amplitude.

We have just seen that in the presence of a damping force $r\dot{x}$ the amplitude decays with time as

$$e^{-rt/2m}$$

so that the energy decay will be proportional to

$$(e^{-rt/2m})^2$$

that is, $e^{-rt/m}$. The larger the value of the damping force r the more rapid the decay of the amplitude and energy. Thus we can use the exponential factor to express the rates at which the amplitude and energy are reduced.

Logarithmic Decrement

This measures the rate at which the *amplitude* dies away. Suppose in the expression

$$x = A\,\mathrm{e}^{-rt/2m}\sin(\omega' t + \phi)$$

we choose

$$\phi = \pi/2$$

and we write

$$x = A_0\,\mathrm{e}^{-rt/2m}\cos\omega' t$$

with $x = A_0$ at $t = 0$. Its behaviour will follow the curve in Figure 2.5.

If the period of oscillation is τ' where $\omega' = 2\pi/\tau'$, then one period later the amplitude is given by

$$A_1 = A_0\,\mathrm{e}^{(-r/2m)\tau'}$$

so that

$$\frac{A_0}{A_1} = \mathrm{e}^{r\tau'/2m} = \mathrm{e}^{\delta}$$

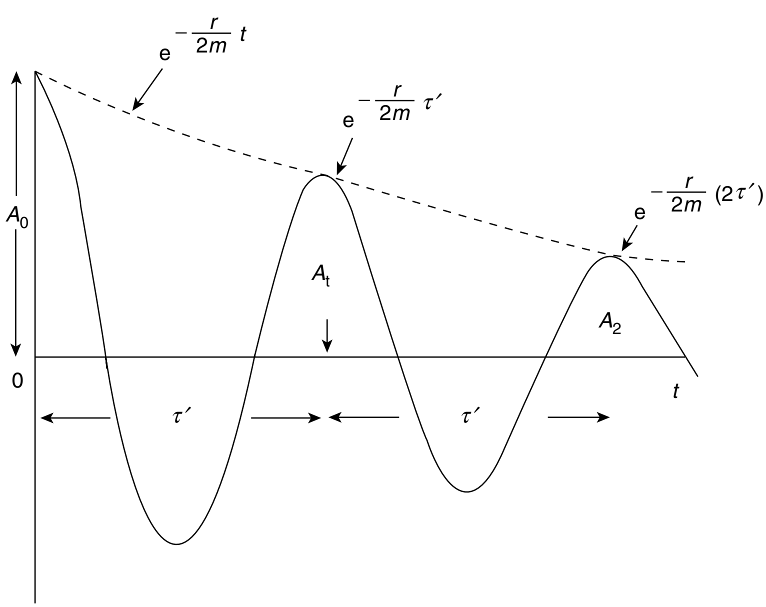


Figure 2.5 The logarithmic ratio of any two amplitudes one period apart is the logarithmic decrement, defined as $\delta = \log_{\mathrm{e}}(A_n/A_{n+1}) = r\tau'/2m$

where

$$\delta = \frac{r}{2m}\tau' = \log_e \frac{A_0}{A_1}$$

is called the *logarithmic decrement.* (Note that this use of δ differs from that in Figure 1.11). The logarithmic decrement δ is the logarithm of the ratio of two amplitudes of oscillation which are separated by one period, the larger amplitude being the numerator since $e^{\delta} > 1$.

Similarly

$$\frac{A_0}{A_2} = e^{r(2\tau')/2m} = e^{2\delta}$$

and

$$\frac{A_0}{A_n} = e^{n\delta}$$

Experimentally, the value of δ is best found by comparing amplitudes of oscillations which are separated by n periods. The graph of

$$\log_e \frac{A_0}{A_n}$$

versus n for different values of n has a slope δ.

Relaxation Time or Modulus of Decay

Another way of expressing the damping effect is by means of the time taken for the amplitude to decay to

$$e^{-1} = 0.368$$

of its original value A_0. This time is called the *relaxation time* or *modulus of decay* and the amplitude

$$A_t = A_0\, e^{-rt/2m} = A_0\, e^{-1}$$

at a time $t = 2m/r$.

Measuring the natural decay in terms of the fraction e^{-1} of the original value is a very common procedure in physics. The time for a natural decay process to reach zero is, of course, theoretically infinite.

(Problem 2.4)

The Quality Factor or Q-value of a Damped Simple Harmonic Oscillator

This measures the rate at which the *energy* decays. Since the decay of the amplitude is represented by

$$A = A_0\, e^{-rt/2m}$$

the decay of energy is proportional to

$$A^2 = A_0^2 \, \mathrm{e}^{(-rt/2m)^2}$$

and may be written

$$E = E_0 \, \mathrm{e}^{(-r/m)t}$$

where E_0 is the energy value at $t = 0$.

The time for the energy E to decay to $E_0 \, \mathrm{e}^{-1}$ is given by $t = m/r$ s during which time the oscillator will have vibrated through $\omega' m/r$ rad.

We define the *quality factor*

$$Q = \frac{\omega' m}{r}$$

as the *number of radians through which the damped system oscillates as its energy decays to*

$$E = E_0 \, \mathrm{e}^{-1}$$

If r is small, then Q is very large and

$$\frac{s}{m} \gg \frac{r^2}{4m^2}$$

so that

$$\omega' \approx \omega_0 = \left(\frac{s}{m} \right)^{1/2}$$

Thus, we write, to a very close approximation,

$$Q = \frac{\omega_0 m}{r}$$

which is a constant of the damped system.

Since r/m now equals ω_0/Q we can write

$$E = E_0 \, \mathrm{e}^{(-r/m)t} = E_0 \, \mathrm{e}^{-\omega_0 t/Q}$$

The fact that Q is a constant $(= \omega_0 m/r)$ implies that the ratio

$$\frac{\text{energy stored in system}}{\text{energy lost per cycle}}$$

is also a constant, for

$$\frac{Q}{2\pi} = \frac{\omega_0 m}{2\pi r} = \frac{\nu_0 m}{r}$$

is the number of *cycles* (or complete oscillations) through which the system moves in decaying to

$$E = E_0 \, \mathrm{e}^{-1}$$

and if

$$E = E_0 \, \mathrm{e}^{(-r/m)t}$$

the energy lost per cycle is

$$-\Delta E = \frac{\mathrm{d}E}{\mathrm{d}t} \, \Delta t = \frac{-r}{m} E \frac{1}{\nu'}$$

where $\Delta t = 1/\nu' = \tau'$, the period of oscillation.

Thus, the ratio

$$\frac{\text{energy stored in system}}{\text{energy lost per cycle}} = \frac{E}{-\Delta E} = \frac{\nu' m}{r} \approx \frac{\nu_0 m}{r}$$
$$= \frac{Q}{2\pi}$$

In the next chapter we shall meet the same quality factor Q in two other roles, the first as a measure of the power absorption bandwidth of a damped oscillator driven near its resonant frequency and again as the factor by which the displacement of the oscillator is amplified at resonance.

Example on the Q-value of a Damped Simple Harmonic Oscillator

An electron in an atom which is freely radiating power behaves as a damped simple harmonic oscillator.

If the radiated power is given by $P = q^2\omega^4 x_0^2/12\pi\varepsilon_0 c^3$ W at a wavelength of 0.6 µm (6000 Å), show that the Q-value of the atom is about 10^8 and that its free radiation lifetime is about 10^{-8}s (the time for its energy to decay to e^{-1} of its original value).

$$\begin{aligned} q &= 1.6 \times 10^{-19}\mathrm{C} \\ 1/4\pi\varepsilon_0 &= 9 \times 10^9 \,\mathrm{m\,F^{-1}} \\ m_e &= 9 \times 10^{-31}\,\mathrm{kg} \\ c &= 3 \times 10^8 \,\mathrm{m\,s^{-1}} \\ x_0 &= \text{maximum amplitude of oscillation} \end{aligned}$$

The radiated power P is $-\nu \, \Delta E$, where $-\Delta E$ is the energy loss per cycle, and the energy of the oscillator is given by $E = \frac{1}{2} m_e \omega^2 x_0^2$.

Thus, $Q = 2\pi E / -\Delta E = \nu\pi m_e \omega^2 x_0^2 / P$, and inserting the values above with $\omega = 2\pi\nu = 2\pi c/\lambda$, where the wavelength λ is given, yields a Q value of $\sim 5 \times 10^7$.

The relation $Q = \omega t$ gives t, the radiation lifetime, a value of $\sim 10^{-8}$ s.

Energy Dissipation

We have seen that the presence of the resistive force reduces the amplitude of oscillation with time as energy is dissipated.

The total energy remains the sum of the kinetic and potential energies

$$E = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} sx^2$$

Now, however, $\mathrm{d}E/\mathrm{d}t$ is not zero but negative because energy is lost, so that

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} sx^2\right) = \dot{x}(m\ddot{x} + sx) \\ &= \dot{x}(-r\dot{x}) \quad \text{for} \quad m\dot{x} + r\dot{x} + sx = 0 \end{aligned}$$

i.e. $\mathrm{d}E/\mathrm{d}t = -r\dot{x}^2$, which is the rate of doing work against the frictional force (dimensions of force $\times$ velocity $=$ force $\times$ distance/time).

(Problems 2.5, 2.6)

Damped SHM in an Electrical Circuit

The force equation in the mechanical oscillator is replaced by the voltage equation in the electrical circuit of inductance, resistance and capacitance (Figure 2.6).

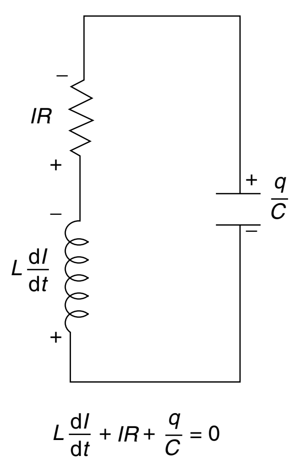


Figure 2.6 Electrical circuit of inductance, capacitance and resistance capable of damped simple harmonic oscillations. The sum of the voltages around the circuit is given from Kirchhoff's law as $L\dfrac{\mathrm{d}I}{\mathrm{d}t} + RI + \dfrac{q}{C} = 0$

We have, therefore,

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + RI + \frac{q}{C} = 0$$

or

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = 0$$

and by comparison with the solutions for x in the mechanical case we know immediately that the charge

$$q = q_0\,\mathrm{e}^{-Rt/2L \pm (R^2/4L^2 - 1/LC)^{1/2}\,t}$$

which, for $1/LC > R^2/4L^2$, gives oscillatory behaviour at a frequency

$$\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

From the exponential decay term we see that R/L has the dimensions of inverse time T^{-1} or ω, so that ωL has the dimensions of R; that is, ωL is measured in ohms.

Similarly, since $\omega^2 = 1/LC, \omega L = 1/\omega C$, so that $1/\omega C$ is also measured in ohms. We shall use these results in the next chapter.

(Problems 2.7, 2.8, 2.9)

Problem 2.1

The heavily damped simple harmonic system of Figure 2.2 is displaced a distance F from its equilibrium position and released from rest. Show that in the expression for the displacement

$$x = \mathrm{e}^{-pt}(F\cosh qt + G\sinh qt)$$

where

$$p = \frac{r}{2m} \quad \text{and} \quad q = \left(\frac{r^2}{4m^2} - \frac{s}{m}\right)^{1/2}$$

that the ratio

$$\frac{G}{F} = \frac{r}{(r^2 - 4ms)^{1/2}}$$

Problem 2.2

Verify that the solution

$$x = (A + Bt)\mathrm{e}^{-rt/2m}$$

satisfies the equation

$$m\ddot{x} + r\dot{x} + sx = 0$$

when

$$r^2/4m^2 = s/m$$

Problem 2.3

The solution for damped simple harmonic motion is given by

$$x = e^{-rt/2m}(C_1 e^{i\omega' t} + C_2 e^{-i\omega' t})$$

If $x = A\cos\phi$ at $t = 0$, find the values of C_1 and C_2 to show that $\dot{x} \approx -\omega' A \sin\phi$ at $t = 0$ only if r/m is very small or $\phi \approx \pi/2$.

Problem 2.4

A capacitance C with a charge q_0 at $t = 0$ discharges through a resistance R. Use the voltage equation $q/C + IR = 0$ to show that the relaxation time of this process is RC s; that is,

$$q = q_0 e^{-t/RC}$$

(Note that t/RC is non-dimensional.)

Problem 2.5

The frequency of a damped simple harmonic oscillator is given by

$$\omega'^2 = \frac{s}{m} - \frac{r^2}{4m^2} = \omega_0^2 - \frac{r^2}{4m^2}$$

(a) If $\omega_0^2 - \omega'^2 = 10^{-6}\omega_0^2$ show that $Q = 500$ and that the logarithmic decrement $\delta = \pi/500$.
(b) If $\omega_0 = 10^6$ and $m = 10^{-10}$ Kg show that the stiffness of the system is $100\,\mathrm{N\,m^{-1}}$, and that the resistive constant r is $2 \times 10^{-7}\ \mathrm{N \cdot sm^{-1}}$.
(c) If the maximum displacement at $t = 0$ is 10^{-2} m, show that the energy of the system is 5×10^{-3} J and the decay to e^{-1} of this value takes 0.5 ms.
(d) Show that the energy loss in the first cycle is $2\pi \times 10^{-5}$ J.

Problem 2.6

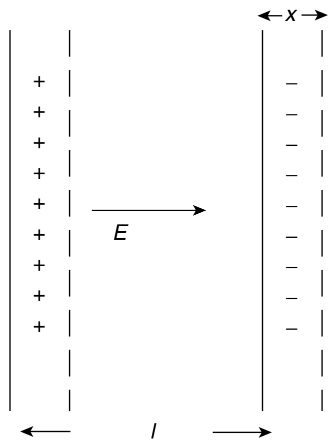
Show that the fractional change in the resonant frequency $\omega_0(\omega_0^2 = s/m)$ of a damped simple harmonic mechanical oscillator is $\approx (8Q^2)^{-1}$ where Q is the quality factor.

Problem 2.7

Show that the quality factor of an electrical LCR series circuit is $Q = \omega_0 L/R$ where $\omega_0^2 = 1/LC$

Problem 2.8

A plasma consists of an ionized gas of ions and electrons of equal number densities $(n_i = n_e = n)$ having charges of opposite sign $\pm e$, and masses m_i and m_e, respectively, where $m_i > m_e$. Relative displacement between the two species sets up a restoring



electric field which returns the electrons to equilibrium, the ions being considered stationary. In the diagram, a plasma slab of thickness l has all its electrons displaced a distance x to give a restoring electric field $E = nex/\varepsilon_0$, where ε_0 is constant. Show that the restoring force per unit area on the electrons is xn^2e^2l/ε_0 and that they oscillate simple harmonically with angular frequency $\omega_e^2 = ne^2/m_e\varepsilon_0$. This frequency is called the electron plasma frequency, and only those radio waves of frequency $\omega > \omega_e$ will propagate in such an ionized medium. Hence the reflection of such waves from the ionosphere.

Problem 2.9

A simple pendulum consists of a mass m at the end of a string of length l and performs small oscillations. The length is very slowly shortened whilst the pendulum oscillates many times at a constant amplitude $l\theta$ where θ is very small. Show that if the length is changed by $-\Delta l$ the work done is $-mg\,\Delta l$ (owing to the elevation of the position of equilibrium) together with an increase in the pendulum energy

$$\Delta E = \left(mg\frac{\overline{\theta^2}}{2} - ml\overline{\dot{\theta}^2} \right)\Delta l$$

where $\overline{\theta^2}$ is the average value of θ^2 during the shortening. If $\theta = \theta_0 \cos \omega t$, show that the energy of the pendulum at any instant may be written

$$E = \frac{ml^2\omega^2\theta_0^2}{2} = \frac{mgl\theta_0^2}{2}$$

and hence show that

$$\frac{\Delta E}{E} = -\frac{1}{2}\frac{\Delta l}{l} = \frac{\Delta\nu}{\nu}$$

that is, E/ν, the ratio of the energy of the pendulum to its frequency of oscillation remains constant during the slowly changing process. (This constant ratio under slowly varying conditions is important in quantum theory where the constant is written as a multiple of Planck's constant, h.)

Summary of Important Results

Damped Simple Harmonic Motion

Equation of motion $m\ddot{x} + r\dot{x} + sx = 0$
Oscillations when

$$\frac{s}{m} > \frac{r^2}{4m^2}$$

Displacement $x = A\,\mathrm{e}^{-rt/2m}\cos(\omega' t + \phi)$ where

$$\omega'^2 = \frac{s}{m} - \frac{r^2}{4m^2}$$

Amplitude Decay

Logarithmic decrement δ—the logarithm of the ratio of two successive amplitudes one period τ' apart

$$\delta = \log_{\mathrm{e}} \frac{A_n}{A_{n+1}} = \frac{r\tau'}{2m}$$

Relaxation Time

Time for amplitude to decay to $A = A_0\,\mathrm{e}^{-rt/2m} = A_0\,\mathrm{e}^{-1}$; that is, $t = 2m/r$

Energy Decay

Quality factor Q is the number of radians during which energy decreases to $E = E_0\,\mathrm{e}^{-1}$

$$Q = \frac{\omega_0 m}{r} = 2\pi \frac{\text{energy stored in system}}{\text{energy lost per cycle}}$$
$$E = E_0\,\mathrm{e}^{-rt/m} = E_0\,\mathrm{e}^{-1} \quad \text{when } Q = \omega_0 t$$

In damped SHM

$$\frac{\mathrm{d}E}{\mathrm{d}t} = (m\ddot{x} + sx)\dot{x} = -r\dot{x}^2 \quad \text{(work rate of resistive force)}$$

For equivalent expressions in electrical oscillators replace m by L, r by R and s by $1/C$. Force equations become voltage equations.