Chapter 3

Special Techniques

3.1 Laplace's Equation

3.1.1 Introduction

The primary task of electrostatics is to find the electric field of a given stationary charge distribution. In principle, this purpose is accomplished by Coulomb's law, in the form of Eq. 2.8:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\mathbf{r}')\,d\tau'. \tag{3.1}$$

Unfortunately, integrals of this type can be difficult to calculate for any but the simplest charge configurations. Occasionally we can get around this by exploiting symmetry and using Gauss's law, but ordinarily the best strategy is first to calculate the *potential*, V, which is given by the somewhat more tractable Eq. 2.29:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'. \tag{3.2}$$

Still, even *this* integral is often too tough to handle analytically. Moreover, in problems involving conductors ρ itself may not be known in advance: since charge is free to move around, the only thing we control directly is the *total* charge (or perhaps the potential) of each conductor.

In such cases it is fruitful to recast the problem in differential form, using Poisson's equation (2.24),

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho, \tag{3.3}$$

which, together with appropriate boundary conditions, is equivalent to Eq. 3.2. Very often, in fact, we are interested in finding the potential in a region where $\rho = 0$. (If $\rho = 0$ *everywhere,* of course, then $V = 0$, and there is nothing further to say—that's not what I mean. There may be plenty of charge *elsewhere,* but we're confining our attention to places where there is no charge.) In this case Poisson's equation reduces to Laplace's equation:

$$\nabla^2 V = 0, \tag{3.4}$$

or, written out in Cartesian coordinates,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{3.5}$$

This formula is so fundamental to the subject that one might almost say electrostatics *is* the study of Laplace's equation. At the same time, it is a ubiquitous equation, appearing in such diverse branches of physics as gravitation and magnetism, the theory of heat, and the study of soap bubbles. In mathematics it plays a major role in analytic function theory. To get a feel for Laplace's equation and its solutions (which are called **harmonic functions**), we shall begin with the one- and two-dimensional versions, which are easier to picture and illustrate all the essential properties of the three-dimensional case (though the one-dimensional example lacks the richness of the other two).

3.1.2 Laplace's Equation in One Dimension

Suppose V depends on only one variable, x. Then Laplace's equation becomes

$$\frac{d^2 V}{dx^2} = 0.$$

The general solution is

$$V(x) = mx + b, \tag{3.6}$$

the equation for a straight line. It contains two undetermined constants (m and b), as is appropriate for a second-order (ordinary) differential equation. They are fixed, in any particular case, by the boundary conditions of that problem. For instance, it might be specified that $V = 4$ at $x = 1$, and $V = 0$ at $x = 5$. In that case $m = -1$ and $b = 5$, so $V = -x + 5$ (see Fig. 3.1).

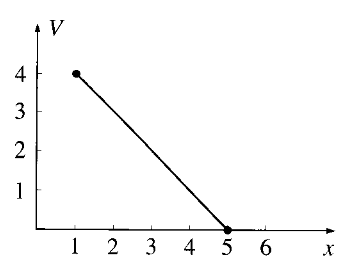

Figure 3.1

I want to call your attention to two features of this result; they may seem silly and obvious in one dimension, where I can write down the general solution explicitly, but the analogs in two and three dimensions are powerful and by no means obvious:

1. $V(x)$ is the *average* of $V(x+a)$ and $V(x-a)$, for any a:

$$V(x) = \tfrac{1}{2}[V(x+a) + V(x-a)].$$

   Laplace's equation is a kind of averaging instruction; it tells you to assign to the point x the average of the values to the left and to the right of x. Solutions to Laplace's equation are, in this sense, *as boring as they could possibly be,* and yet fit the end points properly.

2. Laplace's equation tolerates *no local maxima or minima;* extreme values of V must occur at the end points. Actually, this is a consequence of (1), for if there *were* a local maximum, V at that point would be greater than on either side, and therefore could not be the average. (Ordinarily, you expect the second derivative to be negative at a maximum and positive at a minimum. Since Laplace's equation requires, on the contrary, that the second derivative be zero, it seems reasonable that solutions should exhibit no extrema. However, this is not a *proof,* since there exist functions that have maxima and minima at points where the second derivative vanishes: x^4, for example, has such a minimum at the point $x = 0$.)

3.1.3 Laplace's Equation in Two Dimensions

If V depends on two variables, Laplace's equation becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

This is no longer an *ordinary* differential equation (that is, one involving ordinary derivatives only); it is a *partial* differential equation. As a consequence, some of the simple rules you may be familiar with do not apply. For instance, the general solution to this equation doesn't contain just two arbitrary constants—or, for that matter, *any* finite number—despite the fact that it's a second-order equation. Indeed, one cannot write down a "general solution" (at least, not in a closed form like Eq. 3.6). Nevertheless, it is possible to deduce certain properties common to all solutions.

It may help to have a physical example in mind. Picture a thin rubber sheet (or a soap film) stretched over some support. For definiteness, suppose you take a cardboard box, cut a wavy line all the way around, and remove the top part (Fig. 3.2). Now glue a tightly stretched rubber membrane over the box, so that it fits like a drum head (it won't be a *flat* drumhead, of course, unless you chose to cut the edges off straight). Now, if you lay out coordinates (x, y) on the bottom of the box, the height $V(x, y)$ of the sheet above the point

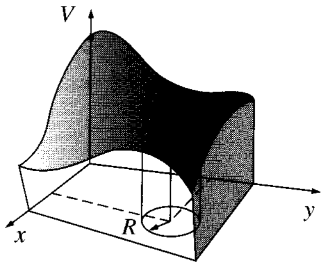

Figure 3.2

(x, y) will satisfy Laplace's equation.[1] (The one-dimensional analog would be a rubber band stretched between two points. Of course, it would form a straight line.)

Harmonic functions in two dimensions have the same properties we noted in one dimension:

1. The value of V at a point (x, y) is the average of those *around* the point. More precisely, if you draw a circle of any radius R about the point (x, y), the average value of V on the circle is equal to the value at the center:

$$V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle}} V \, dl.$$

(This, incidentally, suggests the **method of relaxation** on which computer solutions to Laplace's equation are based: Starting with specified values for V at the boundary, and reasonable guesses for V on a grid of interior points, the first pass reassigns to each point the average of its nearest neighbors. The second pass repeats the process, using the corrected values, and so on. After a few iterations, the numbers begin to settle down, so that subsequent passes produce negligible changes, and a numerical solution to Laplace's equation, with the given boundary values, has been achieved.)[2]

2. V has no local maxima or minima; all extrema occur at the boundaries. (As before, this follows from (1).) Again, Laplace's equation picks the most featureless function possible, consistent with the boundary conditions: no hills, no valleys, just the smoothest surface available. For instance, if you put a ping-pong ball on the stretched rubber sheet of Fig. 3.2, it will roll over to one side and fall off—it will not find a

[1] Actually, the equation satisfied by a rubber sheet is

$$\frac{\partial}{\partial x}\left(g\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(g\frac{\partial V}{\partial y}\right) = 0, \quad \text{where } g = \left[1 + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right]^{-1/2};$$

it reduces (approximately) to Laplace's equation as long as the surface does not deviate too radically from a plane.

[2] See, for example, E. M. Purcell, *Electricity and Magnetism*, 2nd ed., problem 3.30 (p. 119) (New York: McGraw-Hill, 1985).

"pocket" somewhere to settle into, for Laplace's equation allows no such dents in the surface. From a geometrical point of view, just as a straight line is the shortest distance between two points, so a harmonic function in two dimensions minimizes the surface area spanning the given boundary line.

3.1.4 Laplace's Equation in Three Dimensions

In three dimensions I can neither provide you with an explicit solution (as in one dimension) nor offer a suggestive physical example to guide your intuition (as I did in two dimensions). Nevertheless, the same two properties remain true, and this time I will sketch a proof.

1. The value of V at point $\mathbf{r}$ is the average value of V over a spherical surface of radius R centered at $\mathbf{r}$:
$$V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V\, da.$$

2. As a consequence, V can have no local maxima or minima; the extreme values of V must occur at the boundaries. (For if V had a local maximum at $\mathbf{r}$, then by the very nature of maximum I could draw a sphere around $\mathbf{r}$ over which all values of V—and *a fortiori* the average—would be less than at $\mathbf{r}$.)

Proof: Let's begin by calculating the average potential over a spherical surface of radius R due to a *single* point charge q located outside the sphere. We may as well center the sphere at the origin and choose coordinates so that q lies on the z-axis (Fig. 3.3). The potential at a point on the surface is

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}},$$

where

$$\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta,$$

so

$$\begin{aligned} V_{\text{ave}} &= \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int [z^2 + R^2 - 2zR\cos\theta]^{-1/2} R^2 \sin\theta\, d\theta\, d\phi \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\sqrt{z^2 + R^2 - 2zR\cos\theta}\,\Big|_0^\pi \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}[(z+R) - (z-R)] = \frac{1}{4\pi\epsilon_0}\frac{q}{z}. \end{aligned}$$

But this is precisely the potential due to q at the *center* of the sphere! By the superposition principle, the same goes for any *collection* of charges outside the sphere: their average potential over the sphere is equal to the net potential they produce at the center. qed

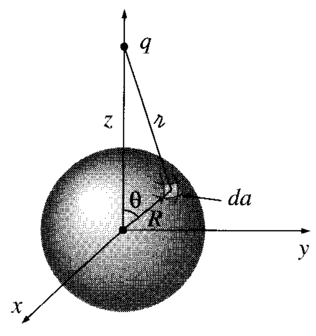

Figure 3.3

Problem 3.1 Find the average potential over a spherical surface of radius R due to a point charge q located *inside* (same as above, in other words, only with $z < R$). (In this case, of course, Laplace's equation does not hold within the sphere.) Show that, in general,

$$V_{\text{ave}} = V_{\text{center}} + \frac{Q_{\text{enc}}}{4\pi\epsilon_0 R},$$

where V_{center} is the potential at the center due to all the *external* charges, and Q_{enc} is the total enclosed charge.

Problem 3.2 In one sentence, justify **Earnshaw's Theorem:** *A charged particle cannot be held in a stable equilibrium by electrostatic forces alone.* As an example, consider the cubical arrangement of fixed charges in Fig. 3.4. It *looks,* off hand, as though a positive charge at the center would be suspended in midair, since it is repelled away from each corner. Where is the leak in this "electrostatic bottle"? [To harness nuclear fusion as a practical energy source it is necessary to heat a plasma (soup of charges particles) to fantastic temperatures—so hot that contact would vaporize any ordinary pot. Earnshaw's theorem says that electrostatic containment is also out of the question. Fortunately, it *is* possible to confine a hot plasma *magnetically.*]

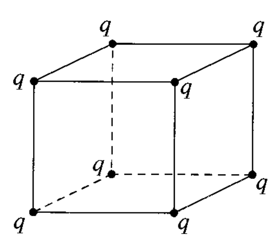

Figure 3.4

Problem 3.3 Find the general solution to Laplace's equation in spherical coordinates, for the case where V depends only on r. Do the same for cylindrical coordinates, assuming V depends only on s.

3.1.5 Boundary Conditions and Uniqueness Theorems

Laplace's equation does not by itself determine V; in addition, a suitable set of boundary conditions must be supplied. This raises a delicate question: What are appropriate boundary conditions, sufficient to determine the answer and yet not so strong as to generate inconsistencies? The one-dimensional case is easy, for here the general solution $V = mx + b$ contains two arbitrary constants, and we therefore require two boundary conditions. We might, for instance, specify the value of the function at the two ends, or we might give the value of the function and its derivative at one end, or the value at one end and the derivative at the other, and so on. But we cannot get away with *just* the value or *just* the derivative at *one* end—this is insufficient information. Nor would it do to specify the derivatives at both ends—this would either be redundant (if the two are equal) or inconsistent (if they are not).

In two or three dimensions we are confronted by a partial differential equation, and it is not so easy to see what would constitute acceptable boundary conditions. Is the shape of a taut rubber membrane, for instance, uniquely determined by the frame over which it is stretched, or, like a canning jar lid, can it snap from one stable configuration to another? The answer, as I think your intuition would suggest, is that V *is* uniquely determined by its value at the boundary (canning jars evidently don't obey Laplace's equation). However, other boundary conditions can also be used (see Prob. 3.4). The *proof* that a proposed set of boundary conditions will suffice is usually presented in the form of a **uniqueness theorem.** There are many such theorems for electrostatics, all sharing the same basic format—I'll show you the two most useful ones.[3]

First uniqueness theorem: The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface $\mathcal{S}$.

Proof: In Fig. 3.5 I have drawn such a region and its boundary. (There could also be "islands" inside, so long as V is given on all their surfaces; also, the outer boundary could be at infinity, where V is ordinarily taken to be zero.) Suppose there were *two* solutions to Laplace's equation:

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0,$$

both of which assume the specified value on the surface. I want to prove that they must be equal. The trick is look at their *difference:*

$$V_3 \equiv V_1 - V_2.$$

[3] I do not intend to prove the *existence* of solutions here—that's a much more difficult job. In context, the existence is generally clear on physical grounds.

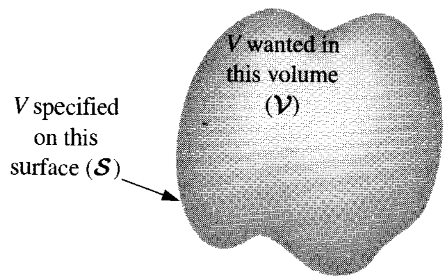

Figure 3.5

This obeys Laplace's equation,

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0,$$

and it takes the value *zero* on all boundaries (since V_1 and V_2 are equal there). But Laplace's equation allows no local maxima or minima—all extrema occur on the boundaries. So the maximum and minimum of V_3 are both zero. Therefore V_3 must be zero everywhere, and hence

$$V_1 = V_2. \quad \text{qed}$$

Example 3.1

Show that the potential is *constant* inside an enclosure completely surrounded by conducting material, provided there is no charge within the enclosure.

Solution: The potential on the cavity wall is some constant, V_0 (that's item (iv), in Sect. 2.5.1), so the potential inside is a function that satisfies Laplace's equation and has the constant value V_0 at the boundary. It doesn't take a genius to think of *one* solution to this problem: $V = V_0$ everywhere. The uniqueness theorem guarantees that this is the *only* solution. (It follows that the *field* inside an empty cavity is zero—the same result we found in Sect. 2.5.2 on rather different grounds.)

The uniqueness theorem is a license to your imagination. It doesn't matter *how* you come by your solution; if (a) it satisfies Laplace's equation and (b) it has the correct value on the boundaries, then it's *right.* You'll see the power of this argument when we come to the method of images.

Incidentally, it is easy to improve on the first uniqueness theorem: I assumed there was no charge inside the region in question, so the potential obeyed Laplace's equation, but we may as well throw in some charge (in which case V obeys Poisson's equation). The argument is the same, only this time

$$\nabla^2 V_1 = -\frac{1}{\epsilon_0}\rho, \qquad \nabla^2 V_2 = -\frac{1}{\epsilon_0}\rho,$$

so

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{1}{\epsilon_0}\rho + \frac{1}{\epsilon_0}\rho = 0.$$

Once again the *difference* ($V_3 \equiv V_1 - V_2$) satisfies Laplace's equation and has the value zero on all boundaries, so $V_3 = 0$ and hence $V_1 = V_2$.

Corollary: The potential in a volume $\mathcal{V}$ is uniquely determined if (a) the charge density throughout the region, and (b) the value of V on all boundaries, are specified.

3.1.6 Conductors and the Second Uniqueness Theorem

The *simplest* way to set the boundary conditions for an electrostatic problem is to specify the value of V on all surfaces surrounding the region of interest. And this situation often occurs in practice: In the laboratory, we have conductors connected to batteries, which maintain a given potential, or to **ground**, which is the experimentalist's word for $V = 0$. However, there are other circumstances in which we do not know the *potential* at the boundary, but rather the *charges* on various conducting surfaces. Suppose I put charge Q_1 on the first conductor, Q_2 on the second, and so on—I'm not telling you how the charge distributes itself over each conducting surface, because as soon as I put it on, it moves around in a way I do not control. And for good measure, let's say there is some specified charge density ρ in the region between the conductors. Is the electric field now uniquely determined? Or are there perhaps a number of different ways the charges could arrange themselves on their respective conductors, each leading to a different field?

Second uniqueness theorem: In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the *total charge* on each conductor is given (Fig. 3.6). (The region as a whole can be bounded by another conductor, or else unbounded.)

Proof: Suppose there are *two* fields satisfying the conditions of the problem. Both obey Gauss's law in differential form in the space between the conductors:

$$\nabla \cdot \mathbf{E}_1 = \frac{1}{\epsilon_0}\rho, \qquad \nabla \cdot \mathbf{E}_2 = \frac{1}{\epsilon_0}\rho.$$

And both obey Gauss's law in integral form for a Gaussian surface enclosing each conductor:

$$\oint_{\substack{i\text{th conducting}\\ \text{surface}}} \mathbf{E}_1 \cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_i, \qquad \oint_{\substack{i\text{th conducting}\\ \text{surface}}} \mathbf{E}_2 \cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_i.$$

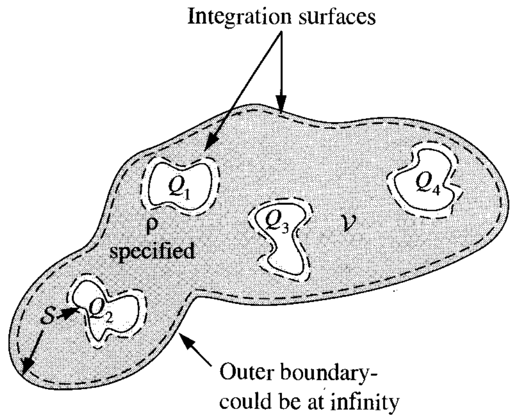

Figure 3.6

Likewise, for the outer boundary (whether this is just inside an enclosing conductor or at infinity),

$$\oint_{\substack{\text{outer}\\ \text{boundary}}} \mathbf{E}_1 \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{tot}}, \qquad \oint_{\substack{\text{outer}\\ \text{boundary}}} \mathbf{E}_2 \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{tot}}.$$

As before, we examine the difference

$$\mathbf{E}_3 \equiv \mathbf{E}_1 - \mathbf{E}_2,$$

which obeys

$$\nabla \cdot \mathbf{E}_3 = 0 \tag{3.7}$$

in the region between the conductors, and

$$\oint \mathbf{E}_3 \cdot d\mathbf{a} = 0 \tag{3.8}$$

over each boundary surface.

Now there is one final piece of information we must exploit: Although we do not know how the charge Q_i distributes itself over the ith conducting surface, we *do* know that each conductor is an equipotential, and hence V_3 is a *constant* (not necessarily the *same* constant) over each conducting surface. (It need not be *zero,* for the potentials V_1 and V_2 may not be equal—all we know for sure is that *both* are *constant* over any given conductor.) Next comes a trick. Invoking product rule number (5), we find that

$$\nabla \cdot (V_3 \mathbf{E}_3) = V_3(\nabla \cdot \mathbf{E}_3) + \mathbf{E}_3 \cdot (\nabla V_3) = -(E_3)^2.$$

Here I have used Eq. 3.7, and $\mathbf{E}_3 = -\nabla V_3$. Integrating this over the entire region between the conductors, and applying the divergence theorem to the left side:

$$\int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{E}_3)\, d\tau = \oint_{\mathcal{S}} V_3 \mathbf{E}_3 \cdot d\mathbf{a} = -\int_{\mathcal{V}} (E_3)^2\, d\tau.$$

The surface integral covers all boundaries of the region in question—the conductors and outer boundary. Now V_3 is a constant over each surface (if the outer boundary is infinity, $V_3 = 0$ there), so it comes outside each integral, and what remains is zero, according to Eq. 3.8. Therefore,

$$\int_{\mathcal{V}} (E_3)^2\, d\tau = 0.$$

But this integrand is never negative; the only way the integral can vanish is if $E_3 = 0$ everywhere. Consequently, $\mathbf{E}_1 = \mathbf{E}_2$, and the theorem is proved.

This proof was not easy, and there is a real danger that the theorem itself will seem more plausible to you than the proof. In case you think the second uniqueness theorem is "obvious," consider this example of Purcell's: Figure 3.7 shows a comfortable electrostatic configuration, consisting of four conductors with charges $\pm Q$, situated so that the plusses are near the minuses. It looks very stable. Now, what happens if we join them in pairs, by tiny wires, as indicated in Fig. 3.8? Since the positive charges are very near negative charges (which is where they *like* to be) you might well guess that *nothing* will happen—the configuration still looks stable.

Well, that sounds reasonable, but it's wrong. The configuration in Fig. 3.8 is *impossible.* For there are now effectively *two* conductors, and the total charge on each is *zero*. *One* possible way to distribute zero charge over these conductors is to have no accumulation of charge anywhere, and hence zero field everywhere (Fig. 3.9). By the second uniqueness theorem, this must be *the* solution: The charge will flow down the tiny wires, canceling itself off.

Figure 3.7

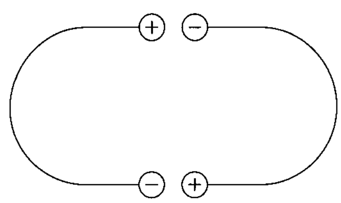

Figure 3.8

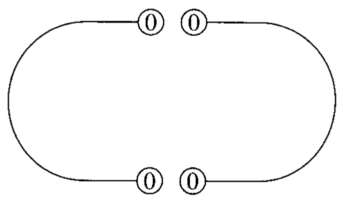

Figure 3.9

Problem 3.4 Prove that the field is uniquely determined when the charge density ρ is given and *either* V *or* the normal derivative $\partial V/\partial n$ is specified on each boundary surface. Do not assume the boundaries are conductors, or that V is constant over any given surface.

Problem 3.5 A more elegant proof of the second uniqueness theorem uses Green's identity (Prob. 1.60c), with $T = U = V_3$. Supply the details.

3.2 The Method of Images

3.2.1 The Classic Image Problem

Suppose a point charge q is held a distance d above an infinite grounded conducting plane (Fig. 3.10). *Question:* What is the potential in the region above the plane? It's not just $(1/4\pi\epsilon_0)q/\mathscr{r}$, for q will induce a certain amount of negative charge on the nearby surface of the conductor; the total potential is due in part to q directly, and in part to this induced charge. But how can we possibly calculate the potential, when we don't know how much charge is induced or how it is distributed?

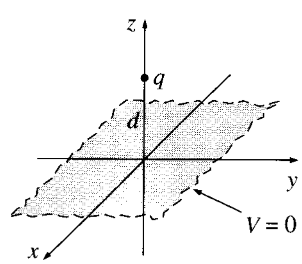

Figure 3.10

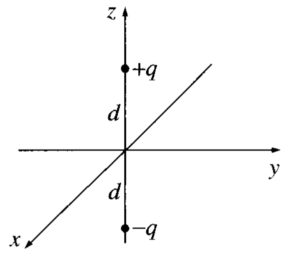

Figure 3.11

From a mathematical point of view our problem is to solve Poisson's equation in the region $z > 0$, with a single point charge q at $(0, 0, d)$, subject to the boundary conditions:

1. $V = 0$ when $z = 0$ (since the conducting plane is grounded), and
2. $V \to 0$ far from the charge (that is, for $x^2 + y^2 + z^2 \gg d^2$).

The first uniqueness theorem (actually, its corollary) guarantees that there is only one function that meets these requirements. If by trick or clever guess we can discover such a function, it's got to be the right answer.

Trick: Forget about the actual problem; we're going to study a *completely different* situation. This new problem consists of *two* point charges, $+q$ at $(0, 0, d)$ and $-q$ at $(0, 0, -d)$, and *no* conducting plane (Fig. 3.11). For this configuration I can easily write down the potential:

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]. \tag{3.9}$$

(The denominators represent the distances from (x, y, z) to the charges $+q$ and $-q$, respectively.) It follows that

1. $V = 0$ when $z = 0$, and
2. $V \to 0$ for $x^2 + y^2 + z^2 \gg d^2$,

and the only charge in the region $z > 0$ is the point charge $+q$ at $(0, 0, d)$. But these are precisely the conditions of the original problem! Evidently the second configuration happens to produce exactly the same potential as the first configuration, in the "upper" region $z \geq 0$. (The "lower" region, $z < 0$, is completely different, but who cares? The upper part is all we need.) *Conclusion:* The potential of a point charge above an infinite grounded conductor is given by Eq. 3.9, for $z \geq 0$.

Notice the crucial role played by the uniqueness theorem in this argument: without it, no one would believe this solution, since it was obtained for a completely different charge distribution. But the uniqueness theorem certifies it: If it satisfies Poisson's equation in the region of interest, and assumes the correct value at the boundaries, then it must be right.

3.2.2 Induced Surface Charge

Now that we know the potential, it is a straightforward matter to compute the surface charge σ induced on the conductor. According to Eq. 2.49,

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n},$$

where $\partial V / \partial n$ is the normal derivative of V at the surface. In this case the normal direction is the z-direction, so

$$\sigma = -\epsilon_0 \left. \frac{\partial V}{\partial z} \right|_{z=0}.$$

From Eq. 3.9,

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0} \left\{ \frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}} \right\},$$

so

$$\sigma(x, y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}. \tag{3.10}$$

As expected, the induced charge is negative (assuming q is positive) and greatest at $x = y = 0$.

While we're at it, let's compute the *total* induced charge

$$Q = \int \sigma \, da.$$

This integral, over the xy plane, could be done in Cartesian coordinates, with $da = dx\,dy$, but it's a little easier to use polar coordinates (r, ϕ), with $r^2 = x^2 + y^2$ and $da = r\,dr\,d\phi$. Then

$$\sigma(r) = \frac{-qd}{2\pi(r^2+d^2)^{3/2}},$$

and

$$Q = \int_0^{2\pi} \int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}} r\,dr\,d\phi = \left. \frac{qd}{\sqrt{r^2+d^2}} \right|_0^{\infty} = -q. \tag{3.11}$$

Evidently the total charge induced on the plane is $-q$, as (with benefit of hindsight) you can perhaps convince yourself it *had* to be.

3.2.3 Force and Energy

The charge q is attracted toward the plane, because of the negative induced charge. Let's calculate the force of attraction. Since the potential in the vicinity of q is the same as in the analog problem (the one with $+q$ and $-q$ but no conductor), so also is the field and, therefore, the force:

$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2d)^2} \hat{\mathbf{z}}. \tag{3.12}$$

Beware: It is easy to get carried away, and assume that *everything* is the same in the two problems. Energy, however, is *not* the same. With the two point charges and no conductor, Eq. 2.42 gives

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}. \tag{3.13}$$

But for a single charge and conducting plane the energy is *half* of this:

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}. \tag{3.14}$$

Why half? Think of the energy stored in the fields (Eq. 2.45):

$$W = \frac{\epsilon_0}{2}\int E^2\,d\tau.$$

In the first case both the upper region ($z > 0$) and the lower region ($z < 0$) contribute—and by symmetry they contribute equally. But in the second case only the upper region contains a nonzero field, and hence the energy is half as great.

Of course, one could also determine the energy by calculating the work required to bring q in from infinity. The force required (to oppose the electrical force in Eq. 3.12) is $(1/4\pi\epsilon_0)(q^2/4z^2)\hat{\mathbf{z}}$, so

$$\begin{aligned} W &= \int_\infty^d \mathbf{F}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_\infty^d \frac{q^2}{4z^2}\,dz \\ &= \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{4z}\right)\Big|_\infty^d = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}. \end{aligned}$$

As I move q toward the conductor, I do work *only on q*. It is true that induced charge is moving in over the conductor, but this costs me nothing, since the whole conductor is at potential zero. By contrast, if I simultaneously bring in *two* point charges (with no conductor), I do work on *both* of them, and the total is twice as great.

3.2.4 Other Image Problems

The method just described is not limited to a single point charge; *any* stationary charge distribution near a grounded conducting plane can be treated in the same way, by introducing its mirror image—hence the name **method of images.** (Remember that the image charges have the *opposite sign;* this is what guarantees that the xy plane will be at potential zero.) There are also some exotic problems that can be handled in similar fashion; the nicest of these is the following.

Example 3.2

A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R (Fig. 3.12). Find the potential outside the sphere.

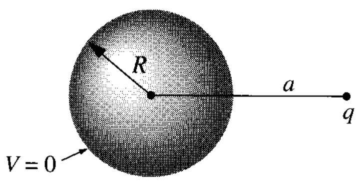

Figure 3.12

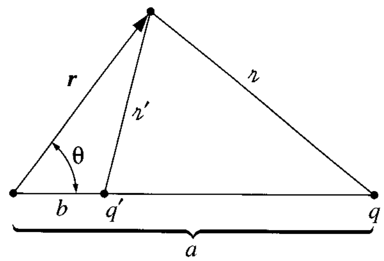

Figure 3.13

Solution: Examine the *completely different* configuration, consisting of the point charge q together with another point charge

$$q' = -\frac{R}{a}q, \tag{3.15}$$

placed a distance

$$b = \frac{R^2}{a} \tag{3.16}$$

to the right of the center of the sphere (Fig. 3.13). No conductor, now—just the two point charges. The potential of this configuration is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right), \tag{3.17}$$

where $\mathscr{r}$ and $\mathscr{r}'$ are the distances from q and q', respectively. Now, it happens (see Prob. 3.7) that this potential vanishes at all points on the sphere, and therefore fits the boundary conditions for our original problem, in the exterior region.

Conclusion: Eq. 3.17 is the potential of a point charge near a grounded conducting sphere. (Notice that b is less than R, so the "image" charge q' is safely inside the sphere—*you cannot put image charges in the region where you are calculating* V; that would change ρ, and you'd be solving Poisson's equation with the wrong source.) In particular, the force of attraction between the charge and the sphere is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}. \tag{3.18}$$

This solution is delightfully simple, but extraordinarily lucky. There's as much art as science in the method of images, for you must somehow think up the right "auxiliary problem" to look at. The first person who solved the problem this way cannot have known in advance what image charge q' to use or where to put it. Presumably, he (she?) started with an *arbitrary* charge at an *arbitrary* point inside the sphere, calculated the potential on the sphere, and then discovered that with q' and b just right the potential on the sphere vanishes. But it is really a miracle that *any* choice does the job—with a cube instead of a sphere, for example, *no* single charge *anywhere* inside would make the potential zero on the surface.

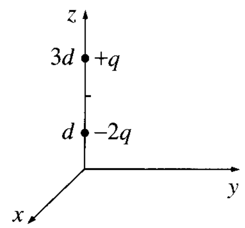

Figure 3.14

Problem 3.6 Find the force on the charge $+q$ in Fig. 3.14. (The xy plane is a grounded conductor.)

Problem 3.7

(a) Using the law of cosines, show that Eq. 3.17 can be written as follows:

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{q}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right], \tag{3.19}$$

where r and θ are the usual spherical polar coordinates, with the z axis along the line through q. In this form it is obvious that $V = 0$ on the sphere, $r = R$.

(b) Find the induced surface charge on the sphere, as a function of θ. Integrate this to get the total induced charge. (What *should* it be?)

(c) Calculate the energy of this configuration.

Problem 3.8 In Ex. 3.2 we assumed that the conducting sphere was grounded ($V = 0$). But with the addition of a second image charge, the same basic model will handle the case of a sphere at *any* potential V_0 (relative, of course, to infinity). What charge should you use, and where should you put it? Find the force of attraction between a point charge q and a *neutral* conducting sphere.

Problem 3.9 A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane. (Let's say the wire runs parallel to the x-axis and directly above it, and the conducting plane is the xy plane.)

(a) Find the potential in the region above the plane.

(b) Find the charge density σ induced on the conducting plane.

Problem 3.10 Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge q, situated as shown in Fig. 3.15. Set up the image configuration, and calculate the potential in this region. What charges do you need, and where should they be located? What is the force on q? How much work did it take to bring q in from infinity? Suppose the planes met at some angle other than 90°; would you still be able to solve the problem by the method of images? If not, for what particular angles *does* the method work?

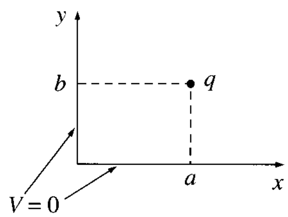

Figure 3.15

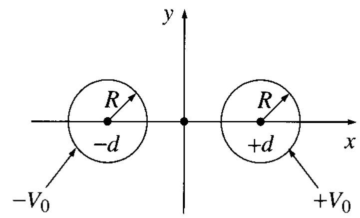

Figure 3.16

! **Problem 3.11** Two long, straight copper pipes, each of radius R, are held a distance $2d$ apart. One is at potential V_0, the other at $-V_0$ (Fig. 3.16). Find the potential everywhere. [*Suggestion:* Exploit the result of Prob. 2.47.]

3.3 Separation of Variables

In this section we shall attack Laplace's equation directly, using the method of **separation of variables**, which is the physicist's favorite tool for solving partial differential equations. The method is applicable in circumstances where the potential (V) or the charge density (σ) is specified on the boundaries of some region, and we are asked to find the potential in the interior. The basic strategy is very simple: *We look for solutions that are* products *of functions, each of which depends on only* one *of the coordinates.* The algebraic details, however, can be formidable, so I'm going to develop the method through a sequence of examples. We'll start with Cartesian coordinates and then do spherical coordinates (I'll leave the cylindrical case for you to tackle on your own, in Prob. 3.23).

3.3.1 Cartesian Coordinates

Example 3.3

Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$ (Fig. 3.17). The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot."

Solution: The configuration is independent of z, so this is really a *two*-dimensional problem. In mathematical terms, we must solve Laplace's equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \tag{3.20}$$

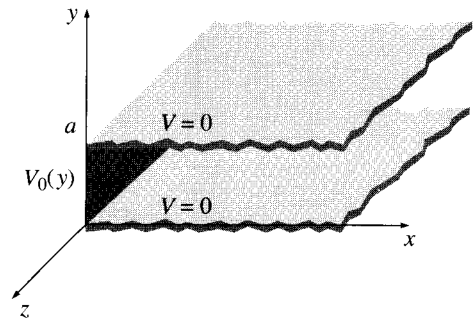

Figure 3.17

subject to the boundary conditions

$$\left.\begin{array}{lll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = V_0(y) \text{ when } x = 0, \\ \text{(iv)} & V \to 0 \text{ as } x \to \infty. \end{array}\right\} \tag{3.21}$$

(The latter, although not explicitly stated in the problem, is necessary on physical grounds: as you get farther and farther away from the "hot" strip at $x = 0$, the potential should drop to zero.) Since the potential is specified on all boundaries, the answer is uniquely determined.

The first step is to look for solutions in the form of products:

$$V(x, y) = X(x)Y(y). \tag{3.22}$$

On the face of it, this is an absurd restriction—the overwhelming majority of solutions to Laplace's equation do *not* have such a form. For example, $V(x, y) = (5x + 6y)$ satisfies Eq. 3.20, but you can't express it as the product of a function x times a function y. Obviously, we're only going to get a tiny subset of all possible solutions by this means, and it would be a *miracle* if one of them happened to fit the boundary conditions of our problem ... But hang on, because the solutions we *do* get are very special, and it turns out that by pasting them together we can construct the general solution.

Anyway, putting Eq. 3.22 into Eq. 3.20, we obtain

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0.$$

The next step is to "separate the variables" (that is, collect all the x-dependence into one term and all the y-dependence into another). Typically, this is accomplished by dividing through by V:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0. \tag{3.23}$$

Here the first term depends only on x and the second only on y; in other words, we have an equation of the form

$$f(x) + g(y) = 0. \tag{3.24}$$

Now, there's only one way this could possibly be true: *f and g must both be constant.* For what if $f(x)$ *changed,* as you vary x—then if we held y fixed and fiddled with x, the sum $f(x) + g(x)$ would *change,* in violation of Eq. 3.24, which says it's always zero. (That's a simple but somehow rather elusive argument; don't accept it without due thought, because the whole method rides on it.) It follows from Eq. 3.23, then, that

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1 \quad \text{and} \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2, \qquad \text{with } C_1 + C_2 = 0. \tag{3.25}$$

One of these constants is positive, the other negative (or perhaps both are zero). In general, one must investigate all the possibilities; however, in our particular problem we need C_1 positive and C_2 negative, for reasons that will appear in a moment. Thus

$$\frac{d^2X}{dx^2} = k^2X, \qquad \frac{d^2Y}{dy^2} = -k^2Y. \tag{3.26}$$

Notice what has happened: A *partial* differential equation (3.20) has been converted into two *ordinary* differential equations (3.26). The advantage of this is obvious—ordinary differential equations are a lot easier to solve. Indeed:

$$X(x) = Ae^{kx} + Be^{-kx}, \qquad Y(y) = C\sin ky + D\cos ky,$$

so that

$$V(x, y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky). \tag{3.27}$$

This is the appropriate separable solution to Laplace's equation; it remains to impose the boundary conditions, and see what they tell us about the constants. To begin at the end, condition (iv) requires that A equal zero.[4] Absorbing B into C and D, we are left with

$$V(x, y) = e^{-kx}(C\sin ky + D\cos ky).$$

Condition (i) now demands that D equal zero, so

$$V(x, y) = Ce^{-kx}\sin ky. \tag{3.28}$$

Meanwhile (ii) yields $\sin ka = 0$, from which it follows that

$$k = \frac{n\pi}{a}, \qquad (n = 1, 2, 3, \ldots). \tag{3.29}$$

(At this point you can see why I chose C_1 positive and C_2 negative: If X were sinusoidal, we could never arrange for it to go to zero at infinity, and if Y were exponential we could not make it vanish at both 0 and a. Incidentally, $n = 0$ is no good, for in that case the potential vanishes *everywhere*. And we have already excluded negative n's.)

That's as far as we can go, using separable solutions, and unless $V_0(y)$ just happens to have the form $\sin(n\pi y/a)$ for some integer n we simply *can't fit* the final boundary condition at $x = 0$. But now comes the crucial step that redeems the method: Separation of variables has given us an *infinite set* of solutions (one for each n), and whereas none of them *by itself* satisfies

[4] I'm assuming k is positive, but this involves no loss of generality—negative k gives the same solution (3.27), only with the constants shuffled ($A \leftrightarrow B$, $C \to -C$). Occasionally (but not in this example) $k = 0$ must also be included (see Prob. 3.47).

the final boundary condition, it is possible to combine them in a way that *does*. Laplace's equation is *linear*, in the sense that if $V_1, V_2, V_3, \ldots$ satisfy it, so does any linear combination, $V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \ldots$, where $\alpha_1, \alpha_2, \ldots$ are arbitrary constants. For

$$\nabla^2 V = \alpha_1 \nabla^2 V_1 + \alpha_2 \nabla^2 V_2 + \ldots = 0\alpha_1 + 0\alpha_2 + \ldots = 0.$$

Exploiting this fact, we can patch together the separable solutions (3.28) to construct a much more general solution:

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a). \tag{3.30}$$

This still satisfies the first three boundary conditions; the question is, can we (by astute choice of the coefficients C_n) fit the last boundary condition?

$$V(0, y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y). \tag{3.31}$$

Well, you may recognize this sum—it's a Fourier sine series. And Dirichlet's theorem[5] guarantees that virtually *any* function $V_0(y)$—it can even have a finite number of discontinuities—can be expanded in such a series.

But how do we actually *determine* the coefficients C_n, buried as they are in that infinite sum? The device for accomplishing this is so lovely it deserves a name—I call it **Fourier's trick**, though it seems Euler had used essentially the same idea somewhat earlier. Here's how it goes: Multiply Eq. 3.31 by $\sin(n'\pi y/a)$ (where n' is a positive integer), and integrate from 0 to a:

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin(n\pi y/a)\, \sin(n'\pi y/a)\, dy = \int_0^a V_0(y) \sin(n'\pi y/a)\, dy. \tag{3.32}$$

You can work out the integral on the left for yourself; the answer is

$$\int_0^a \sin(n\pi y/a)\, \sin(n'\pi y/a)\, dy = \begin{cases} 0, & \text{if } n' \neq n, \\ \dfrac{a}{2}, & \text{if } n' = n. \end{cases} \tag{3.33}$$

Thus all the terms in the series drop out, save only the one where $n' = n$, and the left side of Eq. 3.32, reduces to $(a/2)C_{n'}$. *Conclusion:*[6]

$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a)\, dy. \tag{3.34}$$

That *does* it: Eq. 3.30 is the solution, with coefficients given by Eq. 3.34. As a concrete example, suppose the strip at $x = 0$ is a metal plate with constant potential V_0 (remember, it's insulated from the grounded plates at $y = 0$ and $y = a$). Then

$$C_n = \frac{2V_0}{a} \int_0^a \sin(n\pi y/a)\, dy = \frac{2V_0}{n\pi}(1 - \cos n\pi) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \dfrac{4V_0}{n\pi}, & \text{if } n \text{ is odd.} \end{cases} \tag{3.35}$$

[5]Boas, M., *Mathematical Methods in the Physical Sciences*, 2nd ed. (New York: John Wiley, 1983).

[6]For aesthetic reasons I've dropped the prime; Eq. 3.34 holds for $n = 1, 2, 3, \ldots$, and it doesn't matter (obviously) what letter you use for the "dummy" index.

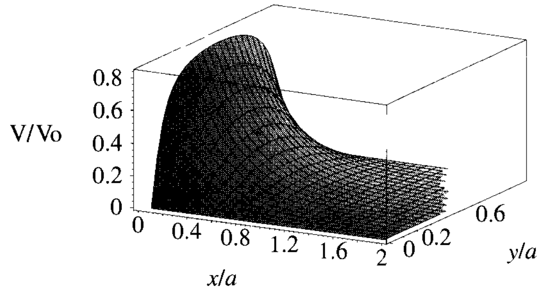

Figure 3.18

Evidently,

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a). \tag{3.36}$$

Figure 3.18 is a plot of this potential; Fig. 3.19 shows how the first few terms in the Fourier series combine to make a better and better approximation to the constant V_0: (a) is $n = 1$ only, (b) includes n up to 5, (c) is the sum of the first 10 terms, and (d) is the sum of the first 100 terms.

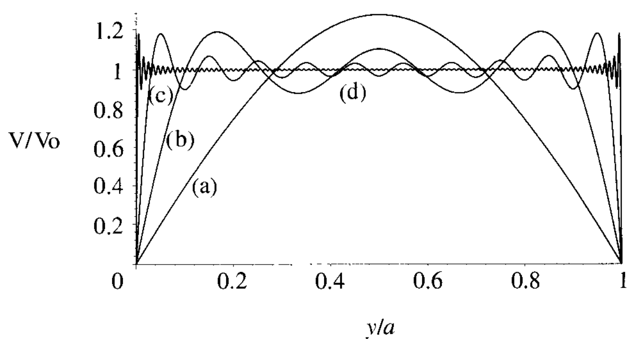

Figure 3.19

Incidentally, the infinite series in Eq. 3.36 can be summed explicitly (try your hand at it, if you like); the result is

$$V(x, y) = \frac{2V_0}{\pi} \tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right). \tag{3.37}$$

In this form it is easy to check that Laplace's equation is obeyed and the four boundary conditions (3.21) are satisfied.

The success of this method hinged on two extraordinary properties of the separable solutions (3.28): **completeness** and **orthogonality**. A set of functions $f_n(y)$ is said to be **complete** if any other function $f(y)$ can be expressed as a linear combination of them:

$$f(y) = \sum_{n=1}^{\infty} C_n f_n(y). \tag{3.38}$$

The functions $\sin(n\pi y/a)$ are complete on the interval $0 \leq y \leq a$. It was this fact, guaranteed by Dirichlet's theorem, that assured us Eq. 3.31 could be satisfied, given the proper choice of the coefficients C_n. (The *proof* of completeness, for a particular set of functions, is an extremely difficult business, and I'm afraid physicists tend to *assume* it's true and leave the checking to others.) A set of functions is **orthogonal** if the integral of the product of any two different members of the set is zero:

$$\int_0^a f_n(y) f_{n'}(y)\, dy = 0 \qquad \text{for } n' \neq n. \tag{3.39}$$

The sine functions are orthogonal (Eq. 3.33); this is the property on which Fourier's trick is based, allowing us to kill off all terms but one in the infinite series and thereby solve for the coefficients C_n. (Proof of orthogonality is generally quite simple, either by direct integration or by analysis of the differential equation from which the functions came.)

Example 3.4

Two infinitely long grounded metal plates, again at $y = 0$ and $y = a$, are connected at $x = \pm b$ by metal strips maintained at a constant potential V_0, as shown in Fig. 3.20 (a thin layer of insulation at each corner prevents them from shorting out). Find the potential inside the resulting rectangular pipe.

Solution: Once again, the configuration is independent of z. Our problem is to solve Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

subject to the boundary conditions

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = V_0 \text{ when } x = b, \\ \text{(iv)} & V = V_0 \text{ when } x = -b. \end{array}\right\} \tag{3.40}$$

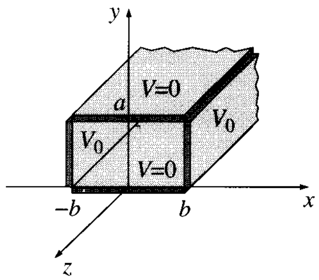

Figure 3.20

The argument runs as before, up to Eq. 3.27:

$$V(x, y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky).$$

This time, however, we cannot set $A = 0$; the region in question does not extend to $x = \infty$, so e^{kx} is perfectly acceptable. On the other hand, the situation is *symmetric* with respect to x, so $V(-x, y) = V(x, y)$, and it follows that $A = B$. Using

$$e^{kx} + e^{-kx} = 2 \cosh kx,$$

and absorbing $2A$ into C and D, we have

$$V(x, y) = \cosh kx\, (C \sin ky + D \cos ky).$$

Boundary conditions (i) and (ii) require, as before, that $D = 0$ and $k = n\pi/a$, so

$$V(x, y) = C \cosh(n\pi x/a)\, \sin(n\pi y/a). \tag{3.41}$$

Because $V(x, y)$ is even in x, it will automatically meet condition (iv) if it fits (iii). It remains, therefore, to construct the general linear combination,

$$V(x, y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi x/a)\, \sin(n\pi y/a),$$

and pick the coefficients C_n in such a way as to satisfy condition (iii):

$$V(b, y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi b/a)\, \sin(n\pi y/a) = V_0.$$

This is the same problem in Fourier analysis that we faced before; I quote the result from Eq. 3.35:

$$C_n \cosh(n\pi b/a) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4V_0}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

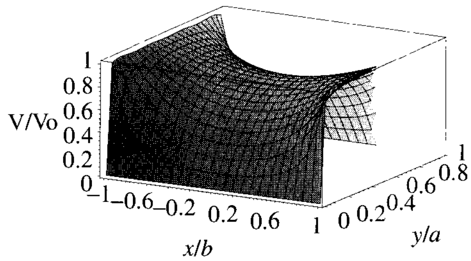

Figure 3.21

Conclusion: The potential in this case is given by

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5...} \frac{1}{n} \frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)} \sin(n\pi y/a). \tag{3.42}$$

This function is shown in Fig. 3.21.

Example 3.5

An infinitely long rectangular metal pipe (sides a and b) is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0(y, z)$, as indicated in Fig. 3.22. Find the potential inside the pipe.

Solution: This is a genuinely three-dimensional problem,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{3.43}$$

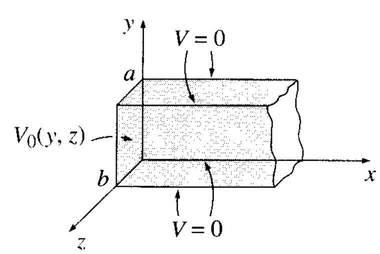

Figure 3.22

subject to the boundary conditions

$$\left.\begin{array}{ll}\text{(i)} & V=0 \text{ when } y=0,\\ \text{(ii)} & V=0 \text{ when } y=a,\\ \text{(iii)} & V=0 \text{ when } z=0,\\ \text{(iv)} & V=0 \text{ when } z=b,\\ \text{(v)} & V\to 0 \text{ as } x\to\infty,\\ \text{(vi)} & V=V_0(y,z) \text{ when } x=0.\end{array}\right\} \tag{3.44}$$

As always, we look for solutions that are products:

$$V(x,y,z)=X(x)Y(y)Z(z). \tag{3.45}$$

Putting this into Eq. 3.43, and dividing by V, we find

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0.$$

It follows that

$$\frac{1}{X}\frac{d^2X}{dx^2}=C_1,\ \frac{1}{Y}\frac{d^2Y}{dy^2}=C_2,\ \frac{1}{Z}\frac{d^2Z}{dz^2}=C_3,\quad \text{with } C_1+C_2+C_3=0.$$

Our previous experience (Ex. 3.3) suggests that C_1 must be positive, C_2 and C_3 negative. Setting $C_2=-k^2$ and $C_3=-l^2$, we have $C_1=k^2+l^2$, and hence

$$\frac{d^2X}{dx^2}=(k^2+l^2)X,\quad \frac{d^2Y}{dy^2}=-k^2Y,\quad \frac{d^2Z}{dz^2}=-l^2Z. \tag{3.46}$$

Once again, separation of variables has turned a *partial* differential equation into *ordinary* differential equations. The solutions are

$$\begin{aligned} X(x) &= Ae^{\sqrt{k^2+l^2}\,x}+Be^{-\sqrt{k^2+l^2}\,x},\\ Y(y) &= C\sin ky+D\cos ky,\\ Z(z) &= E\sin lz+F\cos lz.\end{aligned}$$

Boundary condition (v) implies $A=0$, (i) gives $D=0$, and (iii) yields $F=0$, whereas (ii) and (iv) require that $k=n\pi/a$ and $l=m\pi/b$, where n and m are positive integers. Combining the remaining constants, we are left with

$$V(x,y,z)=Ce^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin(n\pi y/a)\,\sin(m\pi z/b). \tag{3.47}$$

This solution meets all the boundary conditions except (vi). It contains *two* unspecified integers (n and m), and the most general linear combination is a *double* sum:

$$V(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,m}e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin(n\pi y/a)\,\sin(m\pi z/b). \tag{3.48}$$

We hope to fit the remaining boundary condition,

$$V(0,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,m}\sin(n\pi y/a)\,\sin(m\pi z/b)=V_0(y,z), \tag{3.49}$$

by appropriate choice of the coefficients $C_{n,m}$. To determine these constants, we multiply by $\sin(n'\pi y/a)\ \sin(m'\pi z/b)$, where n' and m' are arbitrary positive integers, and integrate:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\int_0^a \sin(n\pi y/a)\ \sin(n'\pi y/a)\,dy\int_0^b \sin(m\pi z/b)\ \sin(m'\pi z/b)\,dz$$

$$= \int_0^a\int_0^b V_0(y,z)\sin(n'\pi y/a)\ \sin(m'\pi z/b)\,dy\,dz.$$

Quoting Eq. 3.33, the left side is $(ab/4)C_{n',m'}$, so

$$C_{n,m} = \frac{4}{ab}\int_0^a\int_0^b V_0(y,z)\sin(n\pi y/a)\ \sin(m\pi z/b)\,dy\,dz. \tag{3.50}$$

Equation 3.48, with the coefficients given by Eq. 3.50, is the solution to our problem.

For instance, if the end of the tube is a conductor at *constant* potential V_0,

$$\begin{aligned} C_{n,m} &= \frac{4V_0}{ab}\int_0^a \sin(n\pi y/a)\,dy\int_0^b \sin(m\pi z/b)\,dz \\ &= \begin{cases} 0, & \text{if } n \text{ or } m \text{ is even,} \\ \dfrac{16V_0}{\pi^2 nm}, & \text{if } n \text{ and } m \text{ are odd.}\end{cases} \end{aligned} \tag{3.51}$$

In this case

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n,m=1,3,5\ldots}^{\infty}\frac{1}{nm}e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin(n\pi y/a)\ \sin(m\pi z/b). \tag{3.52}$$

Notice that the successive terms decrease rapidly; a reasonable approximation would be obtained by keeping only the first few.

Problem 3.12 Find the potential in the infinite slot of Ex. 3.3 if the boundary at $x = 0$ consists of two metal strips: one, from $y = 0$ to $y = a/2$, is held at a constant potential V_0, and the other, from $y = a/2$ to $y = a$, is at potential $-V_0$.

Problem 3.13 For the infinite slot (Ex. 3.3) determine the charge density $\sigma(y)$ on the strip at $x = 0$, assuming it is a conductor at constant potential V_0.

Problem 3.14 A rectangular pipe, running parallel to the z-axis (from $-\infty$ to $+\infty$), has three grounded metal sides, at $y = 0$, $y = a$, and $x = 0$. The fourth side, at $x = b$, is maintained at a specified potential $V_0(y)$.

(a) Develop a general formula for the potential within the pipe.

(b) Find the potential explicitly, for the case $V_0(y) = V_0$ (a constant).

Problem 3.15 A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded (Fig. 3.23). The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box.

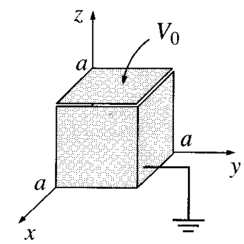

Figure 3.23

3.3.2 Spherical Coordinates

In the examples considered so far, Cartesian coordinates were clearly appropriate, since the boundaries were *planes*. For *round* objects spherical coordinates are more natural. In the spherical system, Laplace's equation reads:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0. \tag{3.53}$$

I shall assume the problem has **azimuthal symmetry**, so that V *is independent of* ϕ;[7] in that case Eq. 3.53 reduces to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0. \tag{3.54}$$

As before, we look for solutions that are products:

$$V(r,\theta)=R(r)\Theta(\theta). \tag{3.55}$$

Putting this into Eq. 3.54, and dividing by V,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0. \tag{3.56}$$

Since the first term depends only on r, and the second only on θ, it follows that each must be a constant:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1),\quad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-l(l+1). \tag{3.57}$$

Here $l(l+1)$ is just a fancy way of writing the separation constant—you'll see in a minute why this is convenient.

[7]The general case, for ϕ-dependent potentials, is treated in all the graduate texts. See, for instance, J. D. Jackson's *Classical Electrodynamics*, 3rd ed., Chapter 3 (New York: John Wiley, 1999).

As always, separation of variables has converted a *partial* differential equation (3.54) into *ordinary* differential equations (3.57). The radial equation,

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R, \tag{3.58}$$

has the general solution

$$R(r) = Ar^l + \frac{B}{r^{l+1}}, \tag{3.59}$$

as you can easily check; A and B are the two arbitrary constants to be expected in the solution of a second-order differential equation. But the angular equation,

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\,\Theta, \tag{3.60}$$

is not so simple. The solutions are **Legendre polynomials** in the variable $\cos\theta$:

$$\Theta(\theta) = P_l(\cos\theta). \tag{3.61}$$

$P_l(x)$ is most conveniently defined by the **Rodrigues formula:**

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l. \tag{3.62}$$

The first few Legendre polynomials are listed in Table 3.1.

$P_0(x)$	=	1
$P_1(x)$	=	x
$P_2(x)$	=	$(3x^2-1)/2$
$P_3(x)$	=	$(5x^3-3x)/2$
$P_4(x)$	=	$(35x^4-30x^2+3)/8$
$P_5(x)$	=	$(63x^5-70x^3+15x)/8$

Table 3.1 Legendre Polynomials

Notice that $P_l(x)$ is (as the name suggests) an lth-order *polynomial* in x; it contains only *even* powers, if l is even, and *odd* powers, if l is odd. The factor in front $(1/2^l l!)$ was chosen in order that

$$P_l(1) = 1. \tag{3.63}$$

The Rodrigues formula obviously works only for nonnegative integer values of l. Moreover, it provides us with only *one* solution. But Eq. 3.60 is *second*-order, and it should possess *two* independent solutions, for *every* value of l. It turns out that these "other solutions" blow up at $\theta = 0$ and/or $\theta = \pi$, and are therefore unacceptable on physical grounds.[8] For instance, the second solution for $l = 0$ is

$$\Theta(\theta) = \ln\left(\tan\frac{\theta}{2}\right). \tag{3.64}$$

You might want to check for yourself that this satisfies Eq. 3.60.

In the case of azimuthal symmetry, then, the most general *separable* solution to Laplace's equation, consistent with minimal physical requirements, is

$$V(r,\theta) = \left(Ar^l + \frac{B}{r^{l+1}}\right) P_l(\cos\theta).$$

(There was no need to include an overall constant in Eq. 3.61 because it can be absorbed into A and B at this stage.) As before, separation of variables yields an infinite set of solutions, one for each l. The *general* solution is the linear combination of separable solutions:

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta).} \tag{3.65}$$

The following examples illustrate the power of this important result.

Example 3.6

The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, of radius R. Find the potential inside the sphere.

Solution: In this case $B_l = 0$ for all l—otherwise the potential would blow up at the origin. Thus,

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta). \tag{3.66}$$

At $r = R$ this must match the specified function $V_0(\theta)$:

$$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta). \tag{3.67}$$

Can this equation be satisfied, for an appropriate choice of coefficients A_l? *Yes:* The Legendre polynomials (like the sines) constitute a complete set of functions, on the interval $-1 \le x \le 1$

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[8] In rare cases where the z axis is for some reason inaccessible, these "other solutions" may have to be considered.

$(0 \leq \theta \leq \pi)$. How do we determine the constants? Again, by Fourier's trick, for the Legendre polynomials (like the sines) are *orthogonal* functions:[9]

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx = \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta = \begin{cases} 0, & \text{if } l' \neq l, \\ \dfrac{2}{2l+1}, & \text{if } l' = l. \end{cases} \tag{3.68}$$

Thus, multiplying Eq. 3.67 by $P_{l'}(\cos\theta)\sin\theta$ and integrating, we have

$$A_{l'} R^{l'} \frac{2}{2l'+1} = \int_0^{\pi} V_0(\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta,$$

or

$$A_l = \frac{2l+1}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta\, d\theta. \tag{3.69}$$

Equation 3.66 is the solution to our problem, with the coefficients given by Eq. 3.69.

In can be difficult to evaluate integrals of the form 3.69 analytically, and in practice it is often easier to solve Eq. 3.67 "by eyeball."[10] For instance, suppose we are told that the potential on the sphere is

$$V_0(\theta) = k \sin^2(\theta/2), \tag{3.70}$$

where k is a constant. Using the half-angle formula, we rewrite this as

$$V_0(\theta) = \frac{k}{2}(1 - \cos\theta) = \frac{k}{2}[P_0(\cos\theta) - P_1(\cos\theta)].$$

Putting this into Eq. 3.67, we read off immediately that $A_0 = k/2$, $A_1 = -k/(2R)$, and all other A_l's vanish. Evidently,

$$V(r,\theta) = \frac{k}{2}\left[r^0 P_0(\cos\theta) - \frac{r^1}{R} P_1(\cos\theta)\right] = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right). \tag{3.71}$$

Example 3.7

The potential $V_0(\theta)$ is again specified on the surface of a sphere of radius R, but this time we are asked to find the potential *outside*, assuming there is no charge there.

Solution: In this case it's the A_l's that must be zero (or else V would not go to zero at ∞), so

$$V(r,\theta) = \sum_{i=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta). \tag{3.72}$$

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[9] M. Boas, *Mathematical Methods in the Physical Sciences,* 2nd ed., Section 12.7 (New York: John Wiley. 1983).

[10] This is certainly true whenever $V_0(\theta)$ can be expressed as a polynomial in $\cos\theta$. The degree of the polynomial tells us the highest l we require, and the leading coefficient determines the corresponding A_l. Subtracting off $A_l R^l P_l(\cos\theta)$ and repeating the process, we systematically work our way down to A_0. Notice that if V_0 is an *even* function of $\cos\theta$, then only even terms will occur in the sum (and likewise for odd functions).

At the surface of the sphere we require that

$$V(R,\theta) = \sum_{i=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0(\theta).$$

Multiplying by $P_{l'}(\cos\theta)\sin\theta$ and integrating—exploiting, again, the orthogonality relation 3.68—we have

$$\frac{B_{l'}}{R^{l'+1}} \frac{2}{2l'+1} = \int_0^\pi V_0(\theta) P_{l'}(\cos\theta)\sin\theta\, d\theta.$$

or

$$B_l = \frac{2l+1}{2} R^{l+1} \int_0^\pi V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta. \tag{3.73}$$

Equation 3.72, with the coefficients given by Eq. 3.73, is the solution to our problem.

Example 3.8

An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. [The field will push positive charge to the "northern" surface of the sphere, leaving a negative charge on the "southern" surface (Fig. 3.24). This induced charge, in turn, distorts the field in the neighborhood of the sphere.] Find the potential in the region outside the sphere.

Solution: The sphere is an equipotential—we may as well set it to zero. Then by symmetry the entire xy plane is at potential zero. This time, however, V does *not* go to zero at large z. In fact, far from the sphere the field is $E_0\hat{\mathbf{z}}$, and hence

$$V \to -E_0 z + C.$$

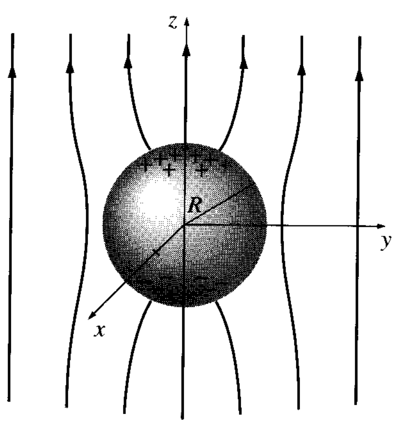


Figure 3.24

Since $V = 0$ in the equatorial plane, the constant C must be zero. Accordingly, the boundary conditions for this problem are

$$\left.\begin{array}{ll}\text{(i)} & V = 0 \quad \text{when } r = R, \\ \text{(ii)} & V \to -E_0 r\cos\theta \quad \text{for } r \gg R.\end{array}\right\} \tag{3.74}$$

We must fit these boundary conditions with a function of the form 3.65.

The first condition yields

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0,$$

or

$$B_l = -A_l R^{2l+1}, \tag{3.75}$$

so

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l\left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right)P_l(\cos\theta).$$

For $r \gg R$, the second term in parentheses is negligible, and therefore condition (ii) requires that

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta.$$

Evidently, only one term is present: $l = 1$. In fact, since $P_1(\cos\theta) = \cos\theta$, we can read off immediately

$$A_1 = -E_0, \quad \text{all other } A_l\text{'s zero.}$$

Conclusion:

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta. \tag{3.76}$$

The first term ($-E_0 r\cos\theta$) is due to the external field; the contribution attributable to the induced charge is evidently

$$E_0\frac{R^3}{r^2}\cos\theta.$$

If you want to know the induced charge density, it can be calculated in the usual way:

$$\sigma(\theta) = -\epsilon_0\frac{\partial V}{\partial r}\bigg|_{r=R} = \epsilon_0 E_0\left(1 + 2\frac{R^3}{r^3}\right)\cos\theta\bigg|_{r=R} = 3\epsilon_0 E_0\cos\theta. \tag{3.77}$$

As expected, it is positive in the "northern" hemisphere ($0 \le \theta \le \pi/2$) and negative in the "southern" ($\pi/2 \le \theta \le \pi$).

Example 3.9

A specified charge density $\sigma_0(\theta)$ is glued over the surface of a spherical shell of radius R. Find the resulting potential inside and outside the sphere.

Solution: You could, of course, do this by direct integration:

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma_0}{\mathscr{r}}\,da,$$

but separation of variables is often easier. For the interior region we have

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad (r \le R) \tag{3.78}$$

(no B_l terms—they blow up at the origin); in the exterior region

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \qquad (r \ge R) \tag{3.79}$$

(no A_l terms—they don't go to zero at infinity). These two functions must be joined together by the appropriate boundary conditions at the surface itself. First, the potential is *continuous* at $r = R$ (Eq. 2.34):

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta). \tag{3.80}$$

It follows that the coefficients of like Legendre polynomials are equal:

$$B_l = A_l R^{2l+1}. \tag{3.81}$$

(To prove that formally, multiply both sides of Eq. 3.80 by $P_{l'}(\cos\theta)\sin\theta$ and integrate from 0 to π, using the orthogonality relation 3.68.) Second, the radial derivative of V suffers a discontinuity at the surface (Eq. 2.36):

$$\left.\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\right|_{r=R} = -\frac{1}{\epsilon_0}\sigma_0(\theta). \tag{3.82}$$

Thus

$$-\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -\frac{1}{\epsilon_0}\sigma_0(\theta),$$

or, using Eq. 3.81:

$$\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\epsilon_0}\sigma_0(\theta). \tag{3.83}$$

From here, the coefficients can be determined using Fourier's trick:

$$A_l = \frac{1}{2\epsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta. \tag{3.84}$$

Equations 3.78 and 3.79 constitute the solution to our problem, with the coefficients given by Eqs. 3.81 and 3.84.

For instance, if

$$\sigma_0(\theta) = k\cos\theta = k P_1(\cos\theta), \tag{3.85}$$

for some constant k, then all the A_l's are zero except for $l = 1$, and

$$A_1 = \frac{k}{2\epsilon_0}\int_0^{\pi} [P_1(\cos\theta)]^2 \sin\theta\, d\theta = \frac{k}{3\epsilon_0}.$$

The potential inside the sphere is therefore

$$V(r,\theta) = \frac{k}{3\epsilon_0} r\cos\theta \qquad (r \le R), \tag{3.86}$$

whereas outside the sphere

$$V(r,\theta) = \frac{kR^3}{3\epsilon_0}\frac{1}{r^2}\cos\theta \qquad (r \ge R). \tag{3.87}$$

In particular, if $\sigma_0(\theta)$ is the induced charge on a metal sphere in an external field $E_0\hat{\mathbf{z}}$, so that $k = 3\epsilon_0 E_0$ (Eq. 3.77), then the potential inside is $E_0 r\cos\theta = E_0 z$, and the *field* is $-E_0\hat{\mathbf{z}}$—exactly right to cancel off the external field, as of course it *should* be. Outside the sphere the potential due to this surface charge is

$$E_0 \frac{R^3}{r^2}\cos\theta,$$

consistent with our conclusion in Ex. 3.8.

Problem 3.16 Derive $P_3(x)$ from the Rodrigues formula, and check that $P_3(\cos\theta)$ satisfies the angular equation (3.60) for $l = 3$. Check that P_3 and P_1 are orthogonal by explicit integration.

Problem 3.17

(a) Suppose the potential is a *constant* V_0 over the surface of the sphere. Use the results of Ex. 3.6 and Ex. 3.7 to find the potential inside and outside the sphere. (Of course, you know the answers in advance—this is just a consistency check on the method.)

(b) Find the potential inside and outside a spherical shell that carries a *uniform* surface charge σ_0, using the results of Ex. 3.9.

Problem 3.18 The potential at the surface of a sphere (radius R) is given by

$$V_0 = k\cos 3\theta,$$

where k is a constant. Find the potential inside and outside the sphere, as well as the surface charge density $\sigma(\theta)$ on the sphere. (Assume there's no charge inside or outside the sphere.)

Problem 3.19 Suppose the potential $V_0(\theta)$ at the surface of a sphere is specified, and there is no charge inside or outside the sphere. Show that the charge density on the sphere is given by

$$\sigma(\theta) = \frac{\epsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta), \tag{3.88}$$

where

$$C_l = \int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta. \tag{3.89}$$

Problem 3.20 Find the potential outside a *charged* metal sphere (charge Q, radius R) placed in an otherwise uniform electric field $\mathbf{E}_0$. Explain clearly where you are setting the zero of potential.

Problem 3.21 In Prob. 2.25 you found the potential on the axis of a uniformly charged disk:

$$V(r, 0) = \frac{\sigma}{2\epsilon_0}(\sqrt{r^2 + R^2} - r).$$

(a) Use this, together with the fact that $P_l(1) = 1$, to evaluate the first three terms in the expansion (3.72) for the potential of the disk at points *off* the axis, assuming $r > R$.

(b) Find the potential for $r < R$ by the same method, using (3.66). [*Note:* You must break the interior region up into two hemispheres, above and below the disk. Do *not* assume the coefficients A_l are the same in both hemispheres.]

Problem 3.22 A spherical shell of radius R carries a uniform surface charge σ_0 on the "northern" hemisphere and a uniform surface charge $-\sigma_0$ on the "southern" hemisphere. Find the potential inside and outside the sphere, calculating the coefficients explicitly up to A_6 and B_6.

• **Problem 3.23** Solve Laplace's equation by separation of variables in *cylindrical* coordinates, assuming there is no dependence on z (cylindrical symmetry). [Make sure you find *all* solutions to the radial equation; in particular, your result must accommodate the case of an infinite line charge, for which (of course) we already know the answer.]

Problem 3.24 Find the potential outside an infinitely long metal pipe, of radius R, placed at right angles to an otherwise uniform electric field $\mathbf{E}_0$. Find the surface charge induced on the pipe. [Use your result from Prob. 3.23.]

Problem 3.25 Charge density

$$\sigma(\phi) = a \sin 5\phi$$

(where a is a constant) is glued over the surface of an infinite cylinder of radius R (Fig. 3.25). Find the potential inside and outside the cylinder. [Use your result from Prob. 3.23.]

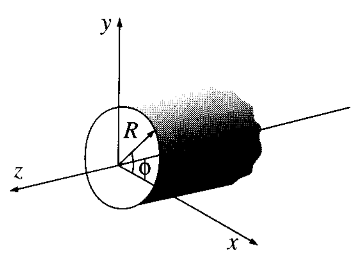


Figure 3.25

3.4 Multipole Expansion

3.4.1 Approximate Potentials at Large Distances

If you are very far away from a localized charge distribution, it "looks" like a point charge. and the potential is—to good approximation—$(1/4\pi\epsilon_0)Q/r$, where Q is the total charge. We have often used this as a check on formulas for V. But what if Q is *zero?* You might reply that the potential is then approximately zero, and of course, you're *right,* in a sense (indeed, the potential at large r is *pretty small* even if Q is *not* zero). But we're looking for something a bit more informative than that.

Example 3.10

A (physical) **electric dipole** consists of two equal and opposite charges ($\pm q$) separated by a distance d. Find the approximate potential at points far from the dipole.

Solution: Let $\mathscr{r}_-$ be the distance from $-q$ and $\mathscr{r}_+$ the distance from $+q$ (Fig. 3.26). Then

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right),$$

and (from the law of cosines)

$$\mathscr{r}_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right).$$

We're interested in the régime $r \gg d$, so the third term is negligible, and the binomial expansion yields

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right).$$

Thus

$$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \cong \frac{d}{r^2}\cos\theta,$$

and hence

$$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}. \tag{3.90}$$

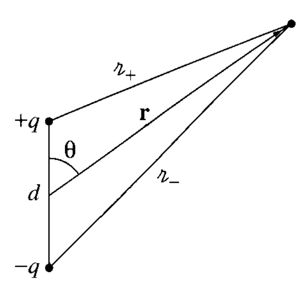


Figure 3.26

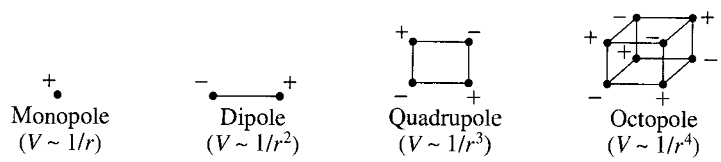


Figure 3.27

Evidently the potential of a dipole goes like $1/r^2$ at large r; as we might have anticipated, it falls off more rapidly than the potential of a point charge. Incidentally, if we put together a pair of equal and opposite *dipoles* to make a **quadrupole**, the potential goes like $1/r^3$; for back-to-back *quadrupoles* (an **octopole**) it goes like $1/r^4$; and so on. Figure 3.27 summarizes this hierarchy; for completeness I have included the electric **monopole** (point charge), whose potential, of course, goes like $1/r$.

Example 3.10 pertained to a very special charge configuration. I propose now to develop a *systematic expansion for the potential of an arbitrary localized charge distribution, in powers of* $1/r$. Figure 3.28 defines the appropriate variables; the potential at $\mathbf{r}$ is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'. \tag{3.91}$$

Using the law of cosines,

$$\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\theta' = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right],$$

or

$$\mathscr{r} = r\sqrt{1+\epsilon} \tag{3.92}$$

where

$$\epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right).$$

For points well outside the charge distribution, ϵ is much less than 1, and this invites a binomial expansion:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \ldots\right), \tag{3.93}$$

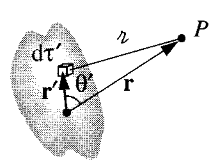


Figure 3.28

or, in terms of r, r', and θ':

$$\begin{aligned}\frac{1}{\mathscr{r}} &= \frac{1}{r}\left[1-\frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r}-2\cos\theta'\right)+\frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r}-2\cos\theta'\right)^2\right.\\ &\left.\quad -\frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r}-2\cos\theta'\right)^3+\dots\right]\\ &= \frac{1}{r}\left[1+\left(\frac{r'}{r}\right)(\cos\theta')+\left(\frac{r'}{r}\right)^2(3\cos^2\theta'-1)/2\right.\\ &\left.\quad +\left(\frac{r'}{r}\right)^3(5\cos^3\theta'-3\cos\theta')/2+\dots\right].\end{aligned}$$

In the last step I have collected together like powers of (r'/r); surprisingly, their coefficients (the terms in parentheses) are Legendre polynomials! The remarkable result[11] is that

$$\frac{1}{\mathscr{r}}=\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta'), \tag{3.94}$$

where θ' is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Substituting this back into Eq. 3.91, and noting that r is a constant, as far as the integration is concerned, I conclude that

$$\boxed{V(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\theta')\rho(\mathbf{r}')\,d\tau',} \tag{3.95}$$

or, more explicitly,

$$\begin{aligned}V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int\rho(\mathbf{r}')\,d\tau'+\frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')\,d\tau'\right.\\ &\left.\quad +\frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta'-\frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau'+\dots\right].\end{aligned} \tag{3.96}$$

This is the desired result—the **multipole expansion** of V in powers of $1/r$. The first term ($n = 0$) is the monopole contribution (it goes like $1/r$); the second ($n = 1$) is the dipole (it goes like $1/r^2$); the third is quadrupole; the fourth octopole; and so on. As it stands, Eq. 3.95 is *exact,* but it is useful primarily as an *approximation* scheme: the lowest nonzero term in the expansion provides the approximate potential at large r, and the successive terms tell us how to improve the approximation if greater precision is required.

---

[11] Incidentally, this affords a second way of obtaining the Legendre polynomials (the first being Rodrigues' formula); $1/\mathscr{r}$ is called the **generating function** for Legendre polynomials.

Problem 3.26 A sphere of radius R, centered at the origin, carries charge density

$$\rho(r, \theta) = k\frac{R}{r^2}(R - 2r)\sin\theta,$$

where k is a constant, and r, θ are the usual spherical coordinates. Find the approximate potential for points on the z axis, far from the sphere.

3.4.2 The Monopole and Dipole Terms

Ordinarily, the multipole expansion is dominated (at large r) by the monopole term:

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}, \tag{3.97}$$

where $Q = \int \rho\, d\tau$ is the total charge of the configuration. This is just what we expected for the approximate potential at large distances from the charge. Incidentally, for a *point* charge *at the origin,* V_{mon} represents the *exact* potential everywhere, not merely a first approximation at large r; in this case all the higher multipoles vanish.

If the total charge is zero, the dominant term in the potential will be the dipole (unless, of course, it *also* vanishes):

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')\, d\tau'.$$

Since θ' is the angle between $\mathbf{r}'$ and $\mathbf{r}$ (Fig. 3.28),

$$r'\cos\theta' = \hat{\mathbf{r}}\cdot\mathbf{r}',$$

and the dipole potential can be written more succinctly:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}\cdot\int \mathbf{r}'\rho(\mathbf{r}')\, d\tau'.$$

This integral, which does not depend on $\mathbf{r}$ at all, is called **dipole moment** of the distribution:

$$\boxed{\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\, d\tau',} \tag{3.98}$$

and the dipole contribution to the potential simplifies to

$$\boxed{V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}.} \tag{3.99}$$

The dipole moment is determined by the geometry (size, shape, and density) of the charge distribution. Equation 3.98 translates in the usual way (Sect. 2.1.4) for point, line, and surface charges. Thus, the dipole moment of a collection of *point* charges is

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}'_i. \tag{3.100}$$

For the "physical" dipole (equal and opposite charges, $\pm q$)

$$\mathbf{p} = q\mathbf{r}'_+ - q\mathbf{r}'_- = q(\mathbf{r}'_+ - \mathbf{r}'_-) = q\mathbf{d}, \tag{3.101}$$

where $\mathbf{d}$ is the vector from the negative charge to the positive one (Fig. 3.29).

Is this consistent with what we got for a *physical* dipole, in Ex. 3.10? Yes: If you put Eq. 3.100 into Eq. 3.99, you recover Eq. 3.90. Notice, however, that this is only the *approximate* potential of the physical dipole—evidently there are higher multipole contributions. Of course, as you go farther and farther away, V_{dip} becomes a better and better approximation, since the higher terms die off more rapidly with increasing r. By the same token, at a fixed r the dipole approximation improves as you shrink the separation d. To construct a "pure" dipole whose potential is given *exactly* by Eq. 3.99, you'd have to let d approach zero. Unfortunately, you then lose the dipole term *too,* unless you simultaneously arrange for q to go to infinity! A *physical* dipole becomes a *pure* dipole, then, in the rather artificial limit $d \to 0$, $q \to \infty$, with the product $qd = p$ held fixed. (When someone uses the word "dipole," you can't always tell whether they mean a *physical* dipole (with finite separation between the charges) or a *pure* (point) dipole. If in doubt, assume that d is small enough (compared to r) that you can safely apply Eq. 3.99.)

Dipole moments are *vectors,* and they add accordingly: if you have two dipoles, $\mathbf{p}_1$ and $\mathbf{p}_2$, the total dipole moment is $\mathbf{p}_1 + \mathbf{p}_2$. For instance, with four charges at the corners of a square, as shown in Fig. 3.30, the net dipole moment is zero. You can see this by combining the charges in pairs (vertically, $\downarrow + \uparrow = 0$, or horizontally, $\rightarrow + \leftarrow = 0$) or by adding up the four contributions individually, using Eq. 3.100. This is a *quadrupole,* as I indicated earlier, and its potential is dominated by the quadrupole term in the multipole expansion.)

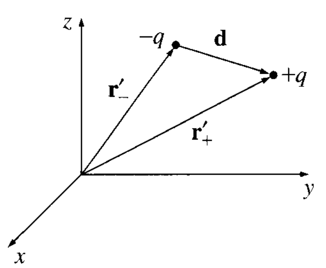


Figure 3.29

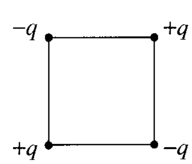


Figure 3.30

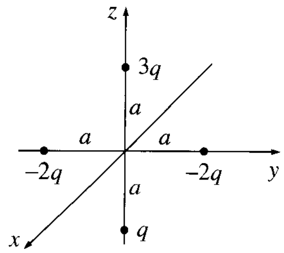


Figure 3.31

Problem 3.27 Four particles (one of charge q, one of charge $3q$, and two of charge $-2q$) are placed as shown in Fig. 3.31, each a distance a from the origin. Find a simple approximate formula for the potential, valid at points far from the origin. (Express your answer in spherical coordinates.)

Problem 3.28 In Ex. 3.9 we derived the exact potential for a spherical shell of radius R, which carries a surface charge $\sigma = k\cos\theta$.

(a) Calculate the dipole moment of this charge distribution.

(b) Find the approximate potential, at points far from the sphere, and compare the exact answer (3.87). What can you conclude about the higher multipoles?

Problem 3.29 For the dipole in Ex. 3.10, expand $1/\imath_{\pm}$ to order $(d/r)^3$, and use this to determine the quadrupole and octopole terms in the potential.

3.4.3 Origin of Coordinates in Multipole Expansions

I mentioned earlier that a point charge at the origin constitutes a "pure" monopole. If it is *not* at the origin, it's no longer a pure monopole. For instance, the charge in Fig. 3.32 has a dipole moment $\mathbf{p} = qd\hat{\mathbf{y}}$, and a corresponding dipole term in its potential. The monopole potential $(1/4\pi\epsilon_0)q/r$ is not quite correct for this configuration; rather, the exact potential is $(1/4\pi\epsilon_0)q/\imath$. The multipole expansion is, remember, a series in inverse powers of r (the distance to the *origin*), and when we expand $1/\imath$ we get *all* powers, not just the first.

So moving the origin (or, what amounts to the same thing, moving the *charge*) can radically alter a multipole expansion. The **monopole moment** Q does not change, since the total charge is obviously independent of the coordinate system. (In Fig. 3.32 the monopole term was unaffected when we moved q away from the origin—it's just that it was no longer the whole story: a dipole term—and for that matter all higher poles—appeared as well.) Ordinarily, the dipole moment *does* change when you shift the origin, but there is an important exception: *If the total charge is zero, then the dipole moment is independent of*

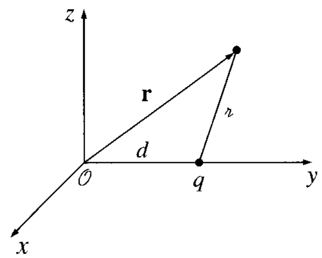


Figure 3.32

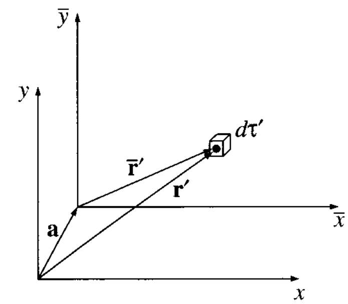


Figure 3.33

the choice of origin. For suppose we displace the origin by an amount **a** (Fig. 3.33). The new dipole moment is then

$$\begin{aligned}\bar{\mathbf{p}} &= \int \bar{\mathbf{r}}'\rho(\mathbf{r}')\,d\tau' = \int (\mathbf{r}' - \mathbf{a})\rho(\mathbf{r}')\,d\tau' \\ &= \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' - \mathbf{a}\int \rho(\mathbf{r}')\,d\tau' = \mathbf{p} - Q\mathbf{a}.\end{aligned}$$

In particular, if $Q = 0$, then $\bar{\mathbf{p}} = \mathbf{p}$. So if someone asks for the dipole moment in Fig. 3.34(a), you can answer with confidence "$q\mathbf{d}$," but if you're asked for the dipole moment in Fig. 3.34(b) the appropriate response would be: "With respect to *what origin*?"

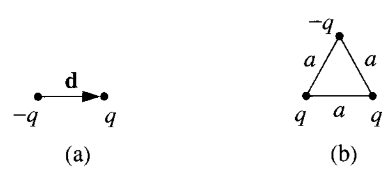


Figure 3.34

Problem 3.30 Two point charges, $3q$ and $-q$, are separated by a distance a. For each of the arrangements in Fig. 3.35, find (i) the monopole moment, (ii) the dipole moment, and (iii) the approximate potential (in spherical coordinates) at large r (include both the monopole and dipole contributions).

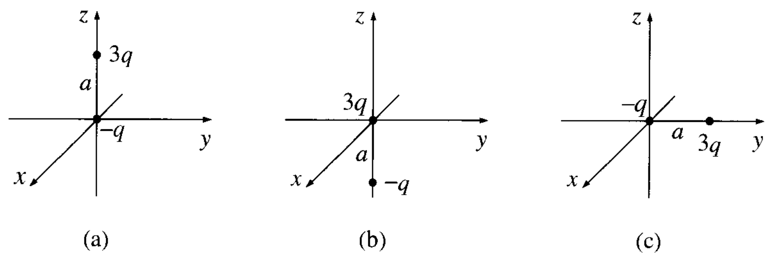


Figure 3.35

3.4.4 The Electric Field of a Dipole

So far we have worked only with *potentials*. Now I would like to calculate the electric *field* of a (pure) dipole. If we choose coordinates so that $\mathbf{p}$ lies at the origin and points in the z direction (Fig. 3.36), then the potential at r, θ is (Eq. 3.99):

$$V_{\text{dip}}(r,\theta) = \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}. \tag{3.102}$$

To get the field, we take the negative gradient of V:

$$\begin{aligned} E_r &= -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}, \\ E_\theta &= -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}, \\ E_\phi &= -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0. \end{aligned}$$

Thus

$$\boxed{\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).} \tag{3.103}$$

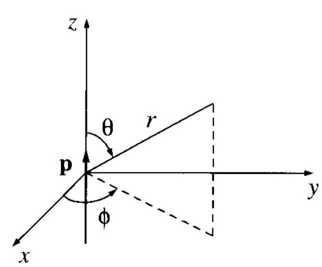
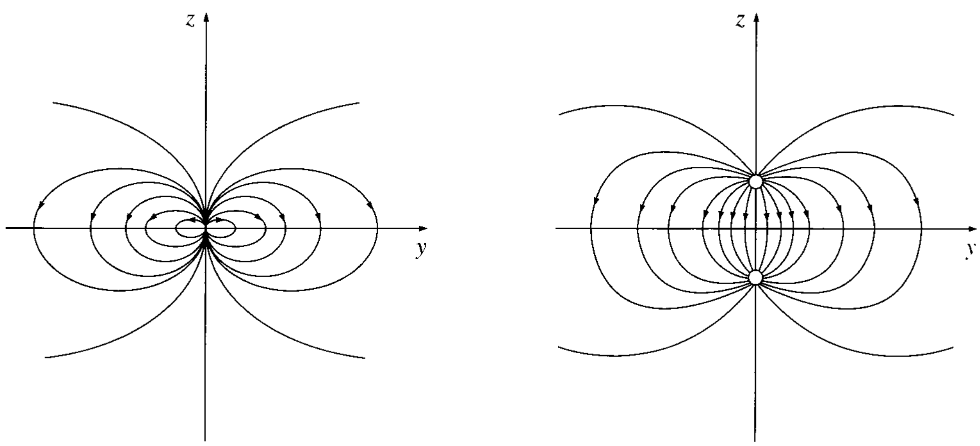


Figure 3.36

This formula makes explicit reference to a particular coordinate system (spherical) and assumes a particular orientation for $\mathbf{p}$ (along z). It can be recast in a coordinate-free form, analogous to the potential in Eq. 3.99—see Prob. 3.33.

Notice that the dipole field falls off as the inverse *cube* of r; the *monopole* field $(Q/4\pi\epsilon_0 r^2)\hat{\mathbf{r}}$ goes as the inverse *square,* of course. Quadrupole fields go like $1/r^4$, octopole like $1/r^5$, and so on. (This merely reflects the fact that monopole *potentials* fall off like $1/r$, dipole like $1/r^2$, quadrupole like $1/r^3$, and so on—the gradient introduces another factor of $1/r$.)

Figure 3.37(a) shows the field lines of a "pure" dipole (Eq. 3.103). For comparison, I have also sketched the field lines for a "physical" dipole, in Fig. 3.37(b). Notice how similar the two pictures become if you blot out the central region; up close, however, they are entirely different. Only for points $r \gg d$ does Eq. 3.103 represent a valid approximation to the field of a physical dipole. As I mentioned earlier, this régime can be reached either by going to large r or by squeezing the charges very close together.[12]



(a) Field of a "pure" dipole

(a) Field of a "physical" dipole

Figure 3.37

Problem 3.31 A "pure" dipole p is situated at the origin, pointing in the z direction.

(a) What is the force on a point charge q at $(a, 0, 0)$ (Cartesian coordinates)?

(b) What is the force on q at $(0, 0, a)$?

(c) How much work does it take to move q from $(a, 0, 0)$ to $(0, 0, a)$?

[12]Even in the limit, there remains an infinitesimal region at the origin where the field of a physical dipole points in the "wrong" direction, as you can see by "walking" down the z axis in Fig. 3.35(b). If you want to explore this subtle and important point, work Prob. 3.42.

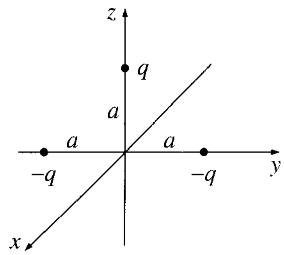


Figure 3.38

Problem 3.32 Three point charges are located as shown in Fig. 3.38, each a distance a from the origin. Find the approximate electric field at points far from the origin. Express your answer in spherical coordinates, and include the two lowest orders in the multipole expansion.

• **Problem 3.33** Show that the electric field of a ("pure") dipole (Eq. 3.103) can be written in the coordinate-free form

$$\boxed{\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\,[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}].} \tag{3.104}$$

More Problems on Chapter 3

Problem 3.34 A point charge q of mass m is released from rest at a distance d from an infinite grounded conducting plane. How long will it take for the charge to hit the plane? [*Answer:* $(\pi d/q)\sqrt{2\pi\epsilon_0 m d}$.]

Problem 3.35 Two infinite parallel grounded conducting planes are held a distance a apart. A point charge q is placed in the region between them, a distance x from one plate. Find the force on q. Check that your answer is correct for the special cases $a \to \infty$ and $x = a/2$. (Obtaining the induced surface is not so easy. See B. G. Dick, *Am. J. Phys.* **41,** 1289 (1973), M. Zahn, *Am. J. Phys.* **44,** 1132 (1976), J. Pleines and S. Mahajan, *Am. J. Phys.* **45,** 868 (1977), and Prob. 3.44 below.)

Problem 3.36 Two long straight wires, carrying opposite uniform line charges $\pm\lambda$, are situated on either side of a long conducting cylinder (Fig. 3.39). The cylinder (which carries no net charge) has radius R, and the wires are a distance a from the axis. Find the potential at point $\mathbf{r}$.

$$\left[\textit{Answer: } V(s,\phi) = \frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{(s^2+a^2+2sa\cos\phi)[(sa/R)^2+R^2-2sa\cos\phi]}{(s^2+a^2-2sa\cos\phi)[(sa/R)^2+R^2+2sa\cos\phi]}\right\}\right]$$

Problem 3.37 A conducting sphere of radius a, at potential V_0, is surrounded by a thin concentric spherical shell of radius b, over which someone has glued a surface charge

$$\sigma(\theta) = k\cos\theta,$$

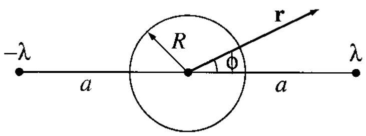


Figure 3.39

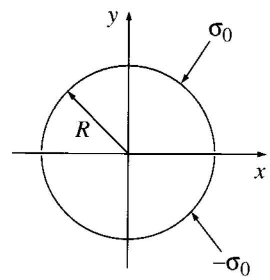


Figure 3.40

where k is a constant, and θ is the usual spherical coordinate.

(a) Find the potential in each region: (i) $r > b$, and (ii) $a < r < b$.

(b) Find the induced surface charge $\sigma_i(\theta)$ on the conductor.

(c) What is the total charge of this system? Check that your answer is consistent with the behavior of V at large r.

$$\left[\textit{Answer: } V(r,\theta) = \begin{cases} aV_0/r + (b^3 - a^3)k\cos\theta/3r^2\epsilon_0, & r \geq b \\ aV_0/r + (r^3 - a^3)k\cos\theta/3r^2\epsilon_0, & r \leq b \end{cases} \right]$$

Problem 3.38 A charge $+Q$ is distributed uniformly along the z axis from $z = -a$ to $z = +a$. Show that the electric potential at a point $\mathbf{r}$ is given by

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0}\frac{1}{r}\left[1 + \frac{1}{3}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \frac{1}{5}\left(\frac{a}{r}\right)^4 P_4(\cos\theta) + \ldots\right],$$

for $r > a$.

Problem 3.39 A long cylindrical shell of radius R carries a uniform surface charge σ_0 on the upper half and an opposite charge $-\sigma_0$ on the lower half (Fig. 3.40). Find the electric potential inside and outside the cylinder.

Problem 3.40 A thin insulating rod, running from $z = -a$ to $z = +a$, carries the indicated line charges. In each case, find the leading term in the multipole expansion of the potential: (a) $\lambda = k\cos(\pi z/2a)$, (b) $\lambda = k\sin(\pi z/a)$, (c) $\lambda = k\cos(\pi z/a)$, where k is a constant.

• **Problem 3.41** Show that the *average* field inside a sphere of radius R, due to all the charge within the sphere, is

$$\mathbf{E}_{\text{ave}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}, \tag{3.105}$$

where $\mathbf{p}$ is the total dipole moment. There are several ways to prove this delightfully simple result. Here's one method:

(a) Show that the average field due to a single charge q at point $\mathbf{r}$ inside the sphere is the same as the field at $\mathbf{r}$ due to a uniformly charged sphere with $\rho = -q/(\frac{4}{3}\pi R^3)$, namely

$$\frac{1}{4\pi\epsilon_0}\frac{1}{(\frac{4}{3}\pi R^3)}\int \frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau',$$

where $\mathscr{r}$ is the vector from $\mathbf{r}$ to $d\tau'$.

(b) The latter can be found from Gauss's law (see Prob. 2.12). Express the answer in terms of the dipole moment of q.

(c) Use the superposition principle to generalize to an arbitrary charge distribution.

(d) While you're at it, show that the average field over the sphere due to all the charges *outside* is the same as the field they produce at the center.

Problem 3.42 Using Eq. 3.103, calculate the average electric field of a dipole, over a spherical volume of radius R, centered at the origin. Do the angular intervals first. [*Note:* You *must* express $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ (see back cover) before integrating. If you don't understand why, reread the discussion in Sect. 1.4.1.] Compare your answer with the general theorem Eq. 3.105. The discrepancy here is related to the fact that the field of a dipole blows up at $r = 0$. The angular integral is zero, but the radial integral is infinite, so we really don't know *what* to make of the answer. To resolve this dilemma, let's say that Eq. 3.103 applies *outside a tiny sphere of radius* ϵ—its contribution to E_{ave} is then *unambiguously* zero, and the whole answer has to come from the field *inside* the ϵ-sphere.

(b) What must the field *inside* the ϵ-sphere be, in order for the general theorem (3.105) to hold? [*Hint:* since ϵ is arbitrarily small, we're talking about something that is infinite at $r = 0$ and whose integral over an infinitesimal volume is finite.] [*Answer:* $-(\mathbf{p}/3\epsilon_0)\delta^3(\mathbf{r})$]

[Evidently, the *true* field of a dipole is

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] - \frac{1}{3\epsilon_0}\mathbf{p}\,\delta^3(\mathbf{r}). \tag{3.106}$$

You may well wonder how we missed the delta-function term when we calculated the field back in Sect. 3.4.4. The answer is that the differentiation leading to Eq. 3.103 is perfectly valid *except* at $r = 0$, but we should have known (from our experience in Sect. 1.5.1) that the point $r = 0$ is problematic. See C. P. Frahm, *Am. J. Phys.* **51**, 826 (1983), or more recently R. Estrada and R. P. Kanwal, *Am. J. Phys.* **63**, 278 (1995). For further details and applications, see D. J. Griffiths, *Am. J. Phys.* **50,** 698 (1982).]

Problem 3.43

(a) Suppose a charge distribution $\rho_1(\mathbf{r})$ produces a potential $V_1(\mathbf{r})$, and some other charge distribution $\rho_2(\mathbf{r})$ produces a potential $V_2(\mathbf{r})$. [The two situations may have nothing in common, for all I care—perhaps number 1 is a uniformly charged sphere and number 2 is a parallel-plate capacitor. Please understand that ρ_1 and ρ_2 are not present *at the same time;* we are talking about two *different problems,* one in which only ρ_1 is present, and another in which only ρ_2 is present.] Prove **Green's reciprocity theorem:**

$$\int_{\text{all space}} \rho_1 V_2\,d\tau = \int_{\text{all space}} \rho_2 V_1\,d\tau.$$

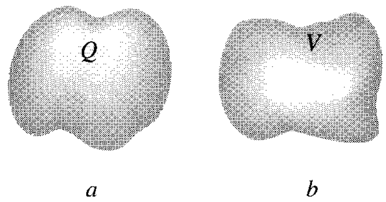


Figure 3.41

[*Hint:* Evaluate $\int \mathbf{E}_1 \cdot \mathbf{E}_2 \, d\tau$ two ways, first writing $\mathbf{E}_1 = -\nabla V_1$ and using integration-by-parts to transfer the derivative to $\mathbf{E}_2$, then writing $\mathbf{E}_2 = -\nabla V_2$ and transferring the derivative to $\mathbf{E}_1$.]

(b) Suppose now that you have two separated conductors (Fig. 3.41). If you charge up conductor a by amount Q (leaving b uncharged) the resulting potential of b is, say, V_{ab}. On the other hand, if you put that same charge Q on conductor b (leaving a uncharged) the potential of a would be V_{ba}. Use Green's reciprocity theorem to show that $V_{ab} = V_{ba}$ (an astonishing result, since we assumed nothing about the shapes or placement of the conductors).

Problem 3.44 Use Green's reciprocity theorem (Prob. 3.43) to solve the following two problems. [*Hint:* for distribution 1, use the actual situation; for distribution 2, remove q, and set one of the conductors at potential V_0.]

(a) Both plates of a parallel-plate capacitor are grounded, and a point charge q is placed between them at a distance x from plate 1. The plate separation is d. Find the induced charge on each plate. [*Answer:* $Q_1 = q(x/d - 1)$; $Q_2 = -qx/d$]

(b) Two concentric spherical conducting shells (radii a and b) are grounded, and a point charge q is placed between them (at radius r). Find the induced charge on each sphere.

Problem 3.45

(a) Show that the quadrupole term in the multipole expansion can be written

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{2r^3} \sum_{i,j=1}^{3} \hat{r}_i \hat{r}_j Q_{ij},$$

where

$$Q_{ij} \equiv \int [3r_i' r_j' - (r')^2 \delta_{ij}] \rho(\mathbf{r}') \, d\tau'.$$

Here

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

is the **Kronecker delta**, and Q_{ij} is the **quadrupole moment** of the charge distribution. Notice the hierarchy:

$$V_{\text{mon}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}; \quad V_{\text{dip}} = \frac{1}{4\pi\epsilon_0} \frac{\sum \hat{r}_i p_i}{r^2}; \quad V_{\text{quad}} = \frac{1}{4\pi\epsilon_0} \frac{\frac{1}{2} \sum \hat{r}_i \hat{r}_j Q_{ij}}{r^3}; \ldots$$

The monopole moment (Q) is a scalar, the dipole moment ($\mathbf{p}$) is a vector, the quadrupole moment (Q_{ij}) is a second-rank tensor, and so on.

(b) Find all nine components of Q_{ij} for the configuration in Fig. 3.30 (assume the square has side a and lies in the xy plane, centered at the origin).

(c) Show that the quadrupole moment is independent of origin if the monopole and dipole moments both vanish. (This works all the way up the hierarchy—the lowest nonzero multipole moment is always independent of origin.)

(d) How would you define the **octopole moment**? Express the octopole term in the multipole expansion in terms of the octopole moment.

Problem 3.46 In Ex. 3.8 we determined the electric field outside a spherical conductor (radius R) placed in a uniform external field $\mathbf{E}_0$. Solve the problem now using the method of images, and check that your answer agrees with Eq. 3.76. [*Hint:* Use Ex. 3.2, but put another charge, $-q$, diametrically opposite q. Let $a \to \infty$, with $(1/4\pi\epsilon_0)(2q/a^2) = -E_0$ held constant.]

! **Problem 3.47**For the infinite rectangular pipe in Ex. 3.4, suppose the potential on the bottom ($y = 0$) and the two sides ($x = \pm b$) is zero, but the potential on the top ($y = a$) is a nonzero constant V_0. Find the potential inside the pipe. [*Note:* This is a rotated version of Prob. 3.14(b), but set it up as in Ex. 3.4 using sinusoidal functions in y and hyperbolics in x. It is an unusual case in which $k = 0$ must be included. Begin by finding the general solution to Eq. 3.26 when $k = 0$. For further discussion see S. Hassani, *Am. J. Phys.* **59**, 470 (1991).]

$\left[Answer: V_0\left(\frac{y}{a} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)}\sin(n\pi y/a)\right)\right]$

! **Problem 3.48**

(a) A long metal pipe of square cross-section (side a) is grounded on three sides, while the fourth (which is insulated from the rest) is maintained at constant potential V_0. Find the net charge per unit length on the side *opposite* to V_0. [*Hint:* Use your answer to Prob. 3.14 or Prob. 3.47.]

(b) A long metal pipe of circular cross-section (radius R) is divided (lengthwise) into four equal sections, three of them grounded and the fourth maintained at constant potential V_0. Find the net charge per unit length on the section opposite to V_0. [*Answer to both (a) and (b):* $\lambda = -\epsilon_0 V_0 \ln 2$][13]

Problem 3.49 An ideal electric dipole is situated at the origin, and points in the z direction, as in Fig. 3.36. An electric charge is released from rest at a point in the xy plane. Show that it swings back and forth in a semi-circular arc, as though it were a pendulum supported at the origin. [This charming result is due to R. S. Jones, *Am. J. Phys.* **63**, 1042 (1995).]

[13]These are special cases of the **Thompson-Lampard theorem**; see J. D. Jackson, *Am. J. Phys.* **67**, 107 (1999).