CHAPTER 2

Limits

CONCEPT OF LIMITS

Suppose $f(x)$ is a real-valued function and c is a real number. The expression $\lim_{x \to c} f(x) = L$ means that $f(x)$ can be as close to L as desired by making x sufficiently close to c. In such a case, we say that the limit of f, as x approaches c, is L. Note that this statement is true even if $f(c) \neq L$. Indeed, the function $f(x)$ need not even be defined at c. Two examples help illustrate this.

Consider $f(x) = \frac{x}{x^2+1}$ as x approaches 2. In this case, $f(x)$ is defined at 2, and it equals its limiting value 0.4.

$f(1.9)$	$f(1.99)$	$f(1.999)$	$f(2)$	$f(2.001)$	$f(2.01)$	$f(2.1)$
0.4121	0.4012	0.4001	$\Rightarrow$ **0.4** $\Leftarrow$	0.3998	0.3988	0.3882

As x approaches 2, $f(x)$ approaches 0.4 and hence we have $\lim_{x \to 2} f(x) = 0.4$. In the case where $f(c) = \lim_{x \to c} f(x)$, f is said to be **continuous** at $x = c$. But it is not always the case.

Consider $g(x) = \begin{cases} \frac{x}{x^2+1}, & \text{if } x \neq 2 \\ 0, & \text{if } x = 2 \end{cases}$

This limit of $g(x)$ as x approaches 2 is 0.4 (just as in $f(x)$), but $\lim_{x \to 2} g(x) \neq g(2)$: g is not continuous at $x = 2$. Or, consider the case where $f(x)$ is undefined at $x = c$.

$f(x) = \frac{x-1}{\sqrt{x}-1}$, in this case as x approaches 1, $f(x)$ is undefined (0/0) at $x = 1$ but the limit equals 2.

$f(0.9)$	$f(0.99)$	$f(0.999)$	$f(1.0)$	$f(1.001)$	$f(1.01)$	$f(1.1)$
1.95	1.99	1.999	$\Rightarrow$ **undefined** $\Leftarrow$	2.001	2.010	2.10

Thus, $f(x)$ can be made arbitrarily close to the limit of 2 just by making x sufficiently close to 1.

Formal Definition of Limit

Karl Weierstrass formally defined limit as follows:

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number (Fig. 2.1).

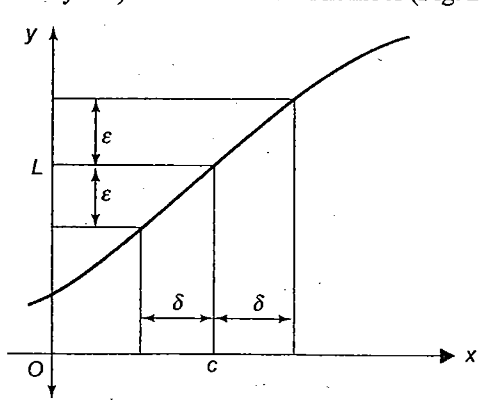


Fig. 2.1

$\lim_{x \to c} f(x) = L$ means that for each real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for all x with $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$ or, symbolically,

$$\forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall x\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$$

Compared to the informal discussion above, the fact that ε can be any arbitrarily small positive number corresponds to being able to bring $f(x)$ as close to L as desired. The δ marks some "sufficiently close" distance for values of x from c such that $f(x)$ stays within a distance less than ε from the limit L.

The formal (ε, δ) definition of limit is called the delta epsilon.

Caution: It should be noted that this definition provides a way to recognize a limit without providing a way to calculate it. One often needs to find limit using informal methods especially when $f(x)$ is discontinuous at c, for example, when f is a ratio with a denominator that becomes 0 at c. One can check that the result actually meets the Weierstrass definition in such cases.

Neighbourhood (NBD) of a Point

Let 'a' be a real number and let δ be a positive real number. Then the set of all real numbers lying between $a - \delta$ and $a + \delta$ is called the neighbourhood of 'a' of radius 'δ' and is denoted by $N_\delta(a)$.

Thus, $N_\delta(a) = (a - \delta, a + \delta) = \{x \in R \mid a - \delta < x < a + \delta\}$

The set $(a - \delta, a)$ is called the left NBD of 'a' and the set $(a, a + \delta)$ is known as the right NBD of 'a'.

Left- and Right-Hand Limits

Let $f(x)$ be a function with domain D and let 'a' be a point such that every NBD of 'a' contains infinitely many points of D. A real number ℓ is called left limit of $f(x)$ at $x = a$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $a - \delta < x < a \Rightarrow |f(x) - \ell| < \varepsilon$

In such a case, we write $\lim_{x \to a^-} f(x) = \ell$.

Thus, $\lim_{x \to a^-} f(x) = \ell$, if $f(x)$ tends to ℓ as x tends to 'a' from the left-hand side.

Similarly, a real number ℓ' is a right limit of $f(x)$ at $x = a$ iff for every $\varepsilon > 0$, there exists $\delta > 0$ such that $a < x < a + \delta \Rightarrow |f(x) - \ell'| < \varepsilon$ and we write $\lim_{x \to a^+} f(x) = \ell'$.

In other words, ℓ' is a right limit of $f(x)$ at $x = a$ iff $f(x)$ tends to ℓ' as x tends to 'a' from the right-hand side.

Existence of Limit

It follows from the discussions made in the previous two sections that $\lim_{x \to a} f(x)$ exists if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exists and both are equal.

Thus, $\lim_{x \to a} f(x)$ exists $\Leftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

For the functions such as $f(x) = \cos^{-1} x$, $\lim_{x \to 1^+} \cos^{-1} x$ does not exist as function is not defined towards right-hand side. However, $\lim_{x \to 1^-} \cos^{-1} x$ exists, and is equal to 0.

Indeterminate Forms

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}$ which seems to be undefined or meaningless. In fact, in many cases this limit exists and has a finite value. The determination of limit in such a case is traditionally referred to as the evaluation of the indeterminate form $\frac{0}{0}$, though literally speaking nothing is indeterminate involved here. Sometimes $\frac{0}{0}$ is referred to as undetermined form or illusory form.

Consider $\lim_{x\to a} \frac{f(x)}{g(x)}$. Let it take $\frac{0}{0}$ form.

$\lim_{x\to a} f(x)$	$\lim_{x\to a} g(x)$	$\lim_{x\to a} \frac{f(x)}{g(x)}$
10^{-100}	10^{-1000}	$10^{900} \to \infty$
10^{-1000}	10^{-100}	$10^{-900} \to 0$
2×10^{-1000}	10^{-1000}	2
10^{-1000}	-3×10^{-1000}	$-1/3$

Thus, $\lim_{x\to a} \frac{f(x)}{g(x)}$ can take any real value or simply $\lim_{x\to a} \frac{f(x)}{g(x)}$ cannot be determined by preliminary methods.

Thus, this form is called indeterminate form.

Other Indeterminate Forms

(i) $\frac{\infty_1}{\infty_2} = \frac{1/\infty_2}{1/\infty_1} = \frac{0}{0}$

(ii) $0\times\infty = \frac{0}{1/\infty} = \frac{0}{0}$

(iii) $y = 0^0 \Rightarrow \log y = \log(0^0) \Rightarrow 0\times\log(0) = 0\times\infty$

(iv) $y = \infty^0 \Rightarrow \log y = \log(\infty^0) \Rightarrow 0\times\log(\infty) = 0\times\infty$

(v) $y = 1^\infty \Rightarrow \log y = \log(1^\infty) \Rightarrow \infty\times\log(1) = \infty\times 0$

(vi) $\infty_1 - \infty_2$ is also an indeterminate form as the ∞_1 and ∞_2 does not necessarily approach to the same infinity.

Difference between Limit of Function at $x = a$ and $f(a)$

Case	$y=f(x)$	Explanation
$\lim_{x\to a} f(x)$ exists but $f(a)$ does not exist	$f(x) = \frac{x^2-a^2}{x-a}$	The value of function at $x = a$ is of the form $\frac{0}{0}$ which is indeterminate, i.e., $f(a)$ does not exist. But $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = 2a$. Hence, $\lim_{x\to a} f(x)$ exists.
$\lim_{x\to a} f(x)$ does not exist but $f(a)$ exists	$f(x) = [x]$ (where [·] represents greatest integer function)	The value of function at $x = n$ ($n \in I$) is n, i.e., $f(n) = n$. But $\lim_{x\to n^-} [x] = n-1$ and $\lim_{x\to n^+} [x] = n$. Hence, $\lim_{x\to n} [x]$ does not exist.
$\lim_{x\to a} f(x)$ and $f(a)$ both exist and are equal	$f(x) = \begin{cases} \sin x, & x<0 \\ x, & x\ge 0 \end{cases}$	The value of function at $x = 0$ is 0, i.e., $f(0) = 0$. Also $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sin x = 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$, i.e., $\lim_{x\to 0} f(x)$ exist.
$\lim_{x\to a} f(x)$ and $f(a)$ both exist but are unequal	$f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x\ne 3 \\ 3, & x=3 \end{cases}$	The value of function at $x=3$ is 3, i.e., $f(3)=3$. Also $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3} = 6$, i.e., $\lim_{x\to 3} f(x)$ exists. But $\lim_{x\to 3} f(x) \ne f(3)$.

Thus, for limit to exist at $x = a$, it is not necessary that function is defined at that point.

Example 2.1 Evaluate the left- and right-hand limits of the function $f(x)=\begin{cases}\dfrac{|x-4|}{x-4}, & x\neq 4\\ 0, & x=4\end{cases}$ at $x=4$.

Sol. L.H.L. of $f(x)$ at $x=4$

$= \lim_{x\to 4^-} f(x)$

$= \lim_{h\to 0} f(4-h)$

$= \lim_{h\to 0}\dfrac{|4-h-4|}{4-h-4} = \lim_{h\to 0}\dfrac{|-h|}{-h}$

$= \lim_{h\to 0}\dfrac{h}{-h} = \lim_{h\to 0} -1 = -1$

R.H.L. of $f(x)$ at $x=4$

$= \lim_{x\to 4^+} f(x)$

$= \lim_{h\to 0} f(4+h)$

$= \lim_{h\to 0}\dfrac{|4+h-4|}{4+h-4}$

$= \lim_{h\to 0}\dfrac{|h|}{h} = \lim_{h\to 0}\dfrac{h}{h} = \lim_{h\to 0} 1 = 1$

Example 2.2 Evaluate the left- and the right-hand limits of the function defined by $f(x)=\begin{cases}1+x^2, & \text{if } 0\le x<1\\ 2-x, & \text{if } x>1\end{cases}$ at $x=1$. Also, show that $\lim_{x\to 1} f(x)$ does not exist.

Sol. L.H.L. of $f(x)$ at $x=1$

$= \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$

$= \lim_{h\to 0}(1+(1-h)^2) = \lim_{h\to 0}(2-2h+h^2)=2$

R.H.L. of $f(x)$ at $x=1$

$= \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$

$= \lim_{h\to 0}[2-(1+h)] = \lim_{h\to 0}(1-h)=1$

Clearly, $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

So, $\lim_{x\to 1} f(x)$ does not exist.

Example 2.3 Let $f(x)=\begin{cases}\cos x, & \text{if } x\ge 0\\ x+k, & \text{if } x<0\end{cases}$. Find the value of constant k, given that $\lim_{x\to 0} f(x)$ exists.

Sol. $\lim_{x\to 0} f(x)$ exists

$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

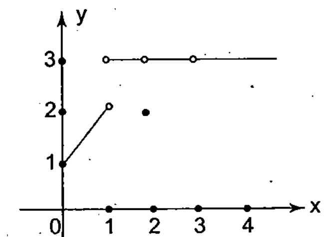
$\Rightarrow \lim_{x\to 0} x+k = \lim_{x\to 0}\cos x$

$\Rightarrow 0+k=\cos 0$

$\Rightarrow k=1$

Concept Application Exercise 2.1

1. If $f(x)=\begin{cases}\dfrac{x-|x|}{x}, & x\neq 0\\ 2, & x=0\end{cases}$ show that $\lim_{x\to 0} f(x)$ does not exist.
2. Show that $\lim_{x\to 0}\dfrac{e^{1/x}-1}{e^{1/x}+1}$ does not exist.
3. Evaluate $\lim_{x\to 0}\dfrac{3x+|x|}{7x-5|x|}$.
4. If $f(x)=\begin{cases}x, & x<0\\ 1, & x=0\\ x^2, & x>0\end{cases}$, then find $\lim_{x\to 0} f(x)$ if exists.
5. Consider the following graph of the function $y=f(x)$. Which of the following is/are correct?



a. $\lim_{x\to 1} f(x)$ does not exist

b. $\lim_{x\to 2} f(x)$ does not exist

c. $\lim_{x\to 3} f(x)=3$

d. $\lim_{x\to 1.99} f(x)$ exists

ALGEBRA OF LIMITS

Let $\lim_{x\to a} f(x)=\ell$ and $\lim_{x\to a} g(x)=m$. If ℓ and m exist, then

1. $\lim_{x\to a}(f\pm g)(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = \ell \pm m$
2. $\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x)\lim_{x\to a} g(x) = \ell m$
3. $\lim_{x\to a}\left(\dfrac{f}{g}\right)(x) = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{\ell}{m}$, provided $m\neq 0$
4. $\lim_{x\to a} k\, f(x) = k.\lim_{x\to a} f(x)$, where k is constant
5. $\lim_{x\to a}|f(x)| = \left|\lim_{x\to a} f(x)\right| = |\ell|$
6. $\lim_{x\to a}(f(x))^{g(x)} = \lim_{x\to a} f(x)^{\lim_{x\to a} g(x)} = \ell^m$
7. $\lim_{x\to a} fog(x) = f\left(\lim_{x\to a} g(x)\right) = f(m)$, only if f is continuous at $g(x)=m$

In particular,

a. $\lim_{x\to a}\log f(x) = \log\left(\lim_{x\to a} f(x)\right) = \log \ell$

b. $\lim_{x \to a} e^{f(x)} = e^{\lim_{x \to a} f(x)} = e^{\ell}$

8. If $\lim_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x \to a} \frac{1}{f(x)} = 0$.

9. If $f(x) \le g(x)$ for every x in the NBD of a, then $\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$.

Points to Remember

1. If $\lim_{x \to c} f(x)g(x)$ exists, then we can have the following cases:

   a. Both $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist. Obviously, then $\lim_{x \to c} f(x)g(x)$ exists.

   b. $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} g(x)$ does not exist.

   Consider $f(x) = x$; $g(x) = \frac{1}{\sin x}$, now $\lim_{x \to 0} f(x) \cdot g(x)$ exists $= 1$. Also $\lim_{x \to 0} f(x) = 0$ exists but $\lim_{x \to 0} g(x)$ does not exist.

   c. Both $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ do not exist.

   Let f be defined as $f(x) = \begin{cases} 1 \text{ if } x \le 0 \\ 2 \text{ if } x > 0 \end{cases}$. Let

   $g(x) = \begin{cases} 2 \text{ if } x \le 0 \\ 1 \text{ if } x > 0 \end{cases}$. Then, $f(x) \cdot g(x) = 2$, and so $\lim_{x \to 0} f(x) \times g(x)$ exists, while $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist.

2. If $\lim_{x \to c} [f(x) + g(x)]$ exists then we can have the following cases:

   a. If $\lim_{x \to c} f(x)$ exists, then $\lim_{x \to c} g(x)$ must exist.

   Proof: This is true as $g = (f + g) - f$.

   Therefore, by the limit theorem, $\lim_{x \to c} g(x) = \lim_{x \to c} (f(x) + g(x)) - \lim_{x \to 0} f(x)$ which exists.

   b. Both $\lim_{x \to 1} f(x)$ and $\lim_{x \to c} g(x)$ do not exist.

   Consider $\lim_{x \to 1} [x]$ and $\lim_{x \to 1} \{x\}$, where $[\cdot]$ and $\{\cdot\}$ represent greatest integer and fractional part functions, respectively. Here both the limits do not exist but $\lim_{x \to 1} ([x] + \{x\}) = \lim_{x \to 1} x = 1$ exists.

Example 2.4 Let $f(x) = \begin{cases} x+1, & x > 0 \\ 2 - x, & x \le 0 \end{cases}$ and

$$g(x) = \begin{cases} x+3, & x < 1 \\ x^2 - 2x - 2, & 1 \le x < 2 \\ x - 5, & x \ge 2 \end{cases}$$

find L.H.L. and R.H.L. of $g(f(x))$ at $x = 0$ and hence find $\lim_{x \to 0} g f(x)$.

Sol. As $x \to 0^- \Rightarrow f(x) \to f(0^-) = 2^+$

$\Rightarrow \lim_{x \to 0^-} g(f(x)) = g(2^+) = -3$

Also as $x \to 0^+ \Rightarrow f(x) \to f(0^+) = 1^+$

$\Rightarrow \lim_{x \to 0^+} g(f(x)) = g(1^+) = -3$

Hence, $\lim_{x \to 0} g(f(x))$ exists and is equal to -3

$\Rightarrow \lim_{x \to 0} g(f(x)) = -3$

Sandwich Theorem for Evaluating Limits

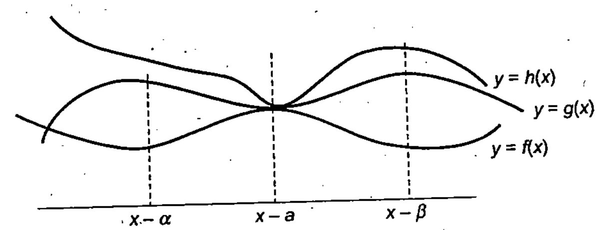


Fig. 2.2

If $f(x) \le g(x) \le h(x)$ $\forall$ $x \in (\alpha, \beta) - \{a\}$ and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ then $\lim_{x \to a} g(x) = L$, where $a \in (\alpha, \beta)$

Remarks

In the **sandwich theorem**, we assume that $f(x) \le g(x) \le h(x)$ for all x near a, "except possibly at a". This means that it is not required that when $x = a$, we have the inequality for the functions. That is, it is not required that $f(a) \le h(a)$. The reason is that we are dealing with limits as x approaches a. So, we have x that is moving closer and closer to a. As long as $f(x) \le g(x) \le h(x)$ is true for all these x, we can be sure the limit, i.e., the point where the function values are heading must behave as the sandwich theorem indicates. In particular, unless we are given extra information about the functions and their values at a, the sandwich theorem does not allow us to make conclusions about functions values at a. So, none of the following claims can be guaranteed by the assumptions in the sandwich theorem:

1. $f(a) = g(a) = h(a)$ [Well, not even $f(a) \le g(a) \le h(a)$]
2. $g(a) = \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$
3. $\lim_{x \to a} g(x) = g(a)$

Example 2.5 Evaluate $\lim_{x \to \infty} \frac{x + 7 \sin x}{-2x + 13}$ using sandwich theorem.

Sol. We know that $-1 \le \sin x \le 1$ for all x.

$\Rightarrow -7 \le 7 \sin x \le 7$

$\Rightarrow x - 7 \le x + 7 \sin x \le x + 7$

Dividing throughout by $-2x + 13$, we get

$$\frac{x-7}{-2x+13} \ge \frac{x+7\sin x}{-2x+13} \ge \frac{x+7}{-2x+13}$$ for all x that are large.

[Why did we switch the inequality signs?]

Now, $$\lim_{x\to\infty}\frac{x-7}{-2x+13} = \lim_{x\to\infty}\frac{1-\frac{7}{x}}{-2+\frac{13}{x}} = \frac{1-0}{-2+0} = -\frac{1}{2}$$

and $$\lim_{x\to\infty}\frac{x+7}{-2x+13} = \lim_{x\to\infty}\frac{1+\frac{7}{x}}{-2+\frac{13}{x}} = \frac{1+0}{-2+0} = -\frac{1}{2}$$

Example 2.6 If [.] denotes the greatest integer function, then find the value of $\lim_{n\to\infty}\frac{[x]+[2x]+\cdots+[nx]}{n^2}$.

Sol. $nx-1<[nx]\le nx$. Putting $n=1, 2, 3, \ldots, n$ and adding them,

$x\Sigma n - n < \Sigma[nx] \le x\Sigma n$

$$\therefore \quad x\frac{\Sigma n}{n^2} - \frac{1}{n} < \frac{\Sigma[nx]}{n^2} \le x.\frac{\Sigma n}{n^2} \qquad (1)$$

Now, $$\lim_{n\to\infty}\left\{x.\frac{\Sigma n}{n^2} - \frac{1}{n}\right\} = x.\lim_{n\to\infty}\frac{\Sigma n}{n^2} - \lim_{n\to\infty}\frac{1}{n} = \frac{x}{2}$$

$$\lim_{n\to\infty}\left\{x.\frac{\Sigma n}{n^2}\right\} = x\lim_{n\to\infty}\frac{\Sigma n}{n^2} = \frac{x}{2}$$

As the two limits are equal by equation (1), $\lim_{n\to\infty}\frac{\Sigma[nx]}{n^2} = \frac{x}{2}$.

Example 2.7 Suppose that f is a function that $2x^2 \le f(x) \le x(x^2+1)$ for all x that are near to 1 but not equal to 1. Show that this fact contains enough information for us to find $\lim_{x\to 1} f(x)$. Also find this limit.

Sol. We see that $\lim_{x\to 1} 2x^2 = 2(1)^2 = 2$

and $\lim_{x\to 1} x(x^2+1) = 1(1^2+1) = 2$

This is enough for us to find $\lim_{x\to 1} f(x)$.

Indeed, it follows from the Sandwich Theorem that

$\lim_{x\to 1} f(x) = 2$.

Example 2.8 Evaluate $\lim_{n\to\infty}\frac{1}{1+n^2} + \frac{1}{2+n^2} + \cdots + \frac{n}{n+n^2}$.

Sol. $$P_n = \frac{1}{1+n^2} + \frac{2}{2+n^2} + \cdots + \frac{n}{n+n^2}$$

Now, $$P_n < \frac{1}{1+n^2} + \frac{2}{1+n^2} + \cdots + \frac{n}{1+n^2}$$

$$= \frac{1}{1+n^2}(1+2+3+\cdots+n)$$

$$= \frac{n(n+1)}{2(1+n^2)}$$

Also, $$P_n > \frac{1}{n+n^2} + \frac{2}{n+n^2} + \frac{3}{n+n^2} + \cdots + \frac{n}{n+n^2}$$

$$= \frac{n(n+1)}{2(n+n^2)}$$

Thus, $$\frac{n(n+1)}{2(n+n^2)} < P_n < \frac{n(n+1)}{2(1+n^2)}$$

$$\Rightarrow \lim_{n\to\infty}\frac{n(n+1)}{2(n+n^2)} < \lim_{n\to\infty} P_n < \lim_{n\to\infty}\frac{n(n+1)}{2(1+n^2)}$$

$$\Rightarrow \lim_{n\to\infty}\frac{1\left(1+\frac{1}{n}\right)}{2\left(\frac{1}{n}+1\right)} < \lim_{n\to\infty} P_n < \lim_{n\to\infty}\frac{1\left(1+\frac{1}{n}\right)}{2\left(\frac{1}{n^2}+1\right)}$$

$$\Rightarrow \frac{1}{2} < \lim_{n\to\infty} P_n < \frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty} P_n = \frac{1}{2}$$

Concept Application Exercise 2.2

Evaluate the following limits using sandwich theorem

1. $\lim_{x\to\infty}\frac{[x]}{x}$, where [.] represents greatest integer function.
2. $\lim_{x\to\infty}\frac{\log_e x}{x}$

USE OF EXPANSIONS IN EVALUATING LIMITS

Some Important Expansions

Sometimes, following expansions are useful in evaluating limits. Students are advised to learn these expansions.

1. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots \; (-1 < x \le 1)$
2. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \cdots \; (-1 < x \le 1)$
3. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$
4. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$

5. $a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \cdots$

6. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$

7. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$

8. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$

Example 2.9 Evaluate $\lim_{x\to 0} \frac{\sin x - x}{x^3}$.

Sol. $\lim_{x\to 0} \frac{\sin x - x}{x^3} \quad \left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - x}{x^3}$$

$$= \lim_{x\to 0}\left[-\frac{1}{3!} + \frac{x^2}{5!} - \cdots\right] = \frac{-1}{3!} = \frac{-1}{6}$$

Example 2.10 Evaluate $\lim_{x\to 0} \frac{5\sin x - 7\sin 2x + 3\sin 3x}{x^2 \sin x}$.

Sol. $\lim_{x\to 0} \frac{5\sin x - 7\sin 2x + 3\sin 3x}{x^2 \sin x}$

$$= \lim_{x\to 0} \frac{5\left(x - \frac{x^3}{3!} + \cdots\right) - 7\left(2x - \frac{(2x)^3}{3!} + \cdots\right) + 3\left(3x - \frac{(3x)^3}{3!} + \cdots\right)}{x^2\left(x - \frac{x^3}{3!} + \cdots\right)}$$

$$= \lim_{x\to 0} \frac{-\frac{5x^3}{3!} + \frac{56x^3}{3!} - \frac{81x^3}{3!}}{x^3\left(1 - \frac{x^2}{3!} + \cdots\right)}$$

$$= \frac{-5 + 56 - 81}{3!} = -5$$

Example 2.11 Evaluate $\lim_{x\to 0} \frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$.

Sol. $(1+x)^{1/x} = e^{\frac{1}{x}\log(1+x)} = e^{\frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)}$

$$= e^{1 - \frac{x}{2} + \frac{x^2}{3} - \cdots} = e \cdot e^{-\frac{x}{2} + \frac{x^2}{3} - \cdots}$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\right) + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^2}{3} \cdots\right)^2 + \cdots\right]$$

$$= e\left[1 - \frac{x}{2} + \frac{11}{24}x^2 - \cdots\right]$$

Hence, $\lim_{x\to 0} \frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2} = \frac{11e}{24}$

Concept Application Exercise 2.3

Evaluate the following limits using the expansion formula of functions.

1. $\lim_{x\to 0}\left(\frac{\sin x - x + \frac{x^3}{6}}{x^5}\right)$

2. $\lim_{x\to 0} \frac{\sin x + \log(1-x)}{x^2}$

3. $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

EVALUATION OF ALGEBRAIC LIMITS

Direct Substitution Method

Consider the following limits: (i) $\lim_{x\to a} f(x)$ (ii) $\lim_{x\to a} \frac{\Phi(x)}{\Psi(x)}$

If $f(a)$ and $\frac{\Phi(a)}{\Psi(a)}$ exist and are fixed real numbers and $\Psi(a) \neq 0$ then we say that $\lim_{x\to a} f(x) = f(a)$ and

$$\lim_{x\to a} \frac{\Phi(x)}{\Psi(x)} = \frac{\Phi(a)}{\Psi(a)}.$$

In other words, if the direct substitution of the point to which the variable tends to, we obtain a fixed real number, then the number obtained is the limit of the function. In fact, if the point to which the variable tends to is a point in the domain of the function, then the value of the function at that point is its limit.

Following examples will illustrate the above method:

1. $\lim_{x\to 1} (3x^2 + 4x + 5) = 3(1)^2 + 4(1) + 5 = 12$

2. $\lim_{x\to 2} \frac{x^2 - 4}{x + 3} = \frac{4 - 4}{2 + 3} = \frac{0}{5} = 0$

Factorization Method

Consider $\lim_{x\to a}\frac{f(x)}{g(x)}$

If by substituting $x = a$, $\frac{f(x)}{g(x)}$ reduces to the form $\frac{0}{0}$, then $(x-a)$ is a factor of both $f(x)$ and $g(x)$. So, we first factorize $f(x)$ and $g(x)$ and then cancel out the common factor to evaluate the limit.

Example 2.12 Evaluate $\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4}$.

Sol. When $x = 2$, the expression $\frac{x^2-5x+6}{x^2-4}$ assumes the indeterminate form $\frac{0}{0}$. Here, $(x-2)$ is a common factor in numerator and denominator. Factorizing the numerator and denominator, we have

$$\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(x-3)}{(x+2)(x-2)}$$

$$=\lim_{x\to 2}\frac{x-3}{x+2}=\frac{2-3}{2+2}=-\frac{1}{4}$$

Example 2.13 Evaluate $\lim_{x\to 1}\left(\frac{2}{1-x^2}+\frac{1}{x-1}\right)$.

Sol. We have $\lim_{x\to 1}\left(\frac{2}{1-x^2}+\frac{1}{x-1}\right)$ $(\infty-\infty$ form)

$$=\lim_{x\to 1}\left(\frac{2}{1-x^2}-\frac{1}{1-x}\right)$$

When $x = 1$, the expression $\frac{2}{1-x^2}-\frac{1}{1-x}$ assumes the form $\infty-\infty$, so we need some simplification to express it in the form $\frac{0}{0}$.

Then,

$$\lim_{x\to 1}\left(\frac{2}{1-x^2}-\frac{1}{1-x}\right)=\lim_{x\to 1}\frac{2-(1+x)}{1-x^2}=\lim_{x\to 1}\frac{1-x}{1-x^2}=\lim_{x\to 1}\frac{1}{1+x}=\frac{1}{2}$$

Example 2.14 Evaluate $\lim_{x\to 1}\frac{x^2+x\log_e x-\log_e x-1}{(x^2-1)}$.

Sol. $\lim_{x\to 1}\frac{x^2+x\log_e x-\log_e x-1}{(x^2-1)}$ $\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 1}\frac{(x-1)(\log_e x+x+1)}{(x+1)(x-1)}\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 1}\frac{\log_e x+x+1}{(x+1)}$$

$$=\frac{\log_e 1+1+1}{1+1}=\frac{0+2}{2}=1$$

Example 2.15 Evaluate $\lim_{x\to\frac{\pi}{4}}\frac{1-\sin 2x}{1+\cos 4x}$.

Sol. $\lim_{x\to\frac{\pi}{4}}\frac{1-\sin 2x}{1+\cos 4x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to\frac{\pi}{4}}\frac{(\sin x-\cos x)^2}{2\cos^2 2x}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{(\sin x-\cos x)^2}{2(\cos^2 x-\sin^2 x)^2}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{1}{2(\cos x+\sin x)^2}=\frac{1}{4}$$

Rationalization Method

This is particularly used when either the numerator or the denominator or both involved expressions consist of squares roots and on substituting the value of x the rational expression takes the form $\frac{0}{0},\frac{\infty}{\infty}$.

Following examples illustrate the procedure.

Example 2.16 Evaluate $\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$.

Sol. When $x = 0$, the expression $\frac{\sqrt{2+x}-\sqrt{2}}{x}$ takes the form $\frac{0}{0}$. Rationalizing the numerator, we have

$$\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$$

$$=\lim_{x\to 0}\frac{\left(\sqrt{2+x}-\sqrt{2}\right)\left(\sqrt{2+x}+\sqrt{2}\right)}{x\left(\sqrt{2+x}+\sqrt{2}\right)}$$

$$=\lim_{x\to 0}\frac{2+x-2}{x(\sqrt{2+x}+\sqrt{2})}$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{2+x}+\sqrt{2}}=\frac{1}{2\sqrt{2}}$$

Example 2.17 Evaluate $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$.

Sol. $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$ $\left(\text{form }\frac{0}{0}\right)$

$$= \lim_{x\to a} \frac{\left(\sqrt{a+2x}-\sqrt{3x}\right)\left(\sqrt{a+2x}+\sqrt{3x}\right)}{\left(\sqrt{3a+x}-2\sqrt{x}\right)\left(\sqrt{3a+x}+2\sqrt{x}\right)} \frac{\left(\sqrt{3a+x}+2\sqrt{x}\right)}{\left(\sqrt{a+2x}+\sqrt{3x}\right)} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to a} \frac{(a+2x-3x)}{(3a+x-4x)} \frac{\left(\sqrt{3a+x}+2\sqrt{x}\right)}{\left(\sqrt{a+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x\to a} \frac{\sqrt{3a+x}+2\sqrt{x}}{3\left(\sqrt{a+2x}+\sqrt{3x}\right)}$$

$$= \frac{\sqrt{3a+a}+2\sqrt{a}}{3\left(\sqrt{a+2a}+\sqrt{3a}\right)} = \frac{1}{3}\times\frac{4\sqrt{a}}{2\sqrt{3a}} = \frac{2}{3\sqrt{3}}$$

Evaluation of Algebraic Limit Using Some Standard Limits

Recall the binomial expansion for any rational power

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots$$

where $|x| < 1$

When x is infinitely small (approaching to zero) such that we can ignore higher powers of x, then we have $(1+x)^n = 1 + nx$ (approximately).

Following theorem will be used to evaluate some algebraic limits:

Theorem: If $n \in Q$, then $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$

Proof: We have $\lim_{x\to a} \frac{x^n - a^n}{x-a}$

$$= \lim_{x\to a^+} \frac{x^n - a^n}{x-a}$$

$$= \lim_{h\to 0} \frac{(a+h)^n - a^n}{a+h-a}$$

$$= \lim_{h\to 0} \frac{a^n\left\{\left(1+\frac{h}{a}\right)^n - 1\right\}}{h}$$

$$= a^n \lim_{h\to 0} \frac{\left\{1+n\frac{h}{a}\right\}-1}{h} \quad [\text{when } x\to 0, (1+x)^n \to 1+nx]$$

$$= a^n \frac{n}{a} = na^{n-1}$$

Example 2.18 Evaluate $\lim_{x\to 2} \frac{x^{10}-1024}{x^5-32}$.

Sol. $$\lim_{x\to 2} \frac{x^{10}-1024}{x^5-32} = \lim_{x\to 2} \frac{x^{10}-2^{10}}{x^5-2^5} = \lim_{x\to 2} \frac{\frac{x^{10}-2^{10}}{x-2}}{\frac{x^5-2^5}{x-2}}$$

$$= \frac{\lim_{x\to 2} \frac{x^{10}-2^{10}}{x-2}}{\lim_{x\to 2} \frac{x^5-2^5}{x-2}} = \frac{10\times 2^{10-1}}{5\times 2^{5-1}} = 64$$

Example 2.19 Evaluate $\lim_{x\to a} \frac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$.

Sol. $$\lim_{x\to a} \frac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$$

$$= \lim_{x\to a} \frac{(x+2)^{5/3}-(a+2)^{5/3}}{(x+2)-(a+2)}$$

$$= \lim_{y\to b} \frac{y^{5/3}-b^{5/3}}{y-b}, \text{ where } x+2=y \text{ and } a+2=b.$$

$$= \frac{5}{3} b^{5/3-1} = \frac{5}{3} b^{2/3} = \frac{5}{3}(a+2)^{2/3}$$

Example 2.20 If $\lim_{x\to 2} \frac{x^n - 2^n}{x-2} = 80$ and $n \in N$, then find the value of n.

Sol. We have $\lim_{x\to 2} \frac{x^n-2^n}{x-2} = 80$

$\Rightarrow n2^{n-1} = 80$

$\Rightarrow n2^{n-1} = 5\times 2^{5-1}$

$\Rightarrow n = 5$

Example 2.21 Evaluate $\lim_{x\to 2} \frac{\sqrt{(x+7)}-3\sqrt{(2x-3)}}{\sqrt[3]{(x+6)}-2\sqrt[3]{(3x-5)}}$.

Sol. We have $L = \lim_{x\to 2} \frac{\sqrt{(x+7)}-3\sqrt{(2x-3)}}{\sqrt[3]{(x+6)}-2\sqrt[3]{(3x-5)}} \quad \left(\frac{0}{0} \text{ form}\right)$

Let $x-2=t$ such that when $x\to 2$, $t\to 0$.

Then $L = \lim_{t\to 0} \frac{(t+9)^{\frac{1}{2}}-3(2t+1)^{\frac{1}{2}}}{(t+8)^{\frac{1}{3}}-2(3t+1)^{\frac{1}{3}}} \quad \left(\frac{0}{0} \text{ form}\right)$

$$= \frac{3}{2}\lim_{t\to 0} \frac{\left(1+\frac{t}{9}\right)^{\frac{1}{2}}-(2t+1)^{\frac{1}{2}}}{\left(1+\frac{t}{8}\right)^{\frac{1}{3}}-(3t+1)^{\frac{1}{3}}} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \frac{3}{2}\lim_{t\to 0}\frac{\frac{1}{2}\frac{t}{9}-(2t)\frac{1}{2}}{\frac{t}{8}\frac{1}{3}-(3t)\frac{1}{3}} = \frac{3}{2}\,\frac{\frac{1}{18}-1}{\frac{1}{24}-1} = \frac{34}{23}$$

Evaluation of Algebraic Limits at Infinity

We know that $\lim\limits_{x\to+\infty}\frac{1}{x}=0$ and $\lim\limits_{x\to+\infty}\frac{1}{x^2}=0$.

Example 2.22 Evaluate $\lim\limits_{x\to\infty}\frac{ax^2+bx+c}{dx^2+ex+f}$.

Sol. Here the expression assumes the form $\frac{\infty}{\infty}$. We notice that the highest power of x in both the numerator and denominator is 2. So we divide each term in both the numerator and denominator by x^2.

$$\therefore \lim_{x\to\infty}\frac{ax^2+bx+c}{dx^2+ex+f} = \lim_{x\to\infty}\frac{a+\frac{b}{x}+\frac{c}{x^2}}{d+\frac{e}{x}+\frac{f}{x^2}} = \frac{a+0+0}{d+0+0}=\frac{a}{d}$$

Example 2.23 Evaluate $\lim\limits_{x\to\infty}\frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$.

Sol. Dividing each term in the numerator and denominator by x, we get

$$\lim_{x\to\infty}\frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$$

$$= \lim_{x\to\infty}\frac{\sqrt{3-1/x^2}-\sqrt{2-1/x^2}}{4+3/x} = \frac{\sqrt{3}-\sqrt{2}}{4}$$

Example 2.24 Evaluate $\lim\limits_{x\to\infty}\sqrt{x}\left(\sqrt{x+c}-\sqrt{x}\right)$

Sol. The given expression is in the form $\infty-\infty$. So we first write it in the rational form $\frac{f(x)}{g(x)}$.

We have $\lim\limits_{x\to\infty}\sqrt{x}\left(\sqrt{x+c}-\sqrt{x}\right)$

$$= \lim_{x\to\infty}\frac{\sqrt{x}\left(\sqrt{x+c}-\sqrt{x}\right)\left(\sqrt{x+c}+\sqrt{x}\right)}{\left(\sqrt{x+c}+\sqrt{x}\right)}$$

$$= \lim_{x\to\infty}\frac{\sqrt{x}(x+c-x)}{\sqrt{x+c}+\sqrt{x}} = \lim_{x\to\infty}\frac{c\sqrt{x}}{\sqrt{x+c}+\sqrt{x}} \quad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty}\frac{c}{\sqrt{1+\frac{c}{x}}+1} \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } \sqrt{x}\,]$$

$$= \frac{c}{\sqrt{1+0}+1} = \frac{c}{2}$$

Example 2.25 Evaluate $\lim\limits_{x\to\infty}\frac{\sqrt{x^2+1}-\sqrt[3]{x^3+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$.

Sol. We have $\lim\limits_{x\to\infty}\frac{\sqrt{x^2+1}-\sqrt[3]{x^3+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$

$$= \lim_{x\to\infty}\frac{\sqrt{1+\frac{1}{x^2}}-\sqrt[3]{1+\frac{1}{x^3}}}{\sqrt[4]{1+\frac{1}{x^4}}-\sqrt[5]{\frac{1}{x}+\frac{1}{x^5}}} = \frac{1-1}{1-0} = 0$$

Example 2.26 Evaluate $\lim\limits_{x\to-\infty}\left(\sqrt{25x^2-3x}+5x\right)$.

Sol.

We have $\lim\limits_{x\to-\infty}\left(\sqrt{25x^2-3x}+5x\right)$ $(\infty-\infty$ form$)$

$$= \lim_{y\to\infty}\left(\sqrt{25y^2+3y}-5y\right), \text{ where } y=-x$$

$$= \lim_{y\to\infty}\frac{25y^2+3y-25y^2}{\sqrt{25y^2+3y}+5y}$$

$$= \lim_{y\to\infty}\frac{3y}{\sqrt{25y^2+3y}+5y}$$

$$= \lim_{y\to\infty}\frac{3}{\sqrt{25+\frac{3}{y}}+5} = \frac{3}{5+5} = \frac{3}{10}$$

Concept Application Exercise 2.4

Evaluate the following limits:

1. $\lim\limits_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$

2. $\lim\limits_{x\to 1}\frac{\sum_{k=1}^{100}x^k-100}{x-1}$

3. $\lim\limits_{x\to\infty}\left[\sqrt{a^2x^2+ax+1}-\sqrt{a^2x^2+1}\right]$

4. $\lim\limits_{x\to a}\frac{\sqrt{3x-a}-\sqrt{x+a}}{x-a}$

5. $\lim\limits_{n\to\infty}\frac{(1^2-2^2+3^2-4^2+5^2+\cdots n \text{ terms})}{n^2}$

6. $\lim\limits_{h\to 0}\left[\frac{1}{h\sqrt[3]{8+h}}-\frac{1}{2h}\right]$

EVALUATION OF TRIGONOMETRIC LIMITS

(i) $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ (where θ is in radians)

Proof: Consider a circle of radius r. Let O be the centre of the circle such that $\angle AOB = \theta$, where θ is measured in radians and its value is very small. Suppose the tangent at A meets OB produced at P. From Fig. 2.3, we have

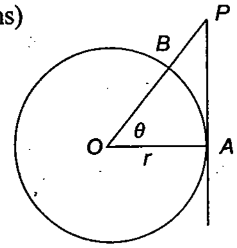


Fig. 2.3

Area of Δ OAB < Area of sector OAB < Area of Δ OAP

$$\Rightarrow \frac{1}{2}\, OA\times OB \sin\theta < \frac{1}{2}\,(OA)^2\,\theta < \frac{1}{2}\, OA\times AP$$

$$\Rightarrow \frac{1}{2}\, r^2 \sin\theta < \frac{1}{2}\, r^2\,\theta < \frac{1}{2}\, r^2 \tan\theta$$

[In $\Delta\, OAP$, $AP = OA\tan\theta$]

$$\Rightarrow \sin\theta < \theta < \tan\theta$$

$$\Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$ [$\because \theta$ is small $\therefore \sin\theta > 0$]

$$\Rightarrow 1 > \frac{\sin\theta}{\theta} > \cos\theta$$

$$\Rightarrow 1 > \lim_{\theta\to 0}\frac{\sin\theta}{\theta} > \lim_{\theta\to 0}\cos\theta \text{ or, } \lim_{\theta\to 0}\cos\theta < \lim_{\theta\to 0}\frac{\sin\theta}{\theta} < 1$$

$$\Rightarrow 1 < \lim_{\theta\to 0}\frac{\sin\theta}{\theta} < 1$$

$$\Rightarrow \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$ (by sandwich theorem)

(ii) $\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1$

We have

$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\,\frac{1}{\cos\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{1}{\cos\theta}$$
$$= (1)(1) = 1$$

(iii) $\lim_{\theta\to a}\frac{\sin(\theta-a)}{\theta-a} = 1$

We have $\lim_{\theta\to a}\frac{\sin(\theta-a)}{\theta-a} = \lim_{h\to 0}\frac{\sin(a+h-a)}{(a+h-a)}$

$$= \lim_{h\to 0}\frac{\sin h}{h} = 1$$

(iv) $\lim_{\theta\to a}\frac{\tan(\theta-a)}{\theta-a} = 1$.

(v) $\lim_{x\to 0}\frac{\sin^{-1}x}{x} = 1$

(vi) $\lim_{x\to 0}\frac{\tan^{-1}x}{x} = 1$

Example 2.27 Evaluate the following limits

a. $\lim_{x\to 0}\frac{\sin 3x}{x}$ **b.** $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$ **c.** $\lim_{x\to 1}\frac{\sin(\log x)}{\log x}$.

Sol.

a. We have $\lim_{x\to 0}\frac{\sin 3x}{x}$

$$= \lim_{x\to 0}\left(3\frac{\sin 3x}{3x}\right) = 3\lim_{x\to 0}\frac{\sin 3x}{3x} = 3(1) = 3$$

$$\left[\because \lim_{x\to 0}\frac{\sin 3x}{3x} = 1\right]$$

b. We have

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)ax}{\left(\frac{\sin bx}{bx}\right)bx} = \frac{a}{b}\,\frac{(1)}{(1)} = \frac{a}{b}$$

c. Given $L = \lim_{x\to 1}\frac{\sin(\log x)}{\log x}$

Let $\log x = t$ then

$$\Rightarrow L = \lim_{t\to 0}\frac{\sin t}{t} = 1$$

Example 2.28 Evaluate $\lim_{x\to 0}\frac{1}{x}\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Sol. We know that $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$, for $-1 \le x \le 1$

$$\Rightarrow \lim_{x\to 0}\frac{1}{x}\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \lim_{x\to 0}\frac{2\tan^{-1}x}{x} = 2$$

Example 2.29 Evaluate $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$.

Sol. We have $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 2}\frac{2\sin^2 x}{x^2} = 2\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2 = 2$$

Example 2.30 Evaluate $\lim_{x\to 0}\frac{\tan x - \sin x}{x^3}$.

Sol. We have $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$ $\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 0}\left(\frac{\sin x-\sin x\cos x}{x^3\cos x}\right)$$

$$=\lim_{x\to 0}\left\{\frac{\sin x(1-\cos x)}{x^3\cos x}\right\}=\lim_{x\to 0}\left\{\frac{\sin x}{x}\,\frac{1-\cos x}{x^2}\,\frac{1}{\cos x}\right\}$$

$$=\left\{\lim_{x\to 0}\frac{\sin x}{x}\right\}\left\{\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2\times 4}\right\}\lim_{x\to 0}\frac{1}{\cos x}$$

$$=\left\{\lim_{x\to 0}\frac{\sin x}{x}\right\}\frac{1}{2}\left\{\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\right\}\lim_{x\to 0}\frac{1}{\cos x}$$

$$=1\times\frac{1}{2}(1)^2\times\frac{1}{1}=\frac{1}{2}$$

Example 2.31 Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2}$.

Sol. We have, $\lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{h\to 0}\frac{1+\cos 2\left(\frac{\pi}{2}+h\right)}{\left[\pi-2\left(\frac{\pi}{2}+h\right)\right]^2}$$

$$=\lim_{h\to 0}\frac{1+\cos(\pi+2h)}{4h^2}$$

$$=\lim_{h\to 0}\frac{1-\cos 2h}{4h^2}$$

$$=\lim_{h\to 0}\frac{2\sin^2 h}{4h^2}=\frac{2}{4}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)^2=\frac{1}{2}$$

Example 2.32 Evaluate $\lim_{x\to\infty}2^{x-1}\tan\left(\frac{a}{2^x}\right)$.

Sol. We have $\lim_{x\to\infty}2^{x-1}\tan\left(\frac{a}{2^x}\right)$

$$=\lim_{x\to\infty}\frac{a}{2}\,\frac{\tan\left(\frac{a}{2^x}\right)}{\left(\frac{a}{2^x}\right)}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\frac{a}{2}\lim_{y\to 0}\frac{\tan y}{y}=\frac{a}{2}\left(\text{where } y=\frac{a}{2^x}\right)$$

Example 2.33 Evaluate $\lim_{x\to 2}\frac{x^2-x-2}{x^2-2x-\sin(x-2)}$.

Sol. $\lim_{x\to 2}\frac{x^2-x-2}{x^2-2x-\sin(x-2)}$ $\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 2}\frac{(x-2)(x+1)}{x(x-2)-\sin(x-2)}$$

$$=\lim_{x\to 2}\frac{(x+1)}{x-\frac{\sin(x-2)}{x-2}}$$

$$=\frac{2+1}{2-1}=3$$

Example 2.34 Evaluate $\lim_{x\to\infty}x\left(\tan^{-1}\frac{x+1}{x+4}-\frac{\pi}{4}\right)$.

Sol. We have $\lim_{x\to\infty}x\left(\tan^{-1}\frac{x+1}{x+4}-\frac{\pi}{4}\right)$

$$=\lim_{x\to\infty}x\left(\tan^{-1}\frac{x+1}{x+4}-\tan^{-1}1\right)$$

$$=\lim_{x\to\infty}x\tan^{-1}\left(\frac{\frac{x+1}{x+4}-1}{1+\frac{x+1}{x+4}}\right)=\lim_{x\to\infty}x\tan^{-1}\left(\frac{-3}{2x+5}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\tan^{-1}\left(\frac{-3}{2x+5}\right)}{\frac{-3}{2x+5}}\right)\left(\frac{-3x}{2x+5}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\tan^{-1}\left(\frac{-3}{2x+5}\right)}{\frac{-3}{2x+5}}\right)\lim_{x\to\infty}\left(\frac{-3x}{2x+5}\right)$$

$$=1\times\lim_{x\to\infty}\left(\frac{-3}{2+\frac{5}{x}}\right)=1\times-\left(\frac{3}{2}\right)=-\frac{3}{2}$$

Example 2.35 Using $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$, prove that the area of circle of radius R is πR^2.

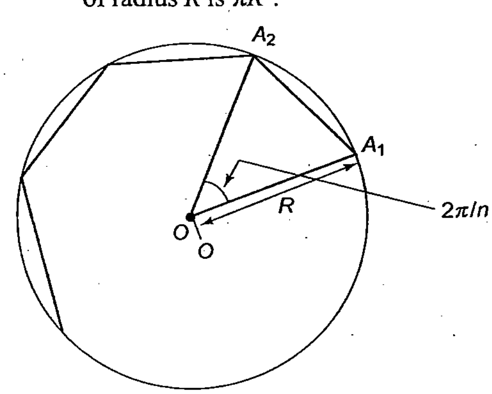


Fig. 2.4

Consider the regular polygon of n sides inscribed in a circle of radius R (Fig. 2.4).

Area of polygon $= n \times$ (area of ΔOA_1A_2)

$$= n \times \frac{1}{2} OA_1 OA_2 \sin(\angle A_1 OA_2)$$

$$= \frac{n}{2} R^2 \sin\left(\frac{2\pi}{n}\right)$$

Now circle is a regular polygon of infinite sides,

Then the area of circle $= \lim_{n\to\infty} \frac{n}{2} R^2 \sin\left(\frac{2\pi}{n}\right)$

$$= \pi R^2 \lim_{n\to\infty} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}$$

$$= \pi R^2$$

Concept Application Exercise 2.5

Evaluate the following limits

1. $\lim_{x\to 0} \frac{\sin x^0}{x}$
2. $\lim_{x\to 0} \frac{1-\cos mx}{1-\cos nx}$
3. $\lim_{x\to \pi/4} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$
4. $\lim_{x\to 0} \frac{\cot 2x - \operatorname{cosec} 2x}{x}$
5. $\lim_{x\to 0} \frac{\tan 2x - x}{3x - \sin x}$
6. $\lim_{h\to 0} \frac{2\left[\sqrt{3}\sin\left(\frac{\pi}{6}+h\right) - \cos\left(\frac{\pi}{6}+h\right)\right]}{\sqrt{3}h(\sqrt{3}\cos h - \sin h)}$
7. $\lim_{n\to\infty} n\cos\left(\frac{\pi}{4n}\right)\sin\left(\frac{\pi}{4n}\right)$ LO
8. $\lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2+\sin x}{x^2+\sin y^2}$, where $(x, y)\to(0, 0)$ along the curve $x = y^2$
9. $\lim_{x\to 0} \frac{\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)}{\sin^{-1} x}$

EVALUATION OF EXPONENTIAL AND LOGARITHMIC LIMITS

In order to evaluate these types of limits, we use the following standard results:

1. $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$

Proof: $\lim_{x\to 0} \frac{a^x - 1}{x}$

$$= \lim_{x\to 0} \frac{\left(1 + \frac{x(\log a)}{1!} + \frac{x^2(\log a)^2}{2!} + \cdots\right) - 1}{x}$$

$$= \lim_{x\to 0} \left(\frac{\log a}{1!} + \frac{x(\log a)^2}{2!} + \cdots\right)$$

$$= \log_e a$$

2. $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ (replace a by e in the above proof)

3. $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

Proof:

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots}{x}$$

$$= \lim_{x\to 0} \left(1 - \frac{x}{2} + \frac{x^2}{3} - \cdots\right) = 1$$

Example 2.36 Evaluate $\lim_{x\to 0} \frac{2^x - 1}{\sqrt{1+x} - 1}$.

Sol. We have $\lim_{x\to 0} \frac{2^x - 1}{\sqrt{1+x} - 1}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 0} \frac{2^x - 1}{\sqrt{1+x} - 1} \frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)}$$

$$= \lim_{x\to 0} \frac{2^x - 1}{x} \lim_{x\to 0} (\sqrt{1+x}+1)$$

$$= (\log 2)\, 2 = \log 4$$

Example 2.37 Evaluate $\lim_{x\to 1} \frac{a^{x-1} - 1}{\sin \pi x}$.

Sol. We have $\lim_{x\to 1} \frac{a^{x-1} - 1}{\sin \pi x}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{h\to 0} \frac{a^{1+h-1} - 1}{\sin \pi(1+h)} = \lim_{h\to 0} \frac{a^h - 1}{-\sin \pi h}$$

$$= \frac{-1}{\pi} \lim_{h\to 0} \left(\frac{a^h - 1}{h}\right) \frac{\pi h}{\sin \pi h} = -\frac{1}{\pi} \log a$$

Example 2.38 Evaluate $\lim\limits_{x\to 0} \frac{10^x - 2^x - 5^x + 1}{x \tan x}$.

Sol. We have $\lim\limits_{x\to 0} \frac{10^x - 2^x - 5^x + 1}{x \tan x}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 0} \frac{5^x \cdot 2^x - 2^x - 5^x + 1}{x \tan x}$$

$$= \lim_{x\to 0} \frac{(5^x - 1)(2^x - 1)}{x \tan x}$$

$$= \lim_{x\to 0} \frac{5^x - 1}{x} \frac{2^x - 1}{x} \frac{x}{\tan x}$$

$$= \lim_{x\to 0} \frac{5^x - 1}{x} \lim_{x\to 0} \frac{2^x - 1}{x} \lim_{x\to 0} \frac{x}{\tan x}$$

$$= (\log 5)(\log 2)(1) = (\log 5)(\log 2)$$

Example 2.39 Evaluate $\lim\limits_{x\to 0} \frac{3^{2x} - 2^{3x}}{x}$.

Sol. We have $\lim\limits_{x\to 0} \frac{3^{2x} - 2^{3x}}{x}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 0} \left\{\left(\frac{3^{2x} - 1}{x}\right) - \left(\frac{2^{3x} - 1}{x}\right)\right\}$$

$$= \lim_{x\to 0} \left(\frac{3^{2x} - 1}{2x} . 2\right) - \lim_{x\to 0} \left(\frac{2^{3x} - 1}{3x} . 3\right)$$

$$= 2 \log 3 - 3 \log 2 = \log 9 - \log 8 = \log \left(\frac{9}{8}\right)$$

Example 2.40 Evaluate $\lim\limits_{x\to 2} \frac{x - 2}{\log_a (x - 1)}$.

Sol. $\lim\limits_{x\to 2} \frac{x - 2}{\log_a (x - 1)}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 2} \frac{x - 2}{\log_a (1 + (x - 2))}$$

$$= \lim_{h\to 0} \frac{h}{\log_a (1 + h)} \quad \text{(Substituting } x - 2 = h)$$

$$= \log_e a$$

ExaSmple 2.41 Evaluate $\lim\limits_{x\to a} \frac{\log x - \log a}{x - a}$.

Sol. Let $x - a = h$, then if $x \to a$, $h \to 0$

$$\Rightarrow \lim_{x\to a} \frac{\log x - \log a}{x - a} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{h\to 0} \frac{\log(a + h) - \log a}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(1 + \frac{h}{a}\right)}{\frac{h}{a} a} = \frac{1}{a}$$

Example 2.42 Evaluate $\lim\limits_{x\to 0} \frac{\log(5 + x) - \log(5 - x)}{x}$.

Sol. We have $\lim\limits_{x\to 0} \frac{\log(5 + x) - \log(5 - x)}{x}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 0} \frac{\log\left\{5\left(1 + \frac{x}{5}\right)\right\} - \log\left\{5\left(1 - \frac{x}{5}\right)\right\}}{x}$$

$$= \lim_{x\to 0} \frac{\left\{\log 5 + \log\left(1 + \frac{x}{5}\right)\right\} - \left\{\log 5 + \log\left(1 - \frac{x}{5}\right)\right\}}{x}$$

$$= \lim_{x\to 0} \frac{\log\left(1 + \frac{x}{5}\right) - \log\left(1 - \frac{x}{5}\right)}{x}$$

$$= \lim_{x\to 0} \frac{1}{5} \frac{\log\left(1 + \frac{x}{5}\right)}{x/5} + \lim_{x\to 0} \frac{\log\left(1 - \frac{x}{5}\right)}{-x/5} \frac{1}{(-5)} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

Example 2.43 Let $P_n = a^{P_{n-1}} - 1$, $\forall\ n = 2, 3 \cdots$ and let $P_1 = a^x - 1$ where $a \in R^+$, then evaluate $\lim\limits_{x\to 0} \frac{P_n}{x}$.

Sol. Clearly, if $P_k \to 0 \Rightarrow P_{k+1} \to 0$

Now, as $x \to 0 \Rightarrow P_1 \to 0 \Rightarrow P_2, P_3, P_4, \cdots P_n \to 0$

$$\Rightarrow \lim_{x\to 0} \frac{P_n}{x} = \lim_{x\to 0} \frac{P_n}{P_{n-1}} \frac{P_{n-1}}{P_{n-2}} \cdots \frac{P_1}{x}$$

Now, $\lim\limits_{x\to 0} \frac{P_k}{P_{k-1}} = \lim\limits_{x\to 0} \frac{a^{P_{k-1}} - 1}{P_{k-1}} = \ln a$

$\Rightarrow$ Required limit $= (\ln a)^n$

Concept Application Exercise 2.6

Evaluate the following limits:

1. $\lim\limits_{x\to\infty} [x(a^{1/x} - 1)], a > 1$
2. $\lim\limits_{x\to 0} \frac{x 2^x - x}{1 - \cos x}$
3. $\lim\limits_{x\to 2} \frac{\sin(e^{x-2} - 1)}{\log(x - 1)}$
4. $\lim\limits_{x\to 0} \frac{e^{x^2} - \cos x}{x^2}$
5. $\lim\limits_{x\to 0} \frac{e^x + e^{-x} - 2}{x^2}$

6. $\lim_{x\to a}\dfrac{\log(x-a)}{\log(e^x-e^a)}$

7. $\lim_{x\to 0}\dfrac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}, a>0$

8. $\lim_{x\to 0}\dfrac{(1-3^x-4^x+12^x)}{\sqrt{(2\cos x+7)}-3}$

9. $\lim_{x\to 0}\dfrac{(729)^x-(243)^x-(81)^x+9^x+3^x-1}{x^3}$ LO

LIMITS OF THE FORM $\lim_{x\to a}(f(x))^{g(x)}$

Form: 0^0, ∞^0

Let $L=\lim_{x\to a}(f(x))^{g(x)}$

$\Rightarrow \log_e L=\log_e\left[\lim_{x\to a}(f(x))^{g(x)}\right]$

$=\lim_{x\to a}\left[\log_e(f(x))^{g(x)}\right]$

$=\lim_{x\to a} g(x)\log_e[f(x)]$

$\Rightarrow L=e^{\lim_{x\to a} g(x)\log_e f(x)}$

Example 2.44 $\lim_{x\to\infty} x^{1/x}$ equals to

a. 0 **b.** 1 **c.** e **d.** ∞.

Sol. $\lim_{x\to\infty} x^{1/x}$

$=e^{\log\lim_{x\to\infty} x^{1/x}}$

$=e^{\lim_{x\to\infty}\log x^{1/x}}$

$=e^{\lim_{x\to\infty}\frac{\log x}{x}}$

($\because$ x increases faster than $\log_e x$ when $x\to\infty$)

$=e^0$

$=1$

Form: 1^∞

1. $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$ or $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

Proof: $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$

$$=\lim_{x\to 0}\left(1+\frac{1}{x}x+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}x^2+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}x^3+\cdots\right)$$

$$=\lim_{x\to 0}\left(1+1+\frac{1(1-x)}{2!}+\frac{1(1-x)(1-2x)}{3!}+\cdots\right)$$

$$=\left(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\right)=e$$

2. $L=\lim_{x\to a} f(x)^{g(x)}$ if $\lim_{x\to a} f(x)=1$ and $\lim_{x\to a} g(x)=\infty$

Then $L=\lim_{x\to a} f(x)^{g(x)}$

$$=\lim_{x\to a}\left(1+(f(x)-1)\right)^{\frac{1}{f(x)-1}(f(x)-1)\times g(x)}$$

$$=\left[\lim_{x\to a}\left(\left(1+(f(x)-1)\right)^{\frac{1}{f(x)-1}}\right)\right]^{\lim_{x\to a}(f(x)-1)\times g(x)}$$

$$=e^{\lim_{x\to a}(f(x)-1)\times g(x)}$$

Example 2.45 Evaluate $\lim_{x\to 0}(1+x)^{\operatorname{cosec} x}$.

Sol. $\lim_{x\to 0}(1+x)^{\operatorname{cosec} x}$

$$=\lim_{x\to 0}\left[(1+x)^{\frac{1}{x}}\right]^{\frac{x}{\sin x}}$$

$$=\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right]^{\lim_{x\to 0}\frac{x}{\sin x}}=e^1$$

Example 2.46 Evaluate $\lim_{x\to 0}(\cos x)^{\cot x}$.

Sol. $\lim_{x\to 0}(\cos x)^{\cot x}$

$$=\lim_{x\to 0}\left[(1+(\cos x-1))^{\frac{1}{\cos x-1}}\right]^{\frac{\cos x-1}{\tan x}}$$

$$=\left[\lim_{x\to 0}(1+(\cos x-1))^{\frac{1}{\cos x-1}}\right]^{\lim_{x\to 0}\frac{\cos x-1}{\tan x}}$$

$$=e^{\lim_{x\to 0}\frac{\cos x-1}{\sin x}\cos x}$$

$$=e^{\lim_{x\to 0}\frac{\cos x-1}{\sin^2 x}\cos x\sin x}$$

$$=e^{\lim_{x\to 0}\frac{\cos x-1}{1-\cos^2 x}\cos x\sin x}$$

$$=e^{\lim_{x\to 0}-\frac{\sin x\cos x}{1+\cos x}}=e^0=1$$

Example 2.47 Evaluate $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\left(\frac{\sin x}{x-\sin x}\right)}$.

Sol. Since $\lim_{x\to 0}\frac{\sin x}{x}=1$ and $\lim_{x\to 0}\frac{\sin x}{x-\sin x}=\lim_{x\to 0}\frac{1}{\left(\frac{x}{\sin x}-1\right)}$

$$=\frac{1}{1-1}=\infty$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\left(\frac{\sin x}{x-\sin x}\right)}=e^{\lim_{x\to 0}\left(\frac{\sin x}{x}-1\right)\left(\frac{\sin x}{(x-\sin x)}\right)}$$

$$=e^{\lim_{x\to 0}-\frac{\sin x}{x}}=e^{-1}=\frac{1}{e}$$

Example 2.48 Evaluate $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{2/x}$; $(a, b, c > 0)$.

Sol. We have $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{2/x}$

$$= e^{\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}-1\right)\frac{2}{x}}$$

$$= e^{\frac{2}{3}\lim_{x\to 0}\left(\frac{a^x+b^x+c^x-3}{x}\right)}$$

$$= e^{\frac{2}{3}\lim_{x\to 0}\left(\frac{a^x-1}{x}+\frac{b^x-1}{x}+\frac{c^x-1}{x}\right)}$$

$$= e^{\frac{2}{3}\left\{\lim_{x\to 0}\frac{a^x-1}{x}+\lim_{x\to 0}\frac{b^x-1}{x}+\lim_{x\to 0}\frac{c^x-1}{x}\right\}}$$

$$= e^{(2/3)\{\ln a+\ln b+\ln c\}} = e^{(2/3)\ln(abc)} = e^{\ln(abc)^{2/3}} = (abc)^{2/3}$$

Example 2.49 The population of a country increases by 2% every year. If it increases k times in a century, then prove that $[k] = 7$, where $[\cdot]$ represents the greatest integer function.

Sol. If the initial number of inhabitant of a given country is A, then after a year the total population will amount to $A + \frac{A}{100}2 = \left(1+\frac{1}{50}\right)A$.

After two years, the population will amount to $\left(1+\frac{1}{50}\right)^2 A$.

After 100 years, it will reach the total of $\left(1+\frac{1}{50}\right)^{100} A$, i.e.,

it will have increased $\left(\left(1+\frac{1}{50}\right)^{50}\right)^2$ times.

Taking into account that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, we can approximately consider that $\left(1+\frac{1}{50}\right)^{50} \approx e$.

Hence after 100 years the population of the country will have increased $e^2 \approx 7.39$ times.

Hence $[k] = [7.39] = 7$.

Concept Application Exercise 2.7

Evaluate the following limits

1. $\lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{x+3}$
2. $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx}$ where a, b, c, and d are positive.
3. $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$
4. $\lim_{x\to 7/2}(2x^2-9x+8)^{\cot(2x-7)}$
5. $\lim_{x\to 0}\left\{\sin^2\left(\frac{\pi}{2-px}\right)\right\}^{\sec^2\left(\frac{\pi}{2-qx}\right)}$

L'HOPITAL'S RULE FOR EVALUATING LIMITS

***Rule*:** If $\lim_{x\to a}\frac{f(x)}{g(x)}$ takes $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form, then,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

where $f'(x) = \frac{df(x)}{dx}$ and $g'(x) = \frac{dg(x)}{dx}$.

Example 2.50 Let $f(x)$ be a twice-differentiable function and $f''(0) = 2$, then evaluate

$$\lim_{x\to 0}\frac{2f(x)-3f(2x)+f(4x)}{x^2}.$$

Sol. The given limit has $\frac{0}{0}$ form.

Using L' Hopital's rule, we have

$$\text{Limit} = \lim_{x\to 0}\frac{2f'(x)-6f'(2x)+4f'(4x)}{2x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{2f''(x)-12f''(2x)+16f''(4x)}{2}$$

(using L'Hopital's rule)

$$= \frac{6f''(0)}{2} = 6$$

Example 2.51 If the graph of the function $y = f(x)$ has a unique tangent at the point $(a, 0)$ through which the graph passes, then evaluate

$$\lim_{x\to a}\frac{\log_e\{1+6f(x)\}}{3f(x)}.$$

Sol. From the question, $f(a) = 0$ and $f(x)$ is differentiable at $x = a$.

$$\therefore \text{limit} = \lim_{x\to a}\frac{\frac{1}{1+6f(x)}\times 6f'(x)}{3f'(x)} = 2\times\frac{1}{1+6f(a)} = 2$$

Example 2.52 Evaluate $\lim_{x\to 0}\log_{\tan^2 x}(\tan^2 2x)$.

Sol. $L = \lim_{x\to 0}\frac{\log(\tan^2 2x)}{\log(\tan^2 x)} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$

Using L' Hopital's rule,

$$\text{We have } L = \lim_{x\to 0}\frac{\left(\frac{1}{\tan^2 2x}2\tan 2x\sec^2 2x\right)\times 2}{\frac{1}{\tan^2 x}2\tan x\cdot\sec^2 x}$$

$$= \lim_{x\to 0} \frac{2\left(\dfrac{1}{\sin 2x \cos 2x}\right)}{\left(\dfrac{1}{\sin x \cos x}\right)} = \lim_{x\to 0} \frac{\left(\dfrac{1}{\sin 2x \cos 2x}\right)}{\left(\dfrac{1}{\sin 2x}\right)}$$

$$= \lim_{x\to 0} \frac{1}{\cos 2x} = 1$$

Example 2.53 Evaluate $\lim_{x\to 0^+} x^m (\log x)^n, m, n \in N.$

Sol. $\lim_{x\to 0+} x^m (\log x)^n = \lim_{x\to 0^+} \frac{(\log x)^n}{x^{-m}} \quad \left(\frac{\infty}{\infty}\text{form}\right)$

$$= \lim_{x\to 0+} \frac{n(\log x)^{(n-1)} \frac{1}{x}}{-mx^{-m-1}} \quad \text{(using L' Hopital's rule)}$$

$$= \lim_{x\to 0+} \frac{n(\log x)^{n-1}}{-mx^{-m}} \quad \left(\frac{\infty}{\infty}\text{form}\right)$$

$$= \lim_{x\to 0+} \frac{n(n-1)(\log x)^{(n-2)} \frac{1}{x}}{(-m)^2 x^{-m-1}} \quad \text{(using L' Hopital's rule)}$$

$$= \lim_{x\to 0+} \frac{n(n-1)(\log x)^{n-2}}{m^2 x^{-m}} \quad \left(\frac{\infty}{\infty}\text{form}\right)$$

...

...

$$= \lim_{x\to 0+} \frac{n!}{(-m)^n x^{-m}} = 0 \text{ (differentiating } N^r \text{ and } D^r\ n \text{ times)}$$

Example 2.54 Evaluate $\lim_{x\to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3}$.

Sol. $\lim_{x\to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3}$

$$= \lim_{x\to 0} \frac{(1+x^2) - \sqrt{1-x^2}}{3x^2\sqrt{1-x^2}(1+x^2)} \quad \text{(using L'Hopital's rule)}$$

$$= \lim_{x\to 0} \frac{(1+x^2)^2 - (1-x^2)}{3x^2\sqrt{1-x^2}(1+x^2)} \times \frac{1}{(1+x^2)+\sqrt{1-x^2}} \quad \text{(Rationalizing)}$$

$$= \lim_{x\to 0} \frac{x^4 + 3x^2}{3x^2\sqrt{1-x^2}(1+x^2)} \times \frac{1}{(1+x^2)+\sqrt{1-x^2}}$$

$$= \lim_{x\to 0} \frac{x^2 + 3}{3\sqrt{1-x^2}(1+x^2)} \times \frac{1}{(1+x^2)+\sqrt{1-x^2}} = 1/2$$

Example 2.55 If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the roots of equation $x^n + nax - b = 0$, show that $(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\cdots(\alpha_1 - \alpha_n) = n(\alpha_1^{n-1} + a)$.

Sol. Since $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the roots of equation $x^n + nax - b = 0$, we have $x^n + nax - b = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$

$$\Rightarrow \frac{x^n + nax - b}{x - \alpha_1} = (x - \alpha_2)(x - \alpha_3)\cdots(x - \alpha_n)$$

$$\Rightarrow \lim_{x\to\alpha_1} \frac{x^n + nax - b}{x - \alpha_1} = \lim_{x\to\alpha_1} [(x - \alpha_2)(x - \alpha_3)\cdots(x - \alpha_n)]$$

$$\Rightarrow \lim_{x\to\alpha_1} \frac{nx^{n-1} + na}{1} = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\cdots(\alpha_1 - \alpha_n)$$

(using L' Hopital's rule on L.H.S.)

$$\Rightarrow (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\cdots(\alpha_1 - \alpha_n) = n\alpha_1^{n-1} + na$$

Concept Application Exercise 2.8

Evaluate the following limits using L'Hopital's Rule

1. $\lim_{x\to 0^+} x^x$
2. $\lim_{x\to \pi/2} \tan x \log \sin x$
3. $\lim_{x\to 0} \frac{\log \cos x}{x}$
4. $\lim_{x\to 0} \frac{2^x - 1}{(1+x)^{1/2} - 1}$
5. $\lim_{x\to \pi/4} (2 - \tan x)^{1/\ln(\tan x)}$
6. If $\lim_{x\to a} \frac{a^x - x^a}{x^x - a^a} = -1$ and $a > 0$, then find the value of a.

FINDING UNKNOWNS WHEN LIMIT IS GIVEN

Example 2.56 If $L = \lim_{x\to 0} \frac{\sin 2x + a \sin x}{x^3}$ be finite, then find the value of a and L.

Sol. $L = \lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3}$

$$= \lim_{x\to 0} \frac{2\sin x \cos x + a \sin x}{x^3}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \frac{(2\cos x + a)}{x^2}$$

$$= \lim_{x\to 0} \frac{2\cos x + a}{x^2}$$

Now D^r tends to 0 when $x \to 0$, then N^r also must tends to zero for which $\lim_{x\to 0} (2\cos x + a) = 0 \Rightarrow a = -2$.

$$\text{Now, } L = \lim_{x\to 0} \frac{2\cos x - 2}{x^2} = -2 \lim_{x\to 0} \frac{2\sin^2 \frac{x}{2}}{x^2} = -1.$$

Example 2.57 If $\lim_{x\to 0} \frac{\cos 4x + a\cos 2x + b}{x^4}$ is finite, find a and b using expansion formula.

Sol. $\lim_{x\to 0} \frac{\cos 4x + a\cos 2x + b}{x^4} = \text{finite}$

Using expansion formula for cos 4x and cos 2x, we get

$$\Rightarrow \lim_{x\to 0} \frac{\left(1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!}\right) + a\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}\right) + b}{x^4} = \text{finite}$$

$$\Rightarrow \lim_{x\to 0} \frac{(1+a+b)+(-8-2a)x^2+\left(\frac{32}{3}+\frac{2}{3}a\right)x^4+\cdots}{x^4}$$

$\Rightarrow\ 1+a+b=0$, and (1)

$-8-2a=0$ (2)

Solving equations (1) and (2) for a and b, we get $a=-4$ and $b=3$

Also, $L=\frac{32}{3}+\frac{2}{3}a=\frac{32-8}{3}=8.$

Example 2.58 Find the values of a and b in order that

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1 \text{ [using L' Hopital's rule].}$$

Sol. $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1 \quad \left(\frac{0}{0}\text{ form}\right)$

$$\Rightarrow \lim_{x\to 0}\frac{1(1+a\cos x)+x(-a\sin x)-b\cos x}{3x^2}=1$$

[using L' Hopital's rule]

Here numerator $\to 1+a-b$ and denominator $\to 0$ and limit is a finite number 1

$\therefore\ 1+a-b=0,$ (1)

[If $1+a-b\neq 0$, then limit will not be finite.]

$$\Rightarrow \lim_{x\to 0}\frac{1+a\cos x-ax\sin x-b\cos x}{3x^2}=1\left(\frac{0}{0}\text{ form}\right)$$

$$\Rightarrow \lim_{x\to 0}\frac{0-a\sin x-a\sin x-ax\cos x+b\sin x}{6x}=1\left(\frac{0}{0}\text{ form}\right)$$

$$\Rightarrow \lim_{x\to 0}\frac{-a\cos x-a\cos x-a\cos x+ax\sin x+b\cos x}{6}=1$$

$$\Rightarrow \frac{-3a+b}{6}=1$$

$\Rightarrow -3a+b=6$ (2)

Solving equations (1) and (2), we get $a=-\frac{5}{2}, b=-\frac{3}{2}.$

Concept Application Exercise 2.9

1. If $\lim_{x\to 0}\frac{ae^x-b}{x}$, then find the values of a and b.
2. If $\lim_{x\to\infty}\left\{\frac{x^2+1}{x+1}-(ax+b)\right\}=0$, then find the values of a and b.
3. If $\lim_{x\to 0}(1+ax+bx^2)^{2/x}=e^3$, then find the values of a and b.

MISCELLANEOUS SOLVED PROBLEMS

1. $\lim_{n\to\infty}(4^n+5^n)^{1/n}$ is equal to

a. 4 b. 5
c. e d. None of these

Sol. b. Given limit $=\lim_{n\to\infty}(4^n+5^n)^{1/n}$

$$=\lim_{n\to\infty}5\left(1+\left(\frac{4}{5}\right)^n\right)^{1/n}=5$$

$$\left(\because \left(\frac{4}{5}\right)^n\to 0 \text{ as } n\to\infty\right)$$

2. $\lim_{x\to\infty}\sqrt{\frac{x+\sin x}{x-\cos x}}$ is equal to

a. 0 b. 1
c. -1 d. None of these

Sol. b. $\lim_{x\to\infty}\sqrt{\left(\frac{x+\sin x}{x-\cos x}\right)}=\lim_{x\to\infty}\sqrt{\left(\frac{1+\frac{\sin x}{x}}{1-\frac{\cos x}{x}}\right)}=\lim_{x\to\infty}\sqrt{1}=1$

[$\because$ both $\lim_{x\to\infty}\frac{\sin x}{x}$ and $\lim_{x\to\infty}\frac{\cos x}{x}$ are equal to 0]

3. $\lim_{n\to\infty}\frac{n^p\sin^2(n!)}{n+1}$, $0<p<1$, is equal to

a. 0 b. ∞
c. 1 d. None of these

Sol. a. $\lim_{n\to\infty}\frac{\sin^2(n!)}{n^{1-p}\left(1+\frac{1}{n}\right)}$

$$=\frac{\text{some number between 0 and 1}}{\infty}=0$$

4. Let $f(x)=\begin{cases}\cos[x], & x\geq 0\\ |x|+a, & x<0,\end{cases}$ then the value of a, so that $\lim_{x\to 0}f(x)$ exists, where $[x]$ denotes the greatest integer function $\leq x$ is equal to

a. 0 b. -1
c. 2 d. 1

Sol. d. Since $\lim_{x\to 0}f(x)$ exists

$\Rightarrow \lim_{x\to 0^-}f(x)=\lim_{x\to 0^+}f(x)$

$\Rightarrow \lim_{h\to 0}f(0-h)=\lim_{h\to 0}f(0+h)$

$\Rightarrow \lim_{h\to 0}|0-h|+a=\lim_{h\to 0}\cos[0+h]$

$\Rightarrow a=\cos 0=1$

$\therefore a=1$

5. If $3-\frac{x^2}{12}\leq f(x)\leq 3+\frac{x^3}{9}$ for all $x\neq 0$, then the value of $\lim_{x\to 0}f(x)$ is equal to

a. 1/3 b. 3
c. -3 d. $-1/3$

Sol. b. According to the question

$$\lim_{x\to 0}\left(3-\frac{x^2}{12}\right)\leq\lim_{x\to 0}f(x)\leq\lim_{x\to 0}\left(3+\frac{x^3}{9}\right)$$

$\Rightarrow (3-0)\leq\lim_{x\to 0}f(x)\leq(3+0)$

Hence $\lim_{x\to 0}f(x)=3$ (from sandwitch theorem).

6. $\lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{2^r}\right]$, where [.] denotes the greatest integer function, is equal to

a. 1 **b.** 0
c. Non-existent **d.** None of these

Sol. b. $\sum_{r=1}^{n}\frac{1}{2^r}=\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\frac{1}{2}\right)}=1-\left(\frac{1}{2}\right)^n$, which tends to 1 as $n\to\infty$ (but in fact always remains less than 1). Thus,

$$\lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{2^r}\right]=0.$$

7. $\lim_{x\to 5\pi/4}[\sin x+\cos x]$, where [.] denotes the greatest integer function, is equal to

a. –2 **b.** –1
c. –3 **d.** None of these

Sol. a. $\sin x+\cos x=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$

For $x\to\frac{5\pi}{4}+0$, $\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\to-\sqrt{2}+0$

and for $x\to\frac{5\pi}{4}-0$

$\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\to-\sqrt{2}+0$

This given limit will be equal to –2.

8. $\lim_{x\to 0}\frac{\sin[\cos x]}{1+[\cos x]}$, where [.] denotes the greatest integer function, is equal to

a. 1 **b.** 0
c. Does not exist **d.** None of these

Sol. b. L.H.L. $=\lim_{x\to 0^-}f(x)=\lim_{h\to 0}\frac{\sin[\cos h]}{1+[\cos h]}=\frac{\sin(0)}{1+0}=0$

$(\because h>0\Rightarrow\cos h<1)$

R.H.L. $=\lim_{x\to 0^+}f(x)=\lim_{h\to 0}\frac{\sin[\cos h]}{1+[\cos h]}=\frac{\sin(0)}{1+0}=0$

$\therefore\ \lim_{x\to 0}\frac{\sin[\cos x]}{1+[\cos x]}=0$

9. $\lim_{x\to 0}\left[\frac{\sin x}{x}\right]$, where [.] denotes the greatest integer function, is equal to

a. 1 **b.** 0
c. Does not exist **d.** None of these

Sol. b. Since $\left|\frac{\sin x}{x}\right|<1$

$\Rightarrow\frac{\sin x}{x}$ tends to 1 forms the values that are less than one as $x\to 0$.

Thus, $\lim_{x\to 0}\left[\frac{\sin x}{x}\right]=0$

10. The value of $\lim_{x\to\pi/6}\frac{2-\sqrt{3}\cos x-\sin x}{(6x-\pi)^2}$ is

a. 1/12 **b.** 1/24
c. 1/36 **d.** 1/48

Sol. c. We have

$$\lim_{x\to\frac{\pi}{6}}\frac{2-\sqrt{3}\cos x-\sin x}{(6x-\pi)^2}=\lim_{h\to 0}\frac{2-\sqrt{3}\cos\left(\frac{\pi}{6}+h\right)-\sin\left(\frac{\pi}{6}+h\right)}{\left[6\left(\frac{\pi}{6}+h\right)-\pi\right]^2}$$

$$=\lim_{h\to 0}\frac{2-\sqrt{3}\left(\cos\frac{\pi}{6}\cos h-\sin\frac{\pi}{6}\sin h\right)-\left(\sin\frac{\pi}{6}\cos h+\cos\frac{\pi}{6}\sin h\right)}{36h^2}$$

$$=\lim_{h\to 0}\frac{2-\frac{3}{2}\cos h+\frac{\sqrt{3}}{2}\sin h-\frac{1}{2}\cos h-\frac{\sqrt{3}}{2}\sin h}{36h^2}$$

$$=\lim_{h\to 0}\frac{2(1-\cos h)}{36h^2}=\frac{1}{18}\lim_{h\to 0}\frac{2\sin^2\left(\frac{h}{2}\right)}{h^2}$$

$$=\frac{1}{9}\lim_{h\to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right)^2\frac{1}{4}=\frac{1}{9}(1)^2\times\frac{1}{4}=\frac{1}{36}$$

11. The value of $\lim_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$ is

a. $\frac{1}{2\sqrt{2}}$ **b.** $\frac{1}{8\sqrt{2}}$
c. $\frac{1}{4\sqrt{2}}$ **d.** $-\frac{1}{4\sqrt{2}}$

Sol. c. We have $\lim_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$

$$=\lim_{x\to 0}\frac{2-(1+\cos x)}{\sin^2 x}\times\frac{1}{\sqrt{2}+\sqrt{1+\cos x}}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{(1+\cos x)(1-\cos x)}\times\lim_{x\to 0}\frac{1}{\sqrt{2}+\sqrt{1+\cos x}}$$

$$=\lim_{x\to 0}\frac{1}{(1+\cos x)}\times\frac{1}{2\sqrt{2}}=\frac{1}{4\sqrt{2}}$$

12. If $a_1 = 1$ and $a_{n+1} = \dfrac{4+3a_n}{3+2a_n}$, $n \geq 1$, and if $\lim_{n\to\infty} a_n = n$, then the value of a is

a. $\sqrt{2}$ b. $-\sqrt{2}$
c. 2 d. None of these

Sol. **a.** We have $a_{n+1} = \dfrac{4+3a_n}{3+2a_n}$

$$\Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{4+3a_n}{3+2a_n}$$

$$\Rightarrow a = \frac{4+3a}{3+2a} \Rightarrow 2a^2 = 4 \Rightarrow a = \sqrt{2} \text{ (where } \lim_{n\to\infty} a_n = a\text{)}$$

($a \neq -\sqrt{2}$ because each $a_n > 0$)

13. Evaluate $\lim_{x\to 0} \dfrac{x-\sin x}{x^3}$ (do not use either L'Hopital's rule or series expansion for $\sin x$), hence evaluate

$$\lim_{x\to 0} \frac{\sin x - x - x\cos x + x^2 \cot x}{x^5}.$$

Sol. $L = \lim_{x\to 0} \dfrac{x-\sin x}{x^3}$

Replace x by $3x$

$$\Rightarrow L = \lim_{x\to 0} \frac{3x - \sin 3x}{(3x)^3}$$

$$= \lim_{x\to 0} \frac{3x-(3\sin x - 4\sin^3 x)}{(3x)^3}$$

$$= \lim_{x\to 0} \frac{3x - 3\sin x}{(3x)^3} + \lim_{x\to 0} \frac{4\sin^3 x}{(3x)^3}$$

$$= \frac{1}{9} \lim_{x\to 0} \frac{x-\sin x}{x^3} + \frac{4}{27} \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^3$$

$$= \frac{1}{9} L + \frac{4}{27}$$

$$\Rightarrow \frac{8}{9} L = \frac{4}{27}$$

$$\Rightarrow L = \frac{1}{6}$$

Also $\lim_{x\to 0} \dfrac{\sin x - x - x\cos x + x^2 \cot x}{x^5}$

$$= \lim_{x\to 0} \frac{(\sin x - x) + x\cot x(x - \sin x)}{x^5}$$

$$= \lim_{x\to 0} \frac{(\sin x - x)(1 - x\cot x)}{x^5}$$

$$= \lim_{x\to 0} \frac{\sin x - x}{x^3} \times \frac{\tan x - x}{x^3} \times \frac{x}{\tan x}$$

$$= \frac{-1}{6} \times \frac{1}{3} \times 1 = \frac{-1}{18} \text{ (Using expansions of } \sin x \text{ and } \tan x\text{)}$$

14. Evaluate $\lim_{n\to\infty} \cos\left(\pi\sqrt{n^2+n}\right)$, when n is an integer.

Sol. $L = \lim_{n\to\infty} \cos\left(\pi\sqrt{n^2+n}\right)$

$$= \lim_{n\to\infty} \pm\cos\left(n\pi - \pi\sqrt{n^2+n}\right)$$

$$= \lim_{n\to\infty} \pm\cos\left(\pi\left(n - \sqrt{n^2+n}\right)\right)$$

$$= \pm \lim_{n\to\infty} \cos\left(\frac{-n\pi}{n+\sqrt{n^2+n}}\right)$$

$$= \pm \lim_{n\to\infty} \cos\left(\frac{n\pi}{n + n\sqrt{1+\frac{1}{n}}}\right)$$

$$= \pm \lim_{n\to\infty} \cos\left(\frac{\pi}{1+\sqrt{1+\frac{1}{n}}}\right)$$

$$= \pm\cos\frac{\pi}{2} \to 0$$

15. Let the variable x_n be determined by the following law of formation:

$$x_0 = \sqrt{a}$$

$$x_1 = \sqrt{a+\sqrt{a}}$$

$$x_2 = \sqrt{a+\sqrt{a+\sqrt{a}}}$$

$$x_3 = \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}$$

$$\vdots$$

Then, find the value of $\lim_{n\to\infty} x_n$.

Sol. We have $x_n^2 = a + x_{n-1}$

$\Rightarrow L^2 = a + L$ (as at infinity $L = \lim_{n\to\infty} x_n$ » $\lim_{n\to\infty} x_{n-1}$)

$\Rightarrow L^2 - L - a = 0$

$\Rightarrow L = \dfrac{1 \pm \sqrt{(1+4a)}}{2}$

$\Rightarrow L = \dfrac{1+\sqrt{(1+4a)}}{2}$ (as according to the question $a > 0$ hence $\dfrac{1-\sqrt{(1+4a)}}{2} < 0$).

16. Evaluate $\lim_{n\to\infty} \prod_{r=3}^{n} \dfrac{r^3-8}{r^3+8}$, where $\prod$ represents the product of function.

Sol. Let $P = \lim_{n\to\infty} \prod_{r=3}^{n} \left(\dfrac{r^3-8}{r^3+8}\right)$

$$= \lim_{n\to\infty} \prod_{r=3}^{n} \left(\frac{r-2}{r+2}\right)\left(\frac{r^2+2r+4}{r^2-2r+4}\right)$$

$$= \lim_{n\to\infty} \prod_{r=3}^{n} \left(\frac{r-2}{r+2}\right) \prod_{r=3}^{n} \left(\frac{r^2+2r+4}{r^2-2r+4}\right)$$

$$= \lim_{n\to\infty} \left\{\frac{1}{5}\frac{2}{6}\frac{3}{7}\frac{4}{8}\frac{5}{9}\cdots\frac{(n-5)}{(n-1)}\frac{(n-4)}{(n)}\frac{(n-3)}{(n+1)}\frac{(n-2)}{(n+2)}\right\} \times$$

$$\left\{\frac{19}{7}\times\frac{28}{12}\times\frac{39}{19}\cdots\frac{(n^2-2n+4)}{(n^2-6n+12)}\times\frac{(n^2+3)}{(n^2-4n+7)}\times\frac{(n^2+2n+4)}{(n^2-2n+4)}\right\}$$

$$= \lim_{n\to\infty} \left\{\frac{1\times2\times3\times4}{(n-1)n(n+1)(n+2)} \times \frac{(n^2+3)(n^2+2n+4)}{7\times12}\right\}$$

$$= \frac{2}{7}\lim_{n\to\infty}\left\{\frac{(n^2+3)(n^2+2n+4)}{(n-1)n(n+1)(n-2)}\right\}$$

$$= \frac{2}{7}\lim_{n\to\infty}\left\{\frac{\left(1+\frac{3}{n^2}\right)\left(1+\frac{2}{n}+\frac{4}{n^2}\right)}{\left(1-\frac{1}{n}\right)1\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}\right\}$$

$$= \frac{2}{7}\frac{(1+0)(1+0+0)}{(1-0)1(1+0)(1+0)} = \frac{2}{7}$$

Hence $P = \dfrac{2}{7}$

17. If $[x]$ denotes the greatest integer $\le x$, then evaluate

$$\lim_{n\to\infty}\frac{1}{n^3}\{[1^2x]+[2^2x]+[3^2x]+\cdots+[n^2x]\}$$

Sol. $\lim_{n\to\infty}\dfrac{1}{n^3}\{[1^2x]+[2^2x]+[3^2x]+\cdots+[n^2x]\}$

$$= \lim_{n\to\infty}\left\{\frac{\sum_{r=1}^{n}[r^2x]}{n^3}\right\} = \lim_{n\to\infty}\left(\frac{\sum_{r=1}^{n} r^2x-\{r^2x\}}{n^3}\right)$$

$$= \lim_{n\to\infty}\left(\frac{x\dfrac{n(n+1)(2n+1)}{6}}{n^3} - \sum_{r=1}^{n}\frac{\{r^2x\}}{n^3}\right)$$

$$= x\frac{(1)(1)(2)}{6} - 0 = \frac{x}{3}$$

18. Find the integral value of n for which the

$\lim_{x\to0}\dfrac{\cos^2 x-\cos x-e^x\cos x+e^x-\dfrac{x^3}{2}}{x^n}$ is a finite non-zero.

Sol. Given that $\lim_{x\to0}\dfrac{\left(\dfrac{x^3}{2}+\dfrac{x^4}{2}+\dfrac{x^5}{12}-\dfrac{x^5}{24}+\cdots\right)-\dfrac{x^3}{2}}{x^n}$

$$= \lim_{x\to0}\frac{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots-1\right)\left[\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}-\cdots\right)\right]-\frac{x^3}{2}}{x^n}$$

$$= \lim_{x\to0}\frac{\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)\left[\left(-x-x^2-\frac{x^3}{3!}-\frac{2x^5}{5!}-\cdots\right)\right]-\frac{x^3}{2}}{x^n}$$

$$= \lim_{x\to0}\frac{\left(\frac{x^3}{2}+\frac{x^4}{2}+\frac{x^5}{12}-\frac{x^5}{24}+\ldots\right)-\frac{x^3}{2}}{x^n} = \text{non-zero if } n=4$$

EXERCISES

Subjective Type

Solutions on page 2.32

1. Evaluate $\lim_{x\to 3\pi/4}\dfrac{1+\sqrt[3]{\tan x}}{1-2\cos^2 x}$.

2. Evaluate $\lim_{x\to 0}\dfrac{e^{\sin x}-(1+\sin x)}{\left(\tan^{-1}(\sin x)\right)^2}$.

3. Evaluate $\lim_{x\to 0}\dfrac{e^x-e^{x\cos x}}{(x+\sin x)}$.

4. If $\lim_{n\to\infty}\dfrac{1}{\left(\sin^{-1}x\right)^n+1}=1$, then find the values of x.

5. Find $\lim_{x\to-\infty}\dfrac{5x+2\cos x}{3x+14}$ using sandwich theorem.

6. If $f(n+1)=\dfrac{1}{2}\left\{f(n)+\dfrac{9}{f(n)}\right\}, n\in N$ and $f(n)>0$ for all $n \in N$, then find $\lim_{x\to\infty} f(n)$.

7. Evaluate $\lim_{x\to 0}\dfrac{8}{x^8}\left\{1-\cos\dfrac{x^2}{2}-\cos\dfrac{x^2}{4}+\cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right\}$.

8. Evaluate

$$\lim_{n\to\infty} n^2\left\{\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\cdots\infty}}}\right\}.$$

9. Evaluate $\lim_{n\to\infty}\left\{\cos\left(\dfrac{x}{2}\right)\cos\left(\dfrac{x}{4}\right)\cos\left(\dfrac{x}{8}\right)\cdots\cos\left(\dfrac{x}{2^n}\right)\right\}$.

10. If x_1 and x_2 are the real and distinct roots of $ax^2+bx+c=0$, then prove that $\lim_{x\to x_1}\left(1+\sin\left(ax^2+bx+c\right)\right)^{\frac{1}{x-x_1}}=e^{a(x_1-x_2)}$.

11. Evaluate $\lim_{x\to\infty} x\left[\tan^{-1}\left(\dfrac{x+1}{x+2}\right)-\tan^{-1}\left(\dfrac{x}{x+2}\right)\right]$.

12. Evaluate $\lim_{x\to 0}\dfrac{2^x-1-x}{x^2}$ without using L'Hopital's rule and expansion of the series.

13. Evaluate $\lim_{x\to 1}\dfrac{\sin\{x\}}{\{x\}}$ if exists, where $\{x\}$ is the fractional part of x.

14. Evaluate $\lim_{x\to 0}\ \{1^{1/\sin^2 x}+2^{1/\sin^2 x}+\dots+n^{1/\sin^2 x}\}^{\sin^2 x}$.

15. Evaluate $\lim_{x\to\infty}\left(\dfrac{a_1^{1/x}+a_2^{1/x}+\cdots+a_n^{1/x}}{n}\right)^{nx}$.

16. Evaluate $\lim_{x\to 0}\dfrac{1-\cos(1-\cos x)}{x^4}$.

17. Evaluate $\lim_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{1}{\sin^2 x}\right)$.

18. Evaluate $\lim_{x\to\infty}\left\{\dfrac{x}{x+\dfrac{\sqrt[3]{x}}{x+\dfrac{\sqrt[3]{x}}{x+\sqrt[3]{x}}\cdots\infty}}\right\}$.

19. Evaluate $\lim_{x\to\pi/2}\dfrac{\sin x-(\sin x)^{\sin x}}{1-\sin x+\log_e \sin x}$.

20. Evaluate $\lim_{\theta\to 0}\dfrac{\cos^2(1-\cos^2(1-\cos^2(\cdots\cos^2(\theta))))\cdots)}{\sin\left(\pi\dfrac{\sqrt{(\theta+4)}-2}{\theta}\right)}$.

21. Evaluate the value of

$$\lim_{x\to\pi/2}\tan^2 x\left(\sqrt{2\sin^2 x+3\sin x+4}-\sqrt{\sin^2 x+6\sin x+2}\right).$$

22. Evaluate $\lim_{x\to 1}\sec\dfrac{\pi}{2^x}\log x$.

23. Evaluate $\lim_{x\to 0}\dfrac{e-(1+x)^{1/x}}{x}$.

24. Evaluate $\lim_{n\to\infty} n^{-n^2}\ \{(n+2^0)(n+2^{-1})(n+2^{-2})\cdots(n+2^{-n+1})\}^n$.

25. Let $f(x)=\lim_{m\to\infty}\left\{\lim_{n\to\infty}\cos^{2m}(n!\pi x)\right\}$, where $x\in R$, then prove that $f(x)=\begin{cases}1, \text{ if } x \text{ is rational}\\ 0, \text{ if } x \text{ is irrational}\end{cases}$.

26. At the endpoint and the midpoint of a circular arc AB tangent lines are drawn, and the point A and B are joined with a chord. Prove that the ratio of the areas of the triangles thus formed tends to 4 as the arc AB decreases infinitely.

27. T_1 is an isosceles triangle in circle C. Let T_2 be another isosceles triangle inscribed in C whose base is one of the equal sides of T_1 and which overlaps the interior of T_1. Similarly, create isosceles triangle T_3 from T_2, T_4, and T_5, and so on. Prove that the triangle T_n approaches an equilateral triangle as $n\to\infty$.

Objective Type

Solutions on page 2.36

Each question has four choices a, b, c, and d, out of which *only one* is correct.

1. If $f(x)=0$ be a quadratic equation such that $f(-\pi)=f(\pi)=0$ and $f\left(\dfrac{\pi}{2}\right)=-\dfrac{3\pi^2}{4}$, then $\lim_{x\to-\pi}\dfrac{f(x)}{\sin(\sin x)}$ is equal to

a. 0 b. π
c. 2π d. None of these

2. If $f(x)=\dfrac{\cos x}{(1-\sin x)^{1/3}}$, then

a. $\lim_{x\to\frac{\pi^-}{2}} f(x)=-\infty$ b. $\lim_{x\to\frac{\pi^+}{2}} f(x)=\infty$

c. $\lim_{x\to\frac{\pi}{2}} f(x)=\infty$ d. None of these

3. $\lim_{x\to-\infty}\dfrac{x^2\tan\frac{1}{x}}{\sqrt{8x^2+7x+1}}$ is equal to

a. $-\dfrac{1}{2\sqrt{2}}$ b. $\dfrac{1}{2\sqrt{2}}$

c. $\dfrac{1}{\sqrt{2}}$ d. Does not exist

4. $\lim_{x\to0}\left[\dfrac{\sin(\text{sgn}(x))}{(\text{sgn}(x))}\right]$, where [·] denotes the greatest integer function, is equal to

a. 0 b. 1
c. –1 d. Does not exist

5. $\lim_{x\to\infty}\dfrac{2+2x+\sin 2x}{(2x+\sin 2x)e^{\sin x}}$ is equal to

a. 0 b. 1
c. –1 d. Does not exists

6. Let $\lim_{x\to0}\dfrac{[x]^2}{x^2}=l$ and $\lim_{x\to0}\dfrac{[x^2]}{x^2}=m$, where [·] denotes greatest integer, then

a. l exists but m does not b. m exists but l does not
c. both l and m exist d. neither l nor m exists

7. $\lim_{x\to1}\dfrac{x\sin(x-[x])}{x-1}$, where [·] denotes the greatest integer function, is equal to

a. 0 b. – 1
c. Non-existent d. None of these

8. $\lim_{x\to0}\left(\dfrac{1+\tan x}{1+\sin x}\right)^{\text{cosec}\, x}$ is equal to

a. e b. $\dfrac{1}{e}$

c. 1 d. None of these

9. $\lim_{x\to\infty}\dfrac{\sin^4 x-\sin^2 x+1}{\cos^4 x-\cos^2 x+1}$ is equal to

a. 0 b. 1

c. $\dfrac{1}{3}$ d. $\dfrac{1}{2}$

10. $\lim_{x\to\infty}\left(\dfrac{x^3}{3x^2-4}-\dfrac{x^2}{3x+2}\right)$ is equal to

a. Does not exist b. 1/3
c. 0 d. 2/9

11. If $f(x)=\dfrac{2}{x-3}$, $g(x)=\dfrac{x-3}{x+4}$ and $h(x)=-\dfrac{2(2x+1)}{x^2+x-12}$, then $\lim_{x\to3}[f(x)+g(x)+h(x)]$ is

a. –2 b. –1

c. $-\dfrac{2}{7}$ d. 0

12. $\lim_{x\to0}\dfrac{x(e^x-1)}{1-\cos x}$ is equal to

a. 0 b. ∞
c. –2 d. 2

13. $\lim_{n\to\infty}\dfrac{n(2n+1)^2}{(n+2)(n^2+3n-1)}$ is equal to

a. 0 b. 2
c. 4 d. ∞

14. The value of $\lim_{x\to\pi}\dfrac{1+\cos^3 x}{\sin^2 x}$ is

a. 1/3 b. 2/3
c. –1/4 d. 3/2

15. $\lim_{n\to\infty} n^2\left(x^{1/n}-x^{1/(n+1)}\right), x>0$, is equal to

a. 0 b. e^x
c. $\log_e x$ d. None of these

16. The value of $\lim_{x\to2}\dfrac{\sqrt{1+\sqrt{2+x}}-\sqrt{3}}{x-2}$ is

a. $\dfrac{1}{8\sqrt{3}}$ b. $\dfrac{1}{4\sqrt{3}}$

c. 0 d. None of these

17. $\lim_{x\to\infty}\dfrac{(2x+1)^{40}(4x-1)^5}{(2x+3)^{45}}$ is equal to

a. 16 b. 24
c. 32 d. 8

18. $\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]$ is equal to

a. 0 b. $\dfrac{1}{2}$

c. log 2 d. e^4

19. $\lim_{x\to\infty}\dfrac{(x+1)^{10}+(x+2)^{10}+\cdots+(x+100)^{10}}{x^{10}+10^{10}}$ is equal to

a. 0 b. 1
c. 10 d. 100

20. $\lim_{x\to 0}\frac{x^a \sin^b x}{\sin(x^c)}$, where $a, b, c \in R \sim \{0\}$, exists and has non-zero value, then

a. $a+c=b$
b. $b+c=a$
c. $a+b=c$
d. None of these

21. $\lim_{x\to \pi/2}\left[x\tan x-\left(\frac{\pi}{2}\right)\sec x\right]$ is equal to

a. 1
b. -1
c. 0
d. None of these

22. If $\lim_{x\to\infty}\left(\frac{x^3+1}{x^2+1}-(ax+b)\right)=2$, then

a. $a=1, b=1$
b. $a=1, b=2$.
c. $a=1, b=-2$
d. None of these

23. The value of $\lim_{x\to 1}(2-x)^{\tan\frac{\pi x}{2}}$ is

a. $e^{-2/\pi}$
b. $e^{1/\pi}$
c. $e^{2/\pi}$
d. $e^{-1/\pi}$

24. $\lim_{x\to 0}\frac{\sin x^n}{(\sin x)^m}$, $(m<n)$ is equal to

a. 1
b. 0
c. n/m
d. None of these

25. $\lim_{x\to 0}\frac{x^4(\cot^4 x-\cot^2 x+1)}{(\tan^4 x-\tan^2 x+1)}$ is equal to

a. 1
b. 0
c. 2
d. None of these

26. $\lim_{x\to\infty}\left(\frac{1}{e}-\frac{x}{1+x}\right)^x$ is equal to

a. $\frac{e}{1-e}$
b. 0
c. $e^{\frac{e}{1-e}}$
d. Does not exist

27. $\lim_{x\to 1}\frac{1-x^2}{\sin 2\pi x}$ is equal to

a. $\frac{1}{2\pi}$
b. $\frac{-1}{\pi}$
c. $\frac{-2}{\pi}$
d. None of these

28. $\lim_{x\to 0}\frac{1}{x}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is equal to

a. 1
b. 0
c. 2
d. None of these

29. $\lim_{x\to\infty}\left(\frac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+1}{2x-1}}$ is equal to

a. 0
b. ∞
c. 1/2
d. None of these

30. $\lim_{x\to\infty}\frac{2\sqrt{x}+3\sqrt[3]{x}+4\sqrt[4]{x}+\cdots+n\sqrt[n]{x}}{\sqrt{(2x-3)}+\sqrt[3]{(2x-3)}+\cdots+\sqrt[n]{(2x-3)}}$ is equal to

a. 1
b. ∞
c. $\sqrt{2}$
d. None of these

31. $\lim_{y\to 0}\frac{(x+y)\sec(x+y)-x\sec x}{y}$ is equal to

a. $\sec x(x\tan x+1)$
b. $x\tan x+\sec x$
c. $x\sec x+\tan x$
d. None of these

32. The value of $\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^m$ is

a. 1
b. e
c. e^{-1}
d. None of these

33. $\lim_{x\to 1}\left[\operatorname{cosec}\frac{\pi x}{2}\right]^{1/(1-x)}$ (where [.] represents the greatest integer function) is equal to

a. 0
b. 1
c. ∞
d. Does not exist

34. $\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)}$ is equal to

a. e
b. e^2
c. e^{-1}
d. 1

35. If $f(x)=\lim_{n\to\infty} n(x^{1/n}-1)$, then for $x>0, y>0$, $f(xy)$ is equal to

a. $f(x)f(y)$
b. $f(x)+f(y)$
c. $f(x)-f(y)$
d. None of these

36. If $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ exists, then

a. both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ must exist

b. $\lim_{x\to a} f(x)$ need not exist but $\lim_{x\to a} g(x)$ exists

c. neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ may exist

d. $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ need not exist

37. If $\lim_{n\to\infty}\frac{n.3^n}{n(x-2)^n+n.3^{n+1}-3^n}=\frac{1}{3}$, then the range of x is (where $n\in N$)

a. $[2,5)$
b. $(1,5)$
c. $(-1,5)$
d. $(-\infty,\infty)$

38. The value of $\lim_{x\to 2}\frac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}$ is

a. 16 b. 8
c. 4 d. 2

39. $\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)^{\alpha}+\sin\frac{1}{n}\right)^n$ (when $\alpha\in Q$) is equal to

a. $e^{-\alpha}$ b. $-\alpha$
c. $e^{1-\alpha}$ d. $e^{1+\alpha}$

40. $f(x)=\frac{\ln\left(x^2+e^x\right)}{\ln\left(x^4+e^{2x}\right)}$. Then $\lim_{n\to\infty}f(x)$ is equal to

a. 1 b. 1/2
c. 2 d. None of these

41. $\lim_{x\to 1}\frac{1+\sin\pi\left(\frac{3x}{1+x^2}\right)}{1+\cos\pi x}$ is equal to

a. 0 b. 1
c. 2 d. 4

42. $\lim_{n\to\infty}\sum_{x=1}^{20}\cos^{2n}(x-10)$ is equal to

a. 0 b. 1
c. 19 d. 20

43. The value of $\lim_{x\to\infty}\frac{(2^{x^n})^{\frac{1}{e^x}}-(3^{x^n})^{\frac{1}{e^x}}}{x^n}$ (where $n\in N$) is

a. $\log n\left(\frac{2}{3}\right)$ b. 0
c. $n\log n\left(\frac{2}{3}\right)$ d. Not defined

44. If $f:(1,2)\to R$ satisfies the inequality

$\frac{\cos(2x-4)-33}{2}<f(x)<\frac{x^2|4x-8|}{x-2}$, $\forall\ x\in(1,2)$, then

$\lim_{x\to 2^-}f(x)$ is

a. 16
b. Cannot be determined from the given information
c. -16
d. Does not exist

45. Let $f(x)=\lim_{n\to\infty}\frac{1}{\left(\frac{3}{\pi}\tan^{-1}2x\right)^{2n}+5}$. Then the set of values of x for which $f(x)=0$ is

a. $|2x|>\sqrt{3}$ b. $|2x|<\sqrt{3}$
c. $|2x|\ge\sqrt{3}$ d. $|2x|\le\sqrt{3}$

46. $\lim_{x\to 0}\left\{(1+x)^{\frac{2}{x}}\right\}$ (where $\{\cdot\}$ denotes the fractional part of x) is equal to

a. e^2-7 b. e^2-8
c. e^2-6 d. None of these

47. $\lim_{x\to 0}\frac{\sin(x^2)}{\ln(\cos(2x^2-x))}$ is equal to

a. 2 b. -2
c. 1 d. -1

48. $\lim_{x\to -1}\frac{1}{\sqrt{|x|-\{-x\}}}$ (where $\{x\}$ denotes the fractional part of x) is equal to

a. Does not exist b. 1
c. ∞ d. $\frac{1}{2}$

49. If $f(x)=\begin{cases}x^n\sin(1/x^2), & x\ne 0\\ 0, & x=0\end{cases}$, $(n\in I)$, then

a. $\lim_{x\to 0}f(x)$ exists for $n>1$
b. $\lim_{x\to 0}f(x)$ exists for $n<0$
c. $\lim_{x\to 0}f(x)$ does not exist for any value of n
d. $\lim_{x\to 0}f(x)$ cannot be determined

50. The value of $\lim_{x\to 1}\left(\frac{p}{1-x^p}-\frac{q}{1-x^q}\right)$; $p,q,\in N$ equals

a. $\frac{p+q}{2}$ b. $\frac{pq}{2}$ c. $\frac{p-q}{2}$ d. $\sqrt{\frac{p}{q}}$

51. $\lim_{x\to -1}\left(\frac{x^4+x^2+x+1}{x^2-x+1}\right)^{\frac{1-\cos(x+1)}{(x+1)^2}}$ is equal to:

a. 1 b. $(2/3)^{1/2}$
c. $(3/2)^{1/2}$ d. $e^{1/2}$

52. The value of $\lim_{x\to 2}\left(\left(\frac{x^3-4x}{x^3-8}\right)^{-1}-\left(\frac{x+\sqrt{2x}}{x-2}-\frac{\sqrt{2}}{\sqrt{x}-\sqrt{2}}\right)^{-1}\right)$ is

a. 1/2 b. 2
c. 1 d. None of these

53. $\lim_{x\to\infty}\frac{e^{1/x^2}-1}{2\tan^{-1}(x^2)-\pi}$ is equal to

a. 1 b. -1
c. $\frac{1}{2}$ d. $-\frac{1}{2}$

54. The value of $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x+\log(1-x)}{x^3}$ is

a. $\dfrac{1}{2}$ b. $-\dfrac{1}{2}$
c. 0 d. None of these

55. $\lim\limits_{x\to 0}\dfrac{\cos(\tan x)-\cos x}{x^4}$ is equal to

a. 1/6 b. −1/3
c. 1/2 d. 1

56. If $x_1 = 3$ and $x_{n+1} = \sqrt{2+x_n}$, $n \ge 1$, then $\lim\limits_{n\to\infty} x_n$ is

a. −1 b. 2
c. $\sqrt{5}$ d. 3

57. $\lim\limits_{x\to 0}\left(\dfrac{1^x+2^x+3^x+\dots+n^x}{n}\right)^{1/x}$ is equal to

a. $(n!)^n$ b. $(n!)^{1/n}$
c. $n!$ d. $ln\,(n!)$

58. The value of the limit $\lim\limits_{x\to 0}\dfrac{a^{\sqrt{x}}-a^{1/\sqrt{x}}}{a^{\sqrt{x}}+a^{1/\sqrt{x}}}$, $a > 1$ is

a. 4 b. 2
c. −1 d. 0

59. Among (i) $\lim\limits_{x\to\infty}\sec^{-1}\left(\dfrac{x}{\sin x}\right)$ and

(ii) $\lim\limits_{x\to\infty}\sec^{-1}\left(\dfrac{\sin x}{x}\right)$

a. (i) exists, (ii) does not exist
b. (i) does not exist, (ii) exists
c. both (i) and (ii) exist
d. neither (i) nor (ii) exists

60. If $\lim\limits_{x\to 0}\dfrac{x^n-\sin x^n}{x-\sin^n x}$ is non-zero finite, then n must be equal

a. 4 b. 1 c. 2 d. 3

61. If $\lim\limits_{x\to -2^-}\dfrac{ae^{1/|x+2|}-1}{2-e^{1/|x+2|}} = \lim\limits_{x\to -2^+}\sin\left(\dfrac{x^4-16}{x^5+32}\right)$, then a is

a. $\sin\dfrac{3}{5}$ b. 2 c. $\sin\dfrac{2}{5}$ d. $\sin\dfrac{1}{5}$

62. $\lim\limits_{x\to\infty}((x+5)\tan^{-1}(x+5)-(x+1)\tan^{-1}(x+1))$ is equal to

a. π b. 2π
c. $\pi/2$ d. None of these

63. $\lim\limits_{x\to 1}\dfrac{(1-x)(1-x^2)\cdots(1-x^{2n})}{\{(1-x)(1-x^2)\cdots(1-x^n)\}^2}$, $n \in N$

a. ${}^{2n}P_n$ b. ${}^{2n}C_n$
c. $(2n)!$ d. None of these

64. The value of $\lim\limits_{x\to 0}\left(\left[\dfrac{100x}{\sin x}\right]+\left[\dfrac{99\sin x}{x}\right]\right)$

(where [.] represents the greatest integral function) is

a. 199 b. 198
c. 0 d. None of these

65. The value of $\lim\limits_{x\to\frac{1}{\sqrt{2}}}\dfrac{x-\cos\left(\sin^{-1}x\right)}{1-\tan\left(\sin^{-1}x\right)}$ is

a. $-\dfrac{1}{\sqrt{2}}$ b. $\dfrac{1}{\sqrt{2}}$ c. $\sqrt{2}$ d. $-\sqrt{2}$

66. The value of $\lim\limits_{x\to 1^-}\dfrac{1-\sqrt{x}}{\left(\cos^{-1}x\right)^2}$ is

a. 4 b. 1/2 c. 2 d. 1/4

67. $\lim\limits_{x\to 0}\left[\min(y^2-4y+11)\dfrac{\sin x}{x}\right]$ (where [.] denotes the greatest integer function) is

a. 5 b. 6
c. 7 d. Does not exist

68. $\lim\limits_{x\to\pi/2}\dfrac{\sin(x\cos x)}{\cos(x\sin x)}$ is equal to

a. 0 b. $p/2$ c. p d. $2p$

69. If $\lim\limits_{x\to 0}(x^{-3}\sin 3x+ax^{-2}+b)$ exists and is equal to 0, then

a. $a=-3$ and $b=9/2$ b. $a=3$ and $b=9/2$
c. $a=-3$ and $b=-9/2$ d. $a=3$ and $b=-9/2$

70. If $\lim\limits_{x\to 0}\dfrac{x^n\sin^n x}{x^n-\sin^n x}$ is non-zero finite, then n is equal to

a. 1 b. 2
c. 3 d. None of these

71. $\lim\limits_{x\to\infty}\dfrac{(1+x+x^2)}{x(\ln x)^3}$ is equal to

a. 2 b. e^2
c. e^{-2} d. None of these

72. $\lim\limits_{x\to 0}\dfrac{(2^m+x)^{1/m}-(2^n+x)^{1/n}}{x}$ is equal to

a. $\dfrac{1}{m2^m}-\dfrac{1}{n2^n}$ b. $\dfrac{1}{m2^m}+\dfrac{1}{n2^n}$
c. $\dfrac{1}{m2^{m-1}}-\dfrac{1}{n2^{n-1}}$ d. $\dfrac{1}{m2^{m-1}}+\dfrac{1}{n2^{n-1}}$

73. $\lim\limits_{x\to 0}\left[\left(1-e^x\right)\dfrac{\sin x}{|x|}\right]$ is (where [·] represents the greatest integer function)

a. −1 b. 1
c. 0 d. does not exist

74. Let $f(x)=\begin{cases}x+1, & x>0\\ 2-x, & x\le 0\end{cases}$ and

$g(x)=\begin{cases}x+3, & x<1\\ x^2-2x-2, & 1\le x<2\\ x-5, & x\ge 2\end{cases}$, then $\lim\limits_{x\to 0} g(f(x))$ is

a. 2 b. 1
c. −3 d. does not exists

75. $\lim_{x\to 1}\dfrac{nx^{n+1}-(n+1)x^n+1}{(e^x-e)\sin\pi x}$, where $n = 100$ is equal to

a. $\dfrac{5050}{\pi e}$ b. $\dfrac{100}{\pi e}$ c. $-\dfrac{5050}{\pi e}$ d. $-\dfrac{4950}{\pi e}$

76. The value of $\lim_{n\to\infty}\left[\dfrac{1}{n}+\dfrac{e^{1/n}}{n}+\dfrac{e^{2/n}}{n}+....+\dfrac{e^{(n-1)/n}}{n}\right]$ is

a. 1 b. 0 c. $e-1$ d. $e+1$

77. The value of $\lim_{n\to\infty}\left[\dfrac{2n}{2n^2-1}\cos\dfrac{n+1}{2n-1}-\dfrac{n}{1-2n}\cdot\dfrac{n(-1)^n}{n^2+1}\right]$ is

a. 1 b. −1
c. 0 d. none of these

78. $\lim_{x\to 0}\dfrac{\log(1+x+x^2)+\log(1-x+x^2)}{\sec x-\cos x}=$

a. −1 b. 1 c. 0 d. 2

79. The value of $\lim_{x\to a}\sqrt{a^2-x^2}\,\cot\dfrac{\pi}{2}\sqrt{\dfrac{a-x}{a+x}}$ is

a. $\dfrac{2a}{\pi}$ b. $-\dfrac{2a}{\pi}$ c. $\dfrac{4a}{\pi}$ d. $-\dfrac{4a}{\pi}$

80. $\lim_{x\to\infty}\dfrac{\cot^{-1}\left(x^{-a}\log_a x\right)}{\sec^{-1}\left(a^x\log_x a\right)}$ $(a > 1)$ is equal to

a. 2 b. 1 c. $\log_a 2$ d. 0

Multiple Correct Answers Type

Solutions on page 2.45

Each question has four choices a, b, c, and d, out of which *one or more* answers are correct.

1. Let $f(x)=\begin{cases}1+\dfrac{2x}{a}, & 0\le x<1\\ ax, & 1\le x<2\end{cases}$. If $\lim_{x\to 1}f(x)$ exists, then a is

a. 1 b. −1 c. 2 d. −2

2. If $f(x) = |x-1| - [x]$, where $[x]$ is the greatest integer less than or equal to x, then

a. $f(1+0)=-1, f(1-0)=0$ b. $f(1+0)=0=f(1-0)$

c. $\lim_{x\to 1}f(x)$ exists d. $\lim_{x\to 1}f(x)$ does not exist

3. If $\lim_{n\to\infty}\left(an-\dfrac{1+n^2}{1+n}\right)=b$, where a is finite number, then

a. $a=1$ b. $a=0$ c. $b=1$ d. $b=-1$

4. If $m, n \in N$, $\lim_{x\to 0}\dfrac{\sin x^n}{(\sin x)^m}$ is

a. 1, if $n=m$ b. 0, if $n>m$
c. ∞, if $n<m$ d. n/m, if $n<m$

5. Which of the following is true ({.} denotes the fractional part of the function)?

a. $\lim_{x\to\infty}\dfrac{\log_e x}{\{x\}}=\infty$ b. $\lim_{x\to 2^+}\dfrac{x}{x^2-x-2}=\infty$

c. $\lim_{x\to -1^-}\dfrac{x}{x^2-x-2}=-\infty$ d. $\lim_{x\to\infty}\dfrac{\log_{0.5}x}{\{x\}}=\infty$

6. If $\lim_{x\to 1}(2-x+a[x-1]+b[1+x])$ exists, then a and b can take the values (where [·] denotes the greatest integer function)

a. $a=1/3, b=1$ b. $a=1, b=-1$
c. $a=9, b=-9$ d. $a=2, b=2/3$

7. $L=\lim_{x\to a}\dfrac{|2\sin x-1|}{2\sin x-1}$, then

a. limit does not exist when $a=\pi/6$
b. $L=-1$ when $a=\pi$
c. $L=1$ when $a=\pi/2$
d. $L=1$ when $a=0$

8. $f(x)=\lim_{n\to\infty}\dfrac{x}{x^{2n}+1}$, then

a. $f(1^+)+f(1^-)=0$ b. $f(1^+)+f(1^-)+f(1)=3/2$
c. $f(-1^+)+f(-1^-)=-1$ d. $f(1^+)+f(-1^-)=0$

9. $\lim_{n\to\infty}\dfrac{-3n+(-1)^n}{4n-(-1)^n}$ is equal to

a. $-\dfrac{3}{4}$ b. 0 if n is even

c. $-\dfrac{3}{4}$ if n is odd d. None of these

10. Given a real-valued function f such that

$$f(x)=\begin{cases}\dfrac{\tan^2\{x\}}{(x^2-[x]^2)} & \text{for } x>0\\ 1 & \text{for } x=0\text{, where } [x] \text{ is the integral}\\ \sqrt{\{x\}\cot\{x\}} & \text{for } x<0\end{cases}$$

part and $\{x\}$ is the fractional part of x, then

a. $\lim_{x\to 0^+}f(x)=1$ b. $\lim_{x\to 0^-}f(x)=\cot 1$

c. $\cot^{-1}\left(\lim_{x\to 0^-}f(x)\right)^2=1$ d. $\tan^{-1}\left(\lim_{x\to 0^+}f(x)\right)=\dfrac{\pi}{4}$

11. If $f(x)=\dfrac{3x^2+ax+a+1}{x^2+x-2}$, then which of the following can be correct

a. $\lim_{x\to 1}f(x)$ exists $\Rightarrow a=-2$
b. $\lim_{x\to -2}f(x)$ exists $\Rightarrow a=13$
c. $\lim_{x\to 1}f(x)=4/3$
d. $\lim_{x\to -2}f(x)=-1/3$

12. $\lim_{n\to\infty} \frac{1}{1+n\sin^2 nx}$ is equal to

a. –1 b. 0 c. 1 d. ∞

13. Let $f(x) = \frac{x^2-9x+20}{x-[x]}$ (where $[x]$ is the greatest integer not greater than x), then

a. $\lim_{x\to 5^-} f(x) = 0$

b. $\lim_{x\to 5^+} f(x) = 1$

c. $\lim_{x\to 5} f(x)$ does not exist

d. None of these

14. Given $\lim_{x\to 0} \frac{f(x)}{x^2} = 2$, where $[\cdot]$ denotes greatest integer function, then

a. $\lim_{x\to 0}[f(x)] = 0$

b. $\lim_{x\to 0}[f(x)] = 1$

c. $\lim_{x\to 0}\left[\frac{f(x)}{x}\right]$ does not exists

d. $\lim_{x\to 0}\left[\frac{f(x)}{x}\right]$ exists

Reasoning Type

Solutions on page 2.47

Each question has four choices a, b, c, and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. if both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.
b. if both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.
c. if STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
d. if STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** $\left[\lim_{x\to 0}\frac{\sin x}{x}\right] = 0$

Statement 2: For $x \in (-\delta, \delta)$, where δ is positive and $\delta \to 0$, $\tan x > x$.

2. **Statement 1:** $\lim_{x\to\alpha} \frac{\sin(f(x))}{x-\alpha}$, where $f(x) = ax^2+bx+c$, is finite and non-zero, then $\lim_{x\to\alpha} \frac{e^{\frac{1}{f(x)}}-1}{e^{\frac{1}{f(x)}}+1}$ does not exist.

Statement 2: $\lim_{x\to\alpha}\frac{\sin(f(x))}{x-\alpha}$ can take finite value only when it takes $\frac{0}{0}$ form.

3. **Statement 1:** $\lim_{x\to 0} \sin^{-1}\{x\}$ does not exist (where $\{.\}$ denotes fractional part function).

Statement 2: $\{x\}$ is discontinuous at $x = 0$.

4. **Statement 1:** If a and b are positive and $[x]$ denotes the greatest integer $\le x$, then $\lim_{x\to 0^+} \frac{x}{a}\left[\frac{b}{x}\right] = \frac{b}{a}$.

Statement 2: $\lim_{x\to\infty} \frac{\{x\}}{x} \to 0$, where $\{x\}$ denotes fractional part of x.

5. **Statement 1:** $\lim_{x\to\infty}\left(\frac{1^2}{x^3}+\frac{2^2}{x^3}+\frac{3^2}{x^3}+\dots+\frac{x^2}{x^3}\right)$

$= \lim_{x\to\infty}\frac{1^2}{x^3} + \lim_{x\to\infty}\frac{2^2}{x^3} + \dots + \lim_{x\to\infty}\frac{x^2}{x^3} = 0$

Statement 2: $\lim_{x\to a} (f_1(x)+f_2(x)+\dots+f_n(x))$

$= \lim_{x\to a} f_1(x) + \lim_{x\to a} f_2(x) + \dots + \lim_{x\to a} f_n(x)$, where $n \in N$.

6. **Statement 1:** $\lim_{x\to 0} \frac{\sqrt{1-\cos 2x}}{x}$ does not exist.

Statement 2: $f(x) = \frac{\sqrt{1-\cos 2x}}{x}$ is not defined at $x = 0$.

7. **Statement 1:** $\lim_{m\to\infty}\lim_{n\to\infty}\{\sin^{2m}(n!\pi x)\} = 0$, $m, n \in N$, when x is rational.

Statement 2: when $n \to \infty$ and x is rational, $n!\,x$ is integer.

8. **Statement 1:**

If $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational}\end{cases}$, then $\lim_{x\to 1/2} f(x)$ does not exist.

Statement 2 : $x \to 1/2$ can be rational or irrational value.

9. **Statement 1:** If $f(x) = \frac{(x-1)(x-2)}{(x-3)(x-4)}$, then $\lim_{x\to-\infty}\sin^{-1} f(x)$ exists, but $\lim_{x\to\infty}\cos^{-1} f(x)$ does not exist.

Statement 2: $\sin^{-1}x$ and $\cos^{-1}x$ are defined for $x \in [-1, 1]$.

10. **Statement 1:** $\lim_{x\to 0}[x]\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$ (where [.] represents the greatest integer function) does not exist.

Statement 2: $\lim_{x\to 0}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$ does not exist.

11. **Statement 1:** If $\lim_{x\to 0}\left(f(x)+\frac{\sin x}{x}\right)$ does not exist, then $\lim_{x\to 0} f(x)$ does not exist.

Statement 2: $\lim_{x\to 0}\frac{\sin x}{x}$ exists and has value 1.

12. **Statement 1:** If $\langle a_n \rangle$ be a sequence such that $a_1 = 1$ and $a_{n+1} = \sin a_n$, then $\lim_{n\to\infty} a_n = 0$.

Statement 2: Since $x > \sin x \ \forall \ x > 0$.

13. **Statement 1**: $\lim_{x\to 0} \log_e\left(\frac{\sin x}{x}\right) = 0$.

Statement 2: $\lim_{x\to a} f(g(x)) = f(\lim_{x\to 0} g(x))$.

Linked Comprehension Type

Solutions on page 2.48

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1 – 3

Let $f(x) = \frac{\sin^{-1}(1-\{x\}) \times \cos^{-1}(1-\{x\})}{\sqrt{2\{x\}} \times (1-\{x\})}$, where $\{x\}$ denotes the fractional part of x

1. $R = \lim_{x\to 0+} f(x)$ is equal to
   a. $\frac{p}{2}$ b. $\frac{\pi}{2\sqrt{2}}$ c. $\frac{\pi}{\sqrt{2}}$ d. $\sqrt{2}\pi$
2. $L = \lim_{x\to 0-} f(x)$ is equal to
   a. $\frac{p}{2}$ b. $\frac{\pi}{2\sqrt{2}}$ c. $\frac{\pi}{\sqrt{2}}$ d. $\sqrt{2}\pi$
3. Which of the following is true?
   a. $\cos L < \cos R$ b. $\tan(2L) > \tan 2R$
   c. $\sin L > \sin R$ d. None of these

For Problems 4 – 6

$A_i = \frac{x - a_i}{|x - a_i|}$, $i = 1, 2, \ldots, n$ and if $a_1 < a_2 < a_3 < \cdots < a_n$.

4. If $1 \le m \le n$, $m \in N$, then the value of $L = \lim_{x\to a_m^-} (A_1 A_2 \cdots A_n)$ is
   a. always 1 b. always – 1
   c. $(-1)^{n-m+1}$ d. $(-1)^{n-m}$
5. If $1 \le m \le n$, $m \in N$, then the value of $R = \lim_{x\to a_m^+} (A_1 A_2 \cdots A_n)$ is
   a. always 1 b. always – 1
   c. $(-1)^{m+1}$ d. $(-1)^{n-m}$
6. If $a_m < a_1$, $m \in N$, then $\lim_{x\to a_m} (A_1 A_2 \cdots A_n)$
   a. Is always equal to – 1. b. Is always equal to + 1.
   c. Does not exist. d. Is equal to 1 or – 1.

For Problems 7 – 9

If $L = \lim_{x\to 0} \frac{\sin x + ae^x + be^{-x} + c\ln(1+x)}{x^3} = \infty$

7. The value of L is
   a. 1/2 b. – 1/3
   c. – 1/6 d. 3
8. Equation $ax^2 + bx + c = 0$ has
   a. real and equal roots
   b. complex roots
   c. unequal positive real roots
   d. unequal roots
9. The solution set of $||x + c| - 2a| < 4b$ is
   a. $[-2, 2]$ b. $[0, 2]$ c. $[-1, 1]$ d. $[-2, 1]$

For Problems 10 – 12

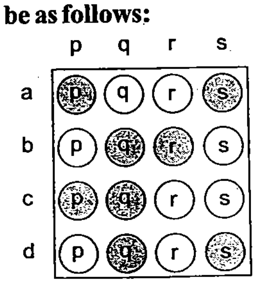
Let $a_1 > a_2 > a_3 \ldots a_n > 1$;
$p_1 > p_2 > p_3 \ldots > p_n > 0$; such that $p_1 + p_2 + p_3 + \cdots + p_n = 1$.
Also $F(x) = (p_1 a_1^x + p_2 a_2^x + \ldots + p_n a_n^x)^{1/x}$

10. $\lim_{x\to 0^+} F(x)$ equals
    a. $p_1 \ln a_1 + p_2 \ln a_2 + \ldots + p_n \ln a_n$
    b. $a_1^{p_1} + a_2^{p_2} + \cdots + a_n^{p_n}$
    c. $a_1^{p_1} \cdot a_2^{p_2} \cdots a_n^{p_n}$
    d. $\sum_{r=1}^{n} a_r p_r$
11. $\lim_{x\to\infty} F(x)$ equals
    a. $\ln a_1$ b. e^{a_n} c. a_1 d. a_n
12. $\lim_{x\to -\infty} f(x)$ equals
    a. $\ln a_n$ b. e^{a_1} c. a_1 d. a_n

Matrix-Match Type

Solutions on page 2.48

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct match are a-p, a-s, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

Column I	Column II
a. If $L = \lim_{x\to -1} \frac{\sqrt[3]{(7-x)} - 2}{(x+1)}$, then $12L =$	p. –2
b. If $L = \lim_{x\to \pi/4} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$, then $-L/4 =$	q. 2

c. If $L=+\lim\limits_{x\to 1}\dfrac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$, then $20L=$	r. 1
d. If $L=\lim\limits_{x\to\infty}\dfrac{\log x^n-[x]}{[x]}$, where $n\in N$, ($[x]$ denotes greatest integer less than or equal to x), then $-2L=$	s. -1

2.

Column I ([·] denotes the greatest integer function)	Column II
a. $\lim\limits_{x\to 0}\left(\left[100\dfrac{\sin x}{x}\right]+\left[100\dfrac{\tan x}{x}\right]\right)$	p. 198
b. $\lim\limits_{x\to 0}\left(\left[100\dfrac{x}{\sin x}\right]+\left[100\dfrac{\tan x}{x}\right]\right)$	q. 199
c. $\lim\limits_{x\to 0}\left(\left[100\dfrac{\sin^{-1}x}{x}\right]+\left[100\dfrac{\tan^{-1}x}{x}\right]\right)$	r. 200
d. $\lim\limits_{x\to 0}\left(\left[100\dfrac{x}{\sin^{-1}x}\right]+\left[100\dfrac{\tan^{-1}x}{x}\right]\right)$	s. 201

3.

Column I	Column II
a. If $\lim\limits_{x\to\infty}(\sqrt{(x^2-x-1)}-ax-b)=0$, where $a>0$, then there exists at least one a and b for which point $(a, 2b)$ lies on the line.	p. $y=-3$
b. If $\lim\limits_{x\to\infty}\dfrac{(1+a^3)+8e^{1/x}}{1+(1-b^3)e^{1/x}}=2$, then there exists at least one a and b for which point (a, b^3) lies on the line.	q. $3x-2y-5=0$
c. If $\lim\limits_{x\to\infty}(\sqrt{(x^4-x^2+1)}-ax^2-b)=0$, then there exists at least one a and b for which point $(a, -4b)$ lies on the line.	r. $15x-2y-11=0$
d. If $\lim\limits_{x\to -a}\dfrac{x^7+a^7}{x+a}=7$, where $a<0$, then there exists at least one a for which point $(-a, 2)$ lies on the line.	s. $y=2$

Integer Type

Solutions on page 2.51

1. The reciprocal of the value of

$$\lim_{n\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$$ is

2. If $f(x)=\begin{cases}x^2+2 & x\ge 2\\ 1-x & x<2\end{cases}$ and $g(x)=\begin{cases}2x & x>1\\ 3-x & x\le 1\end{cases}$, then the value of $\lim\limits_{x\to 1}f(g(x))$ is

3. If $\lim\limits_{x\to 1}(1+ax+bx^2)^{\frac{c}{x-1}}=e^3$, then the value of bc is

4. The value of $\lim\limits_{n\to\infty}\left[\sqrt[3]{(n+1)^2}-\sqrt[3]{(n-1)^2}\right]$ is

5. If $\lim\limits_{x\to 0}\left[1+x+\dfrac{f(x)}{x}\right]^{1/x}=e^3$, then the value of $\ln\left(\lim\limits_{x\to 0}\left[1+\dfrac{f(x)}{x}\right]^{1/x}\right)$ is

6. $\lim\limits_{x\to\infty}f(x)$, where $\dfrac{2x-3}{x}<f(x)<\dfrac{2x^2+5x}{x^2}$, is

7. If $f(x)=\begin{cases}x-1, & x\ge 1\\ 2x^2-2 & x<1\end{cases}$, $g(x)=\begin{cases}x+1, & x>0\\ -x^2+1 & x\le 0\end{cases}$ and $h(x)=|x|$, then find $\lim\limits_{x\to 0}f(g(h(x)))$

8. If $\lim\limits_{x\to\infty}f(x)$ exists and is finite and nonzero and if $\lim\limits_{x\to\infty}\left(f(x)+\dfrac{3f(x)-1}{f^2(x)}\right)=3$, then the value of $\lim\limits_{x\to\infty}f(x)$ is

9. If $L=\lim\limits_{x\to 0}\dfrac{e^{-x^2/2}-\cos x}{x^3\sin x}$ then the value of $1/(3L)$ is

10. If $L=\lim\limits_{x\to 2}\dfrac{(10-x)^{1/3}-2}{x-2}$, then the value of $|1/(4L)|$ is

11. The value of $\lim\limits_{x\to\infty}\dfrac{\log_e(\log_e x)}{e^{\sqrt{x}}}$ is

12. If $L=\lim\limits_{n\to\infty}(2\cdot 3^2\cdot 2^3\cdot 3^4\cdots 2^{n-1}\cdot 3^n)^{\frac{1}{(n^2+1)}}$, then the value of L^4 is

13. If $\lim\limits_{x\to 1}\dfrac{a\sin(x-1)+b\cos(x-1)+4}{x^2-1}=-2$, then $|a+b|$ is

14. Let $\lim\limits_{x\to 1}\dfrac{x^a-ax+a-1}{(x-1)^2}=f(a)$. Then the value of $f(4)$ is

15. The integer n, for which $\lim\limits_{x\to 0}\dfrac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite non-zero number, is

16. If $\lim_{x\to 0} \dfrac{1-\sqrt{\cos 2x}\cdot\sqrt[3]{\cos 3x}\cdot\sqrt[4]{\cos 4x}\cdots\sqrt[n]{\cos nx}}{x^2}$ has the value equal to 10, then the value of n equals

17. $f(x)=\dfrac{3x^2+ax+a+1}{x^2+x-2}$ and $\lim_{x\to -2} f(x)$ exists, then the value of $(a-4)$ is

18. If $L=\lim_{x\to\infty}\left(x-x^2\log_e\left(1+\dfrac{1}{x}\right)\right)$, then the value of $8L$ is

19. Let $S_n=1+2+3+\cdots+n$ and $P_n=\dfrac{S_2}{S_2-1}\cdot\dfrac{S_3}{S_3-1}\cdot\dfrac{S_4}{S_4-1}\cdots\dfrac{S_n}{S_n-1}$, where $n\in N (n\geq 2)$.

Then $\lim_{n\to\infty} P_n=$

20. Let $f''(x)$ be continuous at $x=0$.

If $\lim_{x\to 0}\dfrac{2f(x)-3a\,f(2x)+bf\,(8x)}{\sin^2 x}$ exists and $f(0)\neq 0$, $f'(0)\neq 0$, then the value of $3a/b$ is

Archives

Solutions on page 2.54

Subjective

1. Evaluate $\lim_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}, (a\neq 0)$. (IIT-JEE, 1978)

2. $f(x)$ is the integral of $\dfrac{2\sin x-\sin 2x}{x^3}$, $x\neq 0$, find $\lim_{x\to 0} f'(x)$ [where $f'(x)=\dfrac{df(x)}{dx}$]

(IIT-JEE, 1979)

3. Evaluate $\lim_{h\to 0}\dfrac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$. (IIT-JEE, 1980)

4. Use the formula $\lim_{x\to 0}\dfrac{a^x-1}{x}=\ln a$ to find $\lim_{x\to 0}\dfrac{2^x-1}{(1+x)^{1/2}-1}$.

(IIT-JEE, 1982)

5. Find $\lim_{x\to 0}\{\tan(\pi/4+x)\}^{1/x}$. (IIT-JEE, 1993)

Objective

Fill in the blanks

1. $\lim_{x\to 1}(1-x)\tan\dfrac{\pi x}{2}=$ ______. (IIT-JEE, 1984)

2. If $f(x)=\begin{cases}\sin x, & x\neq n\pi,\ n\in I\\ 2, & \text{otherwise}\end{cases}$ and

$g(x)=\begin{cases}x^2+1, & x\neq 0, 2\\ 4, & x=0\\ 5, & x=2\end{cases}$ then $\lim_{x\to 0} g\{f(x)\}$ is = ______.

(IIT-JEE, 1986)

3. $\lim_{x\to -\infty}\left[\dfrac{x^4\sin\left(\dfrac{1}{x}\right)+x^2}{(1+|x|^3)}\right]=$ ______. (IIT-JEE, 1987)

4. ABC is an isosceles triangle inscribed in a circle of radius r. If $AB=AC$ and h is the altitude from A to BC, then the triangle ABC has perimeter $P=2\left(\sqrt{2hr-h^2}+\sqrt{2hr}\right)$ and area $A=$ ______ and = ______ and also $\lim_{h\to 0}\dfrac{A}{P^3}=$ ______. (IIT-JEE, 1989)

5. $\lim_{x\to\infty}\left(\dfrac{x+6}{x+1}\right)^{x+4}=$ ______. (IIT-JEE, 1990)

6. $\lim_{x\to 0}\left(\dfrac{1+5x^2}{1+3x^2}\right)^{1/x^2}=$ ______. (IIT-JEE, 1996)

7. $\lim_{h\to 0}\dfrac{\ln(1+2h)-2\ln(1+h)}{h^2}=$ ______. (IIT-JEE, 1997)

True or false

1. If $\lim_{x\to a}[f(x)g(x)]$ exists, then both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. (IIT-JEE, 1981)

Multiple choice questions with one correct answer

1. If $f(x)=\sqrt{\dfrac{x-\sin x}{x+\cos^2 x}}$, then $\lim_{x\to\infty} f(x)$ is

a. 0
b. ∞
c. 1
d. None of these

(IIT-JEE, 1979)

2. If $G(x)=-\sqrt{25-x^2}$, then $\lim_{x\to 1}\dfrac{G(x)-G(1)}{x-1}$ is

a. $\dfrac{1}{24}$
b. $\dfrac{1}{5}$
c. $-\sqrt{24}$
d. None of these

(IIT-JEE, 1983)

3. $\lim_{n\to\infty}\left\{\dfrac{1}{1-n^2}+\dfrac{2}{1-n^2}+\cdots+\dfrac{n}{1-n^2}\right\}$ is equal to

a. 0
b. $-\dfrac{1}{2}$
c. $\dfrac{1}{2}$
d. None of these

(IIT-JEE, 1984)

4. If $f(x)=\begin{cases}\dfrac{\sin[x]}{[x]}, & \text{for } [x]\neq 0\\ 0, & \text{for } [x]=0\end{cases}$, where $[x]$ denotes the greatest integer less than or equal to x, then $\lim_{x\to 0} f(x)$ is

a. 1
b. 0
c. -1
d. None of these

(IIT-JEE, 1985)

5. The value of $\lim_{x\to 0}\dfrac{\sqrt{\dfrac{1}{2}(1-\cos 2x)}}{x}$ is

a. 1
b. -1
c. 0
d. None of these

(IIT-JEE, 1991)

6. $\lim_{x\to 1} \frac{\sqrt{1-\cos 2(x-1)}}{x-1}$
 a. exists and it equals $\sqrt{2}$
 b. exists and it equals $-\sqrt{2}$
 c. does not exist because $x-1 \to 0$
 d. does not exist because the left-hand limit is not equal to the right-hand limit **(IIT-JEE, 1998)**

7. $\lim_{x\to 0} \frac{x \tan 2x - 2x \tan x}{(1-\cos 2x)^2}$ is equal to
 a. 2 b. −2
 c. 1/2 d. −1/2 **(IIT-JEE, 1999)**

8. For $x \in R$, $\lim_{x\to\infty} \left(\frac{x-3}{x+2}\right)^x$ is equal to
 a. e b. e^{-1}
 c. e^{-5} d. e^5 **(IIT-JEE, 2000)**

9. $\lim_{x\to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is equal to
 a. $-\pi$ b. π
 c. $\pi/2$ d. 1 **(IIT-JEE, 2001)**

10. The integer n for which $\lim_{x\to 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non-zero number is
 a. 1 b. 2
 c. 3 d. 4 **(IIT-JEE, 2002)**

11. If $\lim_{x\to 0} \frac{((a-n)nx - \tan x)\sin nx}{x^2} = 0$, where n is non-zero real number, then a is
 a. 0 b. $\frac{n+1}{n}$
 c. n d. $n+\frac{1}{n}$ **(IIT-JEE, 2003)**

12. The value of $\lim_{x\to 0} ((\sin x)^{1/x} + (1+x)^{\sin x}) = 0$ where $x > 0$ is
 a. 0 b. −1
 c. 1 d. 2 **(IIT-JEE, 2006)**

13. If $\lim_{x\to 0}[1 + x\ln(1+b^2)]^{1/x} = 2b\sin^2\theta, b > 0$ smf $\theta \in (-\pi, \pi]$, then the value of θ is
 a. $\pm\frac{\pi}{4}$ b. $\pm\frac{\pi}{3}$ c. $\pm\frac{\pi}{6}$ d. $\pm\frac{\pi}{2}$
 (IIT-JEE, 2011)

Multiple choice questions with one or more than one correct answers

1. Let $L = \lim_{x\to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}$, $a > 0$. If L is finite, then
 a. $a = 2$ b. $a = 1$
 c. $L = \frac{1}{64}$ d. $L = \frac{1}{32}$ **(IIT-JEE, 2009)**

ANSWERS AND SOLUTIONS

Subjective Type

1. $\lim_{x\to\frac{3\pi}{4}} \frac{1+(\tan x)^{1/3}}{-(2\cos^2 x - 1)} \frac{(1-(\tan x)^{1/3} + (\tan x)^{2/3})}{(1-(\tan x)^{1/3} + (\tan x)^{2/3})}$

$$\left[\text{Use } a + b = \frac{a^3+b^3}{a^2 - ab + b^2}\right]$$

$$= \lim_{x\to\frac{3\pi}{4}} \frac{(1+\tan x)}{-\cos 2x} \lim_{x\to\frac{3\pi}{4}} \frac{1}{(1-(\tan x)^{1/3} + (\tan x)^{2/3})}$$

$$= \left(-\frac{1}{3}\right) \lim_{x\to\frac{3\pi}{4}} \frac{(1+\tan x)(1+\tan^2 x)}{(1-\tan^2 x)}$$

$$= -\lim_{x\to\frac{3\pi}{4}} \frac{1}{3} \frac{(1+\tan^2 x)}{(1-\tan x)} = \frac{1}{3}$$

2. $\lim_{x\to 0} \frac{e^{\sin x} - (1+\sin x)}{(\tan^{-1}(\sin x))^2}$

$$= \lim_{h\to 0} \frac{e^h - (1+h)}{(\tan^{-1}(h))^2} \quad (\text{where } h = \sin x)$$

$$= \lim_{h\to 0} \frac{\left(1+h+\frac{h^2}{2!}\right) - (1+h)}{(\tan^{-1}(h))^2} = \lim_{h\to 0} \frac{\frac{h^2}{2!}}{(\tan^{-1}(h))^2} = \frac{1}{2}$$

3. $\lim_{x\to 0} \frac{(e^x - 1) - (e^{x\cos x} - 1)}{(x + \sin x)}$

$$= \lim_{x\to 0} \left(\frac{(e^x - 1)}{x\left(1+\frac{\sin x}{x}\right)} - \frac{(e^{x\cos x} - 1)}{x\cos x\left(\sec x + \frac{\sin x}{x\cos x}\right)} \right)$$

$$= \lim_{x\to 0} \frac{(e^x - 1)}{x\left(1+\frac{\sin x}{x}\right)} - \lim_{x\to 0} \frac{(e^{x\cos x} - 1)}{x\cos x\left(\sec x + \frac{\tan x}{x}\right)}$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

4. $\lim_{n\to\infty} \frac{1}{(\sin^{-1} x)^n + 1} = 1$

$\Rightarrow (\sin^{-1}x)^n \to 0$

$\Rightarrow 0 \le \sin^{-1} x < 1$

$\Rightarrow x \in [0, \sin 1)$

5. We know that $-1 \le \cos x \le 1$ for all x.

$\Rightarrow -2 \le 2\cos x \le 2$

$\Rightarrow 5x - 2 \le 5x + 2\cos x \le 5x + 2$

Dividing by $3x - 14$, we get

$$\frac{5x-2}{3x-14} \ge \frac{5x+2\cos x}{3x-14} \ge \frac{5x+2}{3x-14} \qquad \text{(for large negative } x\text{)}$$

Now, $\lim_{x\to-\infty} \frac{5x-2}{3x-14} = \lim_{x\to-\infty} \frac{5x+2}{3x-14} = \frac{5}{3}$

It follows that $\lim_{x\to-\infty} \frac{5x+2\cos x}{3x-14} = \frac{5}{3}$

6. As $n \to \infty$, let $\lim_{n\to\infty} f(n) = f(n+1) = k$

We have $f(n+1) = \frac{1}{2}\left(f(n) + \frac{9}{f(n)}\right)$

$\Rightarrow \lim_{n\to\infty} f(n+1) = \lim_{n\to\infty} \frac{1}{2}\left(f(n) + \frac{9}{f(n)}\right)$

$\Rightarrow k = \frac{1}{2}\left(k + \frac{9}{k}\right) \Rightarrow k^2 = 9$ or $k = 3$

$\Rightarrow \lim_{n\to\infty} f(n) = 3$

7. Let $P = \lim_{x\to0} \frac{8}{x^8}\left\{\left(1-\cos\frac{x^2}{4}\right)\left(1-\cos\frac{x^2}{2}\right)\right\}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to0} \frac{8}{x^8} 4\sin^2\frac{x^2}{8}\sin^2\frac{x^2}{4}$

$= \lim_{x\to0} \frac{32}{64\times16} \frac{\sin^2\frac{x^2}{8}}{\frac{x^4}{64}} \frac{\sin^2\frac{x^2}{4}}{\frac{x^4}{16}} = \frac{1}{32}$

8. Let $P = \lim_{n\to\infty} n^2\left\{\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\cdots\infty}}}\right\}$

Putting $\frac{1}{n} = x$, we get

$$P = \lim_{x\to0} \frac{\sqrt{(1-\cos x)\sqrt{(1-\cos x)\sqrt{(1-\cos x)\cdots\infty}}}}{x^2}$$

$$= \lim_{x\to0} \frac{(1-\cos x)^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\infty}}{x^2}$$

$$= \lim_{x\to0} \frac{(1-\cos x)}{x^2}\frac{(1+\cos x)}{(1+\cos x)}$$

$$= \lim_{x\to0}\left(\frac{\sin x}{x}\right)^2 \lim_{x\to0}\frac{1}{(1+\cos x)} = (1)^2\,\frac{1}{1+1} = \frac{1}{2}$$

9. We know that $\cos\theta\cos2\theta\cos4\theta\cdots\cos2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n\sin\theta}$

Then $\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\cdots\cos\left(\frac{x}{2^n}\right) = \frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$

Then $L = \lim_{n\to\infty}\left\{\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\cdots\cos\left(\frac{x}{2^n}\right)\right\}$

$$= \lim_{n\to\infty} \frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)} = \lim_{n\to\infty} \frac{\frac{\sin x}{x}\left(\frac{x}{2^n}\right)}{\sin\left(\frac{x}{2^n}\right)} = \frac{\sin x}{x}$$

10. $ax^2 + bx + c = a(x-x_1)(x-x_2)$

$\Rightarrow \lim_{x\to x_1} (1+\sin(ax^2+bx+c))^{\frac{1}{x-x_1}}$

$= e^{\lim_{x\to x_1}\frac{\sin(a(x-x_1)(x-x_2))}{(x-x_1)}}$

$= e^{\lim_{x\to x_1}\frac{\sin(a(x-x_1).(x-x_2))}{a(x-x_1)(x-x_2)}.a(x-x_2)} = e^{a(x_1-x_2)}$

11. $\lim_{x\to\infty} x\left[\tan^{-1}\frac{x+1}{x+2} - \tan^{-1}\frac{x}{x+2}\right]$

$$= \lim_{x\to\infty} x\tan^{-1}\left(\frac{\frac{x+1}{x+2}-\frac{x}{x+2}}{1+\frac{x+1}{x+2}\frac{x}{x+2}}\right)$$

$$= \lim_{x\to\infty} x\tan^{-1}\left(\frac{x+2}{2x^2+5x+4}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\tan^{-1}\left(\frac{x+2}{2x^2+5x+4}\right)}{\frac{x+2}{2x^2+5x+4}}\right)\times\frac{x(x+2)}{2x^2+5x+4} = 1\times\frac{1}{2} = \frac{1}{2}$$

12. $L = \lim_{x\to0} \frac{2^x - x - 1}{x^2}$

Let $x = 2t$

$L = \lim_{t\to0} \frac{2^{2t} - 2t - 1}{4t^2}$

$$L=\frac{1}{4}\left[\lim_{t\to 0}\left(\frac{2^t-1}{t}\right)^2+\lim_{t\to 0}2\left(\frac{2^t-t-1}{t^2}\right)\right]$$

$$\Rightarrow L=\frac{1}{4}\left[(\ln 2)^2+2L\right]$$

$$\Rightarrow \frac{L}{2}=\frac{1}{4}(\ln 2)^2$$

$$\Rightarrow L=\frac{1}{2}(\ln 2)^2 \qquad \left(\because \lim_{t\to 0}\frac{2^t-1}{t}=\ln 2\right)$$

13. Let $f(x)=\dfrac{\sin\{x\}}{\{x\}}$

$\therefore$ L.H.L. $=\lim_{x\to 1-}f(x)$

$$=\lim_{h\to 0}f(1-h)$$

$$=\lim_{h\to 0}\frac{\sin\{1-h\}}{\{1-h\}}$$

$$=\lim_{t\to 0}\frac{\sin(1-h)}{(1-h)}$$

$$=\frac{\sin 1}{1}=\sin 1$$

and R.H.L. $=\lim_{h\to 1^+}f(x)$

$$=\lim_{x\to 0}f(1+h)$$

$$=\lim_{h\to 0}\frac{\sin\{1+h\}}{\{1+h\}}$$

$$=\lim_{h\to 0}\frac{\sin h}{h}=1$$

Hence L.H.L. $\neq$ R.H.L.

Hence $\lim_{x\to 1}\dfrac{\sin\{x\}}{\{x\}}$ does not exist.

14. $\lim_{x\to 0}\{1^{1/\sin^2 x}+2^{1/\sin^2 x}+\cdots+n^{1/\sin^2 x}\}^{\sin^2 x}$

Put $\dfrac{1}{\sin^2 x}=t\geq 1$

$\therefore \lim_{t\to\infty}(1^t+2^t+\cdots+n^t)^{1/t}$

$$=\lim_{t\to\infty}(n^t)^{1/t}\left[\left(\frac{1}{n}\right)^t+\left(\frac{2}{n}\right)^t+\cdots+1\right]^{1/t}$$

$$=n\lim_{t\to\infty}\left[\left(\frac{1}{n}\right)^t+\left(\frac{2}{n}\right)^t+\cdots+1\right]^{1/t}$$

$$=n[0+0+\cdots+1]^0=n$$

15. Let $x=\dfrac{1}{y}$, then

$$\lim_{x\to 0}\left(\frac{a_1^{1/x}+a_2^{1/x}+\cdots+a_n^{1/x}}{n}\right)^{nx}$$

$$=\lim_{y\to 0}\left(\frac{a_1^y+a_2^y+\cdots+a_n^y}{n}\right)^{n/y}$$

$$=e^{\lim_{y\to 0}\left(\frac{a_1^y+a_2^y+\cdots+a_n^y-n}{n}\right)\frac{n}{y}}$$

$$=e^{\lim_{y\to 0}\left(\frac{a_1^y-1}{y}+\frac{a_2^y-1}{y}+\cdots+\frac{a_n^y-1}{y}\right)}$$

$$=e^{\log a_1+\log a_2+\cdots+\log a_n}=e^{\log(a_1a_2\cdots a_n)}=a_1a_2a_3\cdots a_n$$

16. $1-\cos(1-\cos x)=2\sin^2\left(\dfrac{1-\cos x}{2}\right)=2\sin^2\left(\sin^2\dfrac{x}{2}\right)$

$$\Rightarrow \lim_{x\to 0}\frac{1-\cos(1-\cos x)}{x^4}$$

$$=\lim_{x\to 0}\frac{2\sin^2\left(\sin^2\frac{x}{2}\right)}{x^4}$$

$$=\lim_{x\to 0}\frac{2\sin^2\left(\sin^2\frac{x}{2}\right)}{\left(\sin^2\frac{x}{2}\right)^2}\times\frac{\sin^4\frac{x}{2}}{\left(\frac{x}{2}\right)^4 16}=\frac{1}{8}$$

17. $\lim_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{1}{\sin^2 x}\right)=\lim_{x\to 0}\left(\dfrac{\sin^2 x-x^2}{x^2\sin^2 x}\right)$

$$=\lim_{x\to 0}\frac{(\sin x+x)(\sin x-x)}{x^2\sin^2 x}$$

$$=\lim_{x\to 0}\frac{\left(\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\cdots\right)+x\right)\left(\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\cdots\right)-x\right)}{x^4\left(\frac{\sin x}{x}\right)^2}$$

$$=\lim_{x\to 0}\frac{-\left(2-\frac{x^2}{3!}+\frac{x^4}{5!}\cdots\right)\left(\frac{1}{3!}-\frac{x^2}{5!}\cdots\right)}{\left(\frac{\sin x}{x}\right)^2}=-\frac{1}{3}$$

18. Let $y=\dfrac{x}{x+\dfrac{\sqrt[3]{x}}{x+\sqrt[3]{x}\cdots\infty}}=\dfrac{x}{x+\dfrac{1}{x^{2/3}}\dfrac{x}{x+\sqrt[3]{x}\cdots\infty}}=\dfrac{x}{x+\dfrac{y}{x^{2/3}}}$

$$\Rightarrow y=\frac{x^{5/3}}{x^{5/3}+y}$$

$$\Rightarrow y^2+(x^{5/3})y-x^{5/3}=0$$

$\therefore y = \dfrac{-x^{5/3} \pm \sqrt{x^{10/3} + 4x^{5/3}}}{2}$

$= \dfrac{-x^{5/3} + \sqrt{x^{10/3} + 4x^{5/3}}}{2}$ $(\because y > 0)$

$= \dfrac{4x^{5/3}}{2(\sqrt{(x^{10/3} + 4x^{5/3})} + x^{5/3})}$

$= \dfrac{2}{\sqrt{\left(1 + \dfrac{4}{x^{5/3}}\right)} + 1}$

$\therefore \lim_{x\to\infty} y = \dfrac{2}{\sqrt{1+0}+1} = \dfrac{2}{2} = 1$

19. Let $t = \sin x$

Then, $P = \lim_{t\to 1} \dfrac{t - t^t}{1 - t + \ln t}$ $\left(\dfrac{0}{0} \text{ form}\right)$

Using L'Hopital's rule

$\therefore P = \lim_{t\to 1} \dfrac{1 - t^t(1 + \ln t)}{0 - 1 + \dfrac{1}{t}}$ $\left(\dfrac{0}{0} \text{ form}\right)$

Again using L'Hopital's rule, then

$P = \lim_{t\to 1} \dfrac{0 - \left\{t^t\left(\dfrac{1}{t}\right) + (1+\ln t)\, t^t\, (1+\ln t)\right\}}{0 - 0 - \dfrac{1}{t^2}} = -\dfrac{(1+1)}{-1} = 2$

20. $\lim_{\theta\to 0} \dfrac{\cos^2(1 - \cos^2(1 - \cos^2(1 \cdots \cos^2\theta))\cdots)}{\sin\left(\dfrac{\pi(\sqrt{\theta+4} - 2)}{\theta}\right)}$

$= \lim_{\theta\to 0} \dfrac{\cos^2(\sin^2(\sin^2 \cdots (\sin^2\theta)\cdots)}{\sin\left(\dfrac{\pi(\sqrt{\theta+4}-2)}{\theta}\right)}$

$= \lim_{\theta\to 0} \dfrac{\cos^2(\sin^2(\sin^2 \cdots (\sin^2\theta)\cdots)}{\sin\left(\pi \lim_{\theta\to 0} \dfrac{\theta}{\theta(\sqrt{\theta+4}+2)}\right)}$

$= \dfrac{\cos^2(0)}{\sin\left(\dfrac{\pi}{4}\right)} = \sqrt{2}$

21. $\lim_{x\to\pi/2} \tan^2 x(\sqrt{2\sin^2 x + 3\sin x + 4} - \sqrt{\sin^2 x + 6\sin x + 2})$

$= \lim_{x\to\pi/2} \tan^2 x \dfrac{(2\sin^2 x + 3\sin x + 4 - \sin^2 x - 6\sin x - 2)}{\sqrt{2\sin^2 x + 3\sin x + 4} + \sqrt{\sin^2 x + 6\sin x + 2}}$

$= \lim_{x\to\pi/2} \dfrac{\tan^2 x(\sin^2 x - 3\sin x + 2)}{\sqrt{2\sin^2 x + 3\sin x + 4} + \sqrt{\sin^2 x + 6\sin x + 2}}$

$= \lim_{x\to\pi/2} \dfrac{\sin^2 x(\sin x - 1)(\sin x - 2)}{(1 - \sin^2 x)(\sqrt{2\sin^2 x + 3\sin x + 4} + \sqrt{\sin^2 x + 6\sin x + 2})}$

$= \lim_{x\to\pi/2} \dfrac{-\sin^2 x(\sin x - 2)}{(1 + \sin x)(\sqrt{2\sin^2 x + 3\sin x + 4} + \sqrt{\sin^2 x + 6\sin x + 2})}$

$= \dfrac{1}{2(\sqrt{9} + \sqrt{9})} = \dfrac{1}{12}$

22. $\lim_{x\to 1} \sec\dfrac{\pi}{2^x} \log x$

$= \lim_{x\to 1} \dfrac{\log x}{\cos\dfrac{\pi}{2^x}}$

$= \lim_{x\to 1} \dfrac{\log(1 + (x-1))}{\sin\left(\dfrac{\pi}{2} - \dfrac{\pi}{2^x}\right)}$

$= \lim_{x\to 1} \dfrac{\dfrac{\log(1+(x-1))}{x-1}(x-1)}{\dfrac{\sin\left(\dfrac{\pi}{2} - \dfrac{\pi}{2^x}\right)}{\left(\dfrac{\pi}{2} - \dfrac{\pi}{2^x}\right)}\left(\dfrac{\pi}{2} - \dfrac{\pi}{2^x}\right)} = \lim_{x\to 1} \dfrac{(x-1)}{\left(\dfrac{\pi}{2} - \dfrac{\pi}{2^x}\right)}$

$\left[\because \lim_{x\to 1} \dfrac{\log(1+(x-1))}{x-1} = 1 \text{ and } \lim_{x\to 1} \dfrac{\sin\left(\dfrac{\pi}{2} - \dfrac{\pi}{2^x}\right)}{\left(\dfrac{\pi}{2} - \dfrac{\pi}{2^x}\right)} = 1\right]$

$= \lim_{x\to 1} \dfrac{(x-1)}{\pi\left(\dfrac{2^{x-1} - 1}{2^x}\right)}$

$= \dfrac{2}{\pi} \lim_{x\to 1} \dfrac{(x-1)}{2^{x-1} - 1} = \dfrac{2}{\pi \log 2}$

23. $\lim_{x\to 0} \dfrac{e - e^{\frac{1}{x}\ln(1+x)}}{x}$

$= \lim_{x\to 0} \dfrac{e - e^{\frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)}}{x}$

$= \lim_{x\to 0} \dfrac{e - e \times e^{-\left(\frac{x}{2} - \frac{x^2}{3} + \frac{x3}{4} \cdots\right)}}{x}$

$$= -e \times \lim_{x\to 0} \frac{\left(e^{-\left(\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}\cdots\right)}-1\right)\left(-\left(\frac{1}{2}-\frac{x}{3}\cdots\right)\right)}{\left(-\left(\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}\cdots\right)\right)} = \frac{e}{2}$$

24. $\lim_{n\to\infty} n^{-n^2}\left[(n+1)\left(n+\frac{1}{2}\right)\cdots\left(n+\frac{1}{2^{n-1}}\right)\right]^n$

$$= \lim_{n\to\infty}\left[\frac{(n+1)\left(n+\frac{1}{2}\right)\cdots\left(n+\frac{1}{2^{n-1}}\right)}{n^n}\right]^n$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\left(\frac{n+\frac{1}{2}}{n}\right)^n\cdots\left(\frac{n+\frac{1}{2^{n-1}}}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{2n}\right)^n\cdots\left(1+\frac{1}{2^{n-1}n}\right)^n \quad (1^\infty \text{ form})$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{2n}\right)^{\frac{2n}{2}}\cdots\left(1+\frac{1}{2^{n-1}n}\right)^{\frac{2^{n-1}.n}{2^{n-1}}}$$

$$= e^1\, e^{1/2}\, e^{1/4}\cdots\left\{\text{using; } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{an} = e^a\right\}$$

$$= e^{(1+1/2+1/4+\cdots)} = e^{\frac{1}{1-\frac{1}{2}}} = e^2$$

25. We know that $0 \le \cos^2(n!\,px) \le 1$.

Hence, $\lim_{m\to\infty}\cos^{2m}(n!\pi x) = 0$ or 1 according to

$0 \le \cos^2(n!\,\pi x) < 1$ or $\cos^2(n!\,\pi x) = 1$

Also, since $n \to \infty$, then $n!\,\pi$ = integer if $x \in Q$ and $n!\,\pi \ne$ integer, if $x \in$ irrational.

Hence, $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$.

26. Δ_1 = Area of $\Delta ABC = R^2\sin\theta(\sec\theta - \cos\theta)$

$\Delta_1 = R^2\tan\theta(1-\cos^2\theta)$

Area of Δ CDE $= \dfrac{R^2(1-\cos\theta)^2}{\cos^2\theta\cdot\tan\theta} = \Delta_2 \quad \begin{bmatrix} CM = R\sec\theta - R \\ DM = CM\cos\theta \end{bmatrix}$

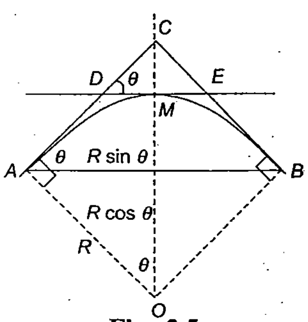


Fig. 2.5

$$\therefore \frac{\Delta_1}{\Delta_2} = \frac{\tan\theta(1-\cos^2\theta)\cos^2\theta\tan\theta}{(1-\cos\theta)^2} = L \text{ (say)}$$

$$\Rightarrow \lim_{\theta\to 0} L = \lim_{\theta\to 0}\frac{(\tan^2\theta)\cos^2\theta(1-\cos\theta)(1+\cos\theta)}{(1-\cos\theta)^2}$$

$$= 1\times 2\lim_{\theta\to 0}\frac{\tan^2\theta}{\theta^2}\times\frac{\theta^2}{1-\cos\theta} = 4$$

27. Let θ be the base angle of T_1, then base angle of T_2 is $\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$.

Base angle of T_3 is $\frac{\pi}{2}-\frac{1}{2}\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$.

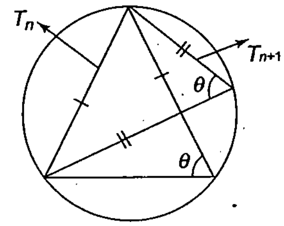


Fig. 2.6

Proceeding in the same way, base angle of T_n is

$$\left(\frac{\pi}{2}-\frac{\pi}{4}+\frac{\pi}{8}-\cdots+\frac{(-1)^{n-1}\theta}{2^{n-1}}\right) \qquad (1)$$

where θ is the base angle of T_1.

Taking limit $n\to\infty$ in equation (1), we have

$$\lim_{n\to\infty}\left(\frac{\pi}{2}-\frac{\pi}{2^2}+\frac{\pi}{2^3}-\cdots+\frac{(-1)^{n-2}\pi}{2^{n-2}}+\frac{(-1)^{n-1}\theta}{2^{n-1}}\right)$$

$$= \frac{\pi/2}{1+\frac{1}{2}} = \frac{\pi}{3}$$

Now, since T_n is isosceles and one of angles approaches to 60° as $n\to\infty \Rightarrow T_n$ is equilateral triangle as $n\to\infty$.

Objective Type

1. c. Given $f(x) = x^2 - \pi^2$

$$\lim_{x\to-\pi}\frac{x^2-\pi^2}{\sin(\sin x)} = \lim_{h\to 0}\frac{(-\pi+h)^2-\pi^2}{\sin(\sin(-\pi+h))} = \lim_{h\to 0}\frac{-2h\pi+h^2}{-\sin(\sin h)}$$

$$= \lim_{h\to 0}\frac{h-2\pi}{\frac{-\sin(\sin h)}{\sin h}\times\frac{\sin h}{h}} = 2\pi$$

2. d. $\lim_{x\to\frac{\pi}{2}}\dfrac{\cos x}{(1-\sin x)^{1/3}} = \lim_{t\to 0}\dfrac{-\sin t}{(1-\cos t)^{1/3}}$

$$= -\lim_{t\to 0}\frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{\left(2\sin^2\frac{t}{2}\right)^{1/3}}$$

$$=-\lim_{t\to 0} 2^{2/3}\cos\frac{t}{2}\left(\sin\frac{t}{2}\right)^{1/3}=0$$

3. a. $$\lim_{x\to-\infty}\frac{x^2\tan\frac{1}{x}}{\sqrt{8x^2+7x+1}}=\lim_{x\to-\infty}\frac{x^2\tan\frac{1}{x}}{-x\sqrt{8+\frac{7}{x}+\frac{1}{x^2}}}$$

$$=-\lim_{x\to-\infty}\frac{\tan\frac{1}{x}}{\frac{1}{x}\sqrt{8+\frac{7}{x}+\frac{1}{x^2}}}=-\frac{1}{2\sqrt{2}}$$

4. a. $$\lim_{x\to 0+}\left[\frac{\sin(\operatorname{sgn} x)}{\operatorname{sgn}(x)}\right]$$

$$=\lim_{x\to 0^+}\left[\frac{\sin 1}{1}\right]$$

$$=0$$

$$=\lim_{x\to 0^-}\left[\frac{\sin(\operatorname{sgn} x)}{\operatorname{sgn}(x)}\right]$$

$$=\lim_{x\to 0^-}\left[\frac{\sin(-1)}{-1}\right]$$

$$=\lim_{x\to 0^-}[\sin 1]$$

$$=0$$

Hence, the given limit is 0.

5. d. The given limit is $\lim_{x\to\infty}\frac{\frac{2}{x}+2+\frac{\sin 2x}{x}}{\left(2+\frac{\sin 2x}{x}\right)e^{\sin x}}$

$$=\frac{0+2+0}{(2+0)\times(\text{a value between }\frac{1}{e}\text{ and }e)}$$

$$\left[\because \lim_{x\to\infty}\sin x\in(-1,1)\right]$$

Hence limit does not exist

6. b. $$\frac{[x]^2}{x^2}=\begin{cases}0 \text{ if } 0<x<1\\ \frac{1}{x^2}\text{ if } -1<x<0\end{cases}\Rightarrow l \text{ does not exist}$$

$$\frac{[x^2]}{x^2}=\begin{cases}0 \text{ if } 0<x<1\\ 0 \text{ if } -1<x<0\end{cases}\Rightarrow m \text{ exists and is equal to 0]}$$

7. c. $$\lim_{x\to 1}\frac{x\sin(x-[x])}{x-1}$$

Now L.H.L. $=\lim_{h\to 0}\frac{(1-h)\sin(1-h-[1-h])}{(1-h)-1}$

$$=\lim_{h\to 0}\frac{(1-h)\sin(1-h)}{-h}=-\infty$$

R.H.L. $=\lim_{h\to 0}\frac{(1+h)\sin(1+h-[1+h])}{(1+h)-1}=\lim_{h\to 0}\frac{(1+h)\sin h}{h}=1$

Hence, the limit does not exist.

8. c. The given limit is $\lim_{x\to 0}[(1+\tan x)^{\operatorname{cosec} x}/(1+\sin x)^{\operatorname{cosec} x}]$

$$=\lim_{x\to 0}[(1+\tan x)^{\cot x}\}^{\sec x}\, x/\{1/(1+\sin x)^{\operatorname{cosec} x}\}]$$

$$=e^{\sec 0}\frac{1}{e}=e\frac{1}{e}=1$$

9. b. $$\lim_{x\to\infty}\frac{\sin^4 x-\sin^2 x+1}{\cos^4 x-\cos^2 x+1}$$

$$=\lim_{x\to\infty}\frac{(1-\cos^2 x)^2-(1-\cos^2 x)+1}{\cos^4 x-\cos^2 x+1}$$

$$=\lim_{x\to\infty}\frac{\cos^4 x-\cos^2 x+1}{\cos^4 x-\cos^2 x+1}$$

$$=1$$

10. d. $$\lim_{x\to\infty}\left(\frac{x^3}{3x^2-4}-\frac{x^2}{3x+2}\right)$$

$$=\lim_{x\to\infty}\frac{x^3(3x+2)-x^2(3x^2-4)}{(3x^2-4)(3x+2)}$$

$$=\lim_{x\to\infty}\frac{2x^3+4x^2}{9x^3+6x^2-12x-8}$$

$$=\lim_{x\to\infty}\frac{2+\frac{4}{x}}{9+\frac{6}{x}-\frac{12}{x^2}-\frac{8}{x^3}}$$

$$=2/9$$

11. c. We have $f(x)+g(x)+h(x)=\frac{x^2-4x+17-4x-2}{x^2+x-12}$

$$=\frac{x^2-8x+15}{x^2+x-12}=\frac{(x-3)(x-5)}{(x-3)(x+4)}$$

$$\therefore \lim_{x\to 3}[f(x)+g(x)+h(x)]=\lim_{x\to 3}\frac{(x-3)(x-5)}{(x-3)(x+4)}=-\frac{2}{7}$$

12. d. $$\lim_{x\to 0}\frac{x\left(e^x-1\right)}{1-\cos x}=\lim_{x\to 0}\frac{2x\left(e^x-1\right)}{4\sin^2\frac{x}{2}}$$

$$=2\lim_{x\to 0}\left[\frac{(x/2)^2}{\sin^2\frac{x}{2}}\right]\left(\frac{e^x-1}{x}\right)=2$$

13. c. $$\lim_{n\to\infty}\frac{n(2n+1)^2}{(n+2)\left(n^2+3n-1\right)}$$

$$= \lim_{n\to\infty} \frac{\left(2+\frac{1}{n}\right)^2}{\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}-\frac{1}{n^2}\right)}$$

$$= \frac{(2+0)^2}{(1+0)(1+0+0)} = 4$$

14. d. We have $\lim_{x\to\pi} \frac{1+\cos^3 x}{\sin^2 x}$

$$= \lim_{x\to\pi} \frac{(1+\cos x)(1-\cos x+\cos^2 x)}{(1-\cos x)(1+\cos x)}$$

$$= \lim_{x\to\pi} \frac{1-\cos x+\cos^2 x}{1-\cos x} = \frac{1+1+1}{1+1} = \frac{3}{2}$$

15. c. $\lim_{n\to\infty} n^2\left(x^{1/n} - x^{\frac{1}{n+1}}\right) = \lim_{n\to\infty} n^2 \cdot x^{\frac{1}{n+1}}\left(x^{\frac{1}{n}-\frac{1}{n+1}} - 1\right)$

$$= \lim_{n\to\infty} x^{\frac{1}{n+1}}\left(x^{\frac{1}{n(n+1)}} - 1\right)n^2$$

$$= \lim_{n\to\infty} x^{\frac{1}{n+1}} \cdot \frac{x^{\frac{1}{n(n+1)}}-1}{\frac{1}{n(n+1)}} \cdot \frac{n^2}{n(n+1)} = 1 \cdot \log_e x \cdot 1 = \log_e x$$

16. a. $\lim_{x\to 2} \frac{\sqrt{1+\sqrt{2+x}}-\sqrt{3}}{x-2}$

$$= \lim_{x\to 2} \frac{1+\sqrt{2+x}-3}{\left(\sqrt{1+\sqrt{2+x}}+\sqrt{3}\right)(x-2)} \text{ (Rationalizing)}$$

$$= \lim_{x\to 2} \frac{\sqrt{2+x}-2}{(\sqrt{1+\sqrt{2+x}}+\sqrt{3})(x-2)}$$

$$= \lim_{x\to 2} \frac{(x-2)}{(\sqrt{1+\sqrt{2+x}}+\sqrt{3})\,(\sqrt{2+x}+2)\,(x-2)}$$

(Rationalizing)

$$= \frac{1}{(2\sqrt{3})\,4} = \frac{1}{8\sqrt{3}}$$

17. c. $\lim_{x\to\infty} \frac{(2x+1)^{40}(4x-1)^5}{(2x+3)^{45}}$

$$= \lim_{x\to\infty} \frac{\left(2+\frac{1}{x}\right)^{40}\left(4-\frac{1}{x}\right)^5}{\left(2+\frac{3}{x}\right)^{45}}$$

(Dividing numerator and denominator by x^{45})

$$= \frac{2^{40}4^5}{2^{45}}$$

$$= 2^5 = 32$$

18. b. $\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x}\right]$

$$= \lim_{x\to\infty} \frac{x+\sqrt{x+\sqrt{x}}-x}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}} \text{ (Rationalizing)}$$

$$\lim_{x\to\infty} \frac{\sqrt{1+x^{-1/2}}}{\sqrt{1+\sqrt{x^{-1}+x^{-3/2}}}+1} = \frac{1}{2}$$

19. d. $\lim_{x\to\infty} \frac{(x+1)^{10}+(x+2)^{10}+\cdots+(x+100)^{10}}{x^{10}+10^{10}}$

$$= \lim_{x\to\infty} \frac{x^{10}\left[\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}+\cdots+\left(1+\frac{100}{x}\right)^{10}\right]}{x^{10}\left[1+\frac{10^{10}}{x^{10}}\right]}$$

$$= 100$$

20. c. $\lim_{x\to 0} \frac{x^a \sin^b x}{\sin x^c}$

$$= \lim_{x\to 0} x^a\left(\frac{\sin x}{x}\right)^b\left(\frac{x^c}{\sin x^c}\right)x^{b-c} = \lim_{x\to 0} x^{a+b-c}$$

This limit will have non-zero value if $a+b=c$.

21. b. $\lim_{x\to\pi/2}\left[x\tan x - \left(\frac{\pi}{2}\right)\sec x\right]$

$$= \lim_{x\to\pi/2} \frac{2x\sin x-\pi}{2\cos x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to\pi/2} \frac{[2\sin x+2x\cos x]}{-2\sin x}$$

(Applying L'Hopital's rule)

$$= -1$$

22. c. $\lim_{x\to\infty}\left(\frac{x^3+1}{x^2+1}-(ax+b)\right) = 2$

$$\Rightarrow \lim_{x\to\infty} \frac{x^3(1-a)-bx^2-ax+(1-b)}{x^2+1} = 2$$

$\Rightarrow 1-a=0$ and $-b=2$

$\Rightarrow a=1, b=-2$

23. c. $\lim_{x\to 1}(2-x)^{\tan\frac{\pi x}{2}}$

$$= \lim_{x\to1}\{1+(1-x)\}^{\tan\frac{\pi x}{2}}$$

$$= e^{\lim_{x\to1}(1-x)\tan\frac{\pi x}{2}}$$

$$= e^{\lim_{x\to1}(1-x)\cot\left(\frac{\pi}{2}-\frac{\pi x}{2}\right)}$$

$$= e^{\lim_{x\to1}\frac{(1-x)}{\tan\left(\frac{\pi}{2}-\frac{\pi x}{2}\right)}}$$

$$= e^{\frac{2}{\pi}\lim_{x\to1}\frac{\frac{\pi}{2}(1-x)}{\tan\left(\frac{\pi}{2}(1-x)\right)}}$$

$$= e^{2/\pi}$$

24. b. $\lim_{x\to0}\frac{\sin x^n}{(\sin x)^m} = \lim_{x\to0}\left(\frac{\sin x^n}{x^n}\right)\left(\frac{x^n}{x^m}\right)\left(\frac{x}{\sin x}\right)^m$

$= \lim_{x\to0} x^{n-m} = 0 \qquad [\because m<n]$

25. a. $\frac{x^4(\cot^4 x-\cot^2 x+1)}{(\tan^4 x-\tan^2 x+1)}$

$= \frac{x^4(1-\tan^2 x+\tan^4 x)}{\tan^4 x(\tan^4 x-\tan^2 x+1)} = \frac{x^4}{\tan^4 x}, x\neq 0$

$\Rightarrow \lim_{x\to0}\frac{x^4(\cot^4 x-\cot^2 x+1)}{(\tan^4 x-\tan^2 x+1)} = \lim_{x\to0}\frac{x^4}{\tan^4 x}=1$

26. d. $\lim_{x\to\infty}\left(\frac{1}{e}-\frac{x}{1+x}\right)^x = \lim_{x\to\infty}\left(\frac{1}{e}-\frac{1}{\frac{1}{x}+1}\right)^x = \left(\frac{1}{e}-1\right)^\infty$

= (some negative value)$^\infty$ which is not defined as base is –ve.

27. b. $\lim_{x\to1}\frac{1-x^2}{\sin 2\pi x}$

$= -\lim_{x\to1}\frac{2\pi(1-x)(1+x)}{2\pi\sin(2\pi-2\pi x)}$

$= -\lim_{x\to1}\frac{(2\pi-2\pi x)}{\sin(2\pi-2\pi x)}\frac{1+x}{2\pi} = \frac{-1}{\pi}$

28. d. We know that $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases}2\tan^{-1}x, & x\geq0\\ -2\tan^{-1}x, & x\leq0\end{cases}$

$\Rightarrow \lim_{x\to0^+}\frac{1}{x}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \lim_{x\to0^+}\frac{2\tan^{-1}x}{x} = 2$, and

$\lim_{x\to0^-}\frac{1}{x}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \lim_{x\to0^+}\left[-\frac{2\tan^{-1}x}{x}\right] = -2$

29. c. $\lim_{x\to\infty}\left(\frac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+1}{2x-1}}$

$= \lim_{x\to\infty}\left(\frac{1+\frac{2}{x}-\frac{1}{x^2}}{2-\frac{3}{x}-\frac{2}{x^2}}\right)^{\frac{2+1/x}{2-1/x}}$

$= 1/2$

30. c. Since the highest degree of x is 1/2, divide numerator and denominator by $\sqrt{x}$, then we have limit $\frac{2}{\sqrt{2}}$ or $\sqrt{2}$.

31. a. $\lim_{y\to0}\left\{\frac{x\{\sec(x+y)-\sec x\}}{y}+\sec(x+y)\right\}$

$= \lim_{y\to0}\left[\frac{x}{y}\left\{\frac{\cos x-\cos(x+y)}{\cos(x+y)\cos x}\right\}\right] + \lim_{y\to0}\sec(x+y)$

$= \lim_{y\to0}\left[\frac{x\,2\sin\left(x+\frac{y}{2}\right)\sin\left(\frac{y}{2}\right)}{y\cos(x+y)\cos x}\right]+\sec x$

$= \lim_{y\to0}\left[\frac{x\sin\left(x+\frac{y}{2}\right)}{\cos(x+y)\cos x}\times\frac{\sin\left(\frac{y}{2}\right)}{\frac{y}{2}}\right]+\sec x$

$= x\tan x\sec x+\sec x$

$= \sec x\,(x\tan x+1)$

32. a. $\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^m$

$= \lim_{m\to\infty}\left[1-\left(1-\cos\frac{x}{m}\right)\right]^m$

$= \lim_{m\to\infty}\left[1-2\sin^2\frac{x}{2m}\right]^m$

$= e^{\lim_{m\to\infty}\left(-2\sin^2\frac{x}{2m}\right)m} = 1$

33. b. $\mathrm{cosec}\frac{\pi x}{2}\to1$ when $x\to1 \Rightarrow \left[\mathrm{cosec}\frac{\pi x}{2}\right]=1$

$\therefore$ limit = 1

34. b. $\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)}$

$= \lim_{n\to\infty}\left(\frac{n(n-1)+1}{n(n-1)-1}\right)^{n(n-1)}$

$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n(n-1)}\right)^{n(n-1)}}{\left(1-\frac{1}{n(n-1)}\right)^{n(n-1)}} = \frac{e}{e^{-1}} = e^2$

35. b. $f(x) = \lim_{n\to\infty} n(x^{1/n} - 1)$

$= \lim_{n\to\infty} \frac{x^{1/n} - 1}{1/n}$

$= \lim_{m\to 0} \frac{x^m - 1}{m}$ (where $\frac{1}{n}$ replaced by m)

$= \ln x$

$\Rightarrow f(xy) = \ln(xy) = \ln x + \ln y = f(x) + f(y)$

36. c. If $f(x) = \sin\left(\frac{1}{x}\right)$ and $g(x) = \frac{1}{x}$, then both $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist, but $\lim_{x\to 0}\frac{f(x)}{g(x)} = 0$ exists.

37. a. $\lim_{n\to\infty} \frac{n\cdot 3^n}{n(x-2)^n + n\cdot 3^{n+1} - 3^n} = \frac{1}{3}$

$\Rightarrow \lim_{n\to\infty} \frac{1}{\frac{(x-2)^n}{3^n} + 3 - \frac{1}{n}}$ (Dividing N^r and D^r by $n \times 3^n$)

For $\lim_{n\to\infty}$ to be equal to 1/3

$\lim_{n\to\infty} \frac{1}{n} \to 0$ (which is true) and $\lim_{n\to\infty}\left(\frac{x-2}{3}\right)^n \to 0$

$\Rightarrow 2 \le x < 5$

38. b. $\lim_{x\to 2} \frac{2^x + 2^{3-x} - 6}{\sqrt{2^{-x}} - 2^{1-x}}$

$= \lim_{x\to 2} \frac{(2^x)^2 - 6\times 2^x + 2^3}{\sqrt{2^x} - 2}$ [Multiplying N^r and D^r by 2^x]

$= \lim_{x\to 2} \frac{(2^x - 4)(2^x - 2)(\sqrt{2^x} + 2)}{(\sqrt{2^x} - 2)(\sqrt{2^x} + 2)}$

$= \lim_{x\to 2} \frac{(2^x - 4)(2^x - 2)(\sqrt{2^x} + 2)}{(2^x - 4)}$

$= \lim_{x\to 2} (2^x - 2)(\sqrt{2^x} + 2) = (2^2 - 2)(2 + 2) = 8$

39. c. 1^∞ form

$L = e^{\lim_{n\to\infty} n\left(\left(\frac{n}{n+1}\right)^\alpha + \sin\frac{1}{n} - 1\right)} = e^{\lim_{n\to\infty} n\sin\frac{1}{n} + \lim_{n\to\infty} n\left(\left(\frac{n}{n+1}\right)^\alpha - 1\right)}$

Consider, $\lim_{n\to\infty} n\left(\left(\frac{n}{n+1}\right)^\alpha - 1\right) = \lim_{n\to\infty} n\left(\left(\frac{1}{1+1/n}\right)^\alpha - 1\right)$

Put $n = \frac{1}{y}$

$= \lim_{y\to 0} \frac{1}{y}\left(\left(\frac{1}{1+y}\right)^\alpha - 1\right) = \lim_{y\to 0} \frac{1-(1+y)^\alpha}{y} = -a$

(Using binomial)

$\therefore L = e^{1-\alpha}$

40. b. $L = \lim_{x\to\infty} \frac{\ln(x^2 + e^x)}{\ln(x^4 + e^{2x})} = \lim_{x\to\infty} \frac{\ln e^x\left(1 + \frac{x^2}{e^x}\right)}{\ln e^{2x}\left(1 + \frac{x^4}{e^{2x}}\right)}$

$= \lim_{x\to\infty} \frac{x + \ln\left(1 + \frac{x^2}{e^x}\right)}{2x + \ln\left(1 + \frac{x^4}{e^{2x}}\right)}$

$= \lim_{x\to\infty} \frac{1 + \frac{1}{x}\ln\left(1 + \frac{x^2}{e^x}\right)}{2 + \frac{1}{x}\ln\left(1 + \frac{x^4}{e^{2x}}\right)}$

Note that as $x\to\infty$ $\frac{x^2}{e^x} \to 0$ and as $x\to\infty$ $\frac{x^2}{e^{2x}} \to 0$

(Using L'Hopital's rule)

Hence $L = \frac{1}{2}$

41. a. $\lim_{x\to 1} \frac{1 + \sin\pi\left(\frac{3x}{1+x^2}\right)}{1 + \cos\pi x}$

$= \lim_{x\to 1} \frac{1 - \cos\left(\frac{3\pi}{2} - \frac{3\pi x}{1+x^2}\right)}{1 - \cos(\pi - \pi x)}$

$= \lim_{x\to 1} \frac{2\sin^2\left(\frac{3\pi}{4} - \frac{3\pi x}{2(1+x^2)}\right)}{2\sin^2\left(\frac{\pi}{2} - \frac{\pi x}{2}\right)}$

$= \lim_{x\to 1} \left(\frac{\frac{3\pi}{4} - \frac{3\pi x}{2(1+x^2)}}{\frac{\pi}{2} - \frac{\pi x}{2}}\right)^2$

$= \lim_{x\to 1} 9\left(\frac{\frac{1}{2} - \frac{x}{1+x^2}}{1-x}\right)^2 = \lim_{x\to 1} 9\left(\frac{x-1}{2(1+x^2)}\right)^2 = 0$

42. b. $\because \lim_{n\to\infty} \cos^{2n} x = \begin{cases} 1, & x = r\pi, r \in I \\ 0, & x \neq r\pi, r \in I \end{cases}$

Here, for $x = 10$, $\lim_{n\to\infty} \cos^{2n}(x-10) = 1$

and in all other cases it is zero.

$\therefore \lim_{n\to\infty} \sum_{x=1}^{\infty} \cos^{2n}(x-10) = 1$

43. b. $L = \lim_{x\to\infty} \dfrac{(2^{x^n})^{\frac{1}{e^x}} - (3^{x^n})^{\frac{1}{e^x}}}{x^n} = \lim_{x\to\infty} \dfrac{(3)^{\frac{x^n}{e^x}}\left(\left(\frac{2}{3}\right)^{\frac{x^n}{e^x}} - 1\right)}{x^n}$

Now, $\lim_{x\to\infty} \dfrac{x^n}{e^x} = \lim_{x\to\infty} \dfrac{n!}{e^x} = 0$ (differentiating numerator and denominator n times for L'Hopital's rule)

Hence $L = \lim_{x\to\infty} (3)^{\frac{x^n}{e^x}} \lim_{x\to\infty} \dfrac{\left(\left(\frac{2}{3}\right)^{\frac{x^n}{e^x}} - 1\right)}{\frac{x^n}{e^x}} \lim_{x\to\infty} \dfrac{1}{e^x}$

$= 1 \times \log(2/3) \times 0 = 0$

44. c. $\dfrac{\cos(2x-4)-33}{2} < f(x) < \dfrac{x^2|4x-8|}{x-2}$

$\Rightarrow \lim_{x\to 2^-} \dfrac{\cos(2x-4)-33}{2} < \lim_{x\to 2^-} f(x) < \lim_{x\to 2^-} \dfrac{x^2|4x-8|}{x-2}$

$\Rightarrow -16 < \lim_{x\to 2^-} f(x) < \lim_{x\to 2^-} \dfrac{x^2(8-4x)}{x-2}$

$\Rightarrow -16 < \lim_{x\to 2^-} f(x) < -16$

$\Rightarrow \lim_{x\to 2^-} f(x) = -16$ (by sandwich theorem)

45. a. Given $g(x) = \lim_{n\to\infty} \dfrac{1}{\left(\frac{3}{\pi}\tan^{-1} 2x\right)^{2n} + 5} = 0$

$\Rightarrow \left[\left(\frac{3}{\pi}\tan^{-1} 2x\right)^2\right]^n \to \infty$

$\Rightarrow \left(\frac{3}{\pi}\tan^{-1} 2x\right)^2 > 1$

$\Rightarrow |\tan^{-1} 2x| > \dfrac{\pi}{3}$

$\Rightarrow \tan^{-1} 2x < -\dfrac{\pi}{3}$ or $\tan^{-1} 2x > \dfrac{\pi}{3}$

$\Rightarrow 2x < -\sqrt{3}$ or $2x > \sqrt{3} \Rightarrow |2x| > \sqrt{3}$

46. a. $(1+x)^{2/x} = (1+x)^{2/x} - [(1+x)^{2/x}]$

Now, $\lim_{x\to 0}(1+x)^{2/x} = e^2$

$\Rightarrow \lim_{x\to 0}\{(1+x)^{2/x}\} = e^2 - [e^2] = e^2 - 7$

47. b. $\lim_{x\to 0} \dfrac{\sin(x^2)}{\ln(\cos(2x^2 - x))}$

$= \lim_{x\to 0} \dfrac{\sin(x^2)}{\log\left(1 - 2\sin^2\left(\frac{2x^2 - x}{2}\right)\right)}$

$= \lim_{x\to 0} \dfrac{\sin(x^2)\, x^2}{x^2 \dfrac{\log\left(1 - 2\sin^2\left(\frac{2x^2-x}{2}\right)\right)}{-2\sin^2\left(\frac{2x^2-x}{2}\right)}\left[-2\sin^2\left(\frac{2x^2-x}{2}\right)\right]}$

$= \lim_{x\to 0} - \dfrac{x^2}{2\dfrac{\sin^2\left(\frac{2x^2-x}{2}\right)}{\left(\frac{2x^2-x}{2}\right)^2}\left(\frac{2x^2-x}{2}\right)^2}$

$= \lim_{x\to 0} - \dfrac{2x^2}{(2x^2-x)^2} = \lim_{x\to 0} - \dfrac{2}{(2x-1)^2} = -2$

48. a. L.H.L. $= \lim_{x\to -1^-} \dfrac{1}{\sqrt{|x| - \{-x\}}} = \lim_{x\to -1^-} \dfrac{1}{\sqrt{-x-(x+2)}}$

$= \lim_{x\to -1^-} \dfrac{1}{\sqrt{-2x-2}} = \infty$

R.H.L. $= \lim_{x\to -1^+} \dfrac{1}{\sqrt{|x| - \{-x\}}} = \lim_{x\to -1^-} \dfrac{1}{\sqrt{-x-(x+1)}}$

$= \lim_{x\to -1^-} \dfrac{1}{\sqrt{-2x-1}} = 1$

Hence, the limit does not exist.

49. a. For $n > 1$,

$\lim_{x\to 0} x^n \sin(1/x^2) = 0 \times$ (any value between – 1 to 1) = 0

For $n < 0$,

$\lim_{x\to 0} x^n \sin(1/x^2) = \infty \times$ (any value between – 1 to 1) = ∞

50. c. $\lim_{x\to 1} \dfrac{p - q + qx^p - px^q}{1 - x^p - x^q + x^{p+q}} \left(\dfrac{0}{0}\right)$

$= \lim_{x\to 1} \dfrac{pqx^{p-1} - pqx^{q-1}}{-px^{p-1} - qx^{q-1} + (p+q)x^{p+q-1}} \left(\dfrac{0}{0}\right)$ (L' Hopital Rule)

$= \lim_{x\to 1} \dfrac{pq(p-1)x^{p-2} - pq(q-1)x^{q-2}}{-p(p-1)x^{p-2} - q(q-1)x^{q-2} + (p+q)(p+q-1)x^{p+q-2}}$ (L' Hopital rule)

$= \dfrac{p-q}{2}$

51. b. $\lim_{x\to -1} \left(\dfrac{x^4 + x^2 + x + 1}{x^2 - x + 1}\right)^{\frac{1-\cos(x+1)}{(x+1)^2}}$

$= \left(\lim_{x\to -1} \dfrac{x^4 + x^2 + x + 1}{x^2 - x + 1}\right)^{\lim_{x\to -1} \frac{1-\cos(x+1)}{(x+1)^2}} = \left(\dfrac{2}{3}\right)^{\lim_{x\to -1} \frac{\sin(x+1)}{2(x+1)}} = \left(\dfrac{2}{3}\right)^{\frac{1}{2}}$

52. a. $\lim_{x\to 2}\left[\left(\frac{x^3-4x}{x^3-8}\right)^{-1}-\left(\frac{\sqrt{x}(\sqrt{x}+\sqrt{2})}{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})}-\frac{\sqrt{2}}{\sqrt{x}-\sqrt{2}}\right)^{-1}\right]$

$= \lim_{x\to 2}\left[\frac{x^2+2x+4}{x(x+2)}-\left(\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x}-\sqrt{2}}\right)^{-1}\right]$

$= \lim_{x\to 2}\left[\frac{x^2+2x+4}{x(x+2)}-1\right]=\frac{12}{8}-1=\frac{1}{2}$

53. d. $\lim_{x\to\infty}\frac{e^{1/x^2}-1}{2\tan^{-1}(x^2)-\pi}$

$= \lim_{t\to 0^+}\frac{e^{t^2}-1}{2\cot^{-1}t^2-\pi}$

$= \lim_{t\to 0^+}\frac{e^{t^2}-1}{-2\tan t^2}$

$= \lim_{t\to 0^+}-\frac{1}{2}\frac{\frac{e^{t^2}-1}{t^2}}{\frac{\tan t^2}{t^2}}=-\frac{1}{2}$

54. b. $\lim_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3}$

$= \lim_{x\to 0}\frac{1+\left(x-\frac{x^3}{3!}+\cdots\right)-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)+\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots\right)}{x^3}$

$= -\frac{1}{3!}-\frac{1}{3}=-\frac{1}{2}$

55. b. $\cos(\tan x)-\cos x = 2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)$

$\Rightarrow \lim_{x\to 0}\frac{\cos(\tan x)-\cos x}{x^4}=\lim_{x\to 0}\frac{2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)}{x^4}$

$= \lim_{x\to 0}\frac{2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)}{x^4\left(\frac{x+\tan x}{2}\right)\left(\frac{x-\tan x}{2}\right)}\left(\frac{x^2-\tan^2 x}{4}\right)$

$= \frac{1}{2}\lim_{x\to 0}\frac{x^2-\tan^2 x}{x^4}$

$= \frac{1}{2}\lim_{x\to 0}\frac{x^2-\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots\right)^2}{x^4}$

$= \frac{1}{2}\lim_{x\to 0}\frac{1}{x^2}\left(1-\left(1+\frac{x^2}{3}+\frac{2}{15}x^4+\cdots\right)^2\right)=-\frac{1}{3}$

56. b. $x_{n+1}=\sqrt{2+x_n}$

$\Rightarrow \lim_{n\to\infty}x_{n+1}=\sqrt{2+\lim_{n\to\infty}x_n}$

$\Rightarrow t=\sqrt{2+t}$ $\quad\left(\because \lim_{x\to\infty}x_{n+1}=\lim_{x\to\infty}x_n=t\right)$

$\Rightarrow t^2-t-2=0$

$\Rightarrow (t-2)(t+1)=0$

$\Rightarrow t=2$ $\quad(\because x_n>0\ \forall\ n \therefore t>0)$

57. b. $\lim_{x\to 0}\left(\frac{1^x+2^x+\cdots+n^x}{n}\right)^{1/x}$

$= e^{\lim_{x\to 0}\left(\frac{1^x-1}{n}+\frac{2^x-1}{n}+\cdots+\frac{n^x-1}{n}\right)\frac{1}{x}}$

$= e^{\lim_{x\to 0}\frac{1}{n}\left\{\frac{1^x-1}{x}+\frac{2^x-1}{x}+\cdots+\frac{n^x-1}{x}\right\}}$

$= e^{\frac{1}{n}[\log 1+\log 2+\cdots+\log n]}$

$= e^{\frac{1}{n}(\log n!)}=e^{\log(n!)^{\frac{1}{n}}}=(n!)^{\frac{1}{n}}$

58. c. $\lim_{x\to 0}\frac{a^{\sqrt{x}}-a^{1/\sqrt{x}}}{a^{\sqrt{x}}+a^{1/\sqrt{x}}},a>1$

Put $x=t^2$

$\therefore\ \lim_{t\to 0}\frac{a^t-a^{1/t}}{a^t+a^{1/t}}$

$\Rightarrow\ \lim_{t\to 0}\frac{a^{t-1/t}-1}{a^{t-1/t}+1}=\frac{a^{-\infty}-1}{a^{-\infty}+1}=\frac{0-1}{0+1}=-1$

59. a.i. $\lim_{x\to\infty}\sec^{-1}\left(\frac{x}{\sin x}\right)$

$= \sec^{-1}\left(\frac{\infty}{\sin\infty}\right)$

$= \sec^{-1}\left(\frac{\infty}{\text{any value between }-1\text{ to }1}\right)$

$= \sec^{-1}(\pm\infty)=\frac{\pi}{2}$

ii. $\lim_{x\to\infty}\sec^{-1}\left(\frac{\sin x}{x}\right)=\sec^{-1}\left(\frac{\sin\infty}{\infty}\right)$

$\sec^{-1}\left(\frac{\text{any value between }-1\text{ to }1}{\infty}\right)$

$\sec^{-1}0=$ not defined

Hence (i) exists but (ii) does not exist.

60. b. For $n=0$, we have $\lim_{x\to 0}\frac{1-\sin 1}{x-1}=\sin 1-1$

For $n=1$, $\lim_{x\to 0}\frac{x-\sin x}{x-\sin x}=1$

For $n=2$, $\lim_{x\to 0}\frac{x^2-\sin^2 x}{x-\sin^2 x}=\lim_{x\to 0}\frac{1-\frac{\sin^2 x}{x^2}}{\frac{1}{x}-\frac{\sin^2 x}{x^2}}$

This does not exist.

For $n=3$ also given limit does not exist.

Hence $n=0$ or 1.

61. c. $\lim_{x\to -2^-}\frac{ae^{1/|x+2|}-1}{2-e^{1/|x+2|}}=\lim_{x\to -2^-}\frac{a-e^{-1/|x+2|}}{2e^{-1/|x+2|}-1}=-a$

$$\lim_{x\to -2^-}\sin\left(\frac{x^4-16}{x^5+32}\right)=\lim_{x\to -2^-}\sin\left(\frac{\frac{x^4-(-2)^4}{x-(-2)}}{\frac{x^5-(-2)^5}{x-(-2)}}\right)$$

$$=\sin\left(-\frac{2}{5}\right)\Rightarrow a=\sin\frac{2}{5}$$

62. b. Given limit is $\lim_{x\to\infty}(x+1)[\tan^{-1}(x+5)-(x+1)]+4\tan^{-1}(x+5)$

$$=\lim_{x\to\infty}\left[(x+1)\tan^{-1}\frac{4}{1+(x+1)(x+5)}+4\tan^{-1}(x+5)\right]$$

$$=\lim_{x\to\infty}\left[(x+1)\frac{\tan^{-1}\frac{4}{x^2+6x+6}}{\left(\frac{4}{x^2+6x+6}\right)}\times\frac{4}{x^2+6x+6}+4\tan^{-1}(x+5)\right]$$

$$=0+4\times\frac{\pi}{2}=2\pi$$

63. b. $\lim_{x\to 1}\frac{(1-x)(1-x^2)\cdots(1-x^{2n})}{\{(1-x)(1-x^2)\cdots(1-x^n)\}^2}$

$$=\lim_{x\to 1}\frac{\left(\frac{1-x}{1-x}\right)\left(\frac{1-x^2}{1-x}\right)\cdots\left(\frac{1-x^{2n}}{1-x}\right)}{\left(\left(\frac{1-x}{1-x}\right)\left(\frac{1-x^2}{1-x}\right)\cdots\left(\frac{1-x^n}{1-x}\right)\right)^2}$$

$$=\frac{1\times 2\times 3\cdots(2n)}{(1\times 2\times 3\cdots n)^2}=\frac{(2n)!}{n!n!}={}^{2n}C_n$$

64. b. We know that $\lim_{x\to 0}\frac{\sin x}{x}\to 1^-$ and $\lim_{x\to 0}\frac{x}{\sin x}\to 1^+$

So, $\lim_{x\to 0}\left[100\frac{x}{\sin x}\right]+\lim_{x\to 0}\left[99\frac{\sin x}{x}\right]$

$=100+98=198$

65. a. Let $\sin^{-1}x=\theta$. Then, $x=\sin\theta$.

Now, $x\to\frac{1}{\sqrt{2}}\Rightarrow\sin\theta\to\frac{1}{\sqrt{2}}\Rightarrow\theta\to\frac{\pi}{4}$

$$\therefore\ \lim_{x\to\frac{1}{\sqrt{2}}}\frac{x-\cos\left(\sin^{-1}x\right)}{1-\tan\left(\sin^{-1}x\right)}$$

$$=\lim_{\theta\to\frac{\pi}{4}}\frac{\sin\theta-\cos\theta}{1-\tan\theta}$$

$$=\lim_{\theta\to\frac{\pi}{4}}\frac{(\sin\theta-\cos\theta)}{(\cos\theta-\sin\theta)}\cos\theta$$

$$=\lim_{\theta\to\frac{\pi}{4}}-\cos\theta=-\frac{1}{\sqrt{2}}$$

66. d. We have $\lim_{x\to 1}\frac{1-\sqrt{x}}{\left(\cos^{-1}x\right)^2}$

$$=\lim_{x\to 1}\frac{\left(1-\sqrt{x}\right)\left(1+\sqrt{x}\right)}{\left(\cos^{-1}x\right)^2\left(1+\sqrt{x}\right)}$$

$$=\lim_{x\to 1}\frac{1-x}{\left(\cos^{-1}x\right)^2\left(1+\sqrt{x}\right)}$$

$$=\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2\left(1+\sqrt{\cos\theta}\right)}1,\text{ where } x=\cos\theta$$

$[\because x\to 1\Rightarrow\cos\theta\to 1\Rightarrow\theta\to 0]$

$$=\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}\frac{1}{\left(1+\sqrt{\cos\theta}\right)}$$

$$=\lim_{\theta\to 0}\frac{2\sin^2\frac{\theta}{2}}{4\frac{\theta^2}{4}}\left(\frac{1}{1+\sqrt{\cos\theta}}\right)$$

$$=\frac{1}{2}\lim_{\theta\to 0}\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2\frac{1}{\left(1+\sqrt{\cos\theta}\right)}=\frac{1}{2}(1)^2\frac{1}{(1+1)}=\frac{1}{4}$$

67. b. $\min(y^2-4y+11)=\min[(y-2)^2+7]=7$

$$\Rightarrow L=\lim_{x\to 0}\left[\min(y^2-4y+11)\frac{\sin x}{x}\right]$$

$$=\lim_{x\to 0}\left[\frac{7\sin x}{x}\right]$$

$=$ [a value slightly lesser than 7] $\quad$ ($|\sin x|<|x|$, when $x\to 0$)

$$\Rightarrow L=\lim_{x\to 0}\left[7\frac{\sin x}{x}\right]=6$$

68. b. $L=\lim_{x\to\pi/2}\frac{\sin(x\cos x)}{\sin\left(\frac{\pi}{2}-x\sin x\right)}$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\sin(x\cos x)}{(x\cos x)}\frac{x\cos x}{\sin\left(\frac{\pi}{2}-x\sin x\right)}\frac{\left(\frac{\pi}{2}-x\sin x\right)}{\left(\frac{\pi}{2}-x\sin x\right)}$$

$$=1\times 1\lim_{x\to\pi/2}\frac{x\cos x}{\left(\frac{\pi}{2}-x\sin x\right)}$$

Put $x = \pi/2 + h$

Then, $L = \lim\limits_{h\to 0} \dfrac{\left(\frac{\pi}{2}+h\right)\cos\left(\frac{\pi}{2}+h\right)}{\frac{\pi}{2}-\left(\frac{\pi}{2}+h\right)\sin\left(\frac{\pi}{2}+h\right)}$

$= \lim\limits_{h\to 0} \dfrac{-\left(\frac{\pi}{2}+h\right)\sin h}{\frac{\pi}{2}(1-\cos h) - h\cos h}$

$= \lim\limits_{h\to 0} \dfrac{-\left(\frac{\pi}{2}+h\right)\left(\frac{\sin h}{h}\right)}{\frac{\pi}{2}\frac{(1-\cos h)}{h} - \cos h}$ (Divide N^r and D^r by h)

$= \dfrac{-\left(\frac{\pi}{2}+0\right)1}{0-1} = \dfrac{\pi}{2}$

69. a. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x^3} + \dfrac{a}{x^2} + b$

$= \lim\limits_{x\to 0} \dfrac{\sin 3x + ax + bx^3}{x^3}$

$= \lim\limits_{x\to 0} \dfrac{3\frac{\sin 3x}{3x} + a + bx^2}{x^2}$

For existence, $(3+a) = 0$

$\Rightarrow a = -3$

$\therefore\ L = \lim\limits_{x\to 0} \dfrac{\sin 3x - 3x + bx^3}{x^3}$

$= 27 \lim\limits_{t\to 0} \dfrac{\sin t - t}{t^3} + b = 0\ (3x = t)$

$= -\dfrac{27}{6} + b = 0$

$\Rightarrow b = \dfrac{9}{2}$

70. b. $\lim\limits_{x\to 0} \dfrac{x^n \sin^n x}{x^n - \sin^n x}$

$\Rightarrow \lim\limits_{x\to 0} \dfrac{x^n\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^n}{x^n - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^n}$

$= \lim\limits_{x\to 0} \dfrac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^n}{1 - \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)^n}$

$= \lim\limits_{x\to 0} \dfrac{x^n\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)^n}{1 - \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)^n}$

For $n = 2$,

$\lim\limits_{x\to 0} \dfrac{x^2\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)^2}{1 - \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)^2}$

$\Rightarrow \lim\limits_{x\to 0} \dfrac{x^2\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)^2}{\left(2 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \cdots\right)}$

$= \dfrac{1(1-0+\cdots)^2}{(2-0+0)\left(\frac{1}{3!} - 0 + \cdots\right)}$

$= 3$

71. d. $\lim\limits_{x\to\infty} \dfrac{1+x+x^2}{x(\ln x)^3} = \lim\limits_{t\to 0^+} \dfrac{t^2+t+1}{t^2\frac{1}{t}\left(\ln\left(\frac{1}{t}\right)\right)^3}$

$= \lim\limits_{t\to 0^+} \dfrac{1+t+t^2}{-t(\ln t)^3} = +\infty$

72. c. $\lim\limits_{x\to 0} \dfrac{(2^m+x)^{1/m} - (2^n+x)^{1/n}}{x}$

$= \lim\limits_{x\to 0} \dfrac{(2^m+x)^{1/m} - 2}{x} - \lim\limits_{x\to 0} \dfrac{(2^n+x)^{1/n} - 2}{x}$

$= \lim\limits_{a\to 2} \dfrac{a-2}{a^m - 2^m} - \lim\limits_{b\to 2} \dfrac{b-2}{b^n - 2^n}$

[Putting $2^m + x = a^m$ and $2^n + x = b^n$]

$= \dfrac{1}{m2^{m-1}} - \dfrac{1}{n2^{n-1}}$

73. a $\lim\limits_{x\to 0^+}\left[\left(1-e^x\right)\dfrac{\sin x}{|x|}\right]$

$= \lim\limits_{x\to 0^+}\left[(0^-)\dfrac{\sin x}{x}\right] = [0^-] = -1.$

$= \lim\limits_{x\to 0^-}\left[\left(1-e^x\right)\dfrac{\sin x}{|x|}\right]$

$= \lim\limits_{x\to 0^-}\left[(0^+)\dfrac{\sin x}{-x}\right] = [0^-] = -1$

Hence $\lim\limits_{x\to 0}\left[\left(1-e^x\right)\dfrac{\sin x}{|x|}\right] = -1$

74. c As $x \to 0^- \Rightarrow f(x) \to f(0^-) = 2^+$

$\Rightarrow \lim\limits_{x\to 0^-} g(f(x)) = g(2^+) = -3$

Also as $x \to 0^+ \Rightarrow f(x) \to f(0^+) = 1^+$

$\Rightarrow \lim_{x\to 0^+} g(f(x)) = g(1^+) = -3$

Hence $\lim_{x\to 0} g(f(x))$ exists and is equal to -3

$\Rightarrow \lim_{x\to 0} g(f(x)) = -3$

75. c $I = \lim_{x\to 1} \dfrac{nx^n(x-1)-(x^n-1)}{(e^x-e)\sin \pi x}$

put $x = 1+h$ so that as $x \to 1$, $h \to 0$

$$\therefore\ I = -\lim_{h\to 0} \frac{h\cdot n(1+h)^n - ((1+h)^n - 1)}{e(e^h-1)\sin \pi h}$$

$$I = -\lim_{x\to 1} \frac{n\cdot h(1+{}^nC_1 h + {}^nC_2 h^2 + {}^nC_3 h^3 + \cdots)}{\pi e(h^2)\left(\dfrac{e^h-1}{h}\right)}$$

$$\frac{-(1+{}^nC_1 h + + {}^nC_2 h^2 + + {}^nC_3 h^3 + \cdots - 1)}{\left(\dfrac{\sin \pi h}{\pi h}\right)}$$

$$= -\frac{n^2 - {}^nC_2}{\pi e} = -\left[\frac{2n^2 - n(n-1)}{2\pi e}\right] = -\frac{n^2+n}{2(\pi e)} = -\frac{n(n+1)}{2(\pi e)}$$

if $n = 100 \Rightarrow \text{I} = -\left(\dfrac{5050}{\pi e}\right)$

76. c $\lim_{n\to\infty}\left[\dfrac{1}{n} + \dfrac{e^{1/n}}{n} + \dfrac{e^{2/n}}{n} + \cdots + \dfrac{e^{(n-1)/n}}{n}\right]$

$$= \lim_{n\to\infty}\left[\frac{1+e^{1/n}+(e^{1/n})^2+\cdots+(e^{1/n})^{n-1}}{n}\right]$$

$$= \lim_{n\to\infty} \frac{1.\left[(e^{1/n})^n - 1\right]}{n(e^{1/n}-1)} = (e-1)\lim_{n\to\infty}\frac{1}{\left(\dfrac{e^{1/n}-1}{1/n}\right)}$$

$= (e-1)\times 1 = (e-1)$.

77. c $\lim_{n\to\infty}\left[\dfrac{2}{2-\dfrac{1}{n^2}}\cdot\dfrac{1}{n}\cos\left(\dfrac{1+1/n}{2-1/n}\right) - \dfrac{1}{\left(\dfrac{1}{n}-2\right)}\cdot\dfrac{(-1)^n}{\left(1+\dfrac{1}{n^2}\right)}\cdot\dfrac{1}{n}\right]$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\frac{2}{2-\dfrac{1}{n^2}}\cdot\cos\left(\frac{1+\dfrac{1}{n}}{2-\dfrac{1}{n}}\right) - \frac{1}{\left(\dfrac{1}{n}-2\right)}\cdot\frac{(-1)^n}{\left(1+\dfrac{1}{n^2}\right)}\right]$$

$$= 0\times\left[\frac{2}{2}\times\cos\frac{1}{2}+\frac{1}{2}\times\frac{1}{1}\right] = 0.$$

78. b $\lim_{x\to 0}\dfrac{\log(1+x+x^2)+\log(1-x+x^2)}{\sec x - \cos x}$

$$= \lim_{x\to 0}\frac{\log\left[(1+x^2)^2 - x^2\right]}{(1-\cos^2 x)/\cos x}$$

$$= \lim_{x\to 0}\frac{\log(1+x^2+x^4)}{\sin x\tan x}$$

$$= \lim_{x\to 0}\frac{\log(1+x^2(1+x^2))}{x^2(1+x^2)}\cdot x^2(1+x^2)\cdot\frac{1}{\dfrac{\sin x}{x}\cdot\dfrac{\tan x}{x}\cdot x^2}$$

$$= 1\cdot\left(\text{as } \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right)$$

79. c $\lim_{x\to a}\sqrt{a^2-x^2}\cot\dfrac{\pi}{2}\sqrt{\dfrac{a-x}{a+x}}$

$$= \lim_{x\to a}\frac{\sqrt{a^2-x^2}}{\tan\dfrac{\pi}{2}\sqrt{\dfrac{a-x}{a+x}}}$$

$$= \frac{2}{\pi}\lim_{x\to a}\frac{\dfrac{\pi}{2}\sqrt{\dfrac{a-x}{a+x}}}{\tan\dfrac{\pi}{2}\sqrt{\dfrac{a-x}{a+x}}}(a+x) = \frac{4a}{\pi}$$

80. b $\lim_{x\to\infty}\dfrac{\cot^{-1}(x^{-a}\log_a x)}{\sec^{-1}(a^x\log_x a)}\ (a>1)$

$$= \lim_{x\to\infty}\frac{\cot^{-1}\left(\dfrac{\log_a x}{x^a}\right)}{\sec^{-1}\left(\dfrac{a^x}{\log_a x}\right)} \quad \text{as } \lim_{x\to\infty}\left(\frac{\log_a x}{x^a}\right)\to 0$$

and $\left(\dfrac{a^x}{\log_a x}\right)\to\infty$ (using L'Hopital rule)

$$\therefore\ I = \frac{\pi/2}{\pi/2} = 1$$

Multiple Correct Answers Type

1. b, c.

R.H. limit $= \lim_{h\to 0} f(1+h) = \lim_{h\to 0} a(1+h) = a$

L.H. limit $= \lim_{h\to 0} f(1+h) = \lim_{h\to 0} a\left\{1+\dfrac{2}{a}(1+h)\right\} = 1+\dfrac{2}{a}$

$\lim_{x\to 1} f(x)$ exists $\Rightarrow$ R.H. limit $=$ L.H. limit $\Rightarrow a = 1+\dfrac{2}{a}$

$\Rightarrow a = 2, -1$

2. a, d.

$f(1+0)=\lim_{h\to0}\{|1+h-1|-[1+h]\}=\lim_{h\to0}\{h-1\}=-1$

$f(1-0)=\lim_{h\to0}\{|1-h-1|-[1-h]\}=\lim_{h\to0}\{h-0\}=0$

3. a, c.

$$\text{Limit}=\lim_{n\to\infty}\frac{an(1+n)-(1+n^2)}{1+n}$$

$$=\lim_{n\to\infty}\frac{(a-1)n^2+an-1}{n+1}$$

$=\infty$ if $a-1\neq0$

If $a-1=0$, limit $=\lim_{n\to\infty}\dfrac{an-1}{n+1}=a=b$

$\therefore\quad a=b=1$

4. a, b, c.

$$L=\lim_{x\to0}\frac{\sin x^n}{(\sin x)^m}=\lim_{x\to0}\frac{\dfrac{\sin x^n}{x^n}x^n}{\dfrac{(\sin x)^m}{x^m}x^m}=\lim_{x\to0}x^{n-m}$$

If $n=m$, then

$L=$ (a very small value near to zero)$^{\text{exactly zero}}=1$

If $n>m$, then

$L=$ (a very small value near to zero)$^{\text{positive integer}}=0$

If $n<m$, then

$L=$ (a very small value near to zero)$^{\text{negative integer}}=\infty$

5. a, b, c.

$$\lim_{x\to\infty}\frac{\log_e x}{\{x\}}=\frac{\text{positive infinity}}{\text{a value between 0 and 1}}=\infty$$

$$\lim_{x\to2^+}\frac{x}{x^2-x-2}=\lim_{x\to2^+}\frac{x}{(x-2)(x+1)}$$

$$=\lim_{h\to0}\frac{2+h}{h(3+h)}=\infty$$

$$\lim_{h\to-1}\frac{x}{x^2-x-2}=-\lim_{h\to-1^-}\frac{x}{(x-2)(x+1)}$$

$$=\lim_{h\to0}\frac{-1-h}{(-3-h)(-h)}=\lim_{h\to0}\frac{1+h}{(3+h)(h)}=-\infty$$

6. b, c.

Since the greatest integer function is discontinuous (sensitive) at integral values of x, then for a given limit to exist both left- and right-hand limit must be equal.

L.H.L. $=\lim_{x\to1^-}(2-x+a[x-1]+b[1+x])$

$=2-1+a(-1)+b(1)=1-a+b$

R.H.L. $=\lim_{x\to1^+}(2-x+a[x-1]+b[1+x])$

$=2-1+a(0)+b(2)=1+2b$

On comparing we have $-a=b$

7. a, b, c.

$$L=\lim_{x\to a}\frac{|2\sin x-1|}{2\sin x-1}$$

For $a=\pi/6$, L.H.L. $=\lim_{x\to\frac{\pi}{6}^-}\dfrac{1-2\sin x}{2\sin x-1}=-1$,

R.H.L. $=\lim_{x\to\frac{\pi}{6}^+}\dfrac{2\sin x-1}{2\sin x-1}=1$

Hence the limit does not exist.

For $a=\pi$, $\lim_{x\to\pi}\dfrac{1-2\sin x}{2\sin x-1}=-1$ (as in neighbourhood of π, $\sin x$ is less than ½).

For $a=\pi$, $\lim_{x\to\pi/2}\dfrac{2\sin x-1}{2\sin x-1}=1$ (as in neighbourhood of $\pi/2$, $\sin x$ approaches to 1).

8. b, c, d.

$$f(x)=\lim_{x\to\infty}\frac{x}{x^{2n}+1}$$

$$=\begin{cases}x, & x^2<1\\ 0, & x^2>1\\ 1/2, & x=1\\ -1/2, & x=-1\end{cases}$$

$\Rightarrow f(1^+)=f(-1^-)=0$

$f(1^-)=1,\ f(-1^+)=-1$

$f(1)=1/2$

9. a, c.

$$\lim_{n\to\infty}\frac{-3+\dfrac{(-1)^n}{n}}{4+\dfrac{(-1)^n}{n}}=\frac{-3}{4}$$

10. a, b, c, d.

We have $\lim_{x\to0^+}f(x)=\lim_{k\to0^+}\dfrac{\tan^2\{x\}}{(x^2-[x]^2)}$

$$=\lim_{x\to0^+}\frac{\tan^2 x}{x^2}=1\qquad(1)$$

$(\because\ x\to0^+;[x]=0\Rightarrow\{x\}=x)$

Also $\lim_{k\to0^-}f(x)=\lim_{x\to0^-}\sqrt{\{x\}\cot\{x\}}=\sqrt{\cot1}\qquad(2)$

$(\because\ x\to0^-;[x]=-1\Rightarrow\{x\}=x+1\Rightarrow\{x\}\to1)$

Also, $\cot^{-1}\left(\lim_{x\to0^-}f(x)\right)^2=\cot^{-1}(\cot1)=1.$

11. a, b, c, d

$$f(x)=\frac{3x^2+ax+a+1}{(x+2)(x-1)}$$

as $x\to1$, $D^r\to0$, hence as $x\to1$, $N^r\to0$

$\therefore\ 3+2a+1=0\ \Rightarrow\ a=-2\Rightarrow$(A)

as $x\to-2$, $D^r\to0$, hence as $x\to-2$, $N^r\to0$

$\therefore\ 12-2a+a+1=0\ \Rightarrow\ a=13\Rightarrow$(B)

Now $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 1} \dfrac{3x^2-2x-1}{(x+2)(x-1)} = \lim\limits_{x\to 1} \dfrac{(3x+1)(x-1)}{(x+2)(x-1)} = \dfrac{4}{3}$

now $\lim\limits_{x\to -2} \dfrac{3x^2+13x+14}{(x+2)(x-1)} = \lim\limits_{x\to -2} \dfrac{(3x+7)(x+2)}{(x+2)(x-1)} = -\dfrac{1}{3}$

12. b, c

Case I $x \neq m\pi$ (m is an integer)

$\lim\limits_{x\to\infty} \dfrac{1}{1+n\sin^2 nx} = \dfrac{1}{\infty} = 0$

Case II $x = m\pi$ (m is an integer)

$\lim\limits_{n\to\infty} \dfrac{1}{1+n\sin^2 nx} = \dfrac{1}{1} = 1.$

13. a, b, c

$= \lim\limits_{x\to 5^-} \dfrac{x^2-9x+20}{x-[x]} = \lim\limits_{x\to 5^-} \dfrac{(x-5)(x-4)}{x-4} = \lim\limits_{x\to 5^-}(x-5) = 0$

$= \lim\limits_{x\to 5^+} \dfrac{x^2-9x+20}{x-[x]} = \lim\limits_{x\to 5^+} \dfrac{(x-5)(x-4)}{x-5} = \lim\limits_{x\to 5^+}(x-4) = 1$

Hence limit does not exist.

14. a, c

Since $x^2 > 0$ and limit equals 2, $f(x)$ must be a positive quantity. Also since $\lim\limits_{x\to 0} \dfrac{f(x)}{x^2} = 2$. The denominator $\to$ zero and limit is finite, therefore $f(x)$ must be approaching to zero or $\lim\limits_{x\to 0}[f(x)] = 0^+$

Hence $\lim\limits_{x\to 0}[f(x)] = 0$

$\lim\limits_{x\to 0^+}\left[\dfrac{f(x)}{x}\right] = \lim\limits_{x\to 0^+}\left[x\dfrac{f(x)}{x^2}\right] = 0$ and $\lim\limits_{x\to 0^-}\left[\dfrac{f(x)}{x}\right]$

$= \lim\limits_{x\to 0^-}\left[x\dfrac{f(x)}{x^2}\right] = -1$

Hence $\lim\limits_{x\to 0}\left[\dfrac{f(x)}{x}\right]$ does not exist.

Reasoning Type

1. b. For $x \in (-\delta, \delta)$, $\sin x < x \Rightarrow \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1^-$

$\Rightarrow \left[\lim\limits_{x\to 0} \dfrac{\sin x}{x}\right] = 0$

Also, $x \in (-\delta, \delta)$, $\tan x > x$, but from this nothing can be said about the relation between $\sin x$ and x.

Hence, both the statements are true but statement 2 is not the correct explanation of statement 1.

2. a. For $\lim\limits_{x\to\alpha} \dfrac{\sin(f(x))}{x-\alpha}$, denominator tends to 0; hence the numerator must also tend to 0 for limit to be finite. Then, α is a root of the equation $ax^2+bx+c=0$ or $f(\alpha)=0$.

Also, consider $f(\alpha^+) \to 0^+$ and $f(\alpha^-) \to 0^-$

$\Rightarrow \lim\limits_{x\to\alpha^+} \dfrac{e^{1/f(x)}-1}{e^{1/f(x)}+1} = \lim\limits_{x\to\alpha^+} \dfrac{1-e^{-1/f(x)}}{1+e^{-1/f(x)}} = 1$

and $\lim\limits_{x\to\alpha^-} \dfrac{e^{1/f(x)}-1}{e^{1/f(x)}+1} = -1$

Thus, both the statements are true and statement 2 is the correct explanation of statement 1.

3. b. Limit of function $y = f(x)$ exists at $x = a$, though it is discontinuous at $x = a$. Consider the function $f(x)$ $= \dfrac{x^2-4}{x-2}$. Here, $f(x)$ is not defined at $x = 2$, but limit of functions exists, as $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} = 4.$

4. a. $L = \lim\limits_{x\to 0^+} \dfrac{x}{a}\left[\dfrac{b}{x}\right]$

$= \lim\limits_{x\to 0^+} \dfrac{x}{a}\left(\dfrac{b}{x} - \left\{\dfrac{b}{x}\right\}\right)$

$= \lim\limits_{x\to 0^+}\left(\dfrac{b}{a} - \dfrac{x}{a}\left\{\dfrac{b}{x}\right\}\right)$

$= \dfrac{b}{a} - \dfrac{b}{a}\lim\limits_{x\to 0^+} \dfrac{\left\{\frac{b}{x}\right\}}{\frac{b}{x}}$

$= \dfrac{b}{a} - \dfrac{b}{a}\lim\limits_{y\to\infty} \dfrac{\{y\}}{y}$ (where $y = \dfrac{b}{x}$ and $b > 0$) $= \dfrac{b}{a}$

Also, if $b < 0$, $L = \dfrac{b}{a} - \dfrac{b}{a}\lim\limits_{y\to-\infty} \dfrac{\{y\}}{y} = \dfrac{b}{a}$

5. d. $\lim\limits_{x\to\infty}\left(\dfrac{1^2}{x^3} + \dfrac{2^2}{x^3} + \dfrac{3^2}{x^3} + \cdots + \dfrac{x^2}{x^3}\right)$

$= \lim\limits_{x\to\infty} \dfrac{x(x+1)(2x+1)}{6x^3} = \dfrac{1}{3}$

6. b. $L = \lim\limits_{x\to 0} \dfrac{\sqrt{1-\cos 2x}}{x} = \lim\limits_{x\to 0} \dfrac{\sqrt{2}\,|\sin x|}{x}$

$\Rightarrow$ L.H.L. $= -\sqrt{2}$ and R.H.L. $= \sqrt{2}$

Hence, the limit of the function does not exist.

Also, statement 2 is true, but it is not the correct explanation of statement 1. As for limit to exist, it is not necessary that function is defined at that point.

7. a. When $n \to \infty$ and x is rational or $x = \dfrac{p}{q}$, where p and q are integers and $q \neq 0$.

$n!x = n! \times \dfrac{p}{q}$ is integer as $n!$ has factor q when $n \to \infty$.

Also, when $n!x$ is integer, $\sin(n!\pi x) = 0 \Rightarrow$ given limit is zero.

8. d. Obviously, statement 2 is true, as on the number line immediate neighbourhood of 1/2 is either rational or irrational, but this does not stop $f(x)$ to have limit at $x = 1/2$.

As $f(1/2) = 1/2, f(1/2^+) = \lim_{x\to 1/2^+} x = 1/2$ (if $1/2^+$ is rational)

or $\lim_{x\to 1/2^+}(1-x) = 1-1/2 = 1/2$ (if $1/2^+$ is irrational)

Hence, $\lim_{x\to 1/2^+} f(x) = 1/2$.

With similar argument, we can prove that

$\lim_{x\to 1/2^-} f(x) = 1/2$. Hence, limit of function exists at $x = 1/2$.

9. a. $\lim_{x\to\infty}\dfrac{(x-1)(x-2)}{(x-3)(x-4)}$

$= \lim_{x\to\infty}\dfrac{x^2-3x+2}{x^2-7x+12}$

$= \lim_{x\to\infty}\dfrac{1-\frac{3}{x}+\frac{2}{x^2}}{1-\frac{7}{x}+\frac{12}{x^2}} \to 1$ (from right-hand side of 1)

Hence $\lim_{x\to\infty}\cos^{-1} f(x)$ does not exist as $\cos^{-1}x$ is defined for $x \in [-1, 1]$.

Also, $\lim_{x\to-\infty}\dfrac{1-\frac{3}{x}+\frac{2}{x^2}}{1-\frac{7}{x}+\frac{12}{x^2}} \to 1$ (from left-hand side of 1)

Hence $\lim_{x\to\infty}\cos^{-1} f(x)$ exists.

10. b. $\lim_{x\to 0^+}[x]\left(\dfrac{e^{1/x}-1}{e^{1/x}+1}\right) = \lim_{h\to 0}[h]\left(\dfrac{1-e^{-1/h}}{1+e^{-1/h}}\right) = 0\times 1 = 0$

$\lim_{x\to 0^-}[x]\left(\dfrac{e^{1/x}-1}{e^{1/x}+1}\right) = \lim_{h\to 0}[-h]\left(\dfrac{e^{-1/h}-1}{e^{-1/h}+1}\right) = -1\times(-1) = 1$

Thus, given limit does not exists. Also $\lim_{x\to 0}\left(\dfrac{e^{1/x}-1}{e^{1/x}+1}\right)$ does not exist, but this cannot be taken as only reason for non-existence of $\lim_{x\to 0}[x]\left(\dfrac{e^{1/x}-1}{e^{1/x}+1}\right)$.

11. a. If $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0}\left(f(x)+\dfrac{\sin x}{x}\right)$ always exists as $\lim_{x\to 0}\dfrac{\sin x}{x}$ exists finitely.

Hence $\lim_{x\to 0} f(x)$ must not exist.

12. a. $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\sin a_n = \lim_{n\to\infty} a_n$

$\Rightarrow \lim_{n\to\infty}(a_n - \sin a_n) = 0$ which is possible only when $\lim_{n\to\infty} a_n = 0$.

13. c. Obviously statement 1 is true, but statement 2 is not always true.

Consider, $f(x) = [x]$ and $g(x) = \sin x$ (where $[\cdot]$ represents greatest integer function).

Here $\lim_{x\to\pi^+}[\sin x] = -1$

and $\lim_{x\to\pi^-}[\sin x] = 0$

$\Rightarrow \lim_{x\to\pi}[\sin x]$ does not exist.

Linked Comprehension Type

For Problems 1–3

1. a, 2. b, 3. d.

Sol.

We have $f(x) = \dfrac{\sin^{-1}(1-\{x\})\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}(1-\{x\})}$

$\therefore \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0}\dfrac{\sin^{-1}(1-\{0+h\})\cos^{-1}(1-\{0+h\})}{\sqrt{2\{0+h\}}(1-\{0+h\})}$

$= \lim_{h\to 0}\dfrac{\sin^{-1}(1-h)\cos^{-1}(1-h)}{\sqrt{2h}(1-h)}$

$= \lim_{h\to 0}\dfrac{\sin^{-1}(1-h)}{(1-h)}\lim_{h\to 0}\dfrac{\cos^{-1}(1-h)}{\sqrt{2h}}$

In second limit put $1-h = \cos\theta$

$= \lim_{h\to 0}\dfrac{\sin^{-1}(1-h)}{(1-h)}\lim_{\theta\to 0}\dfrac{\cos^{-1}(\cos\theta)}{\sqrt{2(1-\cos\theta)}}$

$= \lim_{h\to 0}\dfrac{\sin^{-1}(1-h)}{(1-h)}\lim_{\theta\to 0}\dfrac{\theta}{2\sin(\theta/2)}$ $(\because \theta > 0)$

$= \sin^{-1}1\times 1 = \pi/2$

and $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0}\dfrac{\sin^{-1}(1-\{0-h\})\cos^{-1}(1-\{0-h\})}{\sqrt{2\{0-h\}}(1-\{0-h\})}$

$= \lim_{h\to 0}\dfrac{\sin^{-1}(1+h-1)\cos^{-1}(1+h-1)}{\sqrt{2(-h+1)}(1+h-1)}$

$= \lim_{h\to 0}\dfrac{\sin^{-1}h}{h}\lim_{h\to 0}\dfrac{\cos^{-1}h}{\sqrt{2(1-h)}} = 1\dfrac{\pi/2}{\sqrt{2}} = \dfrac{\pi}{2\sqrt{2}}$

For Problems 4–6

4. c, 5. d, 6. d.

Sol.

We have $A_i = \dfrac{x-a_i}{-(x-a_i)} = -1, i = 1, 2, \ldots, n$ and

$a_1 < a_2 < \cdots < a_{n-1} < a_n$.

Let x be in the left neighbourhood of a_m.

Then $x - a_i < 0$ for $i = m, m+1, \ldots, n$ and $x - a_i > 0$ for $i = 1, 2, \cdots, m-1$. Therefore,

$$A_i = \frac{x-a_i}{-(x-a_i)} = -1 \text{ for } i = m, m+1, \ldots, n \text{ and}$$

$$A_i = \frac{x-a_i}{x-a_i} = 1 \text{ for } i = 1, 2, \ldots, m-1$$

Similarly, if x is in the right neighbourhood of a_m, then $x - a_i < 0$ for $i = m+1, \ldots, n$ and $x - a_i > 0$ for $i = 1, 2, \ldots, m$.

$$\therefore \quad A_i = \frac{x-a_i}{-(x-a_i)} = -1 \text{ for } i = m+1, \ldots, n \text{ and}$$

$$A_i = \frac{x-a_i}{x-a_i} = 1 \text{ for i} = 1, 2, \ldots, m$$

Now, $\lim_{x\to a_m^-} (A_1 A_2 \cdots A_n) = (-1)^{n-m+1}$ and

$$\lim_{x\to a_m^+} (A_1 A_2 \cdots A_n) = (-1)^{n-m}$$

Hence, $\lim_{x\to a_m} (A_1 A_2 \cdots A_n)$ does not exist.

For Problems 7–9

7. b, 8. d, 9. c.

Sol.

$$L = \lim_{x\to 0} \frac{\sin x + ae^x + be^{-x} + c\ln(1+x)}{x^3}$$

$$= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!}\right) + a\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right) + b\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!}\right) + c\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{(a+b) + (1+a-b+c)x + \left(\frac{a}{2} + \frac{b}{2} - \frac{c}{2}\right)x^2 + \left(-\frac{1}{3!} + \frac{a}{3!} - \frac{b}{3!} + \frac{c}{3}\right)x^3}{x^3}$$

$$\Rightarrow a + b = 0,\ 1 + a - b + c = 0,\ \frac{a}{2} + \frac{b}{2} - \frac{c}{2} = 0$$

and $L = -\frac{1}{3!} + \frac{a}{3!} - \frac{b}{3!} + \frac{c}{3}$

Solving the first three equations, we get $c = 0, a = -1/2, b = 1/2$.

Then, $L = -1/3$

Equation $ax^2 + bx + c = 0$ reduces to $x^2 - x = 0 \Rightarrow x = 0$, $1||x+c| - 2a| < 4b$ reduces to $||x| + 1| < 2$.

$\Rightarrow -2 < |x| + 1 < 2$

$\Rightarrow 0 \le |x| < 1$

$\Rightarrow x \in [-1, 1]$

For Problems 10–12

10. c $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} (p_1 a_1^x + p_2 a_2^x + \cdots + p_n a_n^x)^{1/x}$ (1^∞ form)

$$= e^{\lim_{x\to 0}\left(\frac{p_1 a_1^x + p_2 a_2^x + \cdots + p_n a_n^x - 1}{x}\right)}$$

$$= e^{\lim_{x\to 0}(p_1 a_1^x \ln a_1 + p_2 a_2^x \ln a_2 + \cdots + p_n a_n^x \ln a_n)}$$

$$= e^{(p_1 \ln a_1 + p_2 \ln a_2 + \cdots + p_n \ln a_n)}$$

$$= e^{\left(\ln a_1^{p_1} + \ln a_2^{p_2} + \cdots + \ln a_n^{p_n}\right)}$$

$$= e^{\left(\ln a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}\right)}$$

$$= a_1^{p_1} \cdot a_2^{p_2}\, a_3^{p_3} \cdots a_n^{p_n}$$

11. c $\lim_{x\to\infty} F(x) = L = \lim_{x\to\infty} \left(p_1 a_1^x + p_2 a_2^x + \cdots + p_n a_n^x\right)^{1/x}$ (∞^0 form).

$$\therefore \ln L = \lim_{x\to\infty} \frac{\left(p_1 a_1^x + p_2 a_2^x + \cdots + p_n a_n^x\right)}{x}$$

Using L'Hospital's rule

$$\ln L = \lim_{x\to\infty} \frac{p_1 a_1^x \ln a_1 + p_2 a_2^x \ln a_2 + \cdots + p_n a_n^x \ln a_n}{p_1 a_1^x + p_2 a_2^x + \cdots + p_n a_n^x} \quad (1)$$

Dividing by a_1^x and taking limit, we get

$\lim_{x\to\infty} \left(\frac{a_2}{a_1}\right)^x, \left(\frac{a_3}{a_2}\right)^x$, etc. all vanishes as $x \to \infty$

$$\Rightarrow \ln L = \frac{p_1 \ln a_1}{p_1} = \ln a_1$$

hence $\ln L = \ln a_1$

$\Rightarrow L = a_1$

12. d Let $\lim_{x\to-\infty} F(x) = L$

$$\therefore \ln L = \lim_{x\to\infty} \frac{p_1 a_1^x \ln a_1 + p_2 a_2^x \ln a_2 + \cdots + p_n a_n^x \ln a_n}{p_1 a_1^x + p_2 a_2^x + \cdots + p_n a_n^x}$$

Dividing by $(a_n)^x$ and taking $\lim_{x\to-\infty} \left(\frac{a_1}{a_n}\right)^x, \left(\frac{a_2}{a_n}\right)^x$, etc. vanishes.

$$\therefore \ln L = \frac{p_n \ln a_n}{p_n}$$

$\Rightarrow L = a_n$

Matrix-Match Type

1. a → s; b → r; c → p; d → q.

a. Let $x + 1 = h$

Then, $\lim_{x\to -1} \frac{\sqrt[3]{(7-x)} - 2}{(x+1)}$

$$= \lim_{h\to 0} \frac{(8-h)^{1/3} - 2}{h}$$

$$= \lim_{h\to 0} \frac{2\left(1-\frac{h}{8}\right)^{1/3} - 2}{h}$$

$$= 2\lim_{h\to 0} \frac{\left(1-\frac{1}{3}\frac{h}{8}\right)-1}{h}$$

$$= -\frac{1}{12}$$

b. We have $\lim_{x\to\pi/4} \frac{\tan^3 x - \tan x}{\cos(x+\pi/4)}$

$$= \lim_{x\to\pi/4} \frac{\tan x(\tan x - 1)(\tan x + 1)}{\cos(x+\pi/4)}$$

$$= \lim_{x\to\pi/4} \frac{\tan x(\sin x - \cos x)(\tan x + 1)}{\cos x \cos(x+\pi/4)}$$

$$= -\lim_{x\to\pi/4} \frac{\tan x(\cos x - \sin x)(\tan x + 1)}{\cos x \cos(x+\pi/4)}$$

$$= -\sqrt{2}\lim_{x\to\pi/2} \frac{\tan x\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)(\tan x + 1)}{\cos x\cos(x+\pi/4)}$$

$$= -\sqrt{2}\lim_{x\to\pi/4} \frac{\tan x(\tan x+1)}{\cos x}$$

$$= -\sqrt{2}\times 2\times\sqrt{2} = -8$$

c. $\lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$

$$= \lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}$$

$$= \lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(\sqrt{x}-1)(\sqrt{x}+1)}$$

$$= \lim_{x\to 1} \frac{(2x-3)}{(2x+3)(\sqrt{x}+1)}$$

$$= \frac{2-3}{(2+3)(\sqrt{1}+1)}$$

$$= -1/10$$

d. $\lim_{x\to\infty} \frac{\log x^n - [x]}{[x]}$

$$= \lim_{x\to\infty} \frac{\log x^n}{[x]} - \lim_{x\to\infty} \frac{[x]}{[x]}$$

$$= 0 - 1$$

$$= -1$$

2. **a → q; b → r; c → q; d → p.**

We know that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (but a value which is smaller than 1)

$$\Rightarrow \left[\lim_{x\to 0} 100\frac{\sin x}{x}\right] = 99$$

and $\left[\lim_{x\to 0} 100\frac{x}{\sin x}\right] = 100$

(Also $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$ (but a value which is more than 1)

$$\Rightarrow \left[\lim_{x\to 0} 100\frac{\sin^{-1} x}{x}\right] = 100$$

and $\left[\lim_{x\to 0} 100\frac{x}{\sin^{-1} x}\right] = 99$

$\lim_{x\to 0} \frac{\tan x}{x} = 1$ (but a value which is bigger than 1)

$$\Rightarrow \left[\lim_{x\to 0} 100\frac{\tan x}{x}\right] = 100$$

and $\left[\lim_{x\to 0} 100\frac{\tan^{-1} x}{x}\right] = 99$

Hence

a. $\lim_{x\to 0}\left(\left[100\frac{\sin x}{x}\right] + \left[100\frac{\tan x}{x}\right]\right) = 199$

b. $\lim_{x\to 0}\left(\left[100\frac{x}{\sin x}\right] + \left[100\frac{\tan x}{x}\right]\right) = 200$

c. $\lim_{x\to 0}\left(\left[100\frac{\sin^{-1} x}{x}\right] + \left[100\frac{\tan^{-1} x}{x}\right]\right) = 199$

d. $\lim_{x\to 0}\left(\left[100\frac{x}{\sin^{-1} x}\right] + \left[100\frac{\tan^{-1} x}{x}\right]\right) = 198$

3. **a → q; b → p, q, r; c → r, s; d → r, s.**

a. Here, $a > 0$, if $a \le 0$, then limit $= \infty$

$$\therefore \lim_{x\to\infty} \frac{(\sqrt{(x^2-x+1)} - ax - b)(\sqrt{x^2-x+1)} + ax + b)}{(\sqrt{(x^2-x+1)} + ax + b)}$$

$$\Rightarrow \lim_{x\to\infty} \frac{(x^2-x+1)-(ax+b)^2}{\sqrt{(x^2-x+1)} + ax + b} = 0$$

$$\Rightarrow \lim_{x\to\infty} \frac{(1-a^2)x^2 - (1+2ab)x + (1-b^2)}{\sqrt{(x^2-x+1)} + ax + b} = 0$$

$$\Rightarrow \lim_{x\to\infty} \frac{(1-a^2)x-(1+2ab)+\frac{(1-b^2)}{x}}{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+a+\frac{b}{x}}$$

This is possible only when $1-a^2=0$ and $1+2ab=0$

$\therefore\ a=\pm 1$

$\Rightarrow a=1 \qquad (\because a>0) \qquad (1)$

$\Rightarrow b=-1/2$

$\Rightarrow (a, 2b)=(1,-1)$

b. Divide numerator and denominator by $e^{1/x}$, then

$$\lim_{x\to\infty}\frac{(1+a^3)e^{-\frac{1}{x}}+8}{e^{-\frac{1}{x}}+(1-b^3)}=2$$

$$\Rightarrow \frac{0+8}{0+1-b^3}=2$$

$\Rightarrow 1-b^3=4$

$\therefore\ b^3=-3 \Rightarrow b=-3^{1/3}$

Then, $a\in R$

$\Rightarrow (a, b^3)=(a,-3)$

c. $\lim_{x\to\infty}(\sqrt{(x^4-x^2+1)}-ax^2-b)=0$

Put $x=\frac{1}{t}$ $\qquad \therefore \lim_{t\to 0}\left(\sqrt{\left(\frac{1}{t^4}-\frac{1}{t^2}+1\right)}-\frac{a}{t^2}-b\right)=0$

$$\Rightarrow \lim_{t\to 0}\frac{\sqrt{(1-t^2+t^4)}-a-bt^2}{t^2}=0 \qquad (1)$$

Since R.H.S. is finite, numerator must be equal to 0 at $t\to 0$.

$\therefore 1-a=0\ \therefore a=1$

From equation (1), $\lim_{t\to 0}\frac{\sqrt{(1-t^2+t^4)}-1-bt^2}{t^2}=0$

$$\lim_{t\to 0}(-1+t^2)\left(\frac{(1-t^2+t^4)^{1/2}-(1)^{1/2}}{(1-t^2+t^4)-1}\right)=b$$

$$\Rightarrow (-1)\left(\frac{1}{2}\right)=b \Rightarrow a=1, b=-\frac{1}{2}\Rightarrow (a,-4b)=(1,2)$$

d. $\lim_{x\to -a}\frac{x^7-(-a)^7}{x-(-a)}=7\Rightarrow 7a^6=7\Rightarrow a^6=1\Rightarrow a=-1$

Integer Type

1. (2) We have

$$L=\lim_{n\to\infty}\prod_{n=2}^{n}\frac{n^2-1}{n^2}$$

$$=\lim_{n\to\infty}\prod_{n=2}^{n}\frac{n-1}{n}\cdot\prod_{n=2}^{n}\frac{n+1}{n}$$

$$=\lim_{n\to\infty}\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n-1}{n}\right)\left(\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{n+1}{n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\cdot\frac{n+1}{2}=\frac{1}{2}$$

2. (6) $\lim_{x\to 1^+}f(g(x))=f(g(1^+))=f((2^+)=2^2+2=6$

and $\lim_{x\to 1^-}f(g(x))=f(g(1^-))=f(3-1^-)=f(2^+)=2^2+2=6$

Hence $\lim_{x\to 1}f(g(x))=6$

3. (3) $\lim_{x\to 1}(1+ax+bx^2)^{\frac{c}{x-1}}=e^3$

$$\Rightarrow e^{\lim_{x\to 1}(1+ax+bx^2-1)\frac{c}{x-1}}=e^3$$

$$\Rightarrow e^{\lim_{x\to 1}\frac{c(ax+bx^2)}{x-1}}=e^3$$

$$\Rightarrow \lim_{x\to 1}\frac{c(ax+bx^2)}{x-1}=3$$

$$\Rightarrow \lim_{h\to 0}\frac{c(a(1+h)+b(1+h)^2)}{1+h-1}=3$$

$$\Rightarrow \lim_{h\to 0}\frac{(ca+b)+(ac+2b)h+bh^2}{h}=3$$

$\Rightarrow ca+b=0$ and $ac+2b=3$

$\Rightarrow b=3$ and $ac=-3$

Also the form must be 1^∞ for which $a+b=0 \Rightarrow a=-3$ and $c=1$

4. (0) $\lim_{n\to\infty}\left[\sqrt[3]{(n+1)^2}-\sqrt[3]{(n-1)^2}\right]$

$$=\lim_{n\to\infty}n^{2/3}\left[\left(1+\frac{1}{n}\right)^{2/3}-\left(1-\frac{1}{n}\right)^{2/3}\right]$$

$$=\lim_{n\to\infty}n^{2/3}\left[\left(1+\frac{2}{3}\cdot\frac{1}{n}+\frac{\frac{2}{3}\left(\frac{2}{2}-1\right)}{2!}\cdot\frac{1}{n^2}\cdots\right)-\left(1-\frac{2}{3}\cdot\frac{1}{n}+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\cdot\frac{1}{n^2}\cdots\right)\right]$$

$$=\lim_{n\to\infty}n^{2/3}\left[\frac{4}{3}\cdot\frac{1}{n}+\frac{8}{81}\cdot\frac{1}{n^3}+\cdots\right]$$

$$=\lim_{n\to\infty}\left[\frac{4}{3}\cdot\frac{1}{n^{1/3}}+\frac{8}{81}\cdot\frac{1}{n^{7/3}}+\cdots\right]=0$$

5. (2) $\lim_{x\to 0}\left[1+x+\frac{f(x)}{x}\right]^{1/x}=e^3$

$\Rightarrow \lim_{x\to 0} e^{\lim_{x\to 0}\left[1+x+\frac{f(x)}{x}-1\right]\frac{1}{x}}=e^3$

$\Rightarrow \lim_{x\to 0} e^{\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right]}=e^3$

$\Rightarrow \lim_{x\to 0}\frac{f(x)}{x^2}=2$

Now $\lim_{x\to 0}\left[1+\frac{f(x)}{x}\right]^{1/x}=e^{\lim_{x\to 0}\left[1+\frac{f(x)}{x}-1\right]\frac{1}{x}}=e^{\lim_{x\to 0}\frac{f(x)}{x^2}}=e^2$

6. (2) $\lim_{x\to\infty}\frac{2x-3}{x}<\lim_{x\to\infty}f(x)<\lim_{x\to\infty}\frac{2x^2+5x}{x^2}$

$\Rightarrow \lim_{x\to\infty}\frac{2-\frac{3}{x}}{1}<\lim_{x\to\infty}f(x)<\lim_{x\to\infty}\frac{2+\frac{5}{x^2}}{1}$

$\Rightarrow \lim_{x\to\infty}f(x)=2$

7. (0) $\lim_{x\to 0^+}f(g(h(x)))=f(g(0^+))=f(1^+)=0$

$\lim_{x\to 0^-}f(g(h(x)))=f(g(0^+))=f(1^+)=0$

Hence $\lim_{x\to 0}f(g(h(x)))=0$

8. (1) $\lim_{x\to\infty}\left(f(x)+\frac{3f(x)-1}{f^2(x)}\right)=3$

$\Rightarrow \left(\lim_{x\to\infty}f(x)+\frac{3\lim_{x\to\infty}f(x)-1}{\left(\lim_{x\to\infty}f(x)\right)^2}\right)=3$

$\Rightarrow \left(y+\frac{3y-1}{y^2}\right)=3$

$\Rightarrow y^3-3y^2+3y-1=0$

$\Rightarrow (y-1)^3=0$

$\Rightarrow y=1$

9. (4) $\lim_{x\to 0}\frac{e^{-x^2/2}-\cos x}{x^3\sin x}$

$=\lim_{x\to 0}\frac{\left(1-\frac{(x^2/2)}{1!}+\frac{(x^2/2)^2}{2!}\right)-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\right)}{x^3\left(x-\frac{x^3}{3!}\right)}$

$=\lim_{x\to 0}\frac{\left(\frac{x^4}{8}\right)-\left(\frac{x^4}{24}\right)}{x^4\left(1-\frac{x^2}{3!}\right)}=\frac{1}{12}$

10. (3) $\lim_{x\to 2}\frac{(10-x)^{1/3}-2}{x-2}$

$=\lim_{h\to 0}\frac{(8-h)^{1/3}-2}{h}$ (Put $x=2+h$)

$=\lim_{h\to 0}\frac{2\left(1-\frac{h}{8}\right)^{1/3}-2}{h}$

$=2\lim_{h\to 0}\frac{\left(1-\frac{h}{8}\right)^{1/3}-1}{h}$

$=2\lim_{h\to 0}\frac{1-\frac{1}{3}\frac{h}{8}-1}{h}=-\frac{1}{12}$

11. (0) Let $L=\lim_{x\to\infty}\frac{\log_e(\log_e x)}{e^{\sqrt{x}}}=\left(\frac{\infty}{\infty}\text{ form}\right)$

$=\lim_{x\to\infty}\frac{\frac{1}{x\log_e x}}{e^{\sqrt{x}}\frac{1}{2\sqrt{x}}}$

$=\lim_{x\to\infty}\frac{2\sqrt{x}}{e^{\sqrt{x}}x\log_e x}$

$=\lim_{x\to\infty}\frac{2}{e^{\sqrt{x}}\sqrt{x}\log_e x}$

$=0$

12. (6) It is obvious n is even, then

$\lim_{n\to\infty}\left(2^{1+3+5+\cdots+n/2\text{ terms}}\cdot 3^{2+4+6+\cdots+n/2\text{ terms}}\right)^{\frac{1}{(n^2+1)}}$

$=\lim_{n\to\infty}\left(2^{\frac{n^2}{4}}\cdot 3^{\frac{n(n+2)}{4}}\right)^{\frac{1}{(n^2+1)}}$

$=\lim_{n\to\infty}2^{\frac{n^2}{4(n^2+1)}}\cdot 3^{\frac{n(n+2)}{4(n^2+1)}}$

$=2^{\lim_{n\to\infty}\frac{1}{4\left(1+\frac{1}{n^2}\right)}}\cdot 3^{\lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)}{4\left(1+\frac{1}{n^2}\right)}}$

$=2^{\frac{1}{4}}3^{\frac{1}{4}}=(6)^{\frac{1}{4}}$

13. (8) Since RHS is finite quantity

$\therefore$ At $x \to 1$, Numerator must be $= 0$

$\therefore$ $0+b+4=0$

$\therefore$ $b=-4$

Then $\lim\limits_{x\to 1}\dfrac{a\sin(x-1)-4\cos(x-1)+4}{(x^2-1)}=-2$

Put $x=1+h$, Then $\lim\limits_{h\to 0}\dfrac{a\sin h+4(1-\cos h)}{h(2+h)}=-2$

$$\Rightarrow \lim_{h\to 0}\frac{a\left(\frac{\sin h}{h}\right)+4\left(\frac{1-\cos h}{h}\right)}{2+h}=-2$$

$$\Rightarrow \frac{a(1)+0}{2}=-2$$

$\Rightarrow a=-4$

$\Rightarrow |a+b|=8$

14. (6) Put $x=1+h$

Then $f(a)=\lim\limits_{h\to 0}\dfrac{(1+h)^a-a(1+h)+a-1}{h^2}$

$$=\lim_{h\to 0}\frac{\left(1+ah+\frac{a(a-1)}{2!}h^2+\cdots\right)-a-ah+a-1}{h^2}$$

$\therefore \quad f(a)=\dfrac{a(a-1)}{2}$

$\therefore \quad f(4)=6$

15. (3) $L=\lim\limits_{x\to 0}\dfrac{(\cos x-1)(\cos x-e^x)}{x^n}$

$$=-\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)(\cos x-e^x)}{(1+\cos x)x^n}$$

$$=\lim_{x\to 0}\frac{\left(\frac{\sin x}{x}\right)^2\left(\frac{1-\cos x}{x}+\frac{e^x-1}{x}\right)}{x^{n-3}}\,\frac{1}{1+\cos x}$$

If L is finite non-zero, then $n=3$ (as for $n=1,2$, $L=0$ and for $n=4$, $L=\infty$)

16. (6) $L=\lim\limits_{x\to 0}=-\lim\limits_{x\to 0}\dfrac{D\prod\limits_{r=2}^{n}(\cos rx)^{1/r}}{2x}$ (Using L'Hospital's rule)

let $y=\prod\limits_{r=2}^{n}(\cos rx)^{1/r}$

$$\Rightarrow \ln y=\sum_{r=2}^{n}\left(\frac{1}{r}\ln(\cos rx)\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=-\sum_{r=2}^{n}\tan(rx)$$

$$\Rightarrow -Dy=y\sum_{r=2}^{n}\tan(rx)$$

$$\Rightarrow D\prod_{r=2}^{n}(\cos rx)^{1/r}=-y\sum_{r=2}^{n}\tan(rx)$$

$$\Rightarrow L=\lim_{x\to 0}\frac{y\cdot\sum\limits_{r=2}^{n}\tan(rx)}{2x}$$

$$=\frac{1}{2}[2+3+4+\cdots+n]$$

$$=\frac{1}{2}\left[\frac{n(n+1)}{2}-1\right]$$

$$=\frac{n^2+n-2}{4}$$

$$\Rightarrow \frac{n^2+n-2}{4}=10$$

$\Rightarrow n^2+n-42=0$

$\Rightarrow (n+7)(n-6)=0$

$\Rightarrow n=6$

17. (9) $f(x)=\dfrac{3x^2+ax+a+1}{(x+2)(x-1)}$

as $x\to -2$, $D^r\to 0$, hence as $x\to -2$, $N^r\to 0$

$\therefore$ $12-2a+a+1=0 \Rightarrow a=13$

18. (4) Let $x=1/y$

$$\Rightarrow \lim_{x\to\infty}\left(x-x^2\log_e\left(1+\frac{1}{x}\right)\right)$$

$$=\lim_{y\to 0}\left(\frac{1}{y}-\frac{\log_e(1+y)}{y^2}\right)$$

$$=\lim_{y\to 0}\left(\frac{y-\log_e(1+y)}{y^2}\right)$$

$$=\lim_{y\to 0}\left(\frac{y-\left(y-\frac{y^2}{2}\right)}{y^2}\right)=1/2$$

y - y²/2 + y³/3 .....

19. (3) $S_n=\dfrac{n(n+1)}{2}$ and $S_n-1=\dfrac{(n+2)(n-1)}{2}$

$$\therefore \frac{S_n}{S_n-1}=\frac{n(n+1)}{2}\cdot\frac{2}{(n+2)(n-1)}$$

$$\Rightarrow \frac{S_n}{S_n-1}=\left(\frac{n}{n-1}\right)\left(\frac{n+1}{n+2}\right)$$

$$\Rightarrow P_n=\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\cdots\frac{n}{n-1}\right)\left(\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdots\frac{n+1}{n+2}\right)$$

$$\Rightarrow P_n=\left(\frac{n}{1}\right)\left(\frac{3}{n+2}\right)$$

$$\Rightarrow \lim_{n\to\infty}P_n=3$$

20 (7) We have,

$$L=\lim_{x\to 0}\frac{2f(x)-3af(2x)+bf(8x)}{\sin^2 x}$$

For the limit to exist, we have $2f(0)-3af(0)+bf(0)=0$

$\Rightarrow 3a-b=2\ [\because f(0)\neq 0, \text{given}]$ (1)

$$\Rightarrow L=\lim_{x\to 0}\frac{2f'(x)-6a\,f'(2x)+8b\,f'(8x)}{2x}$$

For the limit to exist, we have $2f'(0)-6af'(0)+8bf'(0)=0$

$\Rightarrow 3a-4b=1\ [\because f'(0)\neq 0, \text{given}]$ (2)

Solving equations (1) and (2), we have $a=7/9$ and $b=1/3$.

Archives

Subjective

1. Problems solved in the 'Limit by rationalization method'.

2. $f(x)=\int\frac{2\sin x-\sin 2x}{x^3}dx\ ,x\neq 0$

$$\Rightarrow f'(x)=\frac{2\sin x-\sin 2x}{x^3},x\neq 0$$

$$\Rightarrow \lim_{x\to 0}f'(x)=\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}$$

$$=\lim_{x\to 0}\frac{2\sin x(1-\cos x)(1+\cos x)}{x^3(1+\cos x)}$$

$$=\lim_{x\to 0}2\times\frac{\sin^3 x}{x^3}\times\frac{1}{1+\cos x}=2\times(1)^3\times\frac{1}{2}=1$$

3. $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

$$=\lim_{h\to 0}\frac{a^2[\sin(a+h)-\sin a]+2ah\sin(a+h)+h^2\sin(a+h)}{h}$$

$$=\lim_{h\to 0}\frac{a^2\left[2\cos\left(a+\frac{h}{2}\right)\sin\frac{h}{2}\right]}{2\times\frac{h}{2}}+\lim_{h\to 0}2a\sin(a+h)$$

$$+\lim_{h\to 0}h+\sin(a+h)$$

$$=a^2\cos a+2a\sin a$$

4. $\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}=\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}\times\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$

$$=\lim_{x\to 0}\frac{(2^x-1)(\sqrt{1+x}+1)}{1+x-1}$$

$$=\lim_{x\to 0}\frac{2^x-1}{x}\lim_{x\to 0}(\sqrt{1+x}+1)$$

$$=\ln 2\,(1+1)=2\ln 2$$

5. $\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{1/x}$

$$=\lim_{x\to 0}\left\{\frac{1+\tan x}{1-\tan x}\right\}^{1/x}$$

$$=\frac{\lim_{x\to 0}\left[(1+\tan x)^{1/\tan x}\right]^{\frac{\tan x}{x}}}{\lim_{x\to 0}\left[(1-\tan x)^{-1/\tan x}\right]^{-\frac{\tan x}{x}}}$$

$$=\frac{e}{e^{-1}}=e^2$$

Objective

Fill in the blanks

1. $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}$

$$=\lim_{x\to 1}\frac{(1-x)}{\tan\left(\frac{\pi}{2}-\frac{\pi x}{2}\right)}$$

$$=\frac{2}{\pi}\lim_{x\to 1}\frac{\frac{\pi}{2}(1-x)}{\tan\left(\frac{\pi}{2}(1-x)\right)}$$

$$=\frac{2}{\pi}$$

2. $\lim_{x\to 0^+}g\{f(x)\}=g(f(0^+))=g((\sin 0^+))=g(0^+)=(0)^2+1=1$

$\lim_{x\to 0^-}g\{f(x)\}=g(f(0^-))=g((\sin 0^-))=g(0^-)=(0)^2+1=1$

Hence, $\lim_{x\to 0}g\{f(x)\}=1$.

3. $\lim_{x\to -\infty}\left[\frac{x^4\sin\left(\frac{1}{x}\right)+x^2}{(1+|x|^3)}\right]$

$$=\lim_{x\to -\infty}\left[\frac{x\sin\left(\frac{1}{x}\right)+\frac{1}{x}}{\frac{1}{x^3}-1}\right]$$

$$=\lim_{x\to -\infty}\left[\frac{\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}+\frac{1}{x}}{\frac{1}{x^3}-1}\right]=\frac{1+0}{0-1}=-1$$

4. In $\Delta ABC, AB = AC, AD \perp BC$ (D is a midpoint of BC)

Let r = radius of circumcircle

$\therefore OA = OB = OC = r$

Now, $BD = \sqrt{BO^2 - OD^2} = \sqrt{r^2 - (h-r)^2}$

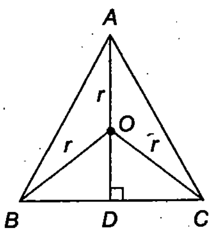


Fig. 2.7

$= \sqrt{2rh - h^2}$

$\therefore BC = 2\sqrt{2rh - h^2}$

$\therefore$ area of $\Delta ABC = \frac{1}{2} \times BC \times AD = h\sqrt{2rh - h^2}$

Also, $\lim_{h\to 0} \frac{A}{p^3} = \frac{h\sqrt{2rh - h^2}}{8\left(\sqrt{2rh - h^2} + \sqrt{2hr}\right)^3}$

$= \lim_{h\to 0} \frac{h^{3/2}\sqrt{2r-h}}{8h^{3/2}\left(\sqrt{2r-h} + \sqrt{2r}\right)^3}$

$= \lim_{h\to 0} \frac{\sqrt{2r-h}}{8\left[\sqrt{2r-h} + \sqrt{2r}\right]^3}$

$= \frac{\sqrt{2r}}{8\left(\sqrt{2r} + \sqrt{2r}\right)^3} = \frac{\sqrt{2r}}{8 \times 8 \times 2r \times \sqrt{2r}} = \frac{1}{128r}$

5. $\lim_{x\to\infty} \left(\frac{x+6}{x+1}\right)^{x+4} = \lim_{x\to\infty} \left(\frac{1+\frac{6}{x}}{1+\frac{1}{x}}\right)^{x+4}$

$= \lim_{x\to\infty} \frac{\left(1+\frac{6}{x}\right)^x}{\left(1+\frac{1}{x}\right)^x} \left(\frac{1+\frac{6}{x}}{1+\frac{1}{x}}\right)^4 \quad \left[\text{Using } \lim_{x\to\infty}\left(1+\frac{\lambda}{x}\right)^x = e^{\lambda}\right]$

$= \frac{e^6}{e}\left(\frac{1}{1}\right)^4 = e^5$

6. $\lim_{x\to 0} \left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2}$

$= \frac{\lim_{x\to 0}\left(1+5x^2\right)^{1/x^2}}{\lim\left(1+3x^2\right)^{1/x^2}}$

$= \frac{\lim_{x\to 0}\left\{\left(1+5x^2\right)^{\frac{1}{5x^2}}\right\}^5}{\lim_{x\to 0}\left\{\left(1+3x^2\right)^{\frac{1}{3x^2}}\right\}^3}$

$= e^{5-3} = e^2$

7. $\lim_{h\to 0} \frac{\ln(1+2h) - 2\ln(1+h)}{h^2}$

$= \lim_{h\to 0} \frac{\ln\left[\frac{(1+2h)}{1+2h+h^2}\right]}{h^2}$

$= \lim_{h\to 0} \ln \frac{\left[1 + \frac{-h^2}{1+2h+h^2}\right]}{\frac{-h^2}{1+2h+h^2}} \times \frac{-1}{1+2h+h^2}$

$= 1 \times \lim_{h\to 0} \frac{-1}{1+2h+h^2} \quad \left[\text{Using } \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right]$

$= -1$

True or false

1. **False**

Consider $f(x) = \frac{|x-a|}{x-a}, g(x) = \frac{x-a}{|x-a|}$ then, $\lim_{x\to a} (f(x) \times g(x))$ exists, but $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist.

$\therefore$ Statement is false.

Multiple choice questions with one correct answer

1. **c.** $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sqrt{\frac{1 - \frac{\sin x}{x}}{1 + \frac{\cos^2 x}{x}}} = \sqrt{\frac{1-0}{1+0}} = 1$

2. **d.** $\lim_{x\to 1} \frac{-\sqrt{25-x^2} - \left(-\sqrt{24}\right)}{x-1}$

$= \lim_{x\to 1} \frac{\sqrt{24} - \sqrt{25-x^2}}{x-1} \times \frac{\sqrt{24} + \sqrt{25-x^2}}{\sqrt{24} + \sqrt{25-x^2}}$

$= \lim_{x\to 1} \frac{x^2 - 1}{(x-1)\left[\sqrt{24} + \sqrt{25-x^2}\right]}$

$= \frac{2}{2\sqrt{24}} = \frac{1}{2\sqrt{6}}$

3. b. $\lim_{n\to\infty}\left(\frac{1}{1-n^2}+\frac{2}{1-n^2}+\cdots+\frac{n}{1-n^2}\right)$

$$=\lim_{n\to\infty}\frac{1+2+3+\cdots+n}{1-n^2}$$

$$=\lim_{n\to\infty}\frac{\frac{n(n+1)}{2}}{1-n^2}$$

$$=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2\left[\frac{1}{n^2}-1\right]}=-1/2$$

4. d. The given function is

$$f(x)=\begin{cases}\frac{\sin[x]}{[x]} & \text{if } x\in(-\infty,0)\cup[1,\infty)\\ 0 & \text{if } x\in[0,1)\end{cases}$$

$$\therefore \lim_{x\to0^-}f(x)=\lim_{h\to0}\frac{\sin[-h]}{[-h]}$$

$$=\lim_{h\to0}\frac{\sin(-1)}{(-1)}=\sin 1$$

$$\lim_{x\to0^+}f(x)=\lim_{h\to0}0=0$$

$$\therefore \lim_{x\to0^-}f(x)\neq\lim_{x\to0^+}f(x)$$

$\therefore \lim_{x\to0}f(x)$ does not exist.

5. d. $\lim_{x\to0}\frac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x}$

$$=\lim_{x\to0}\frac{\sqrt{\frac{1}{2}.2\sin^2 x}}{x}=\lim_{x\to0}\frac{|\sin x|}{x}$$

$$\therefore \text{L.H.L.}=\lim_{h\to0}\frac{|\sin(0-h)|}{0-h}=\lim_{h\to0}\frac{|-\sin h|}{-h}=\lim_{h\to0}\frac{\sin h}{-h}=-1$$

$$\text{R.H.L.}=\lim_{h\to0}\frac{|\sin(0+h)|}{0+h}=\lim_{h\to0}\frac{\sin h}{h}=1$$

As L.H.L. ≠ R.H.L., therefore, the given limit does not exist.

6. d. $\text{L.H.L.}=\lim_{x\to1^-}\frac{\sqrt{1-\cos[2(x-1)]}}{x-1}$

$$=\lim_{x\to1^-}\frac{\sqrt{2\sin^2(x-1)}}{x-1}$$

$$=\sqrt{2}\lim_{x\to1^-}\frac{|\sin(x-1)|}{x-1}$$

$$=\sqrt{2}\lim_{h\to0}\frac{|\sin(-h)|}{-h}=\sqrt{2}\lim_{h\to0}\frac{\sin h}{-h}=-\sqrt{2}$$

Again, $\text{R.H.L.}=\lim_{x\to1^+}\sqrt{2}\frac{|\sin(x-1)|}{x-1}$

$$=\lim_{h\to0}\sqrt{2}\frac{|\sin h|}{h}$$

$$=\lim_{h\to0}\sqrt{2}\frac{\sin h}{h}=\sqrt{2}$$

L.H.L. ≠ R.H.L. Therefore, $\lim_{x\to1}f(x)$ does not exist.

7. c. $\lim_{x\to0}\frac{x\tan 2x-2x\tan x}{4\sin^4 x}$

$$=\lim_{x\to0}\frac{x}{4\sin^4 x}\left[\frac{2\tan x}{1-\tan^2 x}-2\tan x\right]$$

$$=\lim_{x\to0}\frac{x\tan^3 x}{2\sin^4 x\left(1-\tan^2 x\right)}$$

$$=\frac{1}{2}\lim_{x\to0}\frac{x}{\sin x}\frac{1}{\cos^3 x}\frac{1}{1-\tan^2 x}$$

$$=\frac{1}{2}\times1\times\frac{1}{1^3}\times\frac{1}{1-0}=\frac{1}{2}.$$

8. c. $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x=e^{\lim_{x\to\infty}\left[\frac{x-3}{x+2}-1\right]x}$

$$=e^{\lim_{x\to\infty}\left[\frac{-5x}{x+2}\right]}=e^{-5}$$

9. b. $\lim_{x\to0}\frac{\sin\left(\pi\cos^2 x\right)}{x^2}=\lim_{x\to0}\frac{\sin\left(\pi-\pi\cos^2 x\right)}{x^2}$

[$\sin(\pi-\theta)=\sin\theta$]

$$=\lim_{x\to0}\frac{\sin\left(\pi\sin^2 x\right)}{\pi\sin^2 x}\times\frac{\left(\pi\sin^2 x\right)}{x^2}=\pi$$

10. c. $L=\lim_{x\to0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$

$$=-\lim_{x\to0}\frac{(1-\cos x)(1+\cos x)(\cos x-e^x)}{(1+\cos x)x^n}$$

$$=\lim_{x\to0}\frac{\left(\frac{\sin x}{x}\right)^2\left(\frac{1-\cos x}{x}+\frac{e^x-1}{x}\right)}{x^{n-3}}\frac{1}{1+\cos x}$$

L is finite non-zero, then $n=3$ (as for $n=1,2$, $L=0$ and for $n=4$, $L=\infty$)

11. d. Given $\lim_{x\to0}\frac{\left[(a-n)nx-\tan x\right]\sin nx}{x^2}=0$, where a is non-zero number

$$\Rightarrow n\lim_{x\to0}\frac{\sin nx}{nx}\left[\left\{(a-n)n-\frac{\tan x}{x}\right\}\right]=0$$

$$\Rightarrow 1n[(a-n)n-1]=0$$

$$\Rightarrow a=\frac{1}{n}+n$$

12. c. $\lim_{x\to 0}\left[(\sin x)^{1/x}+(1/x)^{\sin x}\right]$

$$= \lim_{x\to 0}(\sin x)^{1/x}+\lim_{x\to 0}\left(\frac{1}{x}\right)^{\sin x}$$

$$= 0 + e^{\lim_{x\to 0}\sin x \log\left(\frac{1}{x}\right)}$$

$$= e^{\lim_{x\to 0}\frac{-\log x}{\operatorname{cosec} x}} = e^{\lim_{x\to 0}\frac{-1/x}{-\operatorname{cosec} x \cot x}} \quad \text{[Using L'Hopital's rule]}$$

$$= e^{\lim_{x\to 0}\frac{\sin x}{x}\tan x} = e^0 = 1$$

13. d. $e^{\ln(1+b^2)} = 2b\sin^2\theta$

$$\Rightarrow \sin^2\theta = \frac{1+b^2}{2b}$$

$$\Rightarrow \sin^2\theta = 1 \text{ as } \frac{1+b^2}{2b} \geq 1$$

$$\theta = \pm\pi/2$$

Multiple choice questions with one or more than one correct answers

1. a, c. $L = \lim_{x\to 0}\dfrac{a-\sqrt{a^2-x^2}-\dfrac{x^2}{4}}{x^4}$

$$= \lim_{x\to 0}\frac{1}{x^2\left(a+\sqrt{a^2-x^2}\right)}-\frac{1}{4x^2}$$

$$= \lim_{x\to 0}\frac{(4-a)-\sqrt{a^2-x^2}}{4x^2\left(a+\sqrt{a^2-x^2}\right)}$$

Numerator $\to 0$ if $a = 2$ and then $L = \dfrac{1}{64}$.