8

Electromagnetic Waves

Earlier chapters have shown that the velocity of waves through a medium is determined by the inertia and the elasticity of the medium. These two properties are capable of storing wave energy in the medium, and in the absence of energy dissipation they also determine the impedance presented by the medium to the waves. In addition, when there is no loss mechanism a pure wave equation with a sine or cosine solution will always be obtained, but this equation will be modified by any resistive or loss term to give an oscillatory solution which decays with time or distance.

These physical processes describe exactly the propagation of electromagnetic waves through a medium. The magnetic inertia of the medium, as in the case of the transmission line, is provided by the inductive property of the medium, i.e. the permeability μ, which has the units of henries per metre. The elasticity or capacitive property of the medium is provided by the permittivity ε, with units of farads per metre. The storage of magnetic energy arises through the permeability μ; the potential or electric field energy is stored through the permittivity ε.

If the material is defined as a dielectric, only μ and ε are effective and a pure wave equation for both the magnetic field vector H and the electric field vector E will result. If the medium is a conductor, having conductivity σ (the inverse of resistivity) with dimensions of siemens per metre or (ohms m)$^{-1}$, in addition to μ and ε, then some of the wave energy will be dissipated and absorption will take place.

In this chapter we will consider first the propagation of electromagnetic waves in a medium characterized by μ and ε only, and then treat the general case of a medium having μ, ε and σ properties.

Maxwell's Equations

Electromagnetic waves arise whenever an electric charge changes its velocity. Electrons moving from a higher to a lower energy level in an atom will radiate a wave of a particular frequency and wavelength. A very hot ionized gas consisting of charged particles will radiate waves over a continuous spectrum as the paths of individual particles are curved in

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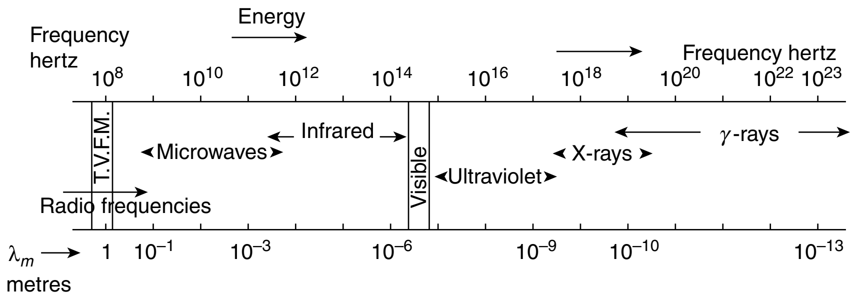


Figure 8.1 Wavelengths and frequencies in the electromagnetic spectrum

mutual collisions. This radiation is called 'Bremsstrahlung'. The radiation of electromagnetic waves from an aerial is due to the oscillatory motion of charges in an alternating current flowing in the aerial.

Figure 8.1 shows the frequency spectrum of electromagnetic waves. All of these waves exhibit the same physical characteristics.

It is quite remarkable that the whole of electromagnetic theory can be described by the four vector relations in Maxwell's equations. In examining these relations in detail we shall see that two are steady state; that is, independent of time, and that two are time-varying.

The two time-varying equations are mathematically sufficient to produce separate wave equations for the electric and magnetic field vectors, E and H, but the steady state equations help to identify the wave nature as transverse.

The first time-varying equation relates the *time* variation of the magnetic induction, $\mu H = B$, with the *space* variation of E; that is

$$\frac{\partial}{\partial t}(\mu H) \text{ is connected with } \frac{\partial E}{\partial z} \text{ (say)}$$

This is nothing but a form of Lenz's or Faraday's Law, as we shall see.

The second time-varying equation states that the *time* variation of εE defines the *space* variation of H, that is

$$\frac{\partial}{\partial t}(\varepsilon E) \text{ is connected with } \frac{\partial H}{\partial z} \text{ (say)}$$

Again we shall see that this is really a statement of Ampere's Law.

These equations show that the variations of E in time and space affect those of H and vice versa. E and H cannot be considered as isolated quantities but are interdependent.

The product εE has dimensions

$$\frac{\text{farads}}{\text{metre}} \times \frac{\text{volts}}{\text{metre}} = \frac{\text{charge}}{\text{area}}$$

This charge per unit area is called the displacement charge $D = \varepsilon E$.

Physically it appears in a dielectric when an applied electric field polarizes the constituent atoms or molecules and charge moves across any plane in the dielectric which

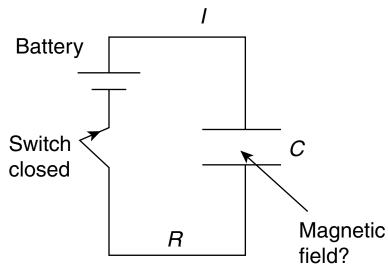


Figure 8.2 In this circuit, when the switch is closed the conduction current charges the condenser. Throughout charging the quantity $\varepsilon\mathbf{E}$ in the volume of the condenser is changing and the displacement current per unit area $\partial/\partial t\ (\varepsilon\mathbf{E})$ is associated with the magnetic field present between the condenser plates

is normal to the applied field direction. If the applied field is varying or alternating with time we see that the dimensions of

$$\frac{\partial\mathbf{D}}{\partial t} = \frac{\partial}{\partial t}(\varepsilon\mathbf{E}) = \frac{\text{charge}}{\text{time} \times \text{area}}$$

current per unit area. This current is called the displacement current. It is comparatively simple to visualize this current in a dielectric where physical charges may move—it is not easy to associate a displacement current with free space in the absence of a material but it may always be expressed as $I_d = \varepsilon(\partial\phi_{\mathrm{E}}/\partial t)$, where ϕ_{E} is the electric field flux through a surface.

Consider what happens in the electric circuit of Figure 8.2 when the switch is closed and the battery begins to charge the condenser C to a potential V. A current I obeying Ohm's Law ($V = IR$) will flow through the connecting leads as long as the condenser is charging and a compass needle or other magnetic field detector placed near the leads will show the presence of the magnetic field associated with that current. But suppose a magnetic field detector (shielded from all outside effects) is placed in the region between the condenser plates where no ohmic or conduction current is flowing. Would it detect a magnetic field? The answer is yes; all the magnetic field effects from a current exist in this region as long as the condenser is charging, that is, as long as the potential difference and the electric field between the condenser plates are changing.

It was Maxwell's major contribution to electromagnetic theory to assert that the existence of a time-changing electric field in free space gave rise to a displacement current. The same result follows from considering the conservation of charge. The flow of charge into any small volume in space must equal that flowing out. If the volume includes the top plate of the condenser the ohmic current through the leads produces the flow into the volume, while the displacement current represents the flow out.

In future, therefore, two different kinds of current will have to be considered:

1. The familar conduction current obeying Ohm's Law ($V = IR$) and
2. The displacement current of density $\partial\mathbf{D}/\partial t$.

In a medium of permeability μ and permittivity ε, but where the conductivity $\sigma = 0$, *the displacement current will be the only current flowing*. In this case a pure wave equation for E and H will follow and there will be no energy loss or attenuation.

When $\sigma \neq 0$ a resistive element allows the conduction current to flow, energy loss will follow, a diffusion term is added to the wave equation and the wave amplitude will attenuate exponentially with distance. We shall see that the relative magnitude of these two currents is frequency-dependent and that their ratio governs whether the medium behaves as a conductor or as a dielectric.

Electromagnetic Waves in a Medium having Finite Permeability μ and Permittivity ε but with Conductivity $\sigma = 0$

We shall consider a system of plane waves and choose the plane xy as that region over which the wave properties are constant. These properties will not vary with respect to x and y and all derivatives $\partial/\partial x$ and $\partial/\partial y$ will be zero.

The first time-varying equation of Maxwell is written in vector notation as

$$\text{curl } \mathbf{E} = \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

This represents three component equations:

$$\left.\begin{aligned} -\mu \frac{\partial}{\partial t} H_x &= \frac{\partial}{\partial y} E_z - \frac{\partial}{\partial z} E_y \\ -\mu \frac{\partial}{\partial t} H_y &= \frac{\partial}{\partial z} E_x - \frac{\partial}{\partial x} E_z \\ -\mu \frac{\partial}{\partial t} H_z &= \frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x \end{aligned}\right\} \tag{8.1}$$

where the subscripts represent the component directions. E_x, E_y and E_z are, respectively, the magnitudes of $\boldsymbol{E}_x \boldsymbol{E}_y$ and $\boldsymbol{E}_z$. Similarly, H_x, H_y and H_z are the magnitudes of $\boldsymbol{H_x H_y}$ and $\boldsymbol{H_z}$. The dimensions of these equations may be written

$$-\frac{\mu H}{\text{time}} = \frac{E}{\text{length}}$$

and multiplying each side by $(\text{length})^2$ gives

$$-\frac{\mu H}{\text{time}} \times \text{area} = E \times \text{length}$$

i.e.

$$\frac{\text{total magnetic flux}}{\text{time}} = \text{volts}$$

This is dimensionally of the form of Lenz's or Faraday's Law.

The second time-varying equation of Maxwell is written in vector notation as

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

This represents three component equations:

$$\left.\begin{aligned} \varepsilon \frac{\partial}{\partial t} E_x &= \frac{\partial}{\partial y} H_z - \frac{\partial}{\partial z} H_y \\ \varepsilon \frac{\partial}{\partial t} E_y &= \frac{\partial}{\partial z} H_x - \frac{\partial}{\partial x} H_z \\ \varepsilon \frac{\partial}{\partial t} E_z &= \frac{\partial}{\partial x} H_y - \frac{\partial}{\partial y} H_x \end{aligned}\right\} \tag{8.2}$$

The dimensions of these equations may be written

$$\frac{\text{current } I}{\text{area}} = \frac{H}{\text{length}}$$

and multiplying both sides by a length gives

$$\frac{\text{current}}{\text{length}} = \frac{I}{\text{length}} = H$$

which is dimensionally of the form of Ampere's Law (i.e. the circular magnetic field at radius r due to the current I flowing in a straight wire is given by $H = I/2\pi r$). Maxwell's first steady state equation may be written

$$\text{div } \mathbf{D} = \nabla \cdot \mathbf{D} = \varepsilon \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = \rho \tag{8.3}$$

where ε is constant and ρ is the charge density. This states that over a small volume element dx dy dz of charge density ρ the change of displacement depends upon the value of ρ.

When $\rho = 0$ the equation becomes

$$\varepsilon \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = 0 \tag{8.3a}$$

so that if the displacement $D = \varepsilon E$ is graphically represented by flux lines which must begin and end on electric charges, the number of flux lines entering the volume element dx dy dz must equal the number leaving it.

The second steady state equation is written

$$\text{div } \mathbf{B} = \nabla \cdot \mathbf{B} = \mu \left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} \right) = 0 \tag{8.4}$$

Again this states that an equal number of magnetic induction lines enter and leave the volume dx dy dz. This is a physical consequence of the non-existence of isolated magnetic poles, i.e. a single north pole or south pole.

Whereas the charge density ρ in equation (8.3) can be positive, i.e. a source of flux lines (or displacement), or negative, i.e. a sink of flux lines (or displacement), no separate source or sink of magnetic induction can exist in isolation, every source being matched by a sink of equal strength.

The Wave Equation for Electromagnetic Waves

Since, with these plane waves, all derivatives with respect to x and y are zero. equations (8.1) and (8.4) give

$$-\mu\frac{\partial H_z}{\partial t} = 0 \quad \text{and} \quad \frac{\partial H_z}{\partial z} = 0$$

therefore, H_z is constant in space and time and because we are considering only the oscillatory nature of H a constant H_z can have no effect on the wave motion. We can therefore put $H_z = 0$. A similar consideration of equations (8.2) and (8.3a) leads to the result that $E_z = 0$.

The absence of variation in H_z and E_z means that the oscillations or variations in H and E occur in directions perpendicular to the z-direction. We shall see that this leads to the conclusion that electromagnetic waves are transverse waves.

In addition to having plane waves we shall simplify our picture by considering only *plane-polarized* waves.

We can choose the electric field vibration to be in either the x or y direction. Let us consider E_x only, with $E_y = 0$. In this case equations (8.1) give

$$-\mu\frac{\partial H_y}{\partial t} = \frac{\partial E_x}{\partial z} \tag{8.1a}$$

and equations (8.2) give

$$\varepsilon\frac{\partial E_x}{\partial t} = -\frac{\partial H_y}{\partial z} \tag{8.2a}$$

Using the fact that

$$\frac{\partial^2}{\partial z \partial t} = \frac{\partial^2}{\partial t \partial z}$$

it follows by taking $\partial/\partial t$ of equation (8.1a) and $\partial/\partial z$ of equation (8.2a) that

$$\frac{\partial^2}{\partial z^2}H_y = \mu\varepsilon\frac{\partial^2}{\partial t^2}H_y \quad \text{(the wave equation for } H_y\text{)}$$

Similarly, by taking $\partial/\partial t$ of (8.2a) and $\partial/\partial z$ of (8.1a), we obtain

$$\frac{\partial^2}{\partial z^2}E_x = \mu\varepsilon\frac{\partial^2}{\partial t^2}E_x \quad \text{(the wave equation for } E_x\text{)}$$

Thus, the vectors E_x and H_y both obey the same wave equation, propagating in the z-direction with the same velocity $v^2 = 1/\mu\varepsilon$. In free space the velocity is that of light, that is, $c^2 = 1/\mu_0\varepsilon_0$, where μ_0 is the permeability of free space and ε_0 is the permittivity of free space.

The solutions to these wave equations may be written, for plane waves, as

$$E_x = E_0 \sin\frac{2\pi}{\lambda}(vt - z)$$
$$H_y = H_0 \sin\frac{2\pi}{\lambda}(vt - z)$$

where E_0 and H_0 are the maximum amplitude values of E and H. Note that the sine (or cosine) solutions means that no attenuation occurs: only displacement currents are involved and there are no conductive or ohmic currents.

We can represent the electromagnetic wave (E_x, H_y) travelling in the z-direction in Figure 8.3, and recall that because E_z and H_z are constant (or zero) the electromagnetic wave is a transverse wave.

The direction of propagation of the waves will always be in the $\mathbf{E}\times\mathbf{H}$ direction; in this case, $\mathbf{E}\times\mathbf{H}$ has magnitude, E_xH_y and is in the z-direction.

This product has the dimensions

$$\frac{\text{voltage} \times \text{current}}{\text{length} \times \text{length}} = \frac{\text{electrical power}}{\text{area}}$$

measured in units of watts per square metre.

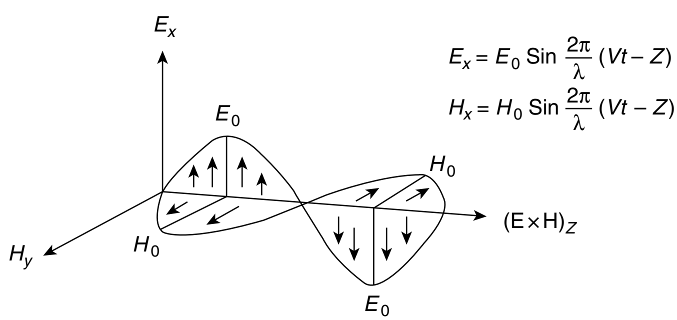


Figure 8.3 In a plane-polarized electromagnetic wave the electric field vector E_x and magnetic field vector H_y are perpendicular to each other and vary sinusoidally. In a non-conducting medium they are in phase. The vector product, $\mathbf{E}\times\mathbf{H}$, gives the direction of energy flow

The vector product, $\boldsymbol{E}\times\boldsymbol{H}$ gives the direction of energy flow. The energy flow per second across unit area is given by the Poynting vector:

$$\frac{1}{2}\boldsymbol{E} \times \boldsymbol{H}^*$$

(Problem 8.1)

Illustration of Poynting Vector

We can illustrate the flow of electromagnetic energy in terms of the Poynting vector by considering the simple circuit of Figure 8.4, where the parallel plate condenser of area A and separation d, containing a dielectric of permittivity ε, is being charged to a voltage V.

Throughout the charging process current flows, and the electric and magnetic field vectors show that the Poynting vector is always directed into the volume Ad occupied by the dielectric.

The capacitance C of the condenser is $\varepsilon A/d$ and the total energy of the condenser at potential V is $\frac{1}{2}CV^2$ joules, which is stored as electrostatic energy. But $V = Ed$, where E is the final value of the electric field, so that the total energy

$$\frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{\varepsilon A}{d}\right)E^2 d^2 = \frac{1}{2}(\varepsilon E^2)Ad$$

where Ad is the volume of the condenser.

The electrostatic energy per unit volume stored in the condenser is therefore $\frac{1}{2}\varepsilon E^2$ and results from the flow of electromagnetic energy during charging.

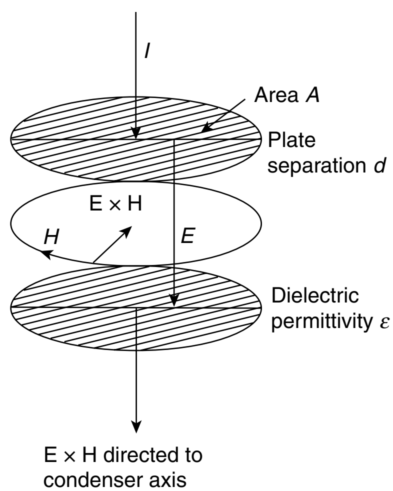


Figure 8.4 During charging the vector $\mathbf{E}\times\mathbf{H}$ is directed into the condenser volume. At the end of the charging the energy is totally electrostatic and equals the product of the condenser volume, Ad, and the electrostatic energy per unit volume, $\frac{1}{2}\varepsilon E^2$

Impedance of a Dielectric to Electromagnetic Waves

If we put the solutions

$$E_x = E_0 \sin \frac{2\pi}{\lambda}(vt - z)$$

and

$$H_y = H_0 \sin \frac{2\pi}{\lambda}(vt - z)$$

in equation (8.1a) where

$$-\mu \frac{\partial H_y}{\partial t} = \frac{\partial E_x}{\partial z}$$

then

$$-\mu v H_y = -E_x, \quad \text{and since} \quad v^2 = \frac{1}{\mu\varepsilon}$$

$$\sqrt{\mu} H_y = \sqrt{\varepsilon} E_x$$

that is

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\varepsilon}} = \frac{E_0}{H_0}$$

which has the dimensions of ohms.

The value $\sqrt{\mu/\varepsilon}$ therefore represents the *characteristic impedance* of the medium to electromagnetic waves (compare this with the equivalent result $V/I = \sqrt{L_0/C_0} = Z_0$ for the transmission line of the previous chapter).

In free space

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\,\Omega$$

so that free space presents an impedance of $376.7\,\Omega$ to electromagnetic waves travelling through it.

It follows from

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\varepsilon}} \quad \text{that} \quad \frac{E_x^2}{H_y^2} = \frac{\mu}{\varepsilon}$$

and therefore

$$\varepsilon E_x^2 = \mu H_y^2$$

Both of these quantities have the dimensions of energy per unit volume, for instance εE_x^2 has dimensions

$$\frac{\text{farads}}{\text{metre}} \times \frac{\text{volts}^2}{\text{metres}^2} = \frac{\text{joules}}{\text{metres}^3}$$

as we saw in the illustration of the Poynting vector. Thus, for a dielectric the electrostatic energy $\frac{1}{2}\varepsilon E_x^2$ per unit volume in an electromagnetic wave equals the magnetic energy per unit volume $\frac{1}{2}\mu H_y^2$ and the total energy is the sum $\frac{1}{2}\varepsilon E_x^2 + \frac{1}{2}\mu H_y^2$.

This gives the instantaneous value of the energy per unit volume and we know that, in the wave,

$$E_x = E_0 \sin(2\pi/\lambda)(vt - z)$$

and

$$H_y = H_0 \sin(2\pi/\lambda)(vt - z)$$

so that the *time average value* of the energy per unit volume is

$$\begin{aligned} \tfrac{1}{2}\varepsilon \bar{E}_x^2 + \tfrac{1}{2}\mu \bar{H}_y^2 &= \tfrac{1}{4}\varepsilon E_0^2 + \tfrac{1}{4}\mu H_0^2 \\ &= \tfrac{1}{2}\varepsilon E_0^2 \,\mathrm{J\,m^{-3}} \end{aligned}$$

Now the amount of energy in an electromagnetic wave which crosses unit area in unit time is called the intensity, I, of the wave and is evidently $(\frac{1}{2}\varepsilon E_0^2)v$ where v is the velocity of the wave.

This gives the time averaged value of the Poynting vector and, for an electromagnetic wave in free space we have

$$I = \tfrac{1}{2}c\varepsilon_0 E_0^2 = \tfrac{1}{2}c\mu_0 H_0^2 \,\mathrm{W\,m^{-2}}$$

(Problems 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, 8.9, 8.10, 8.11)

Electromagnetic Waves in a Medium of Properties μ, ε and σ (where $\sigma \neq 0$)

From a physical point of view the electric vector in electromagnetic waves plays a much more significant role than the magnetic vector, e.g. most optical effects are associated with the electric vector. We shall therefore concentrate our discussion on the electric field behaviour.

In a medium of conductivity $\sigma = 0$ we have obtained the wave equation

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

where the right hand term, rewritten

$$\mu \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}(\varepsilon E_x)\right]$$

shows that we are considering a term

$$\mu \frac{\partial}{\partial t}\left[\frac{\text{displacement current}}{\text{area}}\right]$$

When $\sigma \neq 0$ we must also consider the conduction currents which flow. These currents are given by Ohm's Law as $I = V/R$, and we define the current density; that is, the current per unit area, as

$$J = \frac{I}{\text{Area}} = \frac{1}{R \times \text{Length}} \times \frac{V}{\text{Length}} = \sigma E$$

where σ is the conductivity $1/(R \times \text{Length})$ and E is the electric field. $J = \sigma E$ is another form of Ohm's Law.

With both displacement and conduction currents flowing, Maxwell's second time-varying equation reads, in vector form,

$$\nabla \times \mathbf{H} = \frac{\partial}{\partial t}\mathbf{D} + \mathbf{J} \tag{8.5}$$

each term on the right hand side having dimensions of current per unit area. The presence of the conduction current modifies the wave equation by adding a second term of the same form to its righthand side, namely

$$\mu \frac{\partial}{\partial t}\left(\frac{\text{current}}{\text{area}}\right) \text{ which is } \mu \frac{\partial}{\partial t}(\mathbf{J}) = \mu \frac{\partial}{\partial t}(\sigma \mathbf{E})$$

The final equation is therefore given by

$$\boxed{\frac{\partial^2}{\partial z^2}E_x = \mu\varepsilon \frac{\partial^2}{\partial t^2}E_x + \mu\sigma \frac{\partial}{\partial t}E_x} \tag{8.6}$$

and this equation may be derived formally by writing the component equation of (8.5) as

$$\varepsilon \frac{\partial E_x}{\partial t} + \sigma E_x = -\frac{\partial H_y}{\partial z} \tag{8.5a}$$

together with

$$-\mu \frac{\partial H_y}{\partial t} = \frac{\partial E_x}{\partial z} \tag{8.1a}$$

and taking $\partial/\partial t$ of (8.5a) and $\partial/\partial z$ of (8.1a). We see immediately that the presence of the resistive or dissipation term, which allows conduction currents to flow, will add a diffusion term of the type discussed in the last chapter to the pure wave equation. The product $(\mu\sigma)^{-1}$ is called the magnetic diffusivity, and has the dimensions L^2T^{-1}, as we expect of all diffusion coefficients.

We are now going to look for the behaviour of E_x in this new equation, with the assumption that its time-variation is simple harmonic, so that $E_x = E_0\,\mathrm{e}^{\mathrm{i}\omega t}$. Using this value in equation (8.6) gives

$$\frac{\partial^2 E_x}{\partial z^2} - (\mathrm{i}\omega\mu\sigma - \omega^2\mu\varepsilon)E_x = 0$$

which is in the form of equation (7.5), written

$$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0$$

where $\gamma^2 = \mathrm{i}\omega\mu\sigma - \omega^2\mu\varepsilon$.

We saw in Chapter 7 that this produced a solution with the term $\mathrm{e}^{-\gamma z}$ or $\mathrm{e}^{+\gamma z}$, but we concentrate on the E_x oscillation in the positive z-direction by writing

$$E_x = E_0\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{-\gamma z}$$

In order to assign a suitable value to γ we must go back to equation (8.6) and consider the relative magnitudes of the two right hand side terms. If the medium is a dielectric, only displacement currents will flow. When the medium is a conductor, the ohmic currents of the second term on the right hand side will be dominant. The ratio of the magnitudes of the conduction current density to the displacement current density is the ratio of the two right hand side terms. This ratio is

$$\frac{\mathbf{J}}{\partial\mathbf{D}/\partial t} = \frac{\sigma E_x}{\partial/\partial t(\varepsilon E_x)} = \frac{\sigma E_x}{\partial/\partial t(\varepsilon E_0\,\mathrm{e}^{\mathrm{i}\omega t})} = \frac{\sigma E_x}{\mathrm{i}\omega\varepsilon E_x} = \frac{\sigma}{\mathrm{i}\omega\varepsilon}$$

We see immediately from the presence of i that the phase of the displacement current is 90° ahead of that of the ohmic or conduction current. It is also 90° ahead of the electric field E_x so the displacement current dissipates no power.

For a conductor, where $\mathbf{J} \gg \partial\mathbf{D}/\partial t$, we have $\sigma \gg \omega\varepsilon$, and $\gamma^2 = \mathrm{i}\sigma(\omega\mu) - \omega\varepsilon(\omega\mu)$ becomes

$$\gamma^2 \approx \mathrm{i}\sigma\omega\mu$$

to a high order of accuracy.
Now

$$\sqrt{\mathrm{i}} = \frac{1+\mathrm{i}}{\sqrt{2}}$$

so that

$$\gamma = (1+\mathrm{i})\left(\frac{\omega\mu\sigma}{2}\right)^{1/2}$$

and

$$\begin{aligned} E_x &= E_0\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{-\gamma z} \\ &= E_0\,\mathrm{e}^{-(\omega\mu\sigma/2)^{1/2}z}\,\mathrm{e}^{\mathrm{i}[\omega t-(\omega\mu\sigma/2)^{1/2}z]} \end{aligned}$$

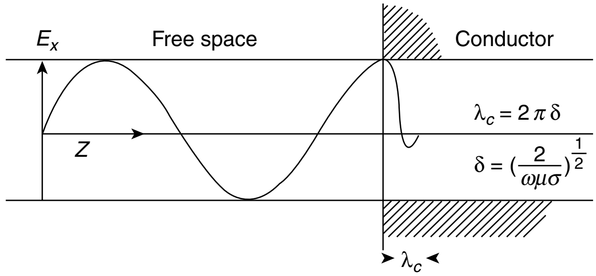


Figure 8.5 Electromagnetic waves in a dielectric strike the plane surface of a conductor, and the electric field vector E_0 is damped to a value $E_0\,e^{-1}$ in a distance of $(2/\omega\mu\sigma)^{1/2}$, the 'skin depth'. This explains the electrical shielding properties of a conductor. λ_c is the wavelength in the conductor

a progressive wave in the positive z-direction with an amplitude decaying with the factor $e^{-(\omega\mu\sigma/2)^{1/2}z}$.

Note that the product $\omega\mu\sigma$ has dimensions L^{-2}.

(Problem 8.12)

Skin Depth

After travelling a distance

$$\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2}$$

in the conductor the electric field vector has decayed to a value $E_x = E_0\,e^{-1}$; this distance is called the *skin depth* (Figure 8.5).

For copper, with $\mu \approx \mu_0$ and $\sigma = 5.8 \times 10^7$ S m^{-1} at a frequency of 60 Hz, $\delta \approx 9$ mm; at 1 MHz, $\delta \approx 6.6 \times 10^{-5}$ m and at 30 000 MHz (radar wavelength of 1 cm), $\delta \approx 3.8 \times 10^{-7}$ m.

Thus, high frequency electromagnetic waves propagate only a very small distance in a conductor. The electric field is confined to a very small region at the surface; significant currents will flow only at the surface and the resistance of the conductor therefore increases with frequency. We see also why a conductor can act to 'shield' a region from electromagnetic waves.

Electromagnetic Wave Velocity in a Conductor and Anomalous Dispersion

The phase velocity of the wave v is given by

$$v = \frac{\omega}{k} = \frac{\omega}{(\omega\mu\sigma/2)^{1/2}} = \omega\delta = \left(\frac{2\omega}{\mu\sigma}\right)^{1/2} = \nu\lambda_c$$

When δ is small, v is small, and the refractive index c/v of a conductor can be very large. We shall see later that this can explain the high optical reflectivities of good conductors. The velocity $v = \omega\delta = 2\pi\nu\delta$, so that λ_c in the conductor is $2\pi\delta$ and can be very small. Since v is a function of the frequency an electrical conductor is a dispersive medium to electromagnetic waves. Moreover, as the table below shows us, $\partial v/\partial\lambda$ is negative, so that the conductor is anomalously dispersive and the group velocity is greater than the wave velocity. Since $c^2/v^2 = \mu\varepsilon/\mu_0\varepsilon_0 = \mu_r\varepsilon_r$, where the subscript *r* defines non-dimensional relative values; that is, $\mu/\mu_0 = \mu_r$, $\varepsilon/\varepsilon_0 = \varepsilon_r$, then for $\mu_r \approx 1$

$$\varepsilon_r v^2 = c^2$$

and

$$\frac{\partial}{\partial\lambda}\varepsilon_r = -\frac{2}{v}\varepsilon_r\frac{\partial v}{\partial\lambda}$$

which confirms our statement in the chapter on group velocity that for $\partial\varepsilon_r/\partial\lambda$ positive a medium is anomalously dispersive. We see too that $c^2/v^2 = \varepsilon_r = n^2$, where n is the refractive index, so that the curve in Figure 3.9 showing the reactive behaviour of the oscillator impedance at displacement resonance is also showing the behaviour of *n*. This relative value of the permittivity is, of course, familiarly known as the dielectric constant when the frequency is low. This identity is lost at higher frequencies because the permittivity is frequency-dependent.

Note that $\lambda_c = 2\pi\delta$ is very small, and that when an electromagnetic wave strikes a conducting surface the electric field vector will drop to about 1% of its surface value in a distance equal to $\frac{3}{4}\lambda_c = 4.6\,\delta$. Effectively, therefore, the electromagnetic wave travels less than one wavelength into the conductor.

Frequency	$\lambda_{\text{free space}}$	δ (m)	$v_{\text{conductor}} = \omega\delta$ (m/s)	Refractive index ($c/v_{\text{conductor}}$)
60	5000 km	9×10^{-3}	3.2	9.5×10^{7}
10^6	300 m	6.6×10^{-5}	4.1×10^{2}	7.3×10^{5}
3×10^{10}	10^{-2} m	3.9×10^{-7}	7.1×10^{4}	4.2×10^{3}

(Problems 8.13, 8.14, 8.15)

When is a Medium a Conductor or a Dielectric?

We have already seen that in any medium having $\mu\varepsilon$ and σ properties the magnitude of the ratio of the conduction current density to the displacement current density

$$\frac{J}{\partial D/\partial t} = \frac{\sigma}{\omega\varepsilon}$$

a non-dimensional quantity.

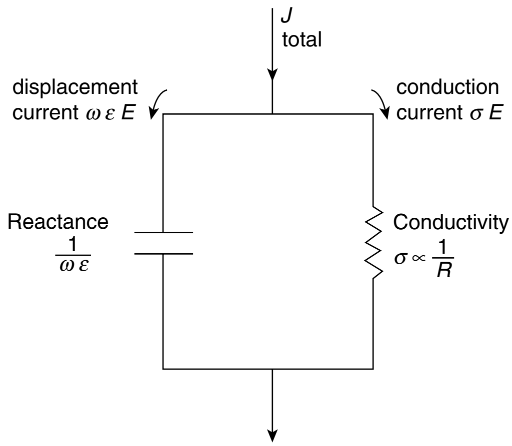


Figure 8.6 A simple circuit showing the response of a conducting medium to an electromagnetic wave. The total current density J is divided by the parallel circuit in the ratio $\sigma/\omega\varepsilon$ (the ratio of the conduction current density to the displacement current density). A large conductance σ (small resistance) gives a large conduction current while a small capacitative reactance $1/\omega\varepsilon$ allows a large displacement current to flow. For a conductor $\sigma/\omega\varepsilon \geq 100$; for a dielectric $\omega\varepsilon/\sigma \geq 100$. Note the frequency dependence of this ratio. At $\omega \approx 10^{20}$ rad/s copper is a dielectric to X-rays

We may therefore represent the medium by the simple circuit in Figure 8.6 where the total current is divided between the two branches, a capacitative branch of reactance $1/\omega\varepsilon$ (ohms-metres) and a resistive branch of conductance σ (siemens/metre). If σ is large the resistivity is small, and most of the current flows through the σ branch and is conductive. If the capacitative reactance $1/\omega\varepsilon$ is so small that it takes most of the current, this current is the displacement current and the medium behaves as a dielectric.

Quite arbitrarily we say that if

$$\frac{J}{\partial D/\partial t} = \frac{\sigma}{\omega\varepsilon} > 100$$

then conduction currents dominate and the medium is a conductor. If

$$\frac{\partial D/\partial t}{J} = \frac{\omega\varepsilon}{\sigma} > 100$$

then displacement currents dominate and the material behaves as a dielectric. Between these values exist a range of quasi-conductors; some of the semi-conductors fall into this category.

The ratio $\sigma/\omega\varepsilon$ is, however, frequency dependent, and a conductor at one frequency may be a dielectric at another.

For copper, which has $\sigma = 5.8 \times 10^7$ S m^{-1} and $\varepsilon \approx \varepsilon_0 = 9 \times 10^{-12}$ F m^{-1},

$$\frac{\sigma}{\omega\varepsilon} \approx \frac{10^{18}}{\text{frequency}}$$

so up to a frequency of 10^{16} Hz (the frequency of ultraviolet light) $\sigma/\omega\varepsilon > 100$, and copper is a conductor. At a frequency of 10^{20} Hz, however (the frequency of X-rays), $\omega\varepsilon/\sigma > 100$, and copper behaves as a dielectric. This explains why X-rays travel distances equivalent to many wavelengths in copper.

Typically, an insulator has $\sigma \approx 10^{-15}$ S m^{-1} and $\varepsilon \approx 10^{-11}$F m^{-1}, which gives

$$\frac{\omega\varepsilon}{\sigma} \approx 10^4\omega$$

so the conduction current is negligible at all frequencies.

Why will an Electromagnetic Wave not Propagate into a Conductor?

To answer this question we need only consider the simple circuit where a condenser C discharges through a resistance R. The voltage equation gives

$$\frac{q}{C} + IR = 0$$

and since $I = \mathrm{d}q/\mathrm{d}t$, we have

$$\frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{q}{RC} \quad \text{or} \quad q = q_0\,\mathrm{e}^{-t/RC}$$

where q_0 is the initial charge.

We see that an electric field will exist between the plates of the condenser only for a time $t \sim RC$ and will disappear when the charge has had time to distribute itself uniformly throughout the circuit. An electric field can only exist in the presence of a non-uniform charge distribution.

If we take a slab of any medium and place a charge of density q at a point within the slab, the medium will behave as an RC circuit and the equation

$$q = q_0\,\mathrm{e}^{-t/RC}$$

becomes

$$q = q_0\,\mathrm{e}^{-\sigma/\omega\varepsilon} \rightarrow q_0\,\mathrm{e}^{-\sigma t/\varepsilon}\left(\frac{\varepsilon \equiv C}{\sigma \equiv 1/R}\right)$$

The charge will distribute itself uniformly in a time $t \sim \varepsilon/\sigma$, and the electric field will be maintained for that time only. The time ε/σ is called the *relaxation time* of the medium (RC time of the electrical circuit) and it is a measure of the maximum time for which an electric field can be maintained before the charge distribution becomes uniform.

Any electric field of a frequency ν, where $1/\nu = t > \varepsilon/\sigma$, will not be maintained; only a high frequency field where $1/\nu = t < \varepsilon/\sigma$ will establish itself.

Impedance of a Conducting Medium to Electromagnetic Waves

The impedance of a lossless medium is a real quantity. For the transmission line of Chapter 7 the characteristic impedance

$$Z_0 = \frac{V_+}{I_+} = \sqrt{\frac{L_0}{C_0}}\,\Omega\,;$$

for an electromagnetic wave in a dielectric

$$Z = \frac{E_x}{H_y} = \sqrt{\frac{\mu}{\varepsilon}}\,\Omega$$

with E_x and H_y in phase.

We saw in the case of the transmission line that when the loss mechanisms of a series resistance R_0 and a shunt conductance G_0 were introduced the impedance became the complex quantity

$$\mathbf{Z} = \sqrt{\frac{R_0 + \mathrm{i}\omega L_0}{G_0 + \mathrm{i}\omega C_0}}$$

We now ask what will be the impedance of a conducting medium of properties μ, ε and σ to electromagnetic waves? If the ratio of E_x to H_y is a complex quantity, it implies that a phase difference exists between the two field vectors.

We have already seen that in a conductor

$$E_x = E_0\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{-\gamma z}$$

where $\gamma = (1+\mathrm{i})\,(\omega\mu\sigma/2)^{1/2}$, and we shall now write $H_y = H_0\,\mathrm{e}^{\mathrm{i}(\omega t - \phi)}\,\mathrm{e}^{-\gamma z}$, suggesting that H_y lags E_x by a phase angle ϕ. This gives the impedance of the conductor as

$$\mathbf{Z}_c = \frac{E_x}{H_y} = \frac{E_0}{H_0}\,\mathrm{e}^{\mathrm{i}\phi}$$

Equation (8.1a) gives

$$\frac{\partial E_x}{\partial z} = -\mu\frac{\partial H_y}{\partial t}$$

so that

$$-\gamma E_x = -\mathrm{i}\omega\mu H_y$$

and

$$\begin{aligned}
\mathbf{Z}_c &= \frac{E_x}{H_y} = \frac{\mathrm{i}\omega\mu}{\gamma} = \frac{\mathrm{i}(\omega\mu)}{(1+\mathrm{i})(\omega\mu\sigma/2)^{1/2}} = \frac{\mathrm{i}(1-\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})}\left(\frac{2\omega\mu}{\sigma}\right)^{1/2} \\
&= \frac{(1+\mathrm{i})}{2}\left(\frac{2\omega\mu}{\sigma}\right)^{1/2} = \frac{1+\mathrm{i}}{\sqrt{2}}\left(\frac{\omega\mu}{\sigma}\right)^{1/2} \\
&= \left(\frac{\omega\mu}{\sigma}\right)^{1/2}\left(\frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}\right) = \left(\frac{\omega\mu}{\sigma}\right)^{1/2}\mathrm{e}^{\mathrm{i}\phi}
\end{aligned}$$

a vector of magnitude $(\omega\mu/\sigma)^{1/2}$ and phase angle $\phi = 45°$. Thus the magnitude

$$Z_c = \frac{E_0}{H_0} = \left(\frac{\omega\mu}{\sigma}\right)^{1/2}$$

and H_y lags E_x by 45°.

We can also express $\mathbf{Z}_c$ by

$$\mathbf{Z}_c = R + \mathrm{i}X = \left(\frac{\omega\mu}{2\sigma}\right)^{1/2} + \mathrm{i}\left(\frac{\omega\mu}{2\sigma}\right)^{1/2}$$

and also write it

$$\begin{aligned}\mathbf{Z}_c &= \frac{1+\mathrm{i}}{\sqrt{2}}\left(\frac{\omega\mu}{\sigma}\right)^{1/2} \\ &= \sqrt{\frac{\mu_0}{\varepsilon_0}\frac{\varepsilon_0}{\varepsilon}\frac{\mu}{\mu_0}\frac{\omega\varepsilon}{\sigma}}\mathrm{e}^{\mathrm{i}\phi}\end{aligned}$$

of magnitude

$$|Z_c| = 376.6\,\Omega\sqrt{\frac{\mu_r}{\varepsilon_r}}\sqrt{\frac{\omega\varepsilon}{\sigma}}$$

At a wavelength $\lambda = 10^{-1}$ m, i.e. at a frequency $\nu = 3000$ MHz, the value of $\omega\varepsilon/\sigma$ for copper is 2.9×10^{-9} and $\mu_r \approx \varepsilon_r \approx 1$. This gives a magnitude $Z_c = 0.02\,\Omega$ at this frequency; for $\sigma = \infty$, $Z_c = 0$, and the electric field vector E_x vanishes, so we can say that when Z_c is small or zero the conductor behaves as a short circuit to the electric field. This sets up large conduction currents and the magnetic energy is increased.

In a dielectric, the impedance

$$Z = \frac{E_x}{H_y} = \sqrt{\frac{\mu}{\varepsilon}}$$

led to the equivalence of the electric and magnetic field energy densities; that is, $\frac{1}{2}\mu H_y^2 = \frac{1}{2}\varepsilon E_x^2$. In a conductor, the magnitude of the impedance

$$Z_c = \left|\frac{E_x}{H_y}\right| = \left(\frac{\omega\mu}{\sigma}\right)^{1/2}$$

so that the ratio of the magnetic to the electric field energy density in the wave is

$$\frac{\frac{1}{2}\mu H_y^2}{\frac{1}{2}\varepsilon E_x^2} = \frac{\mu}{\varepsilon}\frac{\sigma}{\omega\mu} = \frac{\sigma}{\omega\varepsilon}$$

We already know that this ratio is very large for a conductor for it is the ratio of conduction to displacement currents, so that in a conductor the magnetic field energy dominates the electric field energy and increases as the electric field energy decreases.

Reflection and Transmission of Electromagnetic Waves at a Boundary

Normal Incidence

An infinite plane boundary separates two media of impedances Z_1 and Z_2 (real or complex) in Figure 8.7.

The electromagnetic wave normal to the boundary has the components shown where subscripts i, r and t denote incident, reflected and transmitted, respectively. Note that the vector direction $(\mathbf{E}_r \times \mathbf{H}_r)$ must be opposite to that of $(\mathbf{E}_i \times \mathbf{H}_i)$ to satisfy the energy flow condition of the Poynting vector.

The boundary conditions, from electromagnetic theory, are that the components of the field vectors **E** and **H** tangential or parallel to the boundary are continuous across the boundary.

Thus

$$E_i + E_r = E_t$$

and

$$H_i + H_r = H_t$$

where

$$\frac{E_i}{H_i} = Z_1, \ \frac{E_r}{H_r} = -Z_1 \quad \text{and} \quad \frac{E_t}{H_t} = Z_2$$

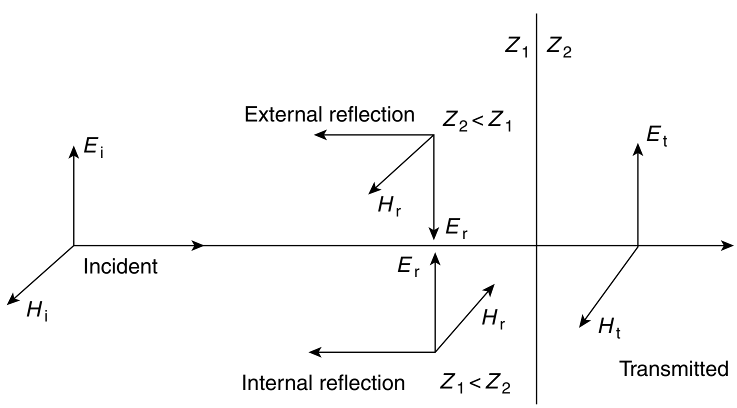


Figure 8.7 Reflection and transmission of an electromagnetic wave incident normally on a plane between media of impedances Z_1 and Z_2. The Poynting vector of the reflected wave $(\mathbf{E} \times \mathbf{H})_r$ shows that either **E** or **H** may be reversed in phase, depending on the relative magnitudes of Z_1 and Z_2

From these relations it is easy to show that the amplitude reflection coefficient

$$R = \frac{E_\mathrm{r}}{E_\mathrm{i}} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

and the amplitude transmission coefficient

$$T = \frac{E_\mathrm{t}}{E_\mathrm{i}} = \frac{2Z_2}{Z_2 + Z_1}$$

in agreement with the reflection and transmission coefficients we have found for the acoustic pressure p (Chapter 6) and voltage V (Chapter 7). If the wave is travelling in air and strikes a perfect conductor of $Z_2 = 0$ at normal incidence then

$$\frac{E_\mathrm{r}}{E_\mathrm{i}} = \frac{Z_2 - Z_1}{Z_2 + Z_1} = -1$$

giving complete reflection and

$$\frac{E_\mathrm{t}}{E_\mathrm{i}} = \frac{2Z_2}{Z_2 + Z_1} = 0$$

Thus, good conductors are very good reflectors of electromagnetic waves, e.g. lightwaves are well reflected from metal surfaces. (See Summary on p. 550.)

Oblique Incidence and Fresnel's Equations for Dielectrics

When the incident wave is oblique and not normal to the infinite boundary of Figure 8.7 we may still use the boundary conditions of the preceding section for these apply to the *tangential* components of **E** and **H** at the boundary and remain valid.

In Figure 8.8(a) **H** is perpendicular to the plane of the paper with tangential components H_i, H_r and H_t but the tangential components of **E** become

$$E_\mathrm{i} \cos\theta,\ E_\mathrm{r} \cos\theta \quad \textit{and} \quad E_\mathrm{t} \cos\phi, \text{ respectively.}$$

In Figure 8.8(b) **E** is perpendicular to the plane of the paper with tangential components E_i, E_r and E_t but the tangential components of **H** become $H_\mathrm{i} \cos\theta$, $H_\mathrm{r} \cos\theta$ and $H_\mathrm{t} \cos\phi$.

Using these components in the expressions for the reflextion and transmission coefficients we have, for Figure 8.8(a)

$$\frac{E_\mathrm{r} \cos\theta}{E_\mathrm{i} \cos\theta} = \frac{E_\mathrm{t} \cos\phi/H_\mathrm{t} - E_\mathrm{i} \cos\theta/H_\mathrm{i}}{E_\mathrm{t} \cos\phi/H_\mathrm{t} + E_\mathrm{i} \cos\theta/H_\mathrm{i}}$$

so

$$R_\parallel = \frac{E_\mathrm{r}}{E_\mathrm{i}} = \frac{Z_2 \cos\phi - Z_1 \cos\theta}{Z_2 \cos\phi + Z_1 \cos\theta}$$

where $R_\parallel$ is the reflection coefficient amplitude when **E** lies in the plane of incidence.

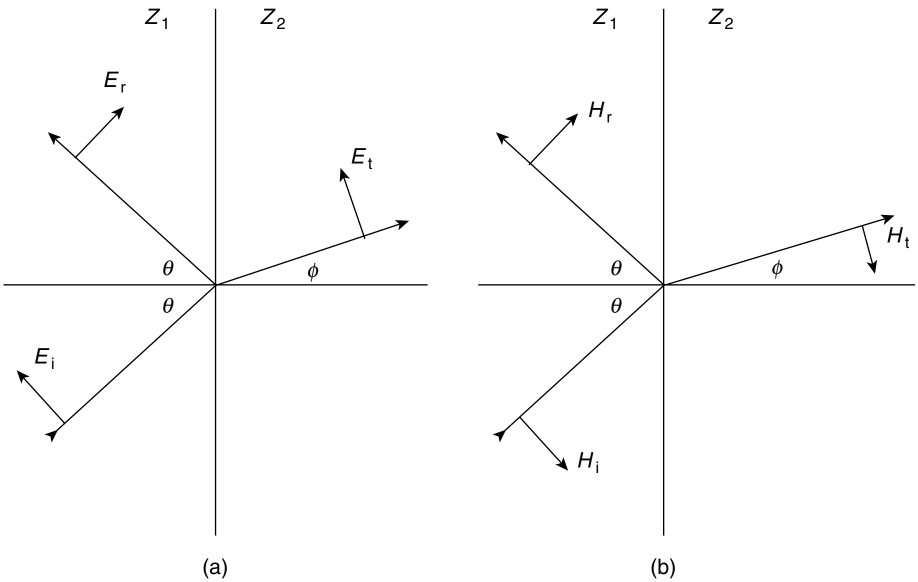


Figure 8.8 Incident, reflected and transmitted components of a plane polarized electromagnetic wave at oblique incidence to the plane boundary separating media of impedances Z_1 and Z_2. The electric vector lies in the plane of incidence in (a) and is perpendicular to the plane of incidence in (b)

For the transmission coefficient in Figure 8.8(a)

$$\frac{E_t \cos\phi}{E_i \cos\theta} = \frac{2E_t \cos\phi/H_t}{E_i \cos\theta/H_i + E_t \cos\phi/H_t}$$

so

$$T_{\parallel} = \frac{E_t}{E_i} = \frac{2Z_2 \cos\theta}{Z_1 \cos\theta + Z_2 \cos\phi}$$

A similar procedure for Figure 8.8(b) where **E** is perpendicular to the plane of incidence yields

$$R_{\perp} = \frac{Z_2 \cos\theta - Z_1 \cos\phi}{Z_2 \cos\theta + Z_1 \cos\phi}$$

and

$$T_{\perp} = \frac{2Z_2 \cos\theta}{Z_2 \cos\theta + Z_1 \cos\phi}$$

Now the relation between the refractive index n of the dielectric and its impedance Z is given by

$$n = \frac{c}{v} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \sqrt{\varepsilon_r} = \frac{Z(\text{free space})}{Z\,(\text{dielectric})}$$

where

$$\frac{\mu}{\mu_0} = \mu_r \approx 1.$$

Hence we have

$$\frac{Z_1}{Z_2} = \frac{n_2}{n_1} = \frac{\sin \theta}{\sin \phi}$$

from Snell's Law and we may write the reflection and transmission amplitude coefficients as

$$R_{\parallel} = \frac{\tan(\phi - \theta)}{\tan(\phi + \theta)}, \qquad T_{\parallel} = \frac{4 \sin \phi \cos \theta}{\sin 2\phi + \sin 2\theta}$$
$$R_{\perp} = \frac{\sin(\phi - \theta)}{\sin(\phi + \theta)}, \qquad T_{\perp} = \frac{2 \sin \phi \cos \theta}{\sin(\phi + \theta)}$$

In this form the expressions for the coefficients are known as Fresnel's Equations. They are plotted in Figure 8.9 for $n_2/n_1 = 1.5$ and they contain several significant features.

When θ is very small and incidence approaches the normal we have $\theta \to 0$ and $\phi \to 0$ so that

$$\sin(\phi - \theta) \sim \tan(\phi - \theta) \sim (\phi - \theta)$$

and

$$R_{\parallel} \sim R_{\perp} \sim \frac{(\phi - \theta)}{(\phi + \theta)} \sim \frac{\dfrac{1}{n_2} - \dfrac{1}{n_1}}{\dfrac{1}{n_2} + \dfrac{1}{n_1}} = \frac{n_1 - n_2}{n_1 + n_2}$$

Thus, the reflected intensity

$$R^2_{\theta \to 0} = \frac{I_r}{I_i} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$
$$\sim 0.4 \quad \text{at an air-glass interface.}$$

We note also that when $\tan(\theta + \phi) = \infty$ and $\theta + \phi = 90^\circ$ then $R_{\parallel} = 0$.

In this case only $R_{\perp}$ is finite and the reflected light is completely plane polarized with the electric vector perpendicular to the plane of incidence. This condition defines the value of the Brewster or polarizing angle θ_B for, when θ and ϕ are complementary $\cos \theta_B = \sin \phi$ so

$$n_1 \sin \theta_B = n_2 \sin \phi = n_2 \cos \theta_B$$

and

$$\tan \theta_B = n_2/n_1$$

which, for air to glass defines $\theta_B = 56^\circ$.

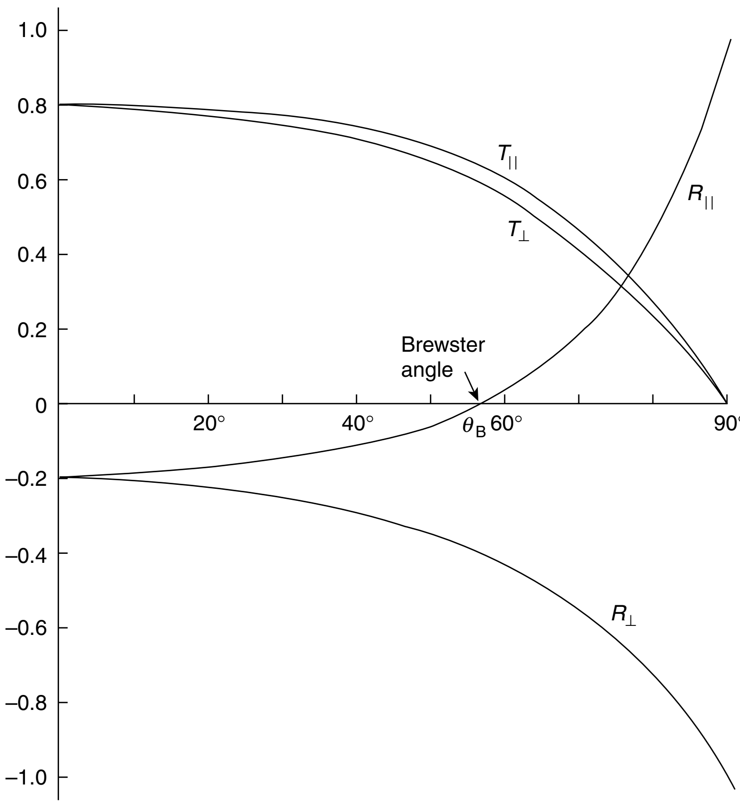


Figure 8.9 Amplitude coefficient R and T of reflection and transmission for $n_2/n_1 = 1.5$. $R_\parallel$ and $T_\parallel$ refer to the case when the electric field vector E lies in the plane of incidence. $R_\perp$ and $T_\perp$ apply when E is perpendicular to the plane of incidence. The Brewster angle θ_B defines $\theta + \phi = 90°$ when $R_\parallel = 0$ and the reflected light is polarized with the E vector perpendicular to the plane of incidence. $R_\parallel$ changes sign (phase) at θ_B. When $\theta < \theta_B$, tan $(\phi - \theta)$ is negative for $n_2/n_1 = 1.5$. When $\theta + \phi \geq 90°$, tan $(\phi + \theta)$ is also negative

A typical modern laboratory use of the Brewster angle is the production of linearly polarized light from a He-Ne laser. If the window at the end of the laser tube is tilted so that the angle of incidence for the emerging light is θ_B and $R_\parallel = 0$, then the light with its electric vector parallel to the plane of incidence is totally transmitted while some of the light with transverse polarization ($R_\perp$) is reflected back into the laser off-axis. If the light makes multiple transits along the length of the tube before it emerges the transmitted beam is strongly polarized in one plane.

More general but less precise uses involve the partial polarization of light reflected from wet road and other shiny surfaces where refractive indices are in the range $n = 1.3 - 1.6$. Polarized windscreens and spectacles are effective in reducing the glare from such reflections.

Reflection from a Conductor (Normal Incidence)

For Z_2 a conductor and Z_1 free space, the refractive index

$$n = \frac{Z_1}{Z_2} = \frac{\beta}{\alpha + \mathrm{i}\alpha}$$

is complex, where

$$\beta = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

and

$$\alpha = \left(\frac{\omega\mu}{2\sigma}\right)^{1/2}$$

A complex refractive index must always be interpreted in terms of absorption because a complex impedance is determined by a complex propagation constant, e.g. here $Z_2 = \mathrm{i}\omega\mu/\gamma$, so that

$$n = \frac{Z_1}{Z_2} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{1}{\mathrm{i}\omega\mu}(1+\mathrm{i})\left(\frac{\omega\mu\sigma}{2}\right)^{1/2} = (1-\mathrm{i})\left(\frac{\sigma}{2\omega\varepsilon_0}\right)^{1/2}$$

where

$$\frac{(\mu\mu_0)^{1/2}}{\mu} \approx 1$$

The ratio $E_\mathrm{r}/E_\mathrm{i}$ is therefore complex (there is a phase difference between the incident and reflected vectors) with a value

$$\frac{E_\mathrm{r}}{E_\mathrm{i}} = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{\alpha + \mathrm{i}\alpha - \beta}{\alpha + \mathrm{i}\alpha + \beta} = \frac{1 - \beta/\alpha + \mathrm{i}}{1 + \beta/\alpha + \mathrm{i}}$$

where $\beta/\alpha \gg 1$.

Since $E_\mathrm{r}/E_\mathrm{i}$ is complex, the value of the reflected intensity $I_\mathrm{r} = (E_\mathrm{r}/E_\mathrm{i})^2$ is found by taking the ratio the squares of the moduli of the numerator and the denominator, so that

$$\begin{aligned} I_\mathrm{r} &= \frac{|E_\mathrm{r}|^2}{|E_\mathrm{i}|^2} = \frac{|Z_2 - Z_1|^2}{|Z_2 + Z_1|^2} = \frac{(1-\beta/\alpha)^2 + 1}{(1+\beta/\alpha)^2 + 1} \\ &= 1 - \frac{4\beta/\alpha}{2 + 2\beta/\alpha + (\beta/\alpha)^2} \to 1 - \frac{4\alpha}{\beta} \quad (\text{for } \beta/\alpha \gg 1) \end{aligned}$$

so that

$$I_r = 1 - 4\left(\frac{\omega\mu}{2\sigma}\right)^{1/2}\left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2} \approx 1 - 2\sqrt{\frac{2\omega\varepsilon_0}{\sigma}}$$

For copper $\sigma = 6 \times 10^7 (\text{ohm m}^{-1})$ and $(2\omega\varepsilon_0/\sigma)^{1/2} \approx 0.01$ at infra-red frequencies. The emission from an electric heater at 10^3K has a peak at $\lambda \approx 2.5 \times 10^{-6}$m. A metal reflector behind the heater filament reflects $\approx 97\%$ of these infra-red rays with 3% entering the metal to be lost as Joule heating between the metal surface and the skin depth. (see Problem 8.20)

(Problems 8.16, 8.17, 8.18, 8.19, 8.20, 8.21, 8.22, 8.23, 8.24)

Electromagnetic Waves in a Plasma

We saw in Problem 1.4 that when an electron in an atom or, quantum mechanically the charge centre of an electron cloud, moves a small distance from its equilibrium position, the charge separation creates an electric field which acts as a linear restoring force and the resulting motion is simple harmonic with an angular frequency ω_0. For a hydrogen atom

$$\omega_0 \approx 4.5 \times 10^{16} \text{ rad s}^{-1}$$

When a steady electric field is applied to a dielectric, the resulting charge separation between an electron and the rest of its atom induces a polarization field of magnitude

$$P = \frac{n_e ex}{\varepsilon_0}$$

where P defines the dipole moment per unit volume. Here, n_e is the electron number density, x is the displacement from equilibrium and ε_0 is the permittivity of free space.

The value of P per unit electric field is called the susceptibility

$$\chi = \frac{n_e ex}{\varepsilon_0 E}$$

and the permittivity of the dielectric is given by

$$\varepsilon = \varepsilon_0(1 + \chi)$$

The relative permittivity or dielectric constant

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = (1 + \chi) = \left(1 + \frac{n_e ex}{\varepsilon_0 E}\right) \tag{8.7}$$

A steady electric field E defines a static susceptibility. An alternating electric field E defines a dynamic susceptibility in which case the relative permittivity.

$$\varepsilon_r = n^2$$

where n is the refractive index of the medium.

There may be resistive or damping effects to the electric field within the medium and it is here that our discussion of the forced damped oscillator on p. 66 becomes significant (see Figure 3.9).

If the electric field is that of an electromagnetic wave of angular frequency ω we have $E = E_0 e^{i\omega t}$ and the value of x in equatin (8.7) is that given by equation (3.2) on p. 67 representing curve (a) in Figure 3.9 where F_0 is now the force Ee acting on each electron.

Equation (8.7) now becomes

$$\varepsilon_r = 1 + \chi = 1 + \frac{n_e e^2 m_e(\omega_0^2 - \omega^2)}{\varepsilon_0[m_e^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2]}$$

where m_e is the electron mass, ω_0 is its harmonic frequency within the atom, ω is the electromagnetic wave frequency and r is the damping constant.

This is the solution given to problem 3.10.

Note that for

$$\omega \ll \omega_0$$
$$\varepsilon_r \approx 1 + \frac{n_e e^2}{\varepsilon_0 m_e \omega_0^2} \tag{8.8}$$

and for

$$\omega \gg \omega_0$$
$$\varepsilon_r \approx 1 - \frac{n_e e^2}{\varepsilon_0 m_e \omega^2} \tag{8.9}$$

The factor $n_e e^2/\varepsilon_0 m_e$ in the second term of ε_r has a particular significance if the material is not a solid but an ionized gas called a plasma. Such a gas consists of ions and electrons of equal number densities $n_i = n_e$ with charges of opposite signs $\pm$e and masses m_i and m_e where $m_i \gg m_e$. Relative displacements between ions and electrons set up a restoring electric field which returns the electrons to equilibrium. The relatively heavy ions are considered as stationary. The result in Figure 8.10 shows a sheet of negative charge $-n_e ex$

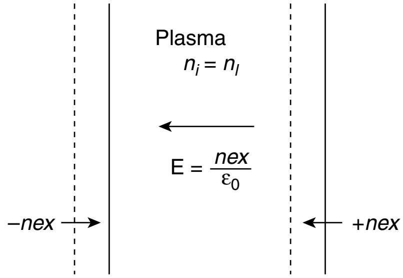


Figure 8.10 In an ionized gas with equal number densities of ions and electrons ($n_i = n_e$) and $m_i \gg m_e$, relative displacements between ions and electrons form thin sheaths of charge $\pm nex$, which generate an electric field $E = nex/\varepsilon_0$ acting on each electron. The motion of each electron is simple harmonic with an electron plasma frequency ω_p where $\omega_p^2 = n_e e^2/\varepsilon_0 m_e$ rad s^{-1}

per unit area on one side of the plasma slab with the stationary ions producing a sheet of positive charge $+n_e ex$ on the other side (where $n_i = n_e$).

This charge separation generates an electric field E in the plasma of magnitude

$$E = \frac{n_e ex}{\varepsilon_0}$$

which produces an electric force $-n_e e^2 x/\varepsilon_0$ acting on each electron in the direction of its equilibrium position.

The equation of motion of each electron is therefore

$$m_e \ddot{x} + \frac{n_e e^2 x}{\varepsilon_0} = 0$$

and the electron motion is simple harmonic with an angular frequency ω_p where

$$\omega_p^2 = \frac{n_e e^2}{\varepsilon_0 m_e}$$

The angular frequency ω_p is called the electron plasma frequency and plays a significant role in the propagation of electromagnetic waves in the plasma.

In the expression for the refractive index

$$\varepsilon_r = n^2 \approx 1 + \frac{\omega_p^2}{\omega_0^2} \tag{8.8}$$

n is real for all values of ω and waves of that frequency will propagate. However, when

$$\varepsilon_r = n^2 \approx 1 - \frac{\omega_p^2}{\omega^2} \tag{8.9}$$

waves will propagate only when $\omega > \omega_p$

When $\omega_p^2/\omega^2 > 1$

$$n^2 = \frac{c^2}{v^2} = \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

is negative and the wave number k is considered to be complex with

$$k = k' - i\alpha.$$

In this case, electromagnetic waves incident on the plasma will be attenuated within the plasma, or if α is large enough, will be reflected at the plasma surface.

The electric field of the wave $E = E_0\,\mathrm{e}^{\mathrm{i}(\omega t - kz)}$ becomes $E = E_0 e^{-\alpha z}\,\mathrm{e}^{\mathrm{i}(\omega t - kz)}$ and is reduced to $E_0 e^{-1}$ when $z = 1/\alpha = \delta$ the penetration depth. When $\alpha \gg k'$, the penetration is extremely small and

$$k^2 \to -\alpha^2 = \left(1 - \frac{\omega_{\mathrm{p}}}{\omega^2}\right)\frac{\omega^2}{c^2}$$

so that

$$\alpha^2 = \frac{\omega_{\mathrm{p}}^2}{c^2}\left(1 - \frac{\omega^2}{\omega_{\mathrm{p}}^2}\right)$$

and

$$\delta = \frac{1}{\alpha} = \frac{c}{\omega_{\mathrm{p}}}\left(1 - \frac{\omega^2}{\omega_{\mathrm{p}}^2}\right)^{-1/2}$$

When

$$\omega \ll \omega_{\mathrm{p}},\ \delta \approx c/\omega_{\mathrm{p}}$$

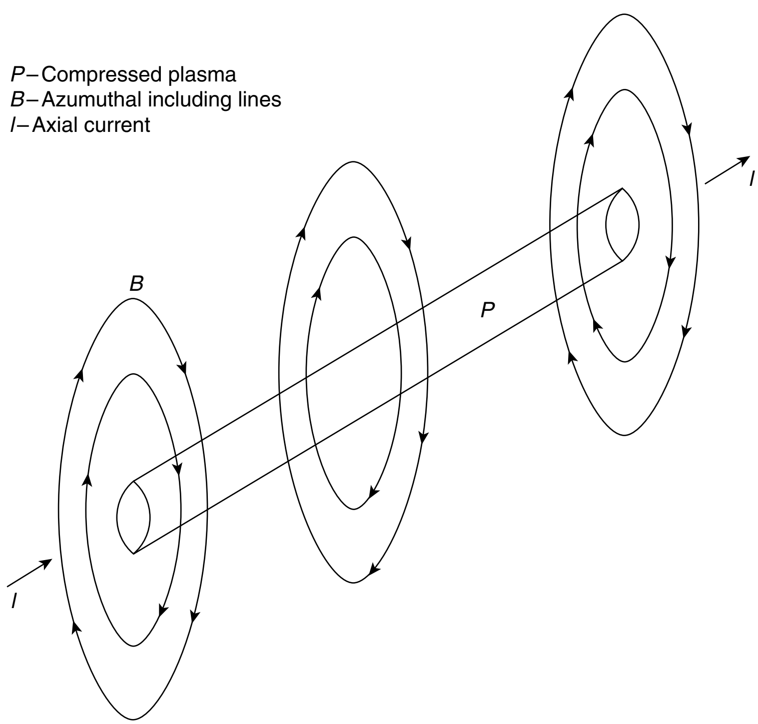


Figure 8.11 The pinch effect. A plasma is formed when a large electrical current I is discharged along the axis of a cylindrical tube of gas. The azimuthal magnetic field lines compress the plasma and when the conductivity of the plasma is very high the penetration of the field lines into the plasma is very small

On a laboratory scale number densities of the order $n_e \approx 10^{-6}$–10^{-10} m^{-3} are produced with electron plasma frequencies in the range $\omega_p \approx 6 \times 10^{10}$–$6 \times 10^{12}$ rad s^{-1}, several orders below that of visible light.

For these values of ω_p, electromagnetic waves have a penetration depth

$$\delta \approx \frac{c}{\omega_p} \approx 5 \times 10^{-3}\text{–}\, 5 \times 10^{-5}\,\mathrm{m}$$

The analysis above provides an experimental method of measuring the electron number density of a plasma using electromagnetic waves as a probe. The angular frequency of the transmitted wave is varied until propagation no longer occurs and a reflected wave is detected.

The rejection of magnetic fields by a plasma is exploited in laboratory experiments on controlled thermonuclear fusion. In these a strong magnetic induction $\mathbf{B}$ is used as the confining mechanism to keep the plasma from the walls of its containing vessel. The magnetic energy per unit volume is given by $B^2/2\mu$ and this has the dimensions of a pressure which opposes and often exceeds that of the hot ionized gas.

The well-known ‘pinch effect’, Figure 8.11, results when a large current is discharged along the axis of gas contained in a cylindrical tube. The current ionizes the gas and its azimuthal field compresses the plasma in the radial direction towards the axis, increasing its temperature even further. Typical magnitudes in such an experiment are $T \sim 10^8$ K and $n_e \sim 10^{21}$ m^{-3}. This corresponds to a pressure of ~ 14 atmospheres which requires a discharge current $\sim 10^3 R$ A where R m is the radius of the cylinder.

Electromagnetic Waves in the Ionosphere

The simple expression

$$n^2 = 1 - \frac{\omega_p^2}{\omega^2} \tag{8.9}$$

for the index of refraction of a plasma is modified by the presence of an external static magnetic field. This situation exists in the ionosphere which consists of bands of low density ionized gas approximately 300 km above the earth and located within the earth’s dipole field of magnetic induction $\mathbf{B}_0$.

A charged particle of velocity $\mathbf{v}$ in such a field experiences an electric field $\mathbf{E} = \mathbf{v} \times \mathbf{B}_0$ and when $\mathbf{v}$ is in the plane perpendicular to $\mathbf{B}_0$ it rotates around the field line with an angular frequency $\omega = eB_0/m$, where e is the particle charge and m is its mass. This is most easily seen by considering the force mv^2/r in a circular orbit balancing the electric force $\varepsilon\mathbf{E} = e \cdot \mathbf{v} \times \mathbf{B}_0$.

From $mv^2/r = ev\mathbf{B}_0$
we have

$$\frac{v}{r} = \frac{eB_0}{m} = 2\pi\left(\frac{v}{2\pi r}\right) = 2\pi f = \omega_{\mathbf{B}}$$

where f is the frequency of precession or the number of orbits per second made by the particle.

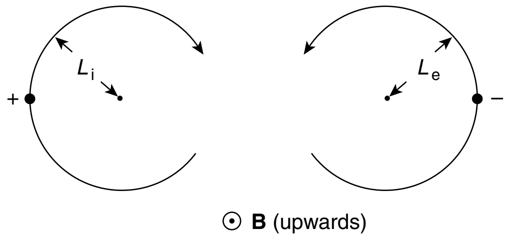


Figure 8.12 Charged particles of velocity **v** perpendicular to a magnetic field line **B** are bound to the field line and orbit around it due to the Lorentz force $e(\mathbf{v} \times \mathbf{B})$. The radius L of the orbit, the Larmor radius, is given by $L = mv/eB$ and the orbital Larmor frequency is $\omega_{\mathbf{B}} = eB/m$ rad s^{-1}

Figure 8.12 shows the direction of motion for positive and negative charges around a magnetic field line which points upwards out of the paper.

We consider the simplest case of electromagnetic wave propagation along the direction $\mathbf{B}_0$ and assume that

- The amplitude of electron motion is small.
- The value of n_e is low enough to neglect collisional damping.
- The magnetic induction $\mathbf{B}_0 \gg$ the magnetic induction of the electromagnetic wave.

If we consider the electric field to be that of a circularly polarized transverse electromagnetic wave, then we may write $\mathbf{E} = E(\mathbf{r}_1 + \mathrm{i}\mathbf{r}_2)$, where $\mathbf{r}_1$ and $\mathbf{r}_2$ are orthogonal (mutually perpendicular) unit vectors and $\mathbf{B}_0$ is along the $\mathbf{r}_3$ direction.

The equation of motion for an electron of velocity **v** is given by

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{E}\,\mathrm{e}^{\mathrm{i}\omega t} + e\mathbf{v} \times \mathbf{B}_0$$

If we take the steady state electron velocity to be of the form

$$\mathbf{v} = v(\mathbf{r}_1 + \mathrm{i}\mathbf{r}_2)\,\mathrm{e}^{\mathrm{i}\omega t}$$

we find that

$$v = \frac{-\mathrm{i}e}{m(\omega \pm \omega_{\mathbf{B}})}E$$

satisfies the equation of motion

This means that the electron precessing around $\mathbf{B}_0$ with an angular frequency $\omega_{\mathbf{B}}$ is driven by a rotating electric field of effective frequency $\omega \pm \omega_{\mathbf{B}}$ depending on the sign of the circular polarization.

Due to the electronic motion there is a current density in the plasma given by

$$\mathbf{J} = n_e e\mathbf{v} = \frac{-\mathrm{i}n_e e^2}{m(\omega \pm \omega_{\mathbf{B}})}\mathbf{E}.$$

In Maxwell's equation

$$\nabla \times \mathbf{H} = \frac{\partial}{\partial t}\mathbf{D} + \mathbf{J} \tag{8.5}$$

we may write, in the absence of $\mathbf{J}$, $\mathbf{D} = \varepsilon_0 \mathbf{E}$ but the presence of $\mathbf{J}$ will modify this and the right hand side of equation (8.5) becomes

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{D} + \mathbf{J} &= \frac{\partial}{\partial t}\varepsilon_0 E\,\mathrm{e}^{\mathrm{i}\omega t} - \frac{\mathrm{i}n_\mathrm{e}e^2}{m(\omega \pm \omega_\mathbf{B})}\mathbf{E} \\ &= \mathrm{i}\omega\varepsilon_0\mathbf{E} - \frac{\mathrm{i}n_\mathrm{e}e^2}{m\varepsilon_0(\omega \pm \omega_\mathbf{B})}\varepsilon_0\mathbf{E} \\ &= \mathrm{i}\omega\varepsilon_0\left(1 - \frac{\omega_\mathrm{p}^2}{\omega(\omega \pm \omega_\mathbf{B})}\right)\mathbf{E} = \mathrm{i}\omega\varepsilon\mathbf{E}\end{aligned}$$

giving

$$\frac{\varepsilon}{\varepsilon_0} = \varepsilon_\mathrm{r} = n_\pm^2 = \left(1 - \frac{\omega_\mathrm{p}^2}{\omega(\omega \pm \omega_\mathbf{B})}\right)$$

We see that the ionosphere is birefringent with two different values of the refractive index, n_+ for the right handed circularly polarized wave and n_- for the left handed incident polarization. These waves propagate at different velocities and their reception by the ionosphere will depend on their polarization. In its lower D layer the ionosphere has an electron number density $n_\mathrm{e} \leq 10^9\,\mathrm{m}^{-3}$ with $\omega_\mathrm{p} \approx 10^6\,\mathrm{rad\,s}^{-1}$ and for the upper F_2 layer, $n_\mathrm{e} \leq 10^{12}\,\mathrm{m}^{-3}$ with $\omega_\mathrm{p} \approx 10^7\,\mathrm{rad\,s}^{-1}$. Taking the value of the earth's magnetic field as 3×10^{-5} T; that is (0.3 G) gives an electron precession frequency $\omega_\mathbf{B} \approx 6 \times 10^6\,\mathrm{rad\,s}^{-1}$.

Figure 8.13 shows the behaviour of n_+^2 and n_-^2 versus $\omega/\omega_\mathbf{B}$ give for the fixed value of $\omega_\mathrm{p}/\omega_\mathbf{B} = 2$. Other values of $\omega_\mathrm{p}/\omega_\mathbf{B}$ give curves of a similar shape. In the wide frequency intervals where n_+^2 and n_-^2 have opposite signs (positive or negative), one state of the circular polarization cannot propagate in the plasma and will be reflected when it strikes the ionosphere. The other wave will be partially transmitted. So, when a linearly polarized wave with $\omega \leq \omega_\mathbf{B}$ in Figure 8.14 is incident on the ionosphere, the reflected wave will be elliptically polarized. The electron number densities in the ionosphere are measured by varying the frequency ω of the transmitted electromagnetic waves until reflection occurs. This method is similar to that used on the laboratory plasmas of the previous section. However, the value of n_e varies in an ionospheric layer. It is found to increase with height until it reaches a maximum, only to fall off rapidly with a further increase in height. The height for a particular value of n_e is measured by timing the interval between the transmitted and reflected wave.

The analysis above explains the main features of radio reception which are:

- Very high frequencies (VHF) are received over relatively short distances only.
- Medium wave (MW) reception is possible over longer distances and improves at night.
- Short wave (SW) reception is possible over very long distances.

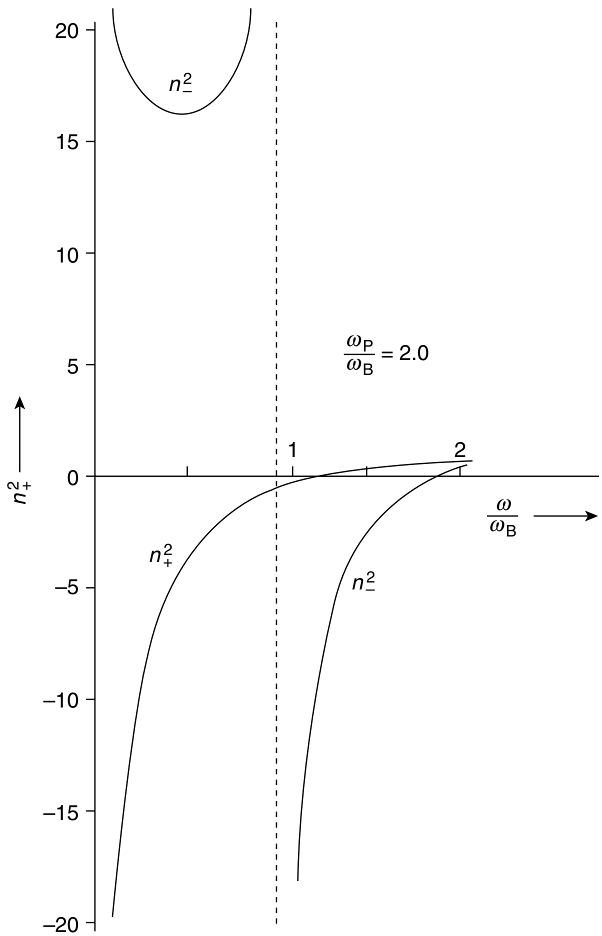


Figure 8.13 The ionospheric plasma is birefringent to electromagnetic waves with different values of the refractive index n_+ for right handed circularly polarized waves and n_- for left handed circularly polarized waves. These values depend upon the ratio of the plasma frequency ω_p to the Larmor frequency $\omega_{\mathbf{B}}$. Graphs of n_+^2 and n_-^2 are shown for a fixed value $\omega_p/\omega_{\mathbf{B}} = 2$ with a horizontal axis $\omega/\omega_{\mathbf{B}}$, where ω is the frequency of the propagating e.m. wave

Very high frequencies are greater than ω_p for both the D and F_2 layers; the waves propagate through both layers without reflection (Figure 8.15). The D layer has a plasma frequency ~300 kHz; that is, a wavelength of ~ 1 km and medium waves with $200 < \lambda < 600$ km are attenuated within it. However, the electron number density in the D layer, sustained by ionizing radiation during the day, drops very sharply after sunset and the medium waves are transmitted to the higher F_2 layer where they are reflected and received over longer distances. The D layer is transparent to short waves, $10 < \lambda < 80$ m, but these are reflected by the layer F_2 allowing long-distance radio reception around the earth.

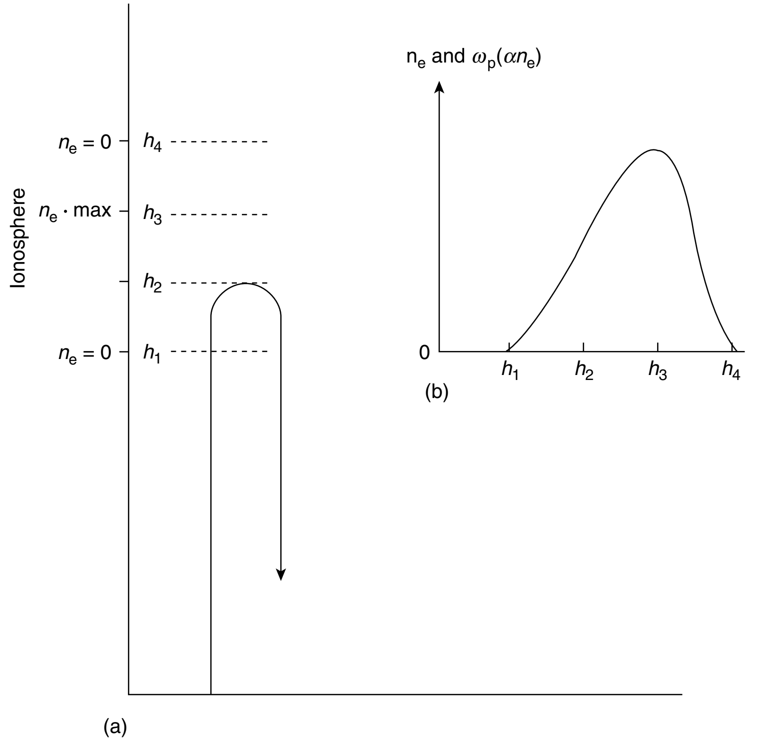


Figure 8.14 (a) The number density n_e of a plasma (in this case the ionosphere) may be measured by a probing electromagnetic wave, the frequency of which is varied until reflection occurs. The time of the wave from transmission to reception is a measure of the height at which reflection occurs. The variation of number density n_e with height h in an ionospheric layer is shown in (b)

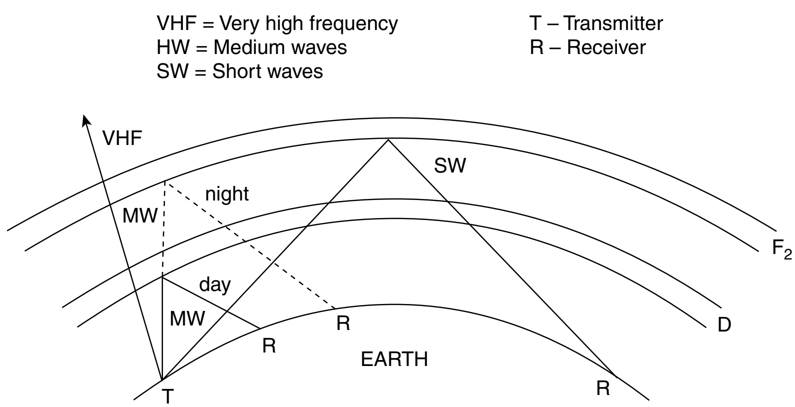


Figure 8.15 Electron number densities in the ionosphere layers D and F_2 govern the pattern of radio reception. Very high frequencies (VHF) penetrate both layers and are received only over short distances Medium waves (MW) are reflected at the D layer during the daytime but are received over longer distances at night when n_e of the D layer drops and medium waves proceed to the F_2 layer before reflection. Short waves (SW) penetrate the D layer to be reflected at the F_2 layer and are received over very long distances

Problem 8.1

The solutions to the e.m. wave equations are given in Figure 8.3 as

$$E_x = E_0 \sin \frac{2\pi}{\lambda}(vt - z)$$

and

$$H_y = H_0 \sin \frac{2\pi}{\lambda}(vt - z)$$

Use equations (8.1a) and (8.2a) to prove that they have the same wavelength and phase as shown in figure.

Problem 8.2

Show that the concept of $B^2/2\mu$ (magnetic energy per unit volume) as a magnetic pressure accounts for the fact that two parallel wires carrying currents in the same direction are forced together and that reversing one current will force them apart. (Consider a point midway between the two wires.) Show that it also explains the motion of a conductor carrying a current which is situated in a steady externally applied magnetic field.

Problem 8.3

At a distance r from a charge e on a particle of mass m the electric field value is $E = e/4\pi\varepsilon_0 r^2$. Show by integrating the electrostatic energy density over the spherical volume of radius a to infinity and equating it to the value mc^2 that the 'classical' radius of the electron is given by

$$a = 1.41 \times 10^{-15}\,\text{m}$$

Problem 8.4

The rate of generation of heat in a long cylindrical wire carrying a current I is I^2R, where R is the resistance of the wire. Show that this joule heating can be described in terms of the flow of energy into the wire from surrounding space and is equal to the product of the Poynting vector and the surface area of the wire.

Problem 8.5

Show that when a current is increasing in a long uniformly wound solenoid of coil radius r the total inward energy flow rate over a length l (the Poynting vector times the surface area $2\pi rl$) gives the time rate of change of the magnetic energy stored in that length of the solenoid.

Problem 8.6

The plane polarized electromagnetic wave (E_x, H_y) of this chapter travels in free space. Show that its Poynting vector (energy flow in watts per squaremetre) is given by

$$S = E_x H_y = c(\tfrac{1}{2}\varepsilon_0 E_x^2 + \tfrac{1}{2}\mu_0 H_y^2) = c\varepsilon_0 E_x^2$$

where c is the velocity of light. The intensity in such a wave is given by

$$I = \overline{S}_{\text{av}} = c\varepsilon_0 \overline{E^2} = \tfrac{1}{2} c\varepsilon_0 E_{\max}^2$$

Show that

$$\overline{S} = 1.327 \times 10^{-3}\, E_{\max}^2$$
$$E_{\max} = 27.45\, \overline{S}^{1/2}\ \text{V m}^{-1}$$
$$H_{\max} = 7.3 \times 10^{-2}\, \overline{S}^{1/2}\ \text{A m}^{-1}$$

Problem 8.7

A light pulse from a ruby laser consists of a linearly polarized wave train of constant amplitude lasting for 10^{-4} s and carrying energy of 0.3 J. The diameter of the circular cross section of the beam is 5×10^{-3} m. Use the results of Problem 8.6 to calculate the energy density in the beam to show that the root mean square value of the electric field in the wave is

$$2.4 \times 10^5\ \text{V m}^{-1}$$

Problem 8.8

One square metre of the earth's surface is illuminated by the sun at normal incidence by an energy flux of 1.35 kW. Show that the amplitude of the electric field at the earth's surface is 1010 V m^{-1} and that the associated magnetic field in the wave has an amplitude of 2.7 A m^{-1} (See Problem 8.6). The electric field energy density $\frac{1}{2}\varepsilon E^2$ has the dimensions of a pressure. Calculate the **radiation** pressure of sunlight upon the earth.

Problem 8.9

If the total power lost by the sun is equal to the power received per unit area of the earth's surface multiplied by the surface area of a sphere of radius equal to the earth sun distance (15×10^7 km), show that the mass per second converted to radiant energy and lost by the sun is 4.2×10^9 kg. (See Problem 8.6.)

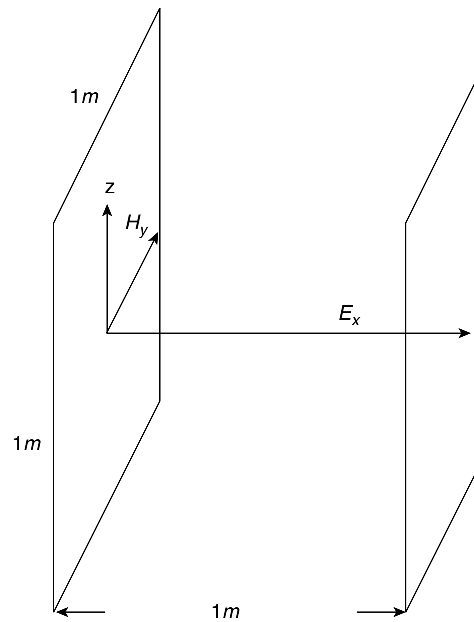
Problem 8.10

A radio station radiates an average power of 10^5 W uniformly over a hemisphere concentric with the station. Find the magnitude of the Poynting vector and the amplitude of the electric and magnetic fields of the plane electromagnetic wave at a point 10 km from the station. (See Problem 8.6)

Problem 8.11

A plane polarized electromagnetic wave propagates along a transmission line consisting of two parallel strips of a perfect conductor containing a medium of permeability μ and permittivity ε. A section of one cubic metre in the figure shows the appropriate field vectors. The electric field E_x generates equal but opposite surface charges on the conductors of magnitude εE_x C m^2. The motion of these surface charges in the direction of wave propagation gives rise to a surface current (as in the discussion associated with Figure 7.1). Show that the magnitude of this current is H_y and that the characteristic impedance of the transmission line is

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\varepsilon}}$$



Problem 8.12

Show that equation (8.6) is dimensionally of the form (per unit area)

$$V = L\frac{\mathrm{d}I}{\mathrm{d}t}$$

where V is a voltage, L is an inductance and I is a current.

Problem 8.13

Show that when a group of electromagnetic waves of nearly equal frequencies propagates in a conducting medium the group velocity is twice the wave velocity.

Problem 8.14

A medium has a conductivity $\sigma = 10^{-1}$ S m^{-1} and a relative permittivity $\varepsilon_r = 50$, which is constant with frequency. If the relative permeability $\mu_r = 1$, is the medium a conductor or a dielectric at a frequency of (a) 50 kHz, and (b) 10^4 MHz?

$$[\varepsilon_0 = (36\pi \times 10^9)^{-1}\,\mathrm{F\,m^{-1}};\ \mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}]$$

Answer: (a) $\sigma/\omega\varepsilon = 720$ (conductor)
(b) $\sigma/\omega\varepsilon = 3.6 \times 10^{-3}$ (dielectric).

Problem 8.15

The electrical properties of the Atlantic Ocean are given by

$$\varepsilon_r = 81, \quad \mu_r = 1, \quad \sigma = 4.3\,\mathrm{S\,m^{-1}}$$

Show that it is a conductor up to a frequency of about 10 MHz. What is the longest electromagnetic wavelength you would expect to propagate under water?

Problem 8.16

Show that when a plane electromagnetic wave travelling in air is reflected normally from a plane conducting surface the transmitted magnetic field value $H_t \approx 2H_i$, and that a magnetic standing wave exists in air with a very large standing wave ratio. If the wave is travelling in a conductor and is reflected normally from a plane conductor–air interface, show that $E_t \approx 2E_i$. Show that these two cases are respectively analogous to a short-circuited and an open-circuited transmission line.

Problem 8.17

Show that in a conductor the average value of the Poynting vector is given by

$$\begin{aligned} S_{\mathrm{av}} &= \tfrac{1}{2} E_0 H_0 \cos 45^\circ \\ &= \tfrac{1}{2} H_0^2 \times (\text{real part of } Z_c)\,\mathrm{W\,m^2} \end{aligned}$$

where E_0 and H_0 are the peak field values. A plane 1000 MHz wave travelling in air with $E_0 = 1\ \mathrm{V\ m^{-1}}$ is incident normally on a large copper sheet. Show firstly that the real part of the conductor impedance is $8.2 \times 10^{-3}\,\Omega$ and then (remembering from Problem 8.16 that H_0 doubles in the conductor) show that the average power absorbed by the copper per square metre is 1.6×10^{-7} W.

Problem 8.18

For a good conductor $\varepsilon_r = \mu_r = 1$. Show that when an electromagnetic wave is reflected normally from such a conducting surface its fractional loss of energy (1−reflection coefficient I_r) is $\approx \sqrt{8\omega\varepsilon/\sigma}$. Note that the ratio of the displacement current density to the conduction current density is therefore a direct measure of the reflectivity of the surface.

Problem 8.19

Using the value of the Poynting vector in the conductor from Problem 8.17, show that the ratio of this value to the value of the Poynting vector in air is $\approx \sqrt{8\omega\varepsilon/\sigma}$, as expected from Problem 8.18.

Problem 8.20

The electromagnetic wave of Problems 8.18 and 8.19 has electric and magnetic field magnitudes in the conductor given by

$$E_x = A\,\mathrm{e}^{-kz}\,\mathrm{e}^{\mathrm{i}(\omega t - kz)}$$

and

$$H_y = A\left(\frac{\sigma}{\omega\mu}\right)^{1/2} \mathrm{e}^{-kz}\,\mathrm{e}^{i(\omega t - kz)}\,\mathrm{e}^{-\mathrm{i}\pi/4}$$

where $k = (\omega\mu\sigma/2)^{1/2}$.

Show that the average value of the Poynting vector in the conductor is given by

$$S_{\mathrm{av}} = \tfrac{1}{2} A^2 \left(\frac{\sigma}{2\omega\mu}\right)^{1/2} \mathrm{e}^{-2kz}\,(\mathrm{W\,m^2})$$

This is the power absorbed per unit area by the conductor. We know, however, that the wave propagates only a distance of the order of the skin depth, so that this power is rapidly transformed. The rate at which it changes with distance is given by $\partial S_{\mathrm{av}}/\partial z$, which gives the energy transformed per unit volume in unit time. Show that this quantity is equal to the conductivity σ times the square of the mean value of the electric field vector $\mathbf{E}$, that is, the joule heating from currents flowing in the surface of the conductor down to a depth of the order of the skin depth.

Problem 8.21

Show that when light travelling in free space is normally incident on the surface of a dielectric of refractive index n the reflected intensity

$$I_{\mathrm{r}} = \left(\frac{E_{\mathrm{r}}}{E_{\mathrm{i}}}\right)^2 = \left(\frac{1-n}{1+n}\right)^2$$

and the transmitted intensity

$$I_{\mathrm{t}} = \frac{Z_{\mathrm{i}} E_{\mathrm{t}}^2}{Z_{\mathrm{t}} E_{\mathrm{i}}^2} = \frac{4n}{(1+n)^2}$$

(Note $I_{\mathrm{r}} + I_{\mathrm{t}} = 1$.)

Problem 8.22

Show that if the medium of Problem 8.21 is glass ($n = 1.5$) then $I_{\mathrm{r}} = 4\%$ and $I_{\mathrm{t}} = 96\%$. If an electromagnetic wave of 100 MHz is normally incident on water ($\varepsilon_{\mathrm{r}} = 81$) show that $I_{\mathrm{r}} = 65\%$ and $I_{\mathrm{t}} = 35\%$.

Problem 8.23

Light passes normally through a glass plate suffering only one air to glass and one glass to air reflection. What is the loss of intensity?

Problem 8.24

A radiating antenna in simplified form is just a length x_0 of wire in which an oscillating current is maintained. The expression for the radiating power is that used on p. 47 for an oscillating electron

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{q^2 \omega^4 x_0^2}{12\pi\varepsilon_0 c^3}$$

where q is the electron charge and ω is the oscillation frequency. The current I in the antenna may be written $I_0 = \omega q$. If $P = \frac{1}{2} R I_0^2$ show that the radiation resistance of the antenna is given by

$$R = \frac{2\pi}{3}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left(\frac{x_0}{\lambda}\right)^2 = 787\left(\frac{x_0}{\lambda}\right)^2 \Omega$$

where λ is the radiated wavelength (an expression valid for $\lambda \gg x_0$).

If the antenna is 30 m long and transmits at a frequency of 5×10^5 H with a root mean square current of 20 A, show that its radiation resistance is $1.97\,\Omega$ and that the power radiated is 400 W. (Verify that $\lambda \gg x_0$.)

Summary of Important Results

Dielectric; μ and $\varepsilon(\sigma = 0)$

Wave equation

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} \qquad \left(v^2 = \frac{1}{\mu\varepsilon}\right)$$

$$\frac{\partial^2 H_y}{\partial z^2} = \mu\varepsilon \frac{\partial^2 H_y}{\partial t^2}$$

Impedance

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\varepsilon}} \quad (376.7\Omega \text{ for free space})$$

Energy density $\frac{1}{2}\varepsilon E_x^2 + \frac{1}{2}\mu H_y^2$

$$\begin{aligned} \textit{Mean energy flow} &= \text{Intensity} = \overline{S} = v(\text{mean energy density}) \\ &= v(\tfrac{1}{2}\varepsilon E_x^2 + \tfrac{1}{2}\mu H_y^2)_{\text{average}} \\ &= v\varepsilon\overline{E_x^2} = \tfrac{1}{2}v\varepsilon E_{x(\max)}^2 \end{aligned}$$

Conductor; μ ε and σ

Add diffusion equation to wave equation for loss effects from σ

$$\frac{\partial^2 E_x^2}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x^2}{\partial t^2} + \mu\sigma \frac{\partial E_x}{\partial t}$$

giving

$$E_x = E_0 \,\mathrm{e}^{-kz}\,\mathrm{e}^{\mathrm{i}(\omega t - kz)}$$

where

$$k^2 = \omega\mu\sigma/2$$

Skin Depth

$$\delta = \frac{1}{k} \quad \text{giving}\, E_x = E_0\,\mathrm{e}^{-1}$$

Criterion for conductor/dielectric behaviour is ratio

$$\frac{\text{conduction current}}{\text{displacement current}} = \frac{\sigma}{\omega\varepsilon} \quad \text{(note frequency dependence)}$$

Impedance Z_c (conductor)

$$\mathbf{Z}_c = \frac{1+\mathrm{i}}{\sqrt{2}}\left(\frac{\omega\mu}{\sigma}\right)^{1/2}$$

with magnitude $Z_c = 376.6\sqrt{\mu_r/\varepsilon_r}\,\sqrt{\omega\varepsilon/\sigma}$ ohms

Reflection and Transmission Coefficients (normal incidence),

$$R = \frac{E_{\mathrm{r}}}{E_{\mathrm{i}}} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \quad (E\text{'s and } Z\text{'s may be complex})$$
$$T = \frac{E_{\mathrm{t}}}{E_{\mathrm{i}}} = \frac{2Z_2}{Z_2 + Z_1}$$

Fresnel's Equations (dielectrics)

$$R_{\parallel} = \frac{\tan(\phi - \theta)}{\tan(\phi + \theta)}, \quad T_{\parallel} = \frac{4 \sin\phi \cos\theta}{\sin 2\phi + \sin 2\theta}$$
$$R_{\perp} = \frac{\sin(\phi - \theta)}{\sin(\phi + \theta)}, \quad T_{\perp} = \frac{2 \sin\phi \cos\theta}{\sin(\phi + \theta)}$$

Refractive Index

$$n = \frac{c}{v} = \frac{Z\,(\text{free space})}{Z\,(\text{dielectric})}$$

Electromagnetic Waves in a Plasma

Low frequency waves propagate, but a high frequency wave $E_0\,\mathrm{e}^{\mathrm{i}\omega t}$ is attenuated or reflected when $\omega < \omega_{\mathrm{p}}$ the plasma frequency, where $\omega_{\mathrm{p}}^2 = n_{\mathrm{e}}e^2/\varepsilon_0 m_{\mathrm{e}}$. ($n_{\mathrm{e}}$ is the electron number density.)

The plasma has a refractive index n, where

$$n^2 = 1 - \omega_{\mathrm{p}}^2/\omega^2$$

when $\omega_{\mathrm{p}} \gg \omega_0$, the wave amplitude $E_0 \to E_0 e^{-1}$ in a skin depth distance

$$\delta = \frac{c}{\omega_{\mathrm{p}}}\left(1 - \frac{\omega^2}{\omega_{\mathrm{p}}^2}\right)^{-1/2} \approx \frac{c}{\omega_{\mathrm{p}}}$$