5 Integration

Let's Study

- Method of Substitution
- Some Special Integrals
- Integration by Parts
- Integration by Partial Fraction

Let's Recall

- Derivatives

5.1.1 Introduction

In this chapter, we shall study the operation which is an inverse process of differentiation. We now want to study the problem : the derivative of a function is given and we have to determine the function. The process of determining such a function is called integration.

Consider the following examples:

(1) Suppose we want to determine a function whose derivative is $3x^2$. Since we know that $\frac{dx^3}{dx} = 3x^2$. Therefore, the required function is $f(x) = x^3$.

x^3 is called integral of $3x^2$ w.r.t.x and this is written as $\int 3x^2\, dx = x^3$.

The symbol $\int$, called the integration sign, was introduced by Leibnitz. 'dx' indicates that the integration is to be taken with respect to the variable 'x'.

(2) Suppose we want to determine a function whose derivative is $\frac{1}{x}$ Since we know that $\frac{d}{dx}(\log x) = \frac{1}{x}$. Therefore, the required function is $\log x$. Using the integral sign, we can write $\int \left(\frac{1}{x}\right) dx = \log x,\ x > 0$.

Let's Learn

5.1.2 Definition: Integral or primitive or antiderivative of a function.

If $f(x)$ and $g(x)$ are two functions such that $\frac{d}{dx}[f(x)] = g(x)$ then $f(x)$ is called an integral of $g(x)$ with respect to x. It is denoted by $\int g(x)\, dx = f(x)$ and read as integral of $g(x)$ w.r.t.x is $f(x)$. Here, we say that $g(x)$ is the integrand.

This process of finding the integral of a function is called integration. Thus, integration is the inverse operation of differentiation.

For, example,

$$\frac{d}{dx}(x^4) = 4x^3$$

$$\therefore \int 4x^3\, dx = x^4$$

But, note that

$$\frac{d}{dx}(x^4 + 5) = 4x^3$$

$$\frac{d}{dx}(x^4 - 8) = 4x^3$$

What is the observation? Can you generalize from the observation?

In general,

$$\frac{d}{dx}(x^4 + c) = 4x^3$$

where, c is any real number.

Hence, in general, we write

$$\therefore \int 4x^3\, dx = x^4 + c$$

The number 'c' is called constant of integration.

Note: (i) From the above discussion, it is clear that integration is an inverse operation of differentiation. Hence integral is also called antiderivative.

(ii) In $\int f(x)dx$, $f(x)$ is the integrand and x is the variable of integration.

(iii) 'I' is used to denote an **integral**.

Integrals of some standard functions.

1	$\frac{d}{dx}x^n = nx^{n-1}$ $\therefore \int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$	$\frac{d}{dx}\left[\frac{(ax+b)^{n+1}}{(n+1)a}\right] = (ax+b)^n$ $\therefore \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$, where, $n \neq -1$
2	$\frac{d}{dx}\log x = \frac{1}{x}$ $\therefore \int \left(\frac{1}{x}\right) dx = \log\lvert x \rvert + c$	$\frac{d}{dx}\log(ax+b) = \frac{1}{ax+b}\frac{d}{dx}(ax+b) = \frac{a}{(ax+b)}$ $\therefore \int \left(\frac{1}{ax+b}\right) dx = \frac{\log\lvert ax+b\rvert}{a} + c$
3	$\frac{d}{dx}a^x = a^x \log a$ $\therefore \int a^x dx = \frac{a^x}{\log a} + c,\ a > 0,\ a \neq 1$	$\frac{d}{dx}a^{px+q} = a^{px+q}(\log a)\frac{d}{dx}(px+q) = a^{px+q}.p\log a$ $\therefore \int a^{px+q} dx = \frac{a^{px+q}}{p\log a} + c,\ a > 0,\ a \neq 1$
4	$\frac{d}{dx}e^x = e^x \log e = e^x$ $\therefore \int e^x dx = e^x + c$	$\frac{d}{dx}e^{px+q} = e^{px+q}\frac{d}{dx}(px+q) = e^{px+q}.p$ $\therefore \int e^{px+q} dx = \frac{e^{px+q}}{p} + c$

Rules of integration:

5.1.3 Theorem 1: If f is a real valued integrable function of x and k is a constant, then

$$\int [k.f(x)]\,dx = k\int f(x)dx$$

Theorem 2: If f and g are real valued integrable functions of x, then

$$\int [f(x)+g(x)]\,dx = \int f(x)dx + \int g(x)dx$$

Theorem 3: If f and g are real valued integrable functions of x, then

$$\int [f(x)-g(x)]\,dx = \int f(x)dx - \int g(x)dx$$

Generalization of (1), (2) and (3)

Corollary 1: If f_1, f_2, ... f_n are real integrable functions of x, and k_1, k_2, ... k_n are scalar constants then

$$\int [k_1 f_1(x) \pm k_2 f_2(x) \pm \ldots \pm k_n f_n(x)]\,dx$$
$$= k_1\int f_1(x)dx \pm k_2\int f_2(x)dx \pm \ldots \pm k_n\int f_n(x)dx$$

Result 1:

$\int f(x)dx = F(x) + c$ then

$$\int f(ax+b)dx = \frac{F(ax+b)}{a} + c$$

SOLVED EXAMPLES

(1) Evaluate $\int (7x-2)^2 dx$

Solution: I $= \frac{(7x-2)^{2+1}}{(2+1)7} + c$

$= \frac{(7x-2)^3}{21} + c$

(2) Evaluate $\int\left[\left(11-\frac{t}{3}\right)^7+(4t+5)^4\right]dt$

Solution : I = $\int\left(11-\frac{t}{3}\right)^7 dt+\int(4t+5)^4 dt$

$$= \frac{\left(11-\frac{t}{3}\right)^{7+1}}{7+1}\times(-3)+\frac{(4t+5)^{4+1}}{4+1}\times\frac{1}{4}+c$$

$$= \frac{-3}{8}\left(11-\frac{t}{3}\right)^8+\frac{1}{20}(4t+5)^5+c$$

(3) Evaluate $\int\left[\frac{1}{(6x+5)^4}-\frac{1}{(8-3x)^9}\right]dx$

Solution : I = $\int(6x+5)^{-4}dx-\int(8-3x)^{-9}dx$

$$= \frac{(6x+5)^{-3}}{-3}\times\frac{1}{6}-\left[\frac{(8-3x)^{-8}}{-8}\right]\times\frac{1}{-3}+c$$

$$= \left(\frac{-1}{18}\right)\frac{1}{(6x+5)^3}-\left(\frac{1}{24}\right)\frac{1}{(8-3x)^8}+c$$

(4) Evaluate $\int\frac{dx}{\sqrt{x}+\sqrt{x-2}}$

Solution: I = $\int\frac{1}{\sqrt{x}+\sqrt{x-2}}\times\frac{\sqrt{x}-\sqrt{x-2}}{\sqrt{x}-\sqrt{x-2}}dx$

$$= \int\frac{\sqrt{x}-\sqrt{x-2}}{x-(x-2)}dx=\frac{1}{2}\int(\sqrt{x}-\sqrt{x-2})dx$$

$$= \frac{1}{2}\left[\int x^{1/2}dx-\int(x-2)^{1/2}dx\right]$$

$$= \frac{1}{2}\left[\frac{x^{3/2}}{3/2}-\frac{(x-2)^{3/2}}{3/2}\right]+c$$

$$= \frac{1}{3}\left[x^{3/2}-(x-2)^{3/2}\right]+c$$

(5) Evaluate: $\int\left(x+\frac{1}{x}\right)^3 dx$

Solution: I = $\int\left(x^3+\frac{1}{x^3}+3x+\frac{3}{x}\right)dx$

$$= \frac{x^4}{4}-\frac{1}{2x^2}+\frac{3x^2}{2}+3\log|x|+c$$

$$= \frac{x^4}{4}+\frac{3x^2}{2}+3\log|x|-\frac{1}{2x^2}+c$$

(6) Evaluate $\int\frac{1}{x^2}(2x+1)^3 dx$

Solution: I = $\int\frac{(8x^3+1+12x^2+6x)}{x^2}dx$

$$= \int\left(8x+12+\frac{6}{x}+\frac{1}{x^2}\right)dx=4x^2+12x+6\log|x|-\frac{1}{x}+c$$

(7) Evaluate $\int\frac{5(x^6+1)}{x^2+1}dx$

Solution: I = $\int\frac{5(x^2+1)(x^4-x^2+1)}{(x^2+1)}dx$

$$= \int 5(x^4-x^2+1)dx=x^5-\frac{5}{3}x^3+5x+c$$

(8) Evaluate $\int x^3\left(2-\frac{3}{x}\right)^2 dx$

Solution: I = $\int x^3\left(4-\frac{12}{x}+\frac{9}{x^2}\right)dx$

$$= \int(4x^3-12x^2+9x)dx$$

$$= 4\frac{x^4}{4}-12\frac{x^3}{3}+9\frac{x^2}{2}+c$$

$$= x^4-4x^3+\frac{9}{2}x^2+c$$

(9) Evaluate $\int\frac{x^3+4x^2-6x+5}{x}dx$

Solution: I = $\int\left(x^2+4x-6+\frac{5}{x}\right)dx$

$$= \int x^2dx+4\int x\,dx-6\int dx+5\int\frac{1}{x}dx$$

$$= \frac{x^3}{3}+4\frac{x^2}{2}-6x+5\log|x|+c$$

(10) Evaluate $\int(e^{a\log x}+e^{x\log a})dx$

Solution: I = $\int(e^{\log_e x^a}+e^{\log_e a^x})dx$

$$= \int(x^a+a^x)dx=\frac{x^{a+1}}{a+1}+\frac{a^x}{\log a}+c$$

(11) Evaluate $\int\left(e^{(1-5t)}+\frac{1}{5t+1}\right)dt$

Solution: I = $\int e^{(1-5t)}dt+\int\left(\frac{1}{5t+1}\right)dt$

$$I=\frac{e^{(1-5t)}}{(-5)}+\left(\frac{\log|5t+1|}{5}\right)+c$$

(12) If $f'(x) = 8x^3 + 3x^2 - 10x - k$, $f(0) = -3$ and $f(-1) = 0$, find $f(x)$

Solution: By the definition of integral

$$f(x)=\int f'(x)dx=\int(8x^3+3x^2-10x-k)dx$$

$$=8\int x^3dx+3\int x^2dx-10\int x\,dx-k\int dx$$

$$=8\frac{x^4}{4}+3\frac{x^3}{3}-10\frac{x^2}{2}-kx+c$$

$$f(x)=2x^4+x^3-5x^2-kx+c$$

Now $f(0) = -3$ gives c = –3

and $f(-1) = 0$ gives $k = 7$

$$f(x)=2x^4+x^3-5x^2-7x-3$$

EXERCISE 5.1

(i) Evaluate $\int\frac{-2}{\sqrt{5x-4}-\sqrt{5x-2}}dx$

(ii) Evaluate $\int\left(1+x+\frac{x^2}{2!}\right)dx$

(iii) Evaluate $\int\frac{3x^3-2\sqrt{x}}{x}dx$

(iv) Evaluate $\int(3x^2-5)^2dx$

(v) Evaluate $\int\frac{1}{x(x-1)}dx$

(vi) If $f'(x) = x^2 + 5$ and $f(0) = -1$, then find the value of $f(x)$.

(vii) If $f'(x) = 4x^3 - 3x^2 + 2x + k$, $f(0) = 1$ and $f(1) = 4$, find $f(x)$

(viii) If $f'(x)=\frac{x^2}{2}-kx+1$, $f(0) = 2$ and $f(3) = 5$, find $f(x)$

5.2 Method of Change of Variable or Method of Substitution

In this method, we reduce the given function to standard form by changing variable x to t, using some suitable substitution $x = \phi(t)$

Theorem 4 : If $x = \phi(t)$ is a differentiable function of t, then

$$\int f(x)dx=\int f[\phi(t)]\,\phi'(t)dt$$

5.2.1 Corollary 1:

$$\int[f(x)]^n f'(x)dx=\frac{[f(x)]^{n+1}}{(n+1)}+c$$

SOLVED EXAMPLES

1. Evaluate $\int\frac{(\log x)^7}{x}dx$

Solution: Put $\log x = t$

$$\therefore\quad \frac{1}{x}dx=dt$$

$$\therefore\quad I=\int t^7dt=\frac{t^{7+1}}{7+1}+c=\frac{1}{8}(\log x)^8+c$$

2. Evaluate $\int\frac{1}{2x+x^{-n}}dx$

Solution: I = $\int\frac{1}{2x+\left(\frac{1}{x^n}\right)}dx$

$$=\int\frac{x^n}{2x^{(n+1)}+1}dx$$

Put $x^{(n+1)}=t$

$$\therefore\quad (n+1)x^n dx=dt$$

$$\therefore\quad x^n dx=\frac{dt}{n+1}$$

$$\therefore\quad I=\int\frac{1}{(2t+1)}\times\frac{dt}{(n+1)}$$

$$=\frac{1}{(n+1)}\int\frac{dt}{(2t+1)}$$

$$=\frac{1}{n+1}\frac{\log|2t+1|}{2}+c$$

$$I=\frac{1}{2(n+1)}\log\left|2x^{n+1}+1\right|+c$$

3. Evaluate $\int \frac{4x-6}{(x^2-3x+5)^{\frac{3}{2}}}dx$

Solution: I = $\int \frac{2(2x-3)}{(x^2-3x+5)^{\frac{3}{2}}}dx$

Put $(x^2-3x+5)=t$

$\therefore \quad (2x-3)dx = dt$

$$I = \int \frac{2dt}{t^{3/2}} = 2\int t^{\left(\frac{-3}{2}\right)}dt$$

$$= \quad 2\left[\frac{t^{\left(\frac{-1}{2}\right)}}{\left(\frac{-1}{2}\right)}\right] = \frac{-4}{\sqrt{t}}+c$$

$$I \;=\; \frac{-4}{\sqrt{x^2-3x+5}}+c$$

4. Evaluate $\int \frac{(x+1)(x+\log x)^4}{-3x}dx$

Solution: I = $\left(-\frac{1}{3}\right)\int (x+\log x)^4\left(\frac{x+1}{x}\right)dx$

$$= \left(-\frac{1}{3}\right)\int (x+\log x)^4\left(1+\frac{1}{x}\right)dx$$

Put $x+\log x = t \therefore \left(1+\frac{1}{x}\right)dx = dt$

$$= \left(-\frac{1}{3}\right)\int (t)^4 dt = \left(-\frac{1}{3}\right)\frac{t^5}{5}+c$$

$$= \left(-\frac{1}{15}\right)(x+\log x)^5+c$$

5.2.2 Corollary 2: $\int\left[\frac{f'(x)}{f(x)}\right]dx = \log f(x)+c$

5. Evaluate $\int \frac{e^{3x}}{e^{3x}+1}dx$

Solution: Put $e^{3x}+1=t$

$\therefore \quad 3e^{3x}dx = dt$

$\therefore \quad e^{3x}dx = \frac{dt}{3}$

$$I = \int \frac{1}{t}\frac{dt}{3} \;=\; \frac{1}{3}\int\frac{1}{t}dt \;=\; \frac{1}{3}\log|t|+c$$

$$= \quad \frac{1}{3}\log\left|e^{3x}+1\right|+c$$

6. Evaluate $\int \frac{1}{x(\log x-1)}dx$

Solution: Put $\log x - 1 = t$

$\frac{1}{x}dx = dt$

$$I = \int \frac{1}{(\log x-1)}\times\frac{1}{x}dx$$

$$\int \frac{1}{t}dt = \log|t|+c = \log|\log x-1|+c$$

7. Evaluate $\int \frac{e^x+1}{e^x+x}dx = \int \frac{\frac{d}{dx}(e^x+x)}{e^x+x}dx$

$$= \quad \log\left|e^x+x\right|+c$$

8. Evaluate $\int \frac{e^{x-1}+x^{e-1}}{e^x+x^e}dx$

Solution: Put $e^x + x^e = t$

$\therefore \quad (e^x+e\,x^{e-1})dx = dt$

$\therefore \quad e(e^{x-1}+x^{e-1})dx = dt$

$\therefore \quad (e^{x-1}+x^{e-1})dx = \frac{dt}{e}$

$$I = \int \frac{1}{t}\frac{dt}{e} = \frac{1}{e}\int\frac{1}{t}dt = \frac{1}{e}\log|t|+c$$

$$= \quad \frac{1}{e}\log\left|e^x+x^e\right|+c$$

9. Evaluate $\int \frac{1}{x\log x.\log(\log x)}dx$

Solution: I = $\int \frac{1}{\log(\log x)}.\frac{1}{x.\log x}dx$

Put $\log\,(\log x) = t$

$\therefore \quad \frac{1}{\log x}\frac{1}{x}dx = dt$

$\therefore \quad \frac{1}{x\log x}dx = dt$

$$I = \int \frac{1}{t} dt = log|t| + c$$

$$= \log|\log(\log x)| + c$$

10. Evaluate $\int \frac{10x^9 + 10^x.\log 10}{10^x + x^{10}} dx$

Solution: Put $10^x + x^{10} = t$

$\therefore \quad (10^x.\log 10 + 10x^9)dx = dt$

$$I = \int \frac{1}{t} dt = log|t| + c$$

$$= log|10^x + x^{10}| + c$$

11. Evaluate $\int \frac{1}{1+e^{-x}} dx = \int \frac{1}{1+\frac{1}{e^x}} dx$

Solution: $I = \int \frac{e^x}{e^x+1} dx = \int \frac{\frac{d}{dx}(e^x+1)}{e^x+1} dx$

$$= log|e^x + 1| + c$$

12. Evaluate $I = \int \frac{e^{2x}-1}{e^{2x}+1} dx$

Solution: $I = \int \frac{e^x(e^x - e^{-x})}{e^x(e^x + e^{-x})} dx$

$$= \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int \frac{\frac{d}{dx}(e^x + e^{-x})}{e^x + e^{-x}} dx$$

$$I = \log|e^x + e^{-x}| + c$$

5.2.3 Corollary 3: $\int \left[\frac{f'(x)}{\sqrt{f(x)}}\right] dx = 2\sqrt{f(x)} + c$

5.2.3 Corollary 4:

$$\int \left[\frac{f'(x)}{\sqrt[n]{f(x)}}\right] dx = \frac{n\sqrt[n]{[f(x)]^{n-1}}}{n-1} + c$$

13. Evaluate : $\int \frac{x^{n-1}}{\sqrt{1+x^n}} dx$

Solution: Put $x^n = t$

$\therefore \quad n x^{n-1} dx = dt$

$\therefore \quad x^{n-1} dx = dt / n$

$$I = \int \frac{1}{\sqrt{1+t}} \frac{dt}{n} = \frac{1}{n} \int (1+t)^{\frac{-1}{2}} dt$$

$$= \frac{1}{n} \cdot \frac{(1+t)^{\left(\frac{1}{2}\right)}}{\left(\frac{1}{2}\right)} + c = \frac{2}{n} \sqrt{1+x^n} + c$$

14. Evaluate $\int \frac{3x^2}{\sqrt{1+x^3}} dx$

Solution: Put $1 + x^3 = t$

$\therefore \quad 3x^2 dx = dt$

$$I = \int \frac{1}{\sqrt{t}} dt \quad 3x^2 dx = dt$$

$$= 2\sqrt{t} + c = 2\sqrt{1+x^3} + c$$

Integral of Type: $\int (ax+b)\sqrt{cx+d}\, dx$

15. Evaluate $\int (2x+1)\sqrt{x-4}\, dx$

Solution: Put $(x-4) = t$

$\therefore \quad dx = dt$

$x = t + 4$

$$I = \int [2(t+4)+1]\sqrt{t}\ dt = \int (2t+9)\sqrt{t}\ dt$$

$$= \int \left(2t^{\frac{3}{2}} + 9t^{\frac{1}{2}}\right) dt = 2\int t^{\frac{3}{2}} dt + 9\int t^{\frac{1}{2}} dt$$

$$= 2\frac{t^{\left(\frac{5}{2}\right)}}{\left(\frac{5}{2}\right)} + 9\frac{t^{\left(\frac{3}{2}\right)}}{\left(\frac{3}{2}\right)} + c = \frac{4}{5}(x-4)^{\frac{5}{2}} + 6(x-4)^{\frac{3}{2}} + c$$

16. Evaluate $\int (5-3x)(2-3x)^{\frac{-1}{2}} dx$

Solution: Put 2 – 3x = t

∴ $-3\,dx = dt$

$dx = -dt/3$ Also $x = (2-t)/3$

$$I = \int\left[5-3\left(\frac{2-t}{3}\right)\right](t)^{\left(\frac{-1}{2}\right)}\left(\frac{-dt}{3}\right)$$

$$= \frac{-1}{3}\int(5-2+t)(t)^{\left(\frac{-1}{2}\right)}dt$$

$$= \frac{-1}{3}\int(3+t)(t)^{\left(\frac{-1}{2}\right)}dt$$

$$= \frac{-1}{3}\int\left(3(t)^{\left(\frac{-1}{2}\right)}+(t)^{\left(\frac{1}{2}\right)}\right)dt$$

$$= \frac{-3}{3}\int t^{\left(\frac{-1}{2}\right)}dt-\frac{1}{3}\int t^{\left(\frac{1}{2}\right)}dt$$

$$= \frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} - \frac{1}{3}\frac{t^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+c$$

$$= -2\sqrt{2-3x}$$

$$-\frac{2}{9}(2-3x)^{\frac{3}{2}}+c$$

17. Evaluate $\int\frac{5x^2+4x+7}{(2x+3)^{\frac{3}{2}}}\,dx$

Solution: Put 2x + 3 = t

∴ $2\,dx = dt$

∴ $dx = \frac{dt}{2}$

Also $x = \frac{(t-3)}{2}$

$$I = \int\frac{5\left(\frac{t-3}{2}\right)^2+4\left(\frac{t-3}{2}\right)+7}{t^{\left(\frac{3}{2}\right)}}\frac{dt}{2}$$

$$= \frac{1}{2}\int\frac{5\left(\frac{t^2-6t+9}{4}\right)+2(t-3)+7}{t^{\left(\frac{3}{2}\right)}}\,dt$$

$$= \frac{1}{2}\int\frac{5t^2-30t+45+8t-24+28}{4t^{\left(\frac{3}{2}\right)}}\,dt$$

$$= \frac{1}{8}\int\frac{5t^2-22t+49}{t^{\left(\frac{3}{2}\right)}}\,dt$$

$$= \frac{1}{8}\int\left(5t^{\left(\frac{1}{2}\right)}-22t^{\left(\frac{-1}{2}\right)}+49t^{\left(\frac{-3}{2}\right)}\right)dt$$

$$= \frac{5}{8}\int t^{\left(\frac{1}{2}\right)}dt-\frac{22}{8}\int t^{\left(\frac{-1}{2}\right)}dt+\frac{49}{8}\int t^{\left(\frac{-3}{2}\right)}\,dt$$

$$= \frac{5}{8}\frac{t^{\left(\frac{3}{2}\right)}}{\left(\frac{3}{2}\right)}-\frac{11}{4}\frac{t^{\left(\frac{1}{2}\right)}}{\left(\frac{1}{2}\right)}+\frac{49}{4}\frac{t^{\left(\frac{-1}{2}\right)}}{\left(\frac{-1}{2}\right)}+c$$

$$= \frac{5}{12}(2x+3)^{3/2}-\frac{11}{2}(2x+3)^{1/2}$$

$$-\int\frac{49}{2}(2x+3)^{-1/2}+c$$

18. Evaluate $\int\frac{x^7}{(1+x^4)^2}dx$

Solution: Let, I = $\int\frac{x^7}{(1+x^4)^2}dx = \int\frac{x^4x^3}{(1+x^4)^2}dx$

Put $1+x^4 = t$

∴ $4x^3dx = dt$

∴ $x^3dx = \frac{dt}{4}$

Also $x^4 = t-1$

$$I = \int\frac{t-1}{(t)^2}\frac{dt}{4} = \frac{1}{4}\int\left(\frac{1}{t}-\frac{1}{t^2}\right)dt$$

$$= \frac{1}{4}\int\left(\frac{1}{t}\right)dt-\frac{1}{4}\int\left(\frac{1}{t^2}\right)dt = \frac{1}{4}\int\left(\frac{1}{t}\right)dt-\frac{1}{4}\int(t^{-2})dt$$

$$= \frac{1}{4}log|t|-\frac{1}{4}.\frac{t^{-1}}{-1}+c = \frac{1}{4}log|t|+\frac{1}{4}\frac{1}{t}+c$$

$$= \frac{1}{4}log\left|1+x^4\right|+\frac{1}{4}\frac{1}{1+x^4}+c$$

EXERCISE 5.2

Evaluate the following.

(i) $\int x\sqrt{1+x^2}\,dx$

(ii) $\int \frac{x^3}{\sqrt{1+x^4}}\,dx$

(iii) $\int (e^x+e^{-x})^2(e^x-e^{-x})\,dx$

(iv) $\int \frac{1+x}{x+e^{-x}}\,dx$

(v) $\int (x+1)(x+2)^7(x+3)\,dx$

(vi) $\int \frac{1}{x\log x}\,dx$

(vii) $\int \frac{x^5}{x^2+1}\,dx$

(viii) $\int \frac{2x+6}{\sqrt{x^2+6x+3}}\,dx$

(ix) $\int \frac{1}{\sqrt{x}+x}\,dx$

(x) $\int \frac{1}{x(x^6+1)}\,dx$

Activities

For each of these integrals, determine a strategy for evaluating. Don't evaluate them, just figure out which technique of integration will work, including what substitutions you will use.

1) $\int \frac{1}{x\log x}\,dx$	2) $\int \frac{3}{x^2+5x+4}\,dx$
3) $\int \frac{x+5}{\sqrt{x^2+5x+7}}\,dx$	4) $\int \frac{e^x}{36-e^{2x}}\,dx$

5.3 Integrals of the form $\int \frac{ae^x+b}{ce^x+d}\,dx$ where $a,b,c,d \in R$

SOLVED EXAMPLES

(1) Evaluate $\int \frac{4e^x-25}{2e^x-5}\,dx$

Put Numerator = A (Denominator + B ($\frac{d}{dx}$ Denominator)

$$4e^x-25 = A(2e^x-5)+B\left[\frac{d}{dx}(2e^x-5)\right]$$

$$= A(2e^x-5)+B(2e^x)$$

$$= (2A+2B)e^x-5A$$

Comparing the coefficients of e^x and constant term on both sides, we get

$2A+2B=4$ & $-25=-5A$

$\therefore$ $A=5$ and $B=-3$

$\therefore$ $4e^x-25=5(2e^x-5)-3(2e^x)$

$$\therefore I = \int \frac{5(2e^x-5)-3(2e^x)}{2e^x-5}\,dx$$

$$= \int\left[5-\frac{3(2e^x)}{2e^x-5}\right]dx$$

$$= 5\int dx-3\int\frac{2e^x}{2e^x-5}\,dx$$

$$= 5x-3\log\left|2e^x-5\right|+c$$

EXERCISE 5.3

Evaluate the following.

1) $\int \frac{3e^{2t}+5}{4e^{2t}-5}\,dt$

2) $\int \frac{20-12e^x}{3e^x-4}\,dx$

3) $\int \frac{3e^x+4}{2e^x-8}\,dt$

4) $\int \frac{2e^x+5}{2e^x+1}\,dt$

5.4.1 Results

1. $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

2. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

3. $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right| + c$

4. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right| + c$

SOLVED EXAMPLES

Evaluate the following.

1. $\int \frac{1}{9x^2-4}\,dx$

Solution: $I = \frac{1}{9}\int \frac{1}{x^2-\left(\frac{2}{3}\right)^2}\,dx$

$$= \left(\frac{1}{9}\right)\frac{1}{2\left(\frac{2}{3}\right)}\log\left|\frac{x-\frac{2}{3}}{x+\frac{2}{3}}\right| + c$$

$$= \frac{1}{12}\log\left|\frac{3x-2}{3x+2}\right| + c$$

2. $\int \frac{1}{16-9x^2}\,dx$

Solution: $I = \frac{1}{9}\int \frac{1}{\frac{16}{9}-x^2}\,dx$

$$= \frac{1}{9}\int \frac{1}{\left(\frac{4}{3}\right)^2-x^2}\,dx$$

$$= \left(\frac{1}{9}\right)\frac{1}{2\left(\frac{4}{3}\right)}\log\left|\frac{\frac{4}{3}+x}{\frac{4}{3}-x}\right| + c$$

$$= \frac{1}{24}\log\left|\frac{4+3x}{4-3x}\right| + c$$

3. $\int \frac{1}{\sqrt{9x^2+25}}\,dx$

Solution: $I = \frac{1}{3}\int \frac{1}{\sqrt{x^2+\frac{25}{9}}}\,dx$

$$= \frac{1}{3}\int \frac{1}{\sqrt{x^2+\left(\frac{5}{3}\right)^2}}\,dx$$

$$= \frac{1}{3}\log\left|x+\sqrt{x^2+\left(\frac{5}{3}\right)^2}\right| + c$$

4. $\int \frac{1}{\sqrt{4x^2-9}}\,dx$

Solution: $I = \frac{1}{2}\int \frac{1}{\sqrt{x^2-\frac{9}{4}}}\,dx$

$$= \frac{1}{2}\int \frac{1}{\sqrt{x^2-\left(\frac{3}{2}\right)^2}}\,dx$$

$$= \frac{1}{2}\log\left|x+\sqrt{x^2-\left(\frac{3}{2}\right)^2}\right| + c$$

5. $\int \frac{1}{x\sqrt{(\log x)^2-5}}\,dx$

Solution: Put $\log x = t \quad \therefore \frac{1}{x}dx = dt$

$$\therefore \quad I = \int \frac{1}{\sqrt{t^2-(\sqrt{5})^2}}\,dt$$

$$= \log\left|t+\sqrt{t^2-(\sqrt{5})^2}\right| + c$$

$$= \log\left|\log x+\sqrt{(\log x)^2-(\sqrt{5})^2}\right| + c$$

5.4.2 Integrals of the form $\int \frac{P(x)}{Q(x)} dx$ where degree (P(x)) ≥ degree (Q(x)).

Method: To evaluate $\int \frac{P(x)}{Q(x)} dx$

1. Divide $P(x)$ by $Q(x)$.

   After dividing $P(x)$ by $Q(x)$ we get quotient $q(x)$ and remainder $r(x)$.

2. Use Dividend = quotient × divisor + remainder

$$P(x) = q(x) \times Q(x) + r(x)$$

$$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$$

$$\int \frac{P(x)}{Q(x)} dx = \int q(x)\, dx + \int \frac{r(x)}{Q(x)} dx$$

3. Using standard integrals, evaluate I.

SOLVED EXAMPLES

1. Evaluate $I = \int \frac{x^3 + x + 1}{x^2 - 1} dx$

Solution: $I = \int \frac{x^3 + x + 1}{x^2 - 1} dx$

$$\begin{array}{r} x = Q \\ D = x^2 - 1 \overline{)\, x^3 + x + 1} \\ -\ x^3 - x \\ \hline +2x + 1 = R \end{array}$$

$$\therefore \quad I = \int \left(Q + \frac{R}{D} \right) dx$$

$$I = \int \left[x + \frac{2x+1}{x^2 - 1} \right] dx$$

$$= \int x\, dx + \int \frac{2x}{x^2 - 1} dx + \int \frac{1}{x^2 - 1} dx$$

$$= \frac{x^2}{2} + \log|x^2 - 1| + \int \frac{1}{x^2 - 1^2} dx + c$$

$$= \frac{x^2}{2} + \log|x^2 - 1| + \frac{1}{2} \log\left|\frac{x-1}{x+1}\right| + c$$

5.4.3 Integrals of the type $\int \frac{1}{ax^2 + bx + c} dx$

In order to find this type of integrals we may use the following steps :

Step 1 : Make the coefficient of x^2 as one if it is not, then $\frac{1}{a} \int \frac{1}{x^2 + \frac{b}{a}x + \frac{c}{a}} dx$

Step 2: Add and subtract the square of the half of coefficient of x that is $\left(\frac{b}{2a}\right)^2$ to complete the square $\frac{1}{a} \int \frac{1}{x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}} dx$ =

$$\frac{1}{a} \int \frac{1}{\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)} dx$$

SOLVED EXAMPLES

Evaluate the following.

1. $\int \frac{1}{2x^2 + x - 1} dx$

Solution: I = $\frac{1}{2} \int \frac{1}{x^2 + \frac{1}{2}x - \frac{1}{2}} dx$

$$= \frac{1}{2} \int \frac{1}{x^2 + \frac{1}{2}x + \frac{1}{16} - \frac{1}{16} - \frac{1}{2}} dx$$

$$= \frac{1}{2} \int \frac{1}{\left(x + \frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2} dx$$

$$= \frac{1}{2} \left[\frac{1}{2\left(\frac{3}{4}\right)} \right] \log \left| \frac{\left(x + \frac{1}{4}\right) - \frac{3}{4}}{\left(x + \frac{1}{4}\right) + \frac{3}{4}} \right| + c$$

$$= \frac{1}{3} \log \left| \frac{x - \frac{1}{2}}{x + 1} \right| + c$$

$$\frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right|+c$$

2. $\int \frac{1}{1+x-x^2}\,dx$

Solution: $I = \int \frac{1}{1+\frac{1}{4}-\frac{1}{4}+x-x^2}\,dx$

$$= \int \frac{1}{\left(1+\frac{1}{4}\right)-\left(x^2-x+\frac{1}{4}\right)}\,dx$$

$$= \int \frac{1}{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$$

$$= \frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\log\left|\frac{\frac{\sqrt{5}}{2}+\left(x-\frac{1}{2}\right)}{\frac{\sqrt{5}}{2}-\left(x-\frac{1}{2}\right)}\right|+c$$

$$= \frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{5}-1+2x}{\sqrt{5}+1-2x}\right|+c$$

3. $\int \frac{e^x}{e^{2x}+6e^x+5}\,dx$

Solution: Put $e^x = t$

$\therefore \quad e^x dx = dt$

$$I = \int \frac{dt}{t^2+6t+5}$$

$$= \int \frac{dt}{t^2+6t+9-9+5}$$

$$= \int \frac{dt}{(t+3)^2-2^2}$$

$$= \frac{1}{2(2)}\log\left|\frac{(t+3)-2}{(t+3)+2}\right|+c$$

$$= \frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right|+c$$

4. $\int \frac{1}{\sqrt{(x-2)(x-3)}}\,dx$

Solution: $I = \int \frac{1}{\sqrt{x^2-5x+6}}\,dx$

$$= \int \frac{1}{\sqrt{x^2-5x+\frac{25}{4}-\frac{25}{4}+6}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(x-\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$$

$$= \log\left|\left(x-\frac{5}{2}\right)+\sqrt{\left(x-\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+c$$

5. $\int \frac{2x+1}{\sqrt{x^2+2x+3}}\,dx$

Solution: $I = \int \frac{(2x+2)-1}{\sqrt{x^2+2x+3}}\,dx$

$$= \int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx - \int \frac{dx}{\sqrt{x^2+2x+3}}$$

$$= 2\sqrt{x^2+2x+3}-\log\left|\frac{1}{\sqrt{(x^2+2x+1)+2}}\right|$$

$$= 2\sqrt{x^2+2x+3}-\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+c$$

6. $\int \frac{x+1}{\sqrt{x^2+3x+2}}\,dx$

Solution: $x+1 = A\frac{d}{dx}(x^2+3x+2)+B$

$x+1 = A(2x+3)+B = 2Ax+3A+B$

$\therefore \quad 2A=1$ and $3A+B=1$ Solving we get

$A = \frac{1}{2}$ and $B = \frac{-1}{2}$

$$\therefore \quad I = \int \frac{\frac{1}{2}(2x+3)-\frac{1}{2}}{\sqrt{x^2+3x+2}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+3}{\sqrt{x^2+3x+2}}\,dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+3x+2}}$$

$$= \frac{2}{2}\sqrt{x^2+3x+2} - \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\,dx$$

$$= \sqrt{x^2+3x+2} - \frac{1}{2}\,log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x+2}\right| + c$$

5.4.4 Integrals reducible to the form

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

To find this type of integrals we use the following steps:

Step 1: Make the coefficients of x^2 as one if it is not, ie $\frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{x^2+\frac{bx}{a}+\frac{c}{a}}}$.

Step 2: Find half of the coefficient of x.

Step 3: Add and subtract $\left(\frac{1}{2}\,coeff.of\ x\right)^2$ inside the square root so that the square root is in the form

$$\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} \ or\ \frac{4ac-b^2}{4a^2} - \left(x+\frac{b}{2a}\right)^2$$

Step 4: Use the suitable standard form for evaluation.

SOLVED EXAMPLES

1. $\int \frac{dx}{x\sqrt{(logx)^2-5}}$

Solution : Put $\log x = t$

$$\therefore \frac{dx}{x} = dt$$

$$I = \int \frac{1}{\sqrt{t^2-5}}\,dt$$

$$= \int \frac{1}{\sqrt{t^2-\left(\sqrt{5}\right)^2}}\,dt$$

$$= log\left|t+\sqrt{t^2-\left(\sqrt{5}\right)^2}\right| + c$$

$$= log\left|log\,x+\sqrt{(log\,x)^2-5}\right| + c$$

2. $\int \frac{x^2dx}{\sqrt{x^6+2x^3+3}}$

Solution: Put $x^3 = t$

$$3x^2dx = dt : x^2dx = \frac{dt}{3}$$

$$I = \int \frac{1}{\sqrt{t^2+2t+3}}\frac{dt}{3}$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{t^2+2t+1+2}}$$

$$= \frac{1}{3}\log\left|(t+1)+\sqrt{t^2+2t+3}\right|$$

$$= \frac{1}{3}log\left|(t+1)+\sqrt{t^2+2t+3}\right| + c$$

$$= \frac{1}{3}log\left|(x^3+1)+\sqrt{x^6+2x^3+3}\right| + c$$

5.4.5 Integrals of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

To find this type of integrals we use the following steps:

Step 1: Write the numerator $px + q$ in the following form

$$px+q = A\frac{d}{dx}(ax^2+bx+c)+B$$

Step 2: Obtain the values of A and B by equating the coefficients of same power of x on both sides.

Step 3: Replace $px + q$ by $A(2ax + b) + B$ in the given integral to get in the form of

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$$

$$= A\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx + B\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

$$= 2A\sqrt{|ax^2+bx+c|} + B\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

SOLVED EXAMPLES

1. $\int \frac{2x+8}{\sqrt{x^2+6x+13}}dx$

Let $2x + 8 = A \frac{d}{dx}(x^2 + 6x + 13) + B$

$2x + 8 = A(2x + 6) + B$

$\therefore \quad A = 1, B = 2$

$= \int \frac{2x+6}{\sqrt{x^2+6x+13}}dx + \int \frac{2}{\sqrt{x^2+6x+13}}dx$

$= \sqrt{x^2+6x+13} + 2\log\left|(x+3)+\sqrt{x^2+6x+13}\right| + c$

(using $\int \frac{f'(x)}{f(x)}dx = 2\sqrt{f(x)} + c$ in the 1st integral)

2. $\int \sqrt{\frac{x+1}{x+2}}\, dx$

Solution: $I = \int \sqrt{\frac{(x+1)(x+1)}{(x+2)(x+1)}}\, dx$

$= \int \frac{x+1}{\sqrt{x^2+3x+2}}dx$

Let $x + 1 = A \frac{d}{dx}(x^2 + 3x + 2) + B$

$= A(2x + 3) + B$

Comparing the coefficient of x, we get

$1 = 2A$ and $1 = 3A + B$

$A = \frac{1}{2}$ and $B = \frac{-1}{2}$

$= \int \frac{x+1}{\sqrt{x^2+3x+2}}dx = \int \frac{\frac{1}{2}(2x+3)-\frac{1}{2}}{\sqrt{x^2+3x+2}}dx$

$= \int \frac{\frac{1}{2}(2x+3)}{\sqrt{x^2+3x+2}}dx - \int \frac{\frac{1}{2}}{\sqrt{x^2+3x+2}}dx$

$= \frac{1}{2}2\sqrt{x^2+3x+2} - \frac{1}{2}\log$

$\left|\left(x+\frac{3}{2}\right)+\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right| + c$

$= \frac{1}{2}\int \frac{(2x+3)}{\sqrt{x^2+3x+2}}dx - \int \frac{\frac{1}{2}}{\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$

$= \sqrt{x^2+3x+3} - \frac{1}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x+2}\right| + c$

3. $\int \frac{2x+1}{\sqrt{x^2+2x+1}}dx$

Solution : Let $2x + 1 = A \frac{d}{dx}(x^2 + 2x + 1) + B$

$= A(2x + 2) + B$

$2x + 1 = 2Ax + (2A + B)$

Comparing the coefficient of x, we get

$2 = 2A$ and $1 = 2A + B$

$A = 1$ and $B = -1$

$I = \int \frac{(2x+2)-1}{\sqrt{x^2+2x+1}}dx$

$= \int \frac{(2x+2)}{\sqrt{x^2+2x+1}}dx - \int \frac{1}{\sqrt{x^2+2x+1}}dx$

$\int \frac{(2x+2)}{\sqrt{x^2+2x+1}}dx - \int \frac{1}{\sqrt{(x+1)^2}}dx$

$= 2\sqrt{x^2+2x+1} - \log|x+1| + c$

4. $\int \sqrt{\frac{1+x}{x}}\, dx$

Solution: $I = \int \sqrt{\frac{(1+x)(1+x)}{x(1+x)}}\, dx$

$$= \int \frac{(1+x)}{\sqrt{x^2+x}}\,dx$$

Let $x + 1 = A\,\frac{d}{dx}(x^2 + x) + B$

$x + 1 \quad = A(2x + 1) + B = 2Ax + (A + B)$

Comparing the coefficient of x, we get

$1 = 2A$ and $1 = A + B$

$A = \frac{1}{2}$ and $B = \frac{1}{2}$

$$\int \frac{x+1}{\sqrt{x^2+x}}\,dx = \int \frac{\frac{1}{2}(2x+1)+\frac{1}{2}}{\sqrt{x^2+x}}\,dx$$

$$= \int \frac{\frac{1}{2}(2x+1)}{\sqrt{x^2+x}}\,dx + \int \frac{\frac{1}{2}}{\sqrt{x^2+x}}\,dx$$

$$= \frac{1}{2}\int \frac{(2x+1)}{\sqrt{x^2+x}}\,dx + \int \frac{\frac{1}{2}}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$$

$$= \frac{1}{2}2\sqrt{x^2+x}+\frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+c$$

$$= \sqrt{x^2+x}+\frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|+c$$

EXERCISE 5.4

Evaluate the following.

1) $\int \frac{1}{4x^2-1}\,dx$

2) $\int \frac{1}{x^2+4x-5}\,dx$

3) $\int \frac{1}{4x^2-20x+17}\,dx$

4) $\int \frac{x}{4x^4-20x^2-3}\,dx$

5) $\int \frac{x^3}{16x^8-25}\,dx$

6) $\int \frac{1}{a^2-b^2x^2}\,dx$

7) $\int \frac{1}{7+6x-x^2}\,dx$

8) $\int \frac{1}{\sqrt{3x^2+8}}\,dx$

9) $\int \frac{1}{\sqrt{x^2+4x+29}}\,dx$

10) $\int \frac{1}{\sqrt{3x^2-5}}\,dx$

11) $\int \frac{1}{\sqrt{x^2-8x-20}}\,dx$

5.5 Integration by Parts.

5.5.1 Theorem 5: If u and v are two functions of x then

$$\int u.v\;dx = u\int v\;dx - \int\left[\int v\;dx.\frac{du}{dx}\right]dx$$

The method of integration by parts is used when the integrand is expressed as a product of two functions, one of which can be differentiated and the other can be integrated conveniently.

Note:

(1) When the integrand is a product of two functions, out of which the second has to be integrated (whose integral is known), hence we should make proper choices of first function and second function.

(2) We can also choose the first function as the function which comes first in the word **'LAE'** where

L - Logarithmic Function

A - The Algebraic Function

E - The Exponential Function

SOLVED EXAMPLES

1. $\int x\, e^{-2x} dx$

Solution: $I = x\int e^{-2x}dx - \int\left[\frac{d}{dx}(x)\int e^{-2x}dx\right]dx$

$= x\frac{e^{-2x}}{-2} - \int 1.\frac{e^{-2x}}{-2}dx + c$

$= \frac{-1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c$

2. $\int \log x\, dx$

Solution: $I = \int(\log x).1\, dx$

$= (\log x)\int 1.dx - \int\left[\frac{d}{dx}(\log x)\int 1.dx\right]dx$

$= x\log x - \int\frac{1}{x}x dx + c$

$= x\log x - \int dx + c$

$= x(\log x - 1) + c$

3. $\int x^3 \log x\, dx$

Solution: $I = \int (\log x)x^3\, dx$

$= (\log x)\int x^3 dx - \int\left[\frac{d}{dx}(\log x)\int x^3 dx\right]dx$

$= \frac{x^4\log x}{4} - \frac{1}{4}\int x^3 dx + c$

$= \frac{x^4\log x}{4} - \frac{1}{4}\frac{x^4}{4} + c$

$= \frac{x^4\log x}{4} - \frac{x^4}{16} + c$

4. $\int\frac{\log(\log x)}{x}\, dx = \int \log(\log x)\ \frac{1}{x}\ dx$

Solution: Put $\log x = t$

$\therefore\ \frac{1}{x}dx = dt$

$I = \int \log t\, dt$

$= \int(\log t).1\, dt$

$= (\log t)\int 1.dt - \int\left[\frac{d}{dt}(\log t)\int 1.dt\right]dt$

$= t\log t - \int\frac{1}{t}t dt + c$

$= t\log t - \int dt + c$

$= t(\log t - 1) + c$

$= (\log x).(\log(\log x) - 1) + c$

5. $\int x.2^{-3x}dx$

Solution: $I = x\int(2^{-3x})dx - \int\left[\frac{d}{dx}x\int(2^{-3x})dx\right]dx$

$= \frac{x(2^{-3x})}{-3(\log 2)} - \int\frac{(2^{-3x})}{-3(\log 2)}dx + c$

$= \frac{x(2^{-3x})}{-3(\log 2)} - \frac{1}{-3(\log 2)}\int(2^{-3x})dx + c$

$= \frac{x(2^{-3x})}{-3(\log 2)} - \frac{1}{-3(\log 2)}\left(\frac{2^{-3x}}{-3(\log 2)}\right) + c$

$= \frac{-x\,2^{-3x}}{3(\log 2)} - \frac{1}{9}\frac{1}{(\log 2)^2}2^{-3x} + c$

Integral of the type $\int e^x\{f(x) + f'(x)\}dx$

These integrals are evaluated by using $\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + c$

1. $\int e^x\left(\frac{x\log x + 1}{x}\right)dx$

Solution: $I = \int e^x\left(\log x + \frac{1}{x}\right)dx$

Put $\log x = f(x)$ $\quad f'(x) = \frac{1}{x}$

$\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + c$

$= e^x\log x + c$

2. $\int e^x\frac{(1+x^2)}{(1+x)^2}dx$

Solution: $I = \int e^x\frac{(x^2-1)+2}{(1+x)^2}dx$

$$= \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right] dx$$

Put $f(x) = \frac{x-1}{x+1}$

$$f'(x) = \frac{2}{(x+1)^2}$$

$$\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + c$$

$$= e^x \left(\frac{x-1}{x+1}\right) + c$$

3. $\int e^x \frac{(x+3)}{(x+4)^2} dx$

Solution: I = $\int e^x \frac{(x+4-1)}{(x+4)^2} dx$

$$= \int e^x \left[\frac{1}{x+4} - \frac{1}{(x+4)^2}\right] dx$$

Put $f(x) = \frac{1}{x+4}$ and $f'(x) = -\frac{1}{(x+4)^2}$

$$\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + c$$

$$= e^x \frac{1}{x+4} + c$$

Integrals of the type $\int \sqrt{x^2 + a^2} dx, \int \sqrt{x^2 - a^2} dx$

$$\int \sqrt{a^2 + x^2} dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} log\left|x + \sqrt{a^2 + x^2}\right| + c$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} log\left|x + \sqrt{x^2 - a^2}\right| + c$$

In order to evaluate integrals of form $\int \sqrt{ax^2 + bx + c}\ dx$ we use the following steps.

Step 1: Make the coefficients of x^2 as one by taking a common.

Step 2: Add and substract $\left(\frac{b}{2a}\right)^2$ in $x^2 + \frac{b}{a}x + \frac{c}{a}$ to get the perfect square

$$\therefore \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$$

After applying these two steps the integrals reduces to one of the following two forms $\int \sqrt{a^2 + x^2}\ dx, \int \sqrt{x^2 - a^2}\ dx$ which can be evaluated easily.

SOLVED EXAMPLES

1. $\int \sqrt{4x^2 + 5}\ dx$

Solution: I = $\int 2\sqrt{x^2 + \frac{5}{4}}\ dx$

$$= 2\int \sqrt{x^2 + \left(\frac{\sqrt{5}}{2}\right)^2}\ dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2 + \frac{5}{4}} + \frac{5/4}{2} log\left|x + \sqrt{x^2 + \frac{5}{4}}\right|\right] + c_1$$

$$= \frac{x}{2}\sqrt{4x^2 + 5} + \frac{5}{4}\log\left|2x + \sqrt{4x^2 + 5}\right| + c$$

2. $\int \sqrt{9x^2 - 4}\ dx$

Solution: I = $\int 3\sqrt{x^2 - \frac{4}{9}}\ dx$

$$= 3\int \sqrt{x^2 - \left(\frac{2}{3}\right)^2}\ dx$$

$$= 3\left[\frac{x}{2}\sqrt{x^2 - \frac{4}{9}}\right] - \frac{4/9}{2} log\left|x + \sqrt{x^2 - \frac{4}{9}}\right| + c_1$$

$$= \frac{x}{2}\sqrt{9x^2 - 4} - \frac{2}{3} log\left|3x + \sqrt{9x^2 - 4}\right| + c$$

3. $\int \sqrt{x^2 - 4x - 5}\, dx$

Solution: I = $\int \sqrt{\left(x^2 - 4x + 4\right) - 9}\, dx$

$$= \int \sqrt{(x-2)^2 - 3^2}\ dx$$

$$= \frac{x-2}{2}\sqrt{x^2 - 4x - 5}$$

$$-\frac{9}{2} log\ \left|(x-2) + \sqrt{x^2 - 4x - 5}\right| + c$$

4. $\int \frac{\sqrt{1+(\log x)^2}}{x} dx$

Solution: $I = \int \sqrt{1+(\log x)^2}\, \frac{1}{x} dx$

Put $\log x = t$

$\therefore \quad \frac{1}{x} dx = dt$

$= \int \sqrt{1+t^2}\, dt$

$= \frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log\left|t+\sqrt{1+t^2}\right| + c$

$= \frac{(\log x)\sqrt{1+(\log x)^2}}{2}$

$+\frac{1}{2}\log\left|(\log x)+\sqrt{1+(\log x)^2}\right| + c$

5. $\int e^x\sqrt{e^{2x}+1}\, dx$

Solution: Let $e^x = t$

$e^x dx = dt$

$I = \int \sqrt{t^2+1}\, dt$

$= \frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\log\left|t+\sqrt{t^2+1}\right| + c$

$= \frac{e^x\sqrt{e^{2x}+1}}{2} + \frac{1}{2}\log\left|e^x+\sqrt{e^{2x}+1}\right| + c$

6. $\int \sqrt{x^2+4x+13}\, dx$

Solution: $I = \int \sqrt{x^2+4x+4+9}\, dx$

$= \int \sqrt{(x+2)^2+3^2}\, dx$

$= \frac{x+2}{2}\sqrt{(x+2)^2+3^2}$

$+\frac{3^2}{2}\log\left|(x+2)+\sqrt{(x+2)^2+3^2}\right| + c$

$= \frac{x+2}{2}\sqrt{x^2+4x+13}$

$+\frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x+13}\right| + c$

7. $\int \sqrt{x^2+x+1}\, dx$

Solution: $I = \int \sqrt{x^2+x+\frac{1}{4}+\frac{3}{4}}\, dx$

$= \int \sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\, dx$

$= \frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}$

$\log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right| + c$

Integrals of the form $\int (px+q)\sqrt{ax^2+bx+c}\, dx$

SOLVED EXAMPLES

$I = \int (3x-2)\sqrt{x^2+x+1}\, dx$

Solution: We express $3x-2 = A\frac{d(x^2+x+1)}{dx} + B$

$3x-2 = A(2x+1)+B$

$= 2Ax+(A+B)$

Comparing coefficients of x and constant term on both sides.

$2A = 3$ and $A + B = -2$

$A = 3/2$ and $B = -7/2$

$\therefore I = \int \left[\frac{3}{2}(2x+1) - \frac{7}{2}\right]\sqrt{x^2+x+1}\, dx$

$= \frac{3}{2}\int (2x+1)\sqrt{x^2+x+1}\, dx$

$-\int \frac{7}{2}\sqrt{x^2+x+1}\, dx$

Let

$I_1 = \int (2x+1)\sqrt{x^2+x+1}\, dx,$

$I_2 = \frac{-7}{2}\int \sqrt{x^2+x+1}\, dx$

Put $x^2+x+1 = t$ in I_1

$\therefore I_1 = \int \sqrt{t}\, dt = \int t^{1/2}\, dt$

$= \frac{t^{3/2}}{3/2} + c$

$$I_1 = \frac{2}{3}(x^2+x+1)^{3/2}+c_1$$

$$I_2 = \frac{-7}{2}\int\sqrt{x^2+x+1}\,dx$$

$$= \frac{-7}{2}\left[\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}\ +\frac{3}{8}log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|\right]+c_2$$

$$I = I_1 + I_2$$

EXERCISE 5.5

Evaluate the following.

1) $\int x\,log\,x$

2) $\int x^2 e^{4x}\,dx$

3) $\int x^2 e^{3x}\,dx$

4) $\int x^3 e^{x^2}\,dx$

5) $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$

6) $\int e^x \frac{x}{(x+1)^2}\,dx$

7) $\int e^x \frac{x-1}{(x+1)^3}\,dx$

8) $\int e^x\left[(logx)^2+\frac{2logx}{x}\right]dx$

9) $\int\left[\frac{1}{logx}-\frac{1}{(logx)^2}\right]dx$

10) $\int\frac{logx}{(1+logx)^2}\,dx$

5.6 Integration by method of Partial Fractions:

5.6.1 Types of Partial Fractions.

(1) If $f(x)$ and $g(x)$ are two polynomials then $f(x)/g(x)$ is a rational function where $g(x)\neq0$.

(2) If degree of $f(x)$ < degree of $g(x)$ then $f(x)/g(x)$ is a proper rational function.

(3) If degree of $f(x) \geq$ degree of $g(x)$ then $f(x)/g(x)$ is improper rational function.

(4) If a function is improper then divide $f(x)$ by $g(x)$ and this rational function can be written in the following form $\frac{f(x)}{g(x)} = Quotient + \frac{Remainder}{g(x)}$ and can be expressed as the sum of partial fractions using following table.

Type	Rational Form	Partial Form
1	$\frac{px\pm q}{(x-a)(x-b)}$	$\frac{A}{x-a}+\frac{B}{x-b}$
2	$\frac{px^2\pm qx\pm r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
3	$\frac{px\pm q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
4	$\frac{px^2\pm qx\pm r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5	$\frac{px^2\pm qx\pm r}{(x-a)^3(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}+\frac{D}{x-b}$
6	$\frac{px^2\pm qx\pm r}{(x-a)(ax^2\pm bx\pm c)}$	$\frac{A}{x-a}+\frac{Bx+C}{ax^2\pm bx\pm c}$ where, $ax^2\pm bx\pm c$ is non factorizable

SOLVED EXAMPLES

1. $\int \frac{x+1}{x^2+5x+6}\,dx$

Solution: I = $\int \frac{x+1}{(x+2)(x+3)}\,dx$

Consider $\frac{x+1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$

$x+1 = A(x+3)+B(x+2)$

Put $x = -2$ and we get A = –1

Pur $x = -3$ and we get B = 2

$\frac{x+1}{(x+2)(x+3)} = \frac{-1}{x+2} + \frac{2}{x+3}$

I = $\int \frac{x+1}{x^2+5x+6}\,dx = \int\left(\frac{-1}{x+2} + \frac{2}{x+3}\right)dx$

= $-\int \frac{dx}{x+2} + 2\int \frac{1}{x+3}\,dx$

= $-log|x+2| + 2log|x+3| + c$

2. $\int \frac{x^2+2}{(x-1)(x+2)(x+3)}\,dx$

Solution: I = Consider

$\frac{x^2+2}{(x-1)(x+2)(x+3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+3}$

$x^2 + 2 = A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2)$

Put $x = 1$ $\quad A = 1/4$

Put $x = -2$ $\quad B = -2$

Put $x = -3$ $\quad C = 11/4$

$\frac{x^2+2}{(x-1)(x+2)(x+3)} = \frac{1/4}{x-1} + \frac{-2}{x+2} + \frac{11/4}{x+3}$

$\int \frac{x^2+2}{(x-1)(x+2)(x+3)}\,dx =$

I = $\int\left(\frac{1/4}{x-1} - \frac{2}{x+2} + \frac{\frac{11}{4}}{x+3}\right)dx$

= $\frac{1}{4}\int \frac{1}{1-x}\,dx - 2\int \frac{1}{x+2}\,dx + \frac{11}{4}\int \frac{1}{x+3}\,dx$

= $\frac{1}{4}\log|x-1| - 2\log|x+2| + \frac{11}{4}\log|x+3| + c$

3. $\int \frac{\log x}{x(1+\log x)(2+\log x)}\,dx$

Solution: Put $\log x = t$

$\frac{1}{x}dx = dt$

$I = \int \frac{t}{(1+t)(2+t)}\,dt$

Consider $\frac{t}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t}$

Put $t = -1$ $\quad$ A = –1

Put $t = -2$ $\quad$ B = 2

$I = \int\left[\frac{-1}{1+t} + \frac{2}{2+t}\right]dt$

= $\int \frac{-1}{1+t}\,dt + 2\int \frac{1}{2+t}\,dt$

= $-\log|t+1| + 2log|t+2| + c$

= $2\log|\log x + 2| - log|logx + 1| + c$

= $\log|(logx + 2)|^2 - \log|(logx + 1)| + c$

4. $\int \frac{x^3 - 4x^2 + 3x + 11}{x^2+5x+6}\,dx$

[D = $x^2 - 5x + 6$) $x^3 - 4x^2 + 3x + 11$ ($x + 1 = Q$

$-(x^3 - 5x^2 + 6x)$

$x^2 - 3x + 11$

$-(x^2 - 5x + 6)$

$2x + 5 = R$

Express $\frac{x^3 - 4x^2 + 3x + 11}{x^2 - 5x + 6} = Q + \frac{R}{D}$

= $(x+1) + \frac{2x+5}{x^2-5x+6}$

$$I = \int \frac{x^3-4x^2+3x+11}{x^2-5x+6}dx$$

$$= \int (x+1)\,dx + \int \frac{2x+5}{x^2-5x+6}\,dx$$

$$= \frac{x^2}{2} + x + \int \frac{2x+5}{x^2-5x+6}\,dx + c_1$$

Express $\frac{2x+5}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$

$2x + 5 = A(x-3) + B(x-2)$

Put $x = 2$ we get $A = -9$

Put $x = 3$ we get $B = 11$

$$\therefore \quad \frac{2x+5}{x^2-5x+6} = \frac{-9}{x-2} + \frac{11}{x-3}$$

$$\therefore \quad I = \frac{x^2}{2} + x + \int \left(\frac{-9}{x-2} + \frac{11}{x-3} \right) dx + c$$

$$\therefore \quad I = \int \frac{x^3-4x^2+3x+11}{x^2-5x+6}dx = \frac{x^2}{2} + x$$

$$-9\log|x-2| + 11\log|x-3| + c$$

5. $\int \frac{3x+1}{(x-2)^2(x+2)}\,dx$

Express

$$\frac{3x+1}{(x-2)^2(x+2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2}$$

$3x + 1 = A(x-2)(x+2) + B(x+2) + C(x-2)^2$

Put $x = 2$ $B = 7/4$

$x = -2$, $C = -5/16$

Comparig Coefficients of x^2 on both sides we get

$A + C = 0 \quad A = 5/16$

$$\frac{3x+1}{(x-2)^2(x+2)} = \frac{\frac{5}{16}}{x-2} + \frac{\frac{7}{4}}{(x-2)^2} + \frac{\frac{-5}{16}}{x+2}$$

$$I = \frac{5}{16}\int \frac{1}{x-2}dx + \frac{7}{4}\int \frac{1}{(x-2)^2}dx - \frac{5}{16}\int \frac{1}{x+2}dx$$

$$I = \frac{5}{16}\log|x-2| - \frac{7}{4}\frac{1}{(x-2)} - \frac{5}{16}\log|x+2| + c$$

EXERCISE 5.6

Evaluate:

1) $\int \frac{2x+1}{(x+1)(x-2)}dx$

2) $\int \frac{2x+1}{x(x-1)(x-4)}dx$

3) $\int \frac{x^2+x-1}{x^2+x-6}dx$

4) $\int \frac{x}{(x-1)^2(x+2)}dx$

5) $\int \frac{3x-2}{(x+1)^2(x+3)}dx$

6) $\int \frac{1}{x(x^5+1)}dx$

7) $\int \frac{1}{x(x^n+1)}dx$

8) $\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx$

Activity

Evaluate: $\int \frac{x-1}{(x-3)(x-2)}dx$

Now, $\frac{x-1}{(x-3)(x-2)} = \frac{[\ ]}{(x-3)} + \frac{[\ ]}{(x-2)}$

There is no indicator of what the numerators should be, so there is work to be done to find them. If we let the numerator be variables, we can use algebra to solve. That is we want to find constants A and B that make equation 2 below true for $x = 2,3$ which are the same constants that make the following equation true.

$$\frac{x-1}{(x-3)(x-2)} = \frac{[A]}{(x-3)} + \frac{[B]}{(x-2)} \qquad (1)$$

$x - 1 = A(x - 2) + B(x - 3) \qquad (2)$

[ ] x + [ ] = [ ]x + [ ] (3)

Note: Two polynomials are equal if corresponding coefficients are equal. For linear functions, this means that $ax + b = cx + d$ for all x exactly when $a = c$ and $b = d$

Alternately, you can evaluate equation (2) for various values of x to get equations relating A and B. Some values of x will be more helpful than others

[ ] = [ ]

[ ] = [ ]

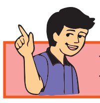
continue solving for the constants A and B.

A = , B =

$$\therefore \quad \frac{x-1}{(x-3)(x-2)} = \frac{[\ ]}{(x-3)} + \frac{[\ ]}{(x-2)}$$

$$\therefore \quad \int \frac{x-1}{(x-3)(x-2)}dx = \int \frac{[\ ]}{(x-3)}dx + \int \frac{[\ ]}{(x-2)}dx$$

I = [ ] + [ ] + c



Let's Remember

Rules of Integration:

1. $\int[f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)dx$
2. $\int k\,f(x)\,dx = k\int f(x)\,dx$; where k is a constant.
3. If $\int f(x)\,dx = g(x) + c$ then,

$$\int f(ax+b)\,dx = \frac{1}{a}g(ax+b) + c;\ a \neq 0$$

Standard Integration Formulae.

1. $\displaystyle\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c;\text{if } n \neq -1$
2. $\displaystyle\int \frac{1}{(ax+b)}dx = \frac{\log|ax+b|}{a} + c$
3. $\displaystyle\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + c$
4. $\displaystyle\int a^{bx+k}\,dx = \frac{a^{bx+k}}{b.\log a} + c$
5. $\displaystyle\int \sqrt{x^2+a^2}\;dx$
$\displaystyle = \frac{x}{2}\int\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$
6. $\displaystyle\int \sqrt{x^2-a^2}\;dx$
$\displaystyle = \int\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$
7. $\displaystyle\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$
8. $\displaystyle\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$
9. $\displaystyle\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c, n \neq -1$
10. $\displaystyle\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$
11. $\displaystyle\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$
12. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + c$
13. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + c$

MISCELLANEOUS EXERCISE - 5

I. Choose the correct alternative from the following.

1) The value of $\int \frac{dx}{\sqrt{1-x}}$ is

a) $2\sqrt{1-x}+c$ b) $-2\sqrt{1-x}+c$

c) $\sqrt{x}+c$ d) $x+c$

2) $\int \sqrt{1+x^2}\, dx =$

a) $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log(x+\sqrt{1+x^2})+c$

b) $\frac{2}{3}(1+x^2)^{3/2}+c$

c) $\frac{1}{3}(1+x^2)+c$ d) $\frac{(x)}{\sqrt{1+x^2}}+c$

3) $\int x^2(3)^{x^3}\, dx =$

a) $(3)^{x^3}+c$ b) $\frac{(3)^{x^3}}{3.\log 3}+c$

c) $\log 3(3)^{x^3}+c$ d) $x^2(3)^{x3}$

4) $\int \frac{x+2}{2x^2+6x+5}\, dx = p\int \frac{4x+6}{2x^2+6x+5}\, dx + \frac{1}{2}\int \frac{dx}{2x^2+6x+5}$ then P = ?

a) $\frac{1}{3}$ b) $\frac{1}{2}$

c) $\frac{1}{4}$ d) 2

5) $\int \frac{dx}{(x-x^2)} =$

a) $\log x - \log(1-x) + c$

b) $\log(1-x^2) + c$

c) $-\log x + \log(1-x) + c$

d) $\log(x-x^2) + c$

6) $\int \frac{dx}{(x-8)(x+7)} =$

a) $\frac{1}{15}\log\left|\frac{x+2}{x-1}\right|+c$

b) $\frac{1}{15}\log\left|\frac{x+8}{x+7}\right|+c$

c) $\frac{1}{15}\log\left|\frac{x-8}{x+7}\right|+c$

d) $(x-8)(x-7)+c$

7) $\int \left(x+\frac{1}{x}\right)^3 dx =$

a) $\frac{1}{4}\left(x+\frac{1}{x}\right)^4+c$

b) $\frac{x^4}{4}+\frac{3x^2}{2}+3\log x-\frac{1}{2x^2}+c$

c) $\frac{x^4}{4}+\frac{3x^2}{2}+3\log x+\frac{1}{x^2}+c$

d) $(x-x^{-1})^3+c$

8) $\int \left(\frac{e^{2x}+e^{-2x}}{e^x}\right) dx =$

a) $e^x-\frac{1}{3e^{3x}}+c$ b) $e^x+\frac{1}{3e^{3x}}+c$

c) $e^{-x}+\frac{1}{3e^{3x}}+c$ d) $e^{-x}+\frac{1}{3e^{3x}}+c$

9) $\int (1-x)^{-2}\, dx =$

a) $(1+x)^{-1}+c$ b) $(1-x)^{-1}+c$

c) $(1-x)^{-1}-1+c$ d) $(1-x)^{-1}+1+c$

10) $\int \frac{(x^3+3x^2+3x+1)}{(x+1)^5} dx$

a) $\frac{-1}{x+1}+c$ b) $\left(\frac{-1}{x+1}\right)^5+c$

c) $log(x+1)+c$ d) $log|x+1|^5+c$

II. Fill in the blanks.

1. $\int \frac{5(x^6+1)}{x^2+1} dx = x^4 + ...x^3 + 5x + c$

2. $\int \frac{x^2+x-6}{(x-2)(x-1)} dx = x + ... + c$

3. If $f'(x) = \frac{1}{x} + x$ and $f(1) = \frac{5}{2}$ then $f(x) = \log x + \frac{x^2}{2} + ...$

4. To find the value of $\int \frac{(1+\log x)\,dx}{x}$ the proper substitution is ....

5. $\int \frac{1}{x^3}\left[\log x^x\right]^2 dx = p(\log x)^3 + c$ then P = ...

III. State whether each of the following is True or False.

1. The proper substitution for $\int x(x^x)^x(2\log x+1)dx$ is $(x^x)^x = t$

2. If $\int x\, e^{2x}\, dx$ is equal to $e^{2x} f(x) + c$ where C is constant of integration then $f(x)$ is $\frac{(2x-1)}{2}$

3. If $\int x\, f(x)\, dx = \frac{f(x)}{2}$ then $f(x) = e^{x^2}$

4. If $\int \frac{(x-1)\,dx}{(x+1)(x-2)} = A\, log|x+1| + B\, log|x-2|$ then A + B = 1

5. For $\int \frac{x-1}{(x+1)^3} e^x dx = e^x f(x) + c$, $f(x) = (x+1)^2$.

IV. Solve the following:

1) Evaluate.

i) $\int \frac{5x^2-6x+3}{2x-3} dx$

ii) $\int (5x+1)^{\frac{4}{9}} dx$

iii) $\int \frac{1}{(2x+3)} dx$

iv) $\int \frac{x-1}{\sqrt{x+4}} dx$

v) If $f'(x) = \sqrt{x}$ and $f(1) = 2$ then find the value of $f(x)$.

vi) $\int |x|\, dx$ if $x < 0$

2) Evaluate.

i) Find the primitive of $\frac{1}{1+e^x}$

ii) $\int \frac{ae^{-x}+be^{-x}}{(ae^{-x}-be^{-x})} dx$

iii) $\int \frac{1}{2x+3x\,logx} dx$

iv) $\int \frac{1}{\sqrt{x}+x} dx$

v) $\int \frac{2e^x-3}{4e^x+1} dx$

3) Evaluate.

i) $\int \frac{dx}{\sqrt{4x^2-5}}$

ii) $\int \frac{dx}{3-2x-x^2}$

iii) $\int \frac{dx}{9x^2-25}$

iv) $\displaystyle\int \frac{e^x}{\sqrt{e^{2x}+4e^x+13}}\,dx$

v) $\displaystyle\int \frac{dx}{x[(logx)^2+4logx-1]}$

vi) $\displaystyle\int \frac{dx}{5-16x^2}$

vii) $\displaystyle\int \frac{dx}{25x-x(logx)^2}$

viii) $\displaystyle\int \frac{e^x}{4e^{2x}-1}\,dx$

4) Evaluate.

i) $\displaystyle\int (logx)^2 dx$

ii) $\displaystyle\int e^x \frac{1+x}{(2+x)^2}\,dx$

iii) $\displaystyle\int xe^{2x}dx$

iv) $\displaystyle\int log(x^2+x)dx$

v) $\displaystyle\int e^{\sqrt{x}}dx$

vi) $\displaystyle\int \sqrt{x^2+2x+5}dx$

vii) $\displaystyle\int \sqrt{x^2-8x+7}dx$

5) Evaluate.

i) $\displaystyle\int \frac{3x-1}{2x^2-x-1}dx$

ii) $\displaystyle\int \frac{2x^3-3x^2-9x+1}{2x^2-x-10}dx$

iii) $\displaystyle\int \frac{(1+\log x)}{x(3+logx)(2+3logx)}dx$

Activities

1) $\displaystyle\int \frac{1}{(x^2-5x+4)}2x\,dx$

Solution: $\displaystyle\frac{2x}{[\quad][\quad]} = \frac{C}{[\quad]} + \frac{D}{[x-4]}$

$\therefore$ $2x = C(x-4) + D(x-1)$

$\therefore$ $C = \square$, $D = \square$

$\therefore$ $\displaystyle\int \frac{}{(x-1)(x-4)} 2x\;dx = \int \left[\frac{}{(x-1)} + \frac{}{(x-4)}\right] dx$

$= \displaystyle\int \frac{}{(x-1)}dx + \int \frac{}{(x-4)}dx$

$= \square + \square + c$

2) $\displaystyle\int x^{13/2}(1+x^{5/2})^{1/2}dx$

Solution: $\int x\square x^{3/2}$ $(1 + x^{5/2})^{1/2}$ dx $=$ $\int(x^{5/2})^2$ $x^{3/2}$ $(1 + x^{5/2})$ dx

let $1 + x^{5/2} = t$

$\square\, dx = \square\, dt$

$I = \dfrac{2}{5}\int (t-1)^2 t^{1/2}\,dt$

$= \dfrac{2}{5}\int (t^2 - 2t + 1)\, t^{1/2}\,dt$

$= \dfrac{2}{5}\int[\square\, dt - \int \square\, dt + \int \square\, dt]$

$= \dfrac{2}{5}\{\square - \square + \square\} + c$

3) $\displaystyle\int \frac{dx}{(x+2)(x^2+1)} =$

$\left(\displaystyle\int \frac{1}{x^2+1}dx = \tan^{-1}x + c\right)$........(given)

Solution: $\displaystyle\frac{1}{(x+2)(x^2+1)} = \frac{\square}{(x+2)} + \frac{Bx+C}{(x^2+1)}$

$\therefore \quad 1 = A(x^2 + 1) + (Bx + C)(x + 2)$

Put $x = -2$ we get $A = \frac{1}{5}$

Now comparing the coefficients of x^2 and constant term, we get

$O = A + B$

and $1 = A + 2C$

$\therefore \quad B = \frac{-1}{5} \quad , \quad C = \frac{2}{5}$

$$\frac{1}{(x+2)(x^2+1)} = \frac{\square}{(x+2)} + \frac{\square x + \square}{(x^2+1)}$$

$$I = \int \frac{dx}{(x+2)} - \int \frac{\square\, x}{x^2+1} dx + \square \int \frac{dx}{x^2+1}$$

$$= \square - \square + \square + c$$

4) If $\int \frac{1}{x^5 + x} dx = f(x) + c = f(x) + C$, then the value of $\int \frac{x^4}{x + x^5} dx$ is equal to ....

$$I = \int \left[\frac{x^4 + 1 -}{x + x^5} \right] dx =$$

$$= \int \frac{1}{x} dx - \int \frac{1}{x^5 + x} dx$$

$I = \quad - \quad + c$

$I = \log x - f(x) + c_1 \quad ....c_1 = -c$

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