08

In this chapter, we will study a specific application of integral to find the areas of bounded regions, area between lines and arcs of circles, parabolas and ellipses in their standard form only.

APPLICATION OF INTEGRALS

CHAPTER CHECKLIST

- Area of Bounded Region
- Area of Symmetrical Region
- Area of Region Bounded by a Curve and a Line

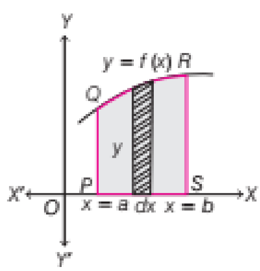
AREA OF BOUNDED REGION

Area of bounded region can be assumed as composed of large number of very thin vertical (or horizontal) strips.

Consider an arbitrary strip of height y and width dx. Then, dA (area of elementary strip) $= y\,dx$. This area is called elementary area, which is located at an arbitrary position within the region, so by adding up the elementary areas of thin strips across the region, we get the required area.

Area of Region Bounded by X-axis, Lines $x = a$, $x = b$ and Curve $y = f(x)$

The area of the region $PQRSP$ bounded by the curve $y = f(x)$, the X-axis and lines $x = a$ and $x = b$, will be the result of adding up the elementary areas of thin vertical strips across the region $PQRSP$.

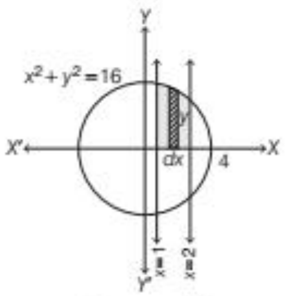


Thus,
$$\text{Area} = \int_a^b dA = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

e.g. The area of the region bounded by the curve $x^2 + y^2 = 16$ and lines $x = 1$ and $x = 2$ in I quadrant is given by

$$\text{Area} = \int_1^2 y\,dx$$
$$= \int_1^2 \sqrt{4^2 - x^2}\,dx \quad \text{[here, we take a vertical strip]}$$

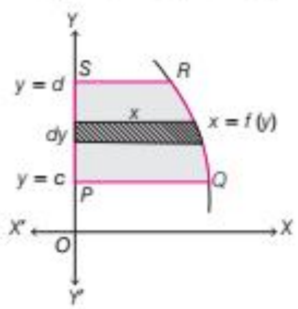
$$A = \int_1^2 x\,dy$$
$$= \int_1^2 \sqrt{4^2 - y^2}\,dy \quad \text{[here, we take a horizontal strip]}$$
$$= \left[\frac{y}{2}\sqrt{16 - y^2} + \frac{4^2}{2}\sin^{-1}\frac{y}{4}\right]_1^2$$



$$=\left[\frac{x}{2}\sqrt{16-x^2}+\frac{4^2}{2}\sin^{-1}\frac{x}{4}\right]_1^2$$
$$=\left[\frac{2}{2}\sqrt{16-4}+8\sin^{-1}\frac{2}{4}-\left(\frac{1}{2}\sqrt{16-1}+8\sin^{-1}\frac{1}{4}\right)\right]$$
$$=\left[2\sqrt{3}+8\times\frac{\pi}{6}-\left(\frac{1}{2}\sqrt{15}+8\sin^{-1}\frac{1}{4}\right)\right]$$
$$=\left[2\sqrt{3}+\frac{4\pi}{3}-\left(\frac{\sqrt{15}}{2}+8\sin^{-1}\frac{1}{4}\right)\right]\text{ sq units}$$

Area of Region Bounded by Y-axis, Lines $y=c$, $y=d$ and the Curve $x=f(y)$

The area of region $PQRSP$ bounded by the curve $x = f(y)$, the Y-axis and lines $y = c$ and $y = d$ is given by

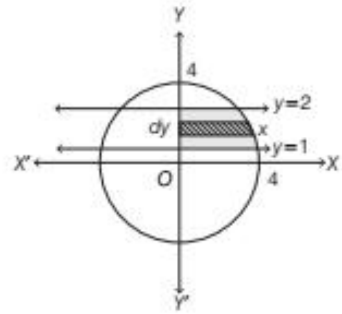


$$\text{Area} = \int_c^d dA = \int_c^d x\,dy = \int_c^d f(y)\,dy$$

e.g. The area of the region bounded by the curve $x^2 + y^2 = 16$ and lines $y = 1$ and $y = 2$ in I quadrant is given by

e.g. The area of the region bounded by the line $2x - y = 2$ and lines $x = \frac{1}{4}$ and $x = \frac{3}{4}$ is given by

$$\text{Area} = \left|\int_{1/4}^{3/4} y\,dx\right|$$

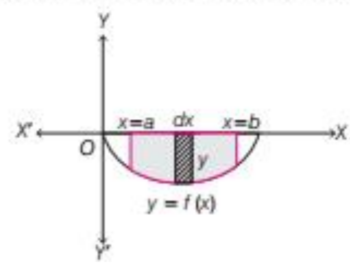


$$=\left[\frac{2}{2}\sqrt{16-4}+8\sin^{-1}\frac{2}{4}-\left(\frac{1}{2}\sqrt{16-1}+8\sin^{-1}\frac{1}{4}\right)\right]$$
$$=\left[2\sqrt{3}+8\times\frac{\pi}{6}-\left(\frac{1}{2}\sqrt{15}+8\sin^{-1}\frac{1}{4}\right)\right]$$
$$=\left[2\sqrt{3}+\frac{4\pi}{3}-\left(\frac{\sqrt{15}}{2}+8\sin^{-1}\frac{1}{4}\right)\right]\text{ sq units}$$

Note The area should be same, either by taken horizontal strip or vertical strip.

Area of Region when Curve is Below the X-axis

If the curve $y = f(x)$ lies below the X-axis, then area bounded by the curve $y = f(x)$, X-axis and the lines $x = a$ and $x = b$ comes out to be negative. But only numerical value of the area is taken into consideration.

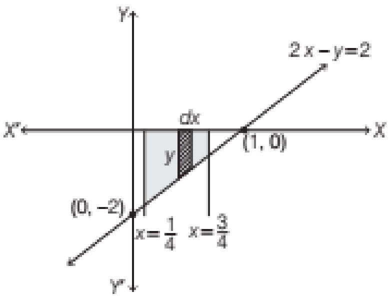


Thus, if the area is negative, then we take its absolute value, i.e. $\left|\int_a^b f(x)\,dx\right|$.

$$\therefore \qquad \text{Area} = \left|\int_a^b y\,dx\right| = \left|\int_a^b f(x)\,dx\right|$$

e.g. The area of the region bounded by the line $3x - y = 3$ and lines $x = \frac{1}{2}$ and $x = 2$ is given by

$$\text{Area} = \left|\int_{1/2}^{1} y\,dx\right| + \int_1^2 y\,dx$$



$$= \left| \int_{1/4}^{3/4} (2x - 2)\, dx \right| \quad [\because 2x - y = 2 \Rightarrow y = 2x - 2]$$

$$= \left| [x^2 - 2x]_{1/4}^{3/4} \right|$$

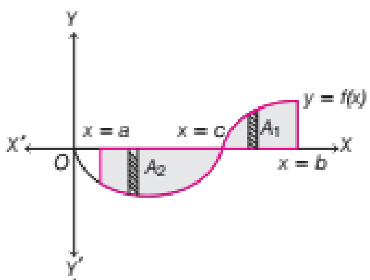
$$= \left| \left[\left(\frac{3}{4}\right)^2 - 2\left(\frac{3}{4}\right) - \left\{ \left(\frac{1}{4}\right)^2 - 2\left(\frac{1}{4}\right) \right\} \right] \right|$$

$$= \left| \left[\frac{9}{16} - \frac{6}{4} - \left(\frac{1}{16} - \frac{1}{2} \right) \right] \right| = \left| \frac{8}{16} - \frac{4}{4} \right|$$

$$= \left| \frac{8 - 16}{16} \right| = \frac{8}{16} = \frac{1}{2} \text{ sq unit}$$

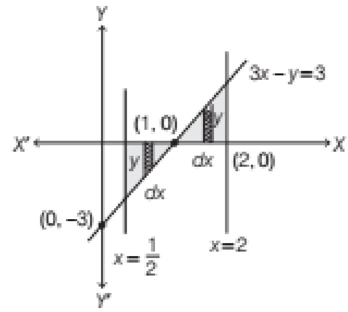
Area of Region when Curve is Above and Below the X-axis

It may happen that some portion of the curve is above the X-axis and some is below the X-axis, which is shown in the figure given below



Here, $A_1 > 0$ and $A_2 < 0$. Therefore, the area A bounded by the curve $y = f(x)$, X-axis and the lines $x = a$ and $x = b$ is given by

$$A = |A_2| + A_1 = \left| \int_a^c f(x)\, dx \right| + \int_c^b f(x)\, dx$$



$$= \left| \int_{1/2}^{1} (3x - 3)\, dx \right| + \int_1^2 (3x - 3)\, dx$$

$$= \left| 3 \left[\frac{x^2}{2} - x \right]_{1/2}^{1} \right| + 3 \left[\frac{x^2}{2} - x \right]_1^2$$

$$= 3 \left| \left[\frac{1}{2} - 1 - \left(\frac{1}{8} - \frac{1}{2} \right) \right] \right| + 3 \left[\frac{4}{2} - 2 - \left(\frac{1}{2} - 1 \right) \right]$$

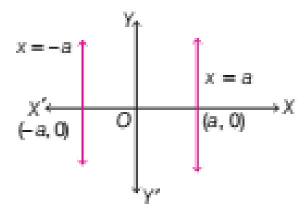
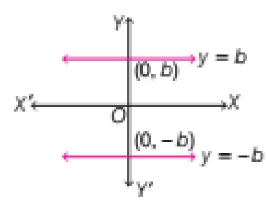
$$= 3 \left| \left[-\frac{1}{2} + \frac{3}{8} \right] \right| + 3 \left[0 + \frac{1}{2} \right]$$

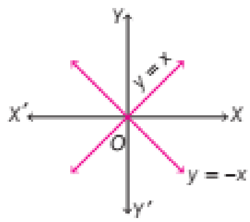
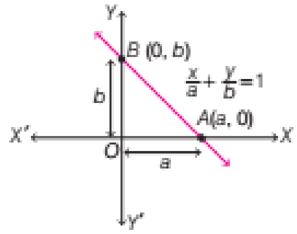
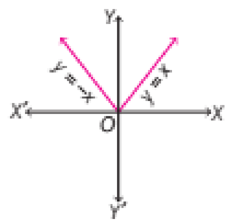
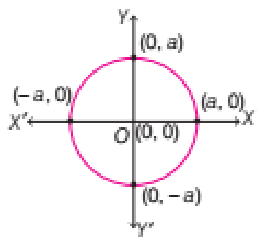
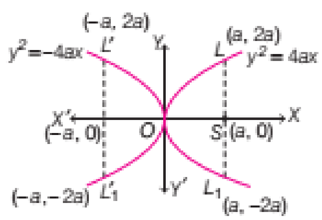
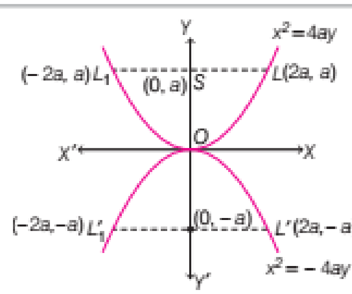
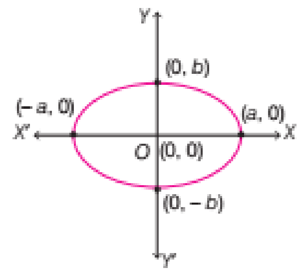
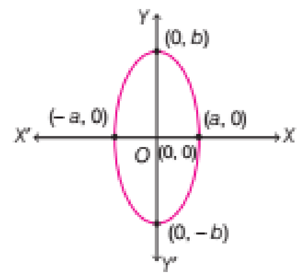
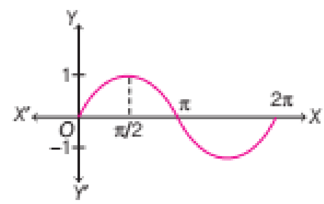
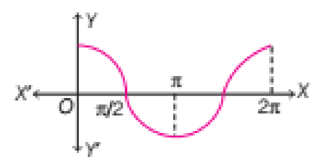
$$= 3 \left| -\frac{1}{8} \right| + \frac{3}{2} = \frac{3}{8} + \frac{3}{2}$$

$$= \frac{3 + 12}{8} = \frac{15}{8} \text{ sq units}$$

Note Area of any bounded region cannot be negative.

Sketches of Some Standard Curves

Equation of curve	Sketch
1. (a) **Straight lines** $x = a$ and $x = -a$, where $a > 0$	
(b) **Straight lines** $y = b$ and $y = -b$, where $b > 0$	

	Equation of curve	Sketch
2.	**Straight lines** $y = x$ and $y = -x$	
3.	**Straight lines** $\frac{x}{a} + \frac{y}{b} = 1, a \neq b$ $x + y = a, a = b$	
4.	**Modulus function** $y = \lvert x \rvert$ $y = \begin{cases} x, \text{ for } x \geq 0 \\ -x, \text{ for } x < 0 \end{cases}$	
5.	**Circle** $x^2 + y^2 = a^2$ **Centre** = (0, 0) **Radius** = a	
6. (a)	**Parabola** $y^2 = 4ax$ (or $y^2 = -4ax$) **Vertex**, O = (0, 0), **Focus**, S = $(a, 0)$ or $(-a, 0)$. **Length of latusrectum** = $4a$	
(b)	**Parabola** $x^2 = 4ay$ (or $x^2 = -4ay$) **Vertex**, O = (0, 0), **Focus**, S = $(0, a)$ or $(0, -a)$. **Length of latusrectum** = $4a$	
7. (a)	**Ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, when $a > b$ **Vertices** = $(\pm a, 0)$, **Centre**, O = (0, 0)	
(b)	**Ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, when $a < b$ **Vertices** = $(0, \pm b)$ **Centre**, O = (0, 0)	
8. (a)	**Sine function** $y = \sin x$	
(b)	**Cosine function** $y = \cos x$	

Working Rule to Find the Area Bounded by a Curve, X-axis and Two Ordinates ($x = a$, $x = b$) or a Curve, Y-axis and Two Abscissae ($y = c$, $y = d$)

For finding the area of bounded region, firstly we identify the region, whose area is to be computed. For this, we draw the rough sketch of given curve and the given lines $x = a$ and $x = b$ (or $y = c$ and $y = d$), which enclose the area with X-axis (or Y-axis). Now, following cases arise

Case I If the bounded area included between two ordinates $x = a$ and $x = b$ and lies above X-axis, then construct an arbitrary vertical strip of height y and width dx and then

Required area $= \int_a^b y\, dx$

If the bounded area lies below the X-axis, then

Required area $= \left| \int_a^b y\, dx \right|$

Case II If the bounded area lies to the right of Y-axis and included between two abscissae $y = c$ and $y = d$, then construct an arbitrary horizontal strip of length x and width dy and then

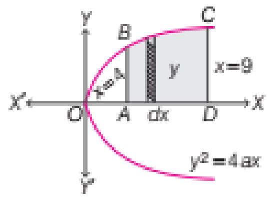
Required area $= \int_c^d x\, dy$

If the bounded area lies to the left of Y-axis, then

Required area $= \left| \int_c^d x\, dy \right|$

EXAMPLE |1| Find the area of the region bounded by the parabola $y^2 = 4ax$, its axis and two ordinates $x = 4$ and $x = 9$ in first quadrant.

Sol. Given equation of parabola is $y^2 = 4ax$, its axis is $Y = 0$ and vertex is (0, 0). Also, given ordinates $x = 4$ and $x = 9$. The bounded region in I quadrant is $ABCDA$.



$\therefore$ Required area $= \int_4^9 y\, dx = \int_4^9 \sqrt{4ax}\, dx$

$[\because y^2 = 4ax \Rightarrow y = \sqrt{4ax}$, as y is in I quadrant$]$

$$= 2\sqrt{a} \int_4^9 \sqrt{x}\, dx = 2\sqrt{a} \left[\frac{x^{3/2}}{3/2} \right]_4^9$$

$$= 2\sqrt{a} \times \frac{2}{3} [(9)^{3/2} - (4)^{3/2}]$$

$$= \frac{4}{3} \sqrt{a} [(3^2)^{3/2} - (2^2)^{3/2}]$$

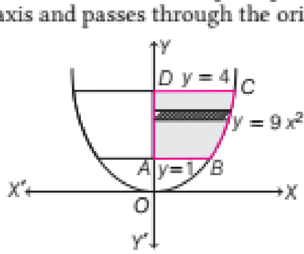
$$= \frac{4}{3} \sqrt{a} [27 - 8] = \frac{4}{3} \sqrt{a} (19) = \frac{76}{3} \sqrt{a} \text{ sq units}$$

Hence, the required area is $\frac{76\sqrt{a}}{3}$ sq units.

EXAMPLE |2| Sketch the region lying in the first quadrant and bounded by $y = 9x^2$, $x = 0$, $y = 1$ and $y = 4$. Find the area of region using integration.

Sol. Given curve is $y = 9x^2 \Rightarrow x^2 = \frac{1}{9} y$ ...(i)

It is a parabolic curve, which open upwards, symmetrical about Y-axis and passes through the origin.



$\therefore$ Required area = Area of bounded region $ABCDA$

$$= \int_1^4 x\, dy = \frac{1}{3} \int_1^4 \sqrt{y}\, dy \qquad \left[\because x^2 = \frac{1}{9} y \Rightarrow x = \frac{1}{3} \sqrt{y} \right]$$

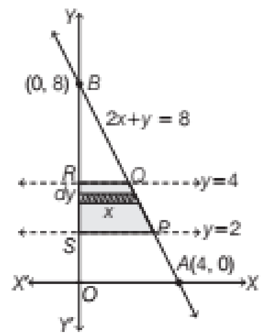
$$= \frac{1}{3} \left[\frac{y^{3/2}}{3/2} \right]_1^4 = \frac{1}{3} \times \frac{2}{3} [(4)^{3/2} - (1)^{3/2}]$$

$$= \frac{2}{9} [(2)^3 - 1] = \frac{14}{9} \text{ sq units}$$

Hence, the required area is $\frac{14}{9}$ sq units.

EXAMPLE |3| Using integration, find the area of region bounded by the line $2x + y = 8$, the Y-axis and the lines $y = 2$ and $y = 4$.

Sol. The given line AB is $2x + y = 8 \Rightarrow x = 4 - \frac{1}{2} y$



Required area = Area of region $PQRSP$

= Area between the line $x = 4 - \frac{1}{2} y$,

the Y-axis and the lines $y = 2$ and $y = 4$

$$= \int_2^4 x\, dy = \int_2^4 \left(4 - \frac{1}{2} y \right) dy$$

$$= \left[4y - \frac{y^2}{4} \right]_2^4 = (16 - 4) - (8 - 1)$$

$$= 12 - 7 = 5 \text{ sq units}$$

Hence, the required area is 5 sq units.

Note Sometimes, lines $x = a$ and $x = b$ (or $y = c$ and $y = d$) are not given to us, then we find the intersection points of curve and axes. If we get the intersection of curve with both axes, then we can consider either vertical strips or horizontal strips for calculating the area of the region.

EXAMPLE |4| Find the area of the region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ in fourth quadrant.

Sol. Given equation of ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$ ...(i)

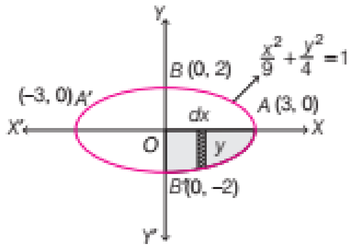
We know that the standard equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$a^2 = 9 \text{ and } b^2 = 4 \Rightarrow a = 3 \text{ and } b = 2$$

Here, we see that $a > b$, so the horizontal ellipse will be formed.



Now, required area = Area of region in IV quadrant

= Area of region $OAB'O$

$$= \left|\int_0^3 y\,dx\right| = \left|\int_0^3 \frac{2\sqrt{9-x^2}}{3}\,dx\right|$$

$$\left[\because \frac{y^2}{4} = 1 - \frac{x^2}{9} \Rightarrow y = \frac{2\sqrt{9-x^2}}{3}\right]$$

$$= \frac{2}{3}\left|\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3\right|$$

$$\left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{2}{3}\left|\left[\frac{3}{2}\sqrt{9-9} + \frac{9}{2}\sin^{-1}\frac{3}{3} - 0 - \frac{9}{2}\sin^{-1}(0)\right]\right|$$

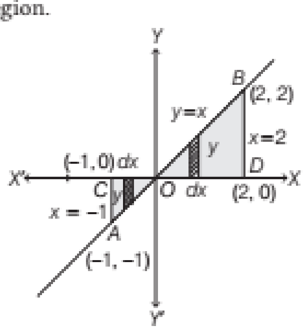
$$= \frac{2}{3}\left|\left[0 + \frac{9}{2}\times\frac{\pi}{2} - 0\right]\right|$$

$$= \frac{2}{3}\times\frac{9\pi}{4} = \frac{3\pi}{2} \text{ sq units}$$

Hence, the required area is $\frac{3\pi}{2}$ sq units.

EXAMPLE |5| Find the area bounded by the line $y = x$, the X-axis and the lines $x = -1$ and $x = 2$.

Sol. We know that $y = x$ is the line passing through the origin and making angle of 45° with the X-axis as shown in the given figure. Now, we have to find the area of the shaded region.



Required area = Area of $OACO$ + Area of $OBDO$

$$= \left|\int_{-1}^0 y\,dx\right| + \int_0^2 y\,dx$$

[$\because$ area $OACO$ is below the X-axis, so we take its absolute value]

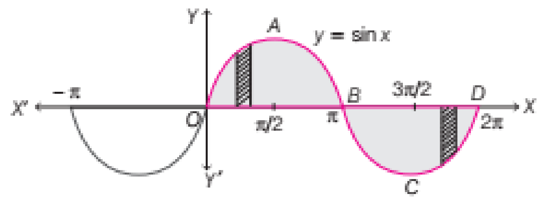
$$= \left|\int_{-1}^0 x\,dx\right| + \int_0^2 x\,dx = \left|\left[\frac{x^2}{2}\right]_{-1}^0\right| + \left[\frac{x^2}{2}\right]_0^2$$

$$= \left|\left[0 - \frac{1}{2}\right]\right| + \left[\frac{4}{2} - 0\right] = \frac{1}{2} + \frac{4}{2} = \frac{5}{2} \text{ sq units}$$

Hence, the required area is 5/2 sq units.

EXAMPLE |6| Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$. **[NCERT]**

Sol. The graph of $y = \sin x$ is shown in the figure below



Clearly, required region is bounded by $y = \sin x$, X-axis and lines $x = 0$ and $x = 2\pi$, which is represented by shaded region.

$\therefore$ Required area = Area of region $OABO$ + Area of region $BCDB$

$$= \int_0^{\pi} \sin x\,dx + \left|\int_{\pi}^{2\pi} \sin x\,dx\right|$$

[since, in region $BCDB$, the graph is below the X-axis, so its area comes out to be negative, therefore we take the absolute value]

$$= [-\cos x]_0^{\pi} + \left|[-\cos x]_{\pi}^{2\pi}\right|$$

$$= (-\cos\pi + \cos 0) + |-\cos 2\pi + \cos\pi|$$

$$= (1+1) + |-1-1|$$

$$= 2 + |-2| = 2 + 2 = 4 \text{ sq units}$$

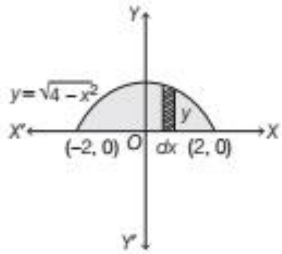
Hence, the required area is 4 sq units.

AREA OF SYMMETRICAL REGION

Sometimes, the bounded region, for which we have to calculate area, is symmetrical about X-axis (i.e. symmetric in I and IV or in II and III quadrants) or Y-axis (i.e. symmetric in I and II or in III and IV quadrants) or both X-axis and Y-axis (i.e. symmetric in all quadrants) or origin (i.e. symmetric in I and III or in II and IV quadrants). Then, firstly we calculate the area of bounded region in one quadrant and then multiply this area by number of quadrants in which region is symmetrical.

EXAMPLE |7| Sketch the region $\{(x, y) : y = \sqrt{4 - x^2}\}$ and X-axis. Find the area of the region using integration. **[NCERT Exemplar]**

Sol. Given region is $\{(x, y): y = \sqrt{4 - x^2}\}$ and X-axis.



We have, $y = \sqrt{4 - x^2} \Rightarrow y^2 = 4 - x^2 \Rightarrow x^2 + y^2 = 4$

This represent the equation of circle having centre (0, 0) and radius 2. But original equation is $y = \sqrt{4 - x^2}$, so y is positive. It means that we have to take a curve above the X-axis.

Thus, only semi-circle is formed above the X-axis.

Since, the region is symmetrical about Y-axis.

$\therefore$ Area of shaded region,

$$A = 2\int_0^2 y\,dx = 2\int_0^2 \sqrt{4 - x^2}\,dx = 2\int_0^2 \sqrt{2^2 - x^2}\,dx$$

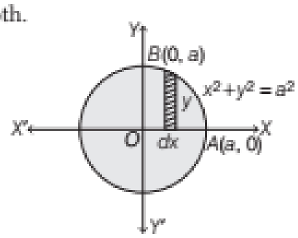
$$= 2\left[\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\cdot\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 2\left[\frac{2}{2}\cdot 0 + 2\cdot\frac{\pi}{2} - \frac{0}{2}\cdot 2 - 2\sin^{-1}(0)\right]$$

$$= 2\left[2\cdot\frac{\pi}{2} + 0\right] = 2\pi \text{ sq units}$$

EXAMPLE |8| Using integration, find the area enclosed by the circle $x^2 + y^2 = a^2$.

Sol. Given equation of circle is $x^2 + y^2 = a^2$, its centre is (0, 0) and radius is a. It cuts the X-axis at $A(a, 0)$ and Y-axis at $B(0, a)$. Also, it is symmetrical about X and Y-axes both.



Clearly, area of region in I quadrant

$$= \int_0^a y\,dx = \int_0^a \sqrt{a^2 - x^2}\,dx \quad \text{[consider vertical strip]}$$

$[\because x^2 + y^2 = a^2 \Rightarrow y = \sqrt{a^2 - x^2}$, as y is in I quadrant$]$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \left[\left(\frac{a}{2}\times 0 + \frac{a^2}{2}\sin^{-1}1\right) - 0\right] = \frac{a^2}{2}\left(\frac{\pi}{2}\right)$$

$$\left[\because \sin^{-1}(1) = \frac{\pi}{2}\right]$$

$$= \frac{\pi a^2}{4}$$

Now, required area = 4 × Area of region in I quadrant

[since, region is symmetrical in all quadrants]

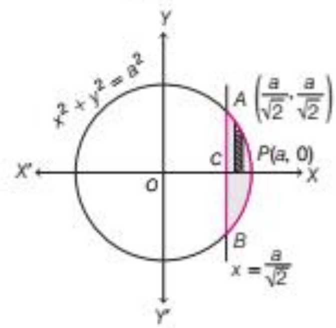
$$= 4 \times \frac{\pi a^2}{4} = \pi a^2 \text{ sq units}$$

EXAMPLE |9| Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut-off by the line $x = \frac{a}{\sqrt{2}}$. **[NCERT]**

Sol. Given equations of circle and line are

$$x^2 + y^2 = a^2 \quad \text{...(i)}$$

and $$x = \frac{a}{\sqrt{2}} \quad \text{...(ii)}$$



Clearly, required region is $APBCA$, which is symmetrical about X-axis. and the x-coordinate of point of intersection curve and line is $\frac{a}{\sqrt{2}}$.

Now, required area = Area of region $APBCA$

= 2 (Area of region $APCA$)

$$= 2\int_{a/\sqrt{2}}^a y\,dx = 2\int_{a/\sqrt{2}}^a \sqrt{a^2 - x^2}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{\frac{a}{\sqrt{2}}}^a$$

$$= 2\left[\left(0 + \frac{a^2}{2}\sin^{-1}\frac{a}{a}\right) - \left\{\frac{a}{2\sqrt{2}}\sqrt{a^2 - \frac{a^2}{2}} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{\sqrt{2}}\cdot\frac{1}{a}\right)\right\}\right]$$

$$= 2\left[\left(\frac{a^2}{2}\sin^{-1}(1)\right) - \frac{a}{2\sqrt{2}}\sqrt{\frac{a^2}{2}} - \frac{a^2}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right]$$

$$= 2\left[\frac{a^2}{2}\cdot\frac{\pi}{2} - \frac{a}{2\sqrt{2}}\cdot\frac{a}{\sqrt{2}} - \frac{a^2}{2}\cdot\frac{\pi}{4}\right]$$

$$= 2\left[\frac{\pi a^2}{4} - \frac{a^2}{4} - \frac{a^2\pi}{8}\right]$$

$$= \frac{\pi a^2}{2} - \frac{a^2}{2} - \frac{\pi a^2}{4} = \frac{2\pi a^2 - 2a^2 - \pi a^2}{4}$$

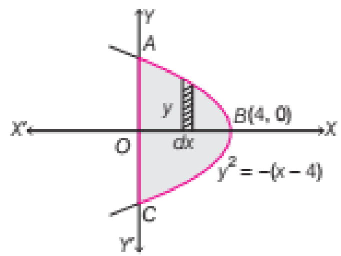
$$= \frac{\pi a^2 - 2a^2}{4} = \frac{a^2(\pi - 2)}{4} \text{ sq units}$$

Hence, the required area is $\frac{a^2(\pi - 2)}{4}$ sq units.

EXAMPLE |10| Find the area bounded by the curve $x = 4 - y^2$ and the Y-axis.

Sol. We have, $x = 4 - y^2$

$\Rightarrow \quad y^2 = -x + 4 \Rightarrow y^2 = -(x - 4)$



which is the parabola of the form $Y^2 = -4aX$.

Its vertex is (4, 0) and it is symmetrical about X-axis.

$\therefore$ Required area = Area of shaded region

$= 2$ (Area of region $OABO$) $= 2\int_0^4 y\,dx = 2\int_0^4 \sqrt{4 - x}\,dx$

$[\because x = 4 - y^2 \Rightarrow y = \sqrt{4 - x}$, as y is in I quadrant$]$

$$= 2\int_0^4 (4 - x)^{1/2}\,dx = 2\left[\frac{-(4 - x)^{3/2}}{3/2}\right]_0^4$$

$$= -2\times\frac{2}{3}[(4 - x)^{3/2}]_0^4 = -\frac{4}{3}[(4 - 4)^{3/2} - (4 - 0)^{3/2}]$$

$$= -\frac{4}{3}[0 - (4)^{3/2}] = \frac{4}{3}\times(4)^{3/2}$$

$$= \frac{4}{3}\times(2^2)^{3/2} = \frac{4}{3}\times 8 = \frac{32}{3} \text{ sq units}$$

Hence, the required area is $\frac{32}{3}$ sq units.

AREA OF REGION BOUNDED BY A CURVE AND A LINE

To find the area of the region bounded by a line and a circle, a line and a parabola, a line and an ellipse, we use the following steps

I. Firstly, we **draw the rough sketch** of given curves and identify the region for which we have to find the area.

II. Find the **intersection point** (or points) of curve and line.

III. Draw perpendicular lines from the intersection points of line and curve to X-axis (or Y-axis).

If these perpendicular lines divide the region (whose area is to be determined) into two (or more) parts, then take two (or more) vertical strips (or horizontal strips), otherwise take one vertical strip (or horizontal strip).

IV. Now, find the area of region by using the suitable formula, i.e.

For one vertical strip Area $= \int_{x=a}^{b} y\,dx$, where y is height of the vertical strip.

For two vertical strips

Area $= \int_{x=a}^{c} y_1\,dx + \int_{x=c}^{b} y_2\,dx$, where y_1 and y_2 represent the heights of vertical strips.

or

For one horizontal strip Area $= \int_c^d x\,dy$, where x is the length of the horizontal strip.

For two horizontal strips Area $= \int_c^e x_1\,dy + \int_e^d x_2\,dy$, where x_1 and x_2 represent the lengths of the horizontal strips.

EXAMPLE |11| Find the area of the region bounded by the curves $x = at^2$ and $y = 2at$ between the ordinates corresponding to $t = 1$ and $t = 2$. **[NCERT Exemplar]**

Sol. Given equations of curves are

$$x = at^2 \quad \text{...(i)}$$

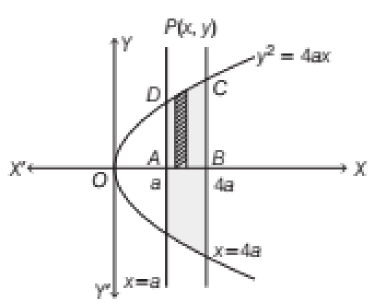
and $$y = 2at \quad \text{...(ii)}$$

Now, for converting the equation in cartesian form, put the value of t from Eq. (ii) in Eq. (i).

i.e. $x = a\left(\frac{y}{2a}\right)^2 \Rightarrow x = a\cdot\frac{y^2}{4a^2} \Rightarrow y^2 = 4ax$ ...(iii)

On putting $t = 1$ and $t = 2$ in Eq. (i), we get $x = a$ and $x = 4a$, respectively.

Now, the region bounded by the curve $y^2 = 4ax$ between $x = a$ and $x = 4a$ is given below



Now, required area = Area of shaded region

$$= 2 \text{ (Area of region } ABCDA)$$

$$= 2\int_a^{4a} y\,dx = 2\int_a^{4a} \sqrt{4ax}\,dx$$

$$= 2 \times 2\int_a^{4a} \sqrt{ax}\,dx = 4\sqrt{a}\int_0^{4a} x^{1/2}\,dx$$

$$= 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_a^{4a} = 4\sqrt{a} \times \frac{2}{3}[(4a)^{3/2} - a^{3/2}]$$

$$= 4\sqrt{a} \times \frac{2}{3}[8a^{3/2} - a^{3/2}]$$

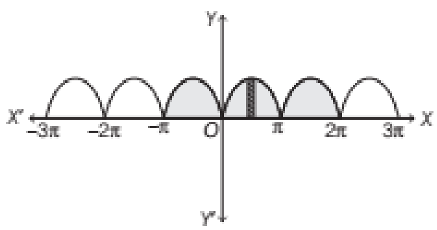
$$= 4\sqrt{a} \times \frac{2}{3} \times 7a^{3/2} = \frac{56a^2}{3} \text{ sq units}$$

Hence, the required area is $\frac{56a^2}{3}$ sq units.

EXAMPLE |12| Draw the graph of the curve $y = |\sin x|$ and find the area bounded by the curve, X-axis and ordinates $x = -\pi$ to 2π.

Sol. We have, $y = |\sin x| = \begin{cases} \sin x, & x \in (0, \pi), (2\pi, 3\pi), \ldots \\ -\sin x, & x \in (-\pi, 0), (\pi, 2\pi), \ldots \end{cases}$

The graph of given curve is shown below



Now, required area = Area of shaded region

= 3 (Area of one shaded region)

[$\because$ all parts are of equal areas]

$$= 3\int_0^{\pi} y\,dx = 3\int_0^{\pi}(\sin x)\,dx = 3[-\cos x]_0^{\pi}$$

$$= -3[\cos \pi - \cos 0] = -3[-1-1] = 6 \text{ sq units}$$

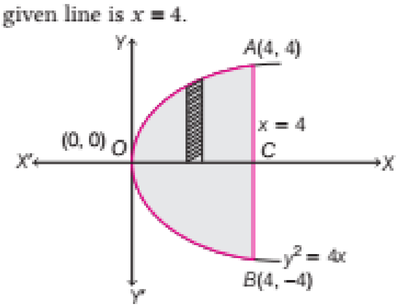
Hence, the required area is 6 sq units.

EXAMPLE |13| Find the area of region bounded by the curve $y^2 = 4x$ and the line $x = 4$.

Sol. Given curve is a parabola, $y^2 = 4x$ ...(i)

which is of the form of $Y^2 = 4aX$ having vertex (0, 0)

and given line is $x = 4$. ...(ii)



It is clear from the figure that the region for which we have to find area is $OBCAO$. Also, the region $OCAO$ is symmetrical about X-axis.

Now, let us find the intersection point of curve and line.

On putting the value of x from Eq. (ii) in Eq. (i), we get

$$y^2 = 4(4) = 16$$

$$\Rightarrow \qquad y = \pm 4$$

Thus, line and curve intersect at two points (4, 4) and (4, −4). So, coordinates of point A are (4, 4), as it is in I quadrant.

Now, area of bounded region $OACBO$

= 2 (Area of region $OACO$)

[since, parabola is symmetrical about the X-axis]

$$= 2\int_0^4 y\,(\text{parabola})\,dx$$

$$[\because y^2 = 4x \Rightarrow y = \sqrt{4x} = 2 \cdot x^{1/2}]$$

$$= 2\int_0^4 2 \cdot x^{1/2}\,dx$$

$$= 4\int_0^4 x^{1/2}dx$$

$$= 4\left[\frac{x^{3/2}}{3/2}\right]_0^4$$

$$= 4 \cdot \frac{2}{3}[x^{3/2}]_0^4$$

$$= \frac{8}{3}[(4)^{3/2} - 0]$$

$$= \frac{8}{3} \times (2^2)^{3/2}$$

$$= \frac{8}{3} \times 8$$

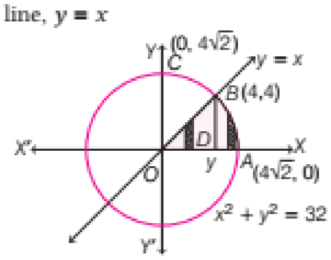
$$= \frac{64}{3} \text{ sq units}$$

Hence, the required area is $\frac{64}{3}$ sq units.

EXAMPLE |14| Find the area of region in the first quadrant enclosed by the X-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. **[Delhi 2020, 2014]**

Sol. We have, circle $x^2 + y^2 = 32$ having centre (0, 0) and radius $4\sqrt{2}$ ...(i)

and the line, $y = x$ ...(ii)



It is clear from the figure that required region is *OABO*.

On putting the value of y from Eq. (ii) in Eq. (i), we get

$$x^2 + x^2 = 32$$

$$\Rightarrow \quad 2x^2 = 32$$

$$\Rightarrow \quad x^2 = \frac{32}{2} = 16$$

$$\Rightarrow \quad x = \pm 4$$

From Eq. (ii), we get $y = \pm 4$

Thus, line and circle intersect at two points (4, 4) and (− 4, − 4).

So, the coordinates of B are (4, 4) .

[since, it is in I quadrant]

Also, circle cuts the X-axis at $A\ (4\sqrt{2}, 0)$ and Y-axis at $C\ (0, 4\sqrt{2})$ in I quadrant.

[$\because 4\sqrt{2}$ is radius of a circle]

Here, we have to draw two vertical strips, as perpendicular line drawn from intersection point to the X-axis, divides the region into two parts. Now, first strip is drawn in region *ODBO* and then limit is taken from 0 to 4. Second strip is drawn in region *DABD* and then limit is taken from 4 to $4\sqrt{2}$.

Now, area of region *ODBO* $= \int_0^4 y\, dx$

[where, y is the height of vertical strip]

$$= \int_0^4 x\, dx = \left[\frac{x^2}{2}\right]_0^4$$

$$= \frac{(4)^2}{2} - 0 = 8 \text{ sq units}$$

and area of region *DABD* $= \int_4^{4\sqrt{2}} y\, dx$

[where, y is the height of vertical strip in this region]

$$= \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\, dx = \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2}\, dx$$

$$= \left[\frac{x}{2}\sqrt{(4\sqrt{2})^2 - x^2} + \frac{(4\sqrt{2})^2}{2} \times \sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}}$$

$$= \left[\left\{\frac{4\sqrt{2}}{2}\sqrt{(4\sqrt{2})^2 - (4\sqrt{2})^2} + \frac{32}{2}\sin^{-1}\left(\frac{4\sqrt{2}}{4\sqrt{2}}\right)\right\} - \left\{\frac{4}{2}\sqrt{(4\sqrt{2})^2 - (4)^2} + \frac{32}{2}\sin^{-1}\left(\frac{4}{4\sqrt{2}}\right)\right\}\right]$$

$$= 2\sqrt{2} \times 0 + 16\sin^{-1}(1) - 2\sqrt{32 - 16} - 16\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= 16 \cdot \left(\frac{\pi}{2}\right) - 2\sqrt{16} - 16 \cdot \left(\frac{\pi}{4}\right)$$

$$= 8\pi - 8 - 4\pi$$

$$= 4\pi - 8$$

$\therefore$ Required area

$$= \text{Area of region } ODBO + \text{Area of region } DABD$$

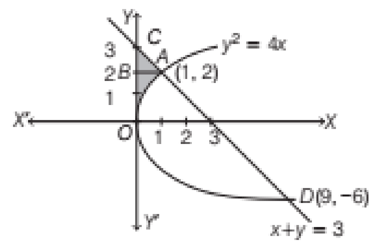
$$= 8 + 4\pi - 8$$

$$= 4\pi \text{ sq units}$$

Hence, the area of required region is 4π sq units.

EXAMPLE |15| Find the area bounded between the curve $y^2 = 4x$, line $x + y = 3$ and Y-axis.

Sol. Given curve $y^2 = 4x$ is a parabola having vertex (0, 0) and open right side and the given line is $x + y = 3$ or $\frac{x}{3} + \frac{y}{3} = 1$, which intersects both coordinate axes at (3, 0) and (0, 3).



It is clear from the figure that the required region is *OACBO*.

Now, the point of intersection of line $x + y = 3$ and curve is given by

$$y^2 = 4(3 - y)$$

$$\Rightarrow \quad y^2 + 4y - 12 = 0$$

$$\Rightarrow \quad y^2 + 6y - 2y - 12 = 0$$

$$\Rightarrow \quad y(y + 6) - 2(y + 6) = 0$$

$$\Rightarrow \quad (y - 2)(y + 6) = 0$$

$$\Rightarrow \quad y = 2, -6$$

When $y = 2$, then $x + 2 = 3 \Rightarrow x = 1$

When $y = -6$, then $x - 6 = 3 \Rightarrow x = 9$

So, the points of intersection are $A(1, 2)$ and $D(9, -6)$.

Now, required area = Area of shaded region *OABO* + Area of shaded region *ABCA*

$$= \int_0^2 x\,(\text{parabola})\, dy + \int_2^3 x\,(\text{line})\, dy$$

$$= \int_0^2 \left(\frac{y^2}{4}\right) dy + \int_2^3 (3 - y)\, dy = \frac{1}{4}\left[\frac{y^3}{3}\right]_0^2 + \left[3y - \frac{y^2}{2}\right]_2^3$$

$$= \frac{1}{12}[(2)^3 - (0)^3] + \left[9 - \frac{9}{2} - \left(6 - \frac{4}{2}\right)\right]$$

$$= \frac{1}{12}(8) + \left(3 - \frac{5}{2}\right) = \frac{8}{12} + \frac{1}{2}$$

$$= \frac{8 + 6}{12} = \frac{14}{12} = \frac{7}{6} \text{ sq units}$$

Hence, the required area is $\frac{7}{6}$ sq units.

EXAMPLE |16| Find the area of the region $\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$.

Sol. Given equations of curve and lines are

$$y = x^2 + 1 \quad \text{or} \quad x^2 = y - 1 \quad \text{...(i)}$$

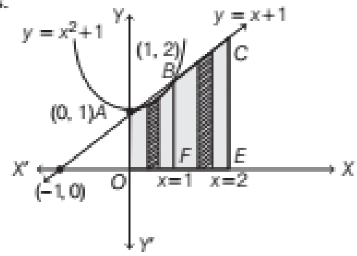
$$y = x + 1 \quad \text{...(ii)}$$

and $$x = 2 \quad \text{...(iii)}$$

Here, Eq. (i) represents a parabola with vertex $A(0, 1)$ and axis along the positive direction of Y-axis.

Eq. (ii) represents a line which intersects the coordinate axes at $(0, 1)$ and $(-1, 0)$.

Eq. (iii) represents a line which is perpendicular to X-axis.



As, $0 \le y \le x^2 + 1$, therefore it represents the area below the parabola and above the X-axis.

Similarly, $0 \le y \le x + 1$ represents the area below the line AB and above X-axis and $0 \le x \le 2$ represents the area between the parallel lines $x = 0$ and $x = 2$.

$\therefore$ The region common to

$$\left.\begin{array}{l} 0 \le y \le x^2 + 1 \\ 0 \le y \le x + 1 \\ \text{and } 0 \le x \le 2 \end{array}\right\} \text{is shown as shaded region } OABCEFO.$$

Now, solving Eqs. (i) and (ii), we get

$$x^2 + 1 = x + 1$$

$$\Rightarrow \quad x^2 - x = 0$$

$$\Rightarrow \quad x(x - 1) = 0 \Rightarrow x = 0, 1$$

When $x = 0$, then $y = 0 + 1 = 1$ [using Eq. (ii)]

When $x = 1$, then $y = 1 + 1 = 2$ [using Eq. (ii)]

So, the points of intersection are $A(0, 1)$ and $B(1, 2)$.

$\therefore$ Required area = Area of shaded region $OABCEFO$

$$= \text{Area of region } OABFO + \text{Area of region } FECBF$$

$$= \int_0^1 y(\text{parabola})\, dx + \int_1^2 y(\text{line})\, dx$$

$$= \int_0^1 (x^2 + 1)\, dx + \int_1^2 (x + 1)\, dx$$

$$= \left[\frac{x^3}{3} + x\right]_0^1 + \left[\frac{x^2}{2} + x\right]_1^2$$

$$= \left[\left(\frac{1}{3} + 1\right) - 0\right] + \left[\left(\frac{4}{2} + 2\right) - \left(\frac{1}{2} + 1\right)\right]$$

$$= \frac{4}{3} + \left[4 - \frac{3}{2}\right]$$

$$= \frac{4}{3} + \frac{5}{2}$$

$$= \frac{8 + 15}{6}$$

$$= \frac{23}{6} \text{ sq units}$$

Hence, the required area is $\frac{23}{6}$ sq units.

SUMMARY

- **Area of Region Bounded by X-axis, Lines $x = a$, $x = b$ and Curve $y = f(x)$**

$$\text{Area} = \int_a^b dA = \int_a^b y\, dx = \int_a^b f(x)\, dx$$

- **Area of Region Bounded by Y-axis, Lines $y = c$, $y = d$ and Curve $x = f(y)$**

$$\text{Area} = \int_c^d dA = \int_c^d x\, dy = \int_c^d f(y)\, dy$$

- **Area of Region when Curve is Below the X-axis** If the curve $y = f(x)$ lies below X-axis, then area bounded by the curve $y = f(x)$, X-axis and the lines $x = a$ and $x = b$ come out to be negative. If the area is negative, then we take its absolute value.

$$\therefore \quad \text{Area} = \left|\int_a^b y\, dx\right| = \left|\int_a^b f(x)\, dx\right|$$

- **Area of Region when Curve is Above and Below the X-axis** If the area A bounded by the curve $y = f(x)$, X-axis and the lines $x = a$ and $x = b$, is such that $A = A_1 + A_2$ where $A_2 < 0$ and $A_1 > 0$, then

$$A = |A_2| + A_1 = \left|\int_a^c f(x)\, dx\right| + \int_c^b f(x)\, dx.$$

- **Area of the Region Bounded by a Curve and a Line**
  We use the following steps

  ***Step* I** Firstly, we draw the rough sketch and line, of given curve and identify the region for which we have to find the area.

  ***Step* II** Find the intersection point (or points) of curve and line and check whether region is symmetrical or not.

  ***Step* III** If the region is symmetrical, then draw a vertical strip (or horizontal strip) in the required area of I quadrant and take suitable limits for x, say a and b (or for y, say c and d). But if the region is not symmetric, then draw a vertical strip (or horizontal strip) in the required area and take suitable limits for x (or y). Sometimes, we need to draw two vertical (or horizontal) strips in the area of I quadrant (or required area).

Step IV Now, find area of region by using suitable formula, i.e. for one vertical strip,

Area $= \int_{x=a}^{b} y\,dx$, where y is height of vertical strip.

For two vertical strips, Area $= \int_{x=a}^{c} y_1\,dx + \int_{x=c}^{b} y_2\,dx$

[here, y_1 and y_2 represent the heights of vertical strips] or For one horizontal strip, area $\int_c^d X\,dy$, where X is the length of the horizontal strip.

For two horizontal strip,

Area $= \int_c^e X_1\,dy + \int_e^d X_2\,dy$, where X_1 and X_2 represent the lengths of the horizontal strips.

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1 The area of the region bounded by the curve $y = x + 1$ and the lines $x = 2$, $x = 3$, is

(a) $\frac{7}{2}$ sq units (b) $\frac{9}{2}$ sq units

(c) $\frac{11}{2}$ sq units (d) $\frac{13}{2}$ sq units

2 The area of the region bounded by the curve $x = 2y + 3$ and the lines $y = 1$, $y = -1$ is

(a) 4 sq units (b) $\frac{3}{2}$ sq units

(c) 6 sq units (d) 8 sq units

3 Area of the region bounded by the curve $y = \cos x$ between $x = 0$ and $x = \pi$ is

(a) 2 sq units (b) 4 sq units

(c) 3 sq units (d) 1 sq unit

4 The area of the region bounded by the curve $y = \sin x$ between the ordinates $x = 0$, $x = \frac{\pi}{2}$ and the X-axis is

(a) 2 sq units (b) 4 sq units

(c) 3 sq units (d) 1 sq unit

5 The area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$ is

(a) $5\sqrt{3}$ sq units (b) $4\sqrt{3}$ sq units

(c) $8\sqrt{3}$ sq units (d) None of these

(a) 8 sq units (b) 16 sq units

(c) 32 sq units (d) None of these

8 The area bounded by the curve $y = x|x|$, X-axis and the coordinates $x = -1$ and $x = 1$ is given by

(a) 0 sq unit (b) $\frac{1}{3}$ sq unit

(c) $\frac{2}{3}$ sq unit (d) $\frac{4}{3}$ sq unit

9 Let the straight line $x = b$ divide the area enclosed by $y = (1 - x)^2$, $y = 0$ and $x = 0$ into two parts $R_1(0 \le x \le b)$ and $R_2(b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then, b equals

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

10 If a curve $y = a\sqrt{x} + bx$ passes through the point (1, 2) and the area bounded by the curve, line $x = 4$ and X-axis is 8 sq units, then

(a) $a = 3, b = -1$ (b) $a = 3, b = 1$

(c) $a = -3, b = 1$ (d) $a = -3, b = -1$

LONG ANSWER Type Questions

11 Find the area of the region bounded by the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. **[NCERT Exemplar]**

12 Find the area bounded by the curve $y = \cos x$

6 The line $x = \frac{\pi}{4}$ divides the area of the region bounded by $y = \sin x$, $y = \cos x$ and X-axis $\left(0 \le x \le \frac{\pi}{2}\right)$ into two regions of areas A_1 and A_2. Then, $A_1 : A_2$ is equal to

(a) 4 : 1 (b) 3 : 1

(c) 2 : 1 (d) 1 : 1

7 Area of the region bounded by the curve $y = |x + 1| + 1$, $x = -3$, $x = 3$ and $y = 0$ is

16 Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the Y-axis in the first quadrant. **[NCERT]**

17 Find the area bounded by the curve $y = x^3$, the X-axis and the ordinates $x = -2$ and $x = 1$. **[NCERT]**

18 Find the area of the region lying in the first quadrant and bounded by $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$. **[NCERT]**

19 The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$. Find the value of a. **[NCERT]**

20 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **[NCERT]**

21 Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the coordinates $x = 0$ and $x = ae$, where $b^2 = a^2(1 - e^2)$ and $e < 1$. **[NCERT]**

22 Find the area of the minor segment of the circle $x^2 + y^2 = a^2$ cut-off by the line $x = \frac{a}{2}$. **[NCERT Exemplar]**

23 Using integration find the area of the region. $\{(x, y): 0 \le y \le x^2, 0 \le y \le x, 0 \le x \le 2\}$ **[Delhi 2020]**

24 Using integration, find the area of the region in the first quadrant enclosed by the line $x + y = 2$, the parabola $y^2 = x$ and the X-axis. **[CBSE 2021 (Term I)]**

25 Using integration, find the area of the region. $\{(x, y): 0 \le y \le \sqrt{3}x, x^2 + y^2 \le 4\}$ **[CBSE 2021 (Term I)]**

26 Find the area of the parabola $y^2 = 4ax$ bounded

between $x = 0$ and $x = 2\pi$. **[NCERT Exemplar]**

13 Find the area under the curve $y = \sqrt{3x + 4}$ between $x = 0$, $x = 4$ and the X-axis.

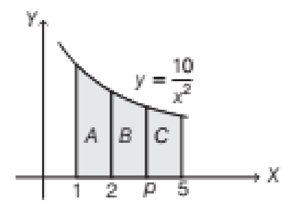
14 Draw a rough sketch of the curve $y = \sqrt{x - 1}$ in the interval [1, 5]. Find the area under the curve and the lines $x = 1$ and $x = 5$. **[NCERT Exemplar]**

15 Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the X-axis in the first quadrant. **[NCERT]**

29 Find the area of the region bounded by the line $y = 3x + 2$, the X-axis and the ordinates $x = -1$ and $x = 1$. **[NCERT]**

30 The figure shows the part of the curve $y = \frac{10}{x^2}$.

Find the area of region A. Also, find the value of p, for which region B and region C are equal in area.



31 Using integration, find the area of the region in the first quadrant enclosed by the X-axis, the line $y = x$ and the circle $x^2 + y^2 = 18$. **[All India 2014C]**

32 Find the area of the region in the first quadrant enclosed by the Y-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$, using integration. **[Delhi 2015C]**

33 Sketch the graph of $y = |x + 3|$ and evaluate the area under the curve $y = |x + 3|$ above X-axis and between $x = -6$ to $x = 0$. **[All India 2011]**

34 Draw a rough sketch of the given curve $y = 1 + |x + 1|$, $x = -3$, $x = 3$ and $y = 0$ and find the area of the region bounded by them, using integration. **[Delhi 2014C; NCERT Exemplar]**

35 Draw a rough sketch of the curve $y = |x - 2|$. Find the area under the curve and line $x = 0$ and $x = 4$.

36 Find the area bounded by the Y-axis, $y = \cos x$ and $y = \sin x$, when $0 \le x \le \frac{\pi}{2}$. **[NCERT]**

by its latusrectum. [NCERT]

27 Find the area of the region bounded by $y=-1$, $y=2$, $x=y^3$ and $x=0$.

28 Find the area bounded by the curve $y=x|x|$, X-axis and the ordinates $x=-3$ and $x=3$.

37 Find the area enclosed by the curves $x=3\cos t$ and $y=2\sin t$. [NCERT Exemplar]

38 Using integration, find the area of the region bounded between the two circles $x^2+y^2=9$ and $(x-3)^2+y^2=9$. [All India 2020]

| ANSWERS |

1. (a)	**2.** (c)	**3.** (a)	**4.** (d)	**5.** (c)
6. (d)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (a)
11. 20π sq units	**12.** 4 sq units	**13.** $\frac{112}{9}$ sq units	**14.** $\frac{16}{3}$ sq units	
15. $4(4-\sqrt{2})$ sq units	**16.** $\frac{8}{3}[4-\sqrt{2}]$ sq units	**17.** $\frac{17}{4}$ sq units	**18.** $\frac{7}{3}$ sq units	
19. $a=(4)^{2/3}$	**20.** πab sq units	**21.** $ab[e\sqrt{1-e^2}+\sin^{-1}(e)]$ sq units		
22. $\frac{a^2}{12}(4\pi-3\sqrt{3})$ sq units	**23.** $\frac{11}{6}$ sq units	**24.** $\frac{7}{6}$ sq units	**25.** $\frac{2\pi}{3}$ sq units	
26. $\frac{8}{3}a^2$ sq units	**27.** $\frac{17}{4}$ sq units	**28.** 18 sq units	**29.** $\frac{13}{3}$ sq units	
30. $\frac{20}{7}$	**31.** $\frac{9\pi}{4}$ sq units	**32.** 4π sq units	**33.** 9 sq units	
34. 16 sq units	**35.** 4 sq units	**36.** $(\sqrt{2}-1)$ sq units	**37.** 6π sq units	
38. $2\left[-\frac{9\sqrt{3}}{4}+3\pi\right]$ sq units				