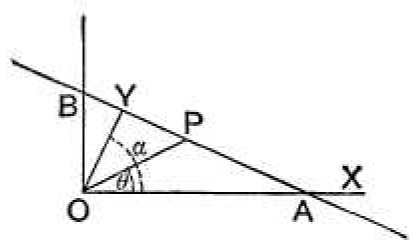
Chapter 5

THE STRAIGHT LINE

: POLAR EQUATIONS. OBLIQUE COORDINATES. MISCELLANEOUS PROBLEMS. MISCELLANEOUS PROBLEMS. LOCI.

88. *To find the general equation to a straight line in polar coordinates.*

Let p be the length of the perpendicular OY from the origin upon the straight line, and let this perpendicular make an angle α with the initial line.



Let P be any point on the line and let its coordinates be r and θ.

The equation required will then be the relation between r, θ, p, and α.

From the triangle OYP we have

$$p = r\cos YOP = r\cos(\alpha - \theta) = r\cos(\theta - \alpha).$$

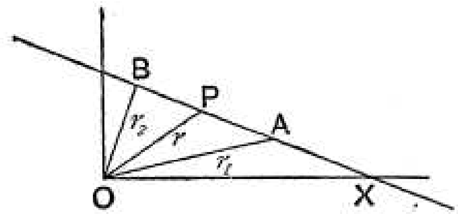
The required equation is therefore

$$r\cos(\theta - \alpha) = p.$$

[On transforming to Cartesian coordinates this equation becomes the equation of Art. 53.]

89. *To find the polar equation of the straight line joining the points whose coordinates are* (r_1, θ_1) *and* (r_2, θ_2).

Let A and B be the two given points and P any point on the line joining them whose coordinates are r and θ.



Then, since

$\triangle AOB = \triangle AOP + \triangle POB$,

we have

$$\tfrac{1}{2} r_1 r_2 \sin AOB = \tfrac{1}{2} r_1 r \sin AOP + \tfrac{1}{2} r r_2 \sin POB,$$

i.e. $$r_1 r_2 \sin(\theta_2 - \theta_1) = r_1 r \sin(\theta - \theta_1) + r r_2 \sin(\theta_2 - \theta),$$

i.e. $$\frac{\sin(\theta_2 - \theta_1)}{r} = \frac{\sin(\theta - \theta_1)}{r_2} + \frac{\sin(\theta_2 - \theta)}{r_1}.$$

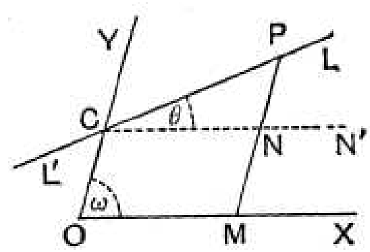
OBLIQUE COORDINATES.

90. In the previous chapter we took the axes to be rectangular. In the great majority of cases rectangular axes are employed, but in some cases oblique axes may be used with advantage.

In the following articles we shall consider the propositions in which the results for oblique axes are different from those for rectangular axes. The propositions of Arts. 50 and 62 are true for oblique, as well as rectangular, coordinates.

91. *To find the equation to a straight line referred to axes inclined at an angle* ω.

Let LPL' be a straight line which cuts the axis of Y at a distance c from the origin and is inclined at an angle θ to the axis of x.



Let P be any point on the straight line. Draw PNM parallel to the axis of y to meet OX in M, and let it meet the straight line through C parallel to the axis of x in the point N.

Let P be the point (x, y), so that

$$CN = OM = x, \text{ and } NP = MP - OC = y - c.$$

Since $\angle CPN = \angle PNN' - \angle PCN' = \omega - \theta$, we have

$$\frac{y-c}{x} = \frac{NP}{CN} = \frac{\sin NCP}{\sin CPN} = \frac{\sin\theta}{\sin(\omega-\theta)}.$$

Hence $$y = x\frac{\sin\theta}{\sin(\omega-\theta)} + c \quad \text{.................(1).}$$

This equation is of the form

$$y = mx + c,$$

where

$$m = \frac{\sin\theta}{\sin(\omega-\theta)} = \frac{\sin\theta}{\sin\omega\cos\theta - \cos\omega\sin\theta} = \frac{\tan\theta}{\sin\omega - \cos\omega\tan\theta},$$

and therefore $$\tan\theta = \frac{m\sin\omega}{1+m\cos\omega}.$$

In oblique coordinates the equation

$$y = mx + c$$

therefore represents a straight line which is inclined at an angle

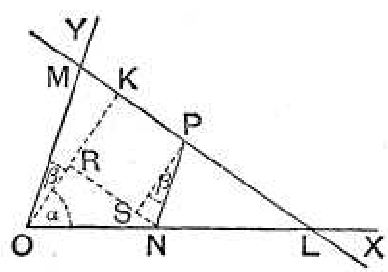
$$\tan^{-1}\frac{m\sin\omega}{1+m\cos\omega}$$

to the axis of x.

Cor. From (1), by putting in succession θ equal to 90° and $90^\circ + \omega$, we see that the equations to the straight lines, passing through the origin and perpendicular to the axes of x and y, are respectively $y = -\dfrac{x}{\cos\omega}$ and $y = -x\cos\omega$.

92. *The axes being oblique, to find the equation to the straight line, such that the perpendicular on it from the origin is of length p and makes angles α and β with the axes of x and y.*

Let LM be the given straight line and OK the perpendicular on it from the origin.



Let P be *any* point on the straight line; draw the ordinate PN and draw NR perpendicular to OK and PS perpendicular to NR.

Let P be the point (x, y), so that $ON = x$ and $NP = y$.

The lines NP and OY are parallel.

Also OK and SP are parallel, each being perpendicular to NR.

Thus $\angle SPN = \angle KOM = \beta$.

We therefore have

$$p = OK = OR + SP = ON \cos\alpha + NP \cos\beta = x\cos\alpha + y\cos\beta.$$

Hence $x\cos\alpha + y\cos\beta - p = 0$,

being the relation which holds between the coordinates of any point on the straight line, is the required equation.

93. *To find the angle between the straight lines*

$$y = mx + c \ \ and \ \ y = m'x + c',$$

the axes being oblique.

If these straight lines be respectively inclined at angles θ and θ' to the axis of x, we have, by the last article,

$$\tan\theta = \frac{m\sin\omega}{1 + m\cos\omega} \text{ and } \tan\theta' = \frac{m'\sin\omega}{1 + m'\cos\omega}.$$

The angle required is $\theta \sim \theta'$.

Now $$\tan(\theta - \theta') = \frac{\tan\theta - \tan\theta'}{1 + \tan\theta . \tan\theta'}$$

$$= \frac{\dfrac{m\sin\omega}{1 + m\cos\omega} - \dfrac{m'\sin\omega}{1 + m'\cos\omega}}{1 + \dfrac{m\sin\omega}{1 + m\cos\omega} \dfrac{m'\sin\omega}{1 + m'\cos\omega}}$$

$$= \frac{m\sin\omega(1 + m'\cos\omega) - m'\sin\omega(1 + m\cos\omega)}{(1 + m\cos\omega)(1 + m'\cos\omega) + mm'\sin^2\omega}$$

$$= \frac{(m - m')\sin\omega}{1 + (m + m')\cos\omega + mm'}.$$

The required angle is therefore

$$\tan^{-1}\frac{(m - m')\sin\omega}{1 + (m + m')\cos\omega + mm'}.$$

Cor. 1. The two given lines are parallel if $\mathbf{m = m'}$.

Cor. 2. The two given lines are perpendicular if

$$\mathbf{1 + (m + m')\cos\omega + mm' = 0}.$$

94. If the straight lines have their equations in the form

$$Ax + By + C = 0 \text{ and } A'x + B'y + C' = 0,$$

then $$m = -\frac{A}{B} \text{ and } m' = -\frac{A'}{B'}.$$

Substituting these values in the result of the last article the angle between the two lines is easily found to be

$$\tan^{-1} \frac{A'B - AB'}{AA' + BB' - (AB' + A'B) \cos \omega} \sin \omega.$$

The given lines are therefore parallel if

$$A'B - AB' = 0.$$

They are perpendicular if

$$AA' + BB' = (AB' + A'B) \cos \omega.$$

95. Ex. *The axes being inclined at an angle of* 30°, *obtain the equations to the straight lines which pass through the origin and are inclined at* 45° *to the straight line* $x + y = 1$.

Let either of the required straight lines be $y = mx$.

The given straight line is $y = -x + 1$, so that $m' = -1$.

We therefore have

$$\frac{(m - m') \sin \omega}{1 + (m + m') \cos \omega + mm'} = \tan (\pm 45°),$$

where $m' = -1$ and $\omega = 30°$.

This equation gives $\dfrac{m + 1}{2 + (m - 1)\sqrt{3} - 2m} = \pm 1$.

Taking the upper sign we obtain $m = -\dfrac{1}{\sqrt{3}}$.

Taking the lower sign we have $m = -\sqrt{3}$.

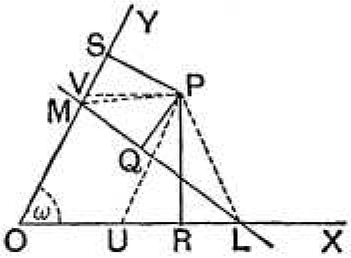
The required equations are therefore

$$y = -\sqrt{3}x \text{ and } y = -\frac{1}{\sqrt{3}}x,$$

i.e. $$y + \sqrt{3}x = 0 \text{ and } \sqrt{3}y + x = 0.$$

96. *To find the length of the perpendicular from the point* (x', y') *upon the straight line* $Ax + By + C = 0$, *the axes being inclined at an angle* ω, *and the equation being written so that* C *is a negative quantity.*

Let the given straight line meet the axes in L and M, so that $OL = -\frac{C}{A}$ and $OM = -\frac{C}{B}$.



Let P be the given point (x', y'). Draw the perpendiculars PQ, PR, and PS on the given line and the two axes.

Taking O and P on opposite sides of the given line, we then have

$$\triangle LPM + \triangle MOL = \triangle OLP + \triangle OPM,$$

i.e. $\quad PQ \, . \, LM + OL \, . \, OM \sin \omega = OL \, . \, PR + OM \, . \, PS \ldots (1).$

Draw PU and PV parallel to the axes of y and x, so that $PU = y'$ and $PV = x'$.

Hence $\qquad PR = PU \sin PUR = y' \sin \omega,$

and $\qquad PS = PV \sin PVS = x' \sin \omega.$

Also

$$LM = \sqrt{OL^2 + OM^2 - 2OL \, . \, OM \cos \omega}$$

$$= \sqrt{\frac{C^2}{A^2} + \frac{C^2}{B^2} - 2\frac{C^2}{AB} \cos \omega} = -C\sqrt{\frac{1}{A^2} + \frac{1}{B^2} - \frac{2 \cos \omega}{AB}},$$

since C is a negative quantity.

On substituting these values in (1), we have

$$PQ \times (-C) \times \sqrt{\frac{1}{A^2} + \frac{1}{B^2} - \frac{2 \cos \omega}{AB}} + \frac{C^2}{AB} \sin \omega$$

$$= -\frac{C}{A} \, . \, y' \sin \omega - \frac{C}{B} \, . \, x' \sin \omega,$$

so that $$PQ = \frac{\mathbf{Ax' + By' + C}}{\sqrt{\mathbf{A^2 + B^2 - 2AB \cos \omega}}} \, . \, \mathbf{\sin \omega}.$$

Cor. If $\omega = 90°$, *i.e.* if the axes be rectangular, we have the result of Art. 75.

EXAMPLES IX

1. The axes being inclined at an angle of 60°, find the inclination to the axis of x of the straight lines whose equations are

(1) $y = 2x + 5$,

and (2) $2y = (\sqrt{3} - 1)x + 7$.

2. The axes being inclined at an angle of 120°, find the tangent of the angle between the two straight lines

$$8x + 7y = 1 \text{ and } 28x - 73y = 101.$$

3. With oblique coordinates find the tangent of the angle between the straight lines

$$y = mx + c \text{ and } my + x = d.$$

4. If $y = x \tan \frac{11\pi}{24}$ and $y = x \tan \frac{19\pi}{24}$ represent two straight lines at right angles, prove that the angle between the axes is $\frac{\pi}{4}$.

5. Prove that the straight lines $y + x = c$ and $y = x + d$ are at right angles, whatever be the angle between the axes.

6. Prove that the equation to the straight line which passes through the point (h, k) and is perpendicular to the axis of x is

$$x + y \cos \omega = h + k \cos \omega.$$

7. Find the equations to the sides and diagonals of a regular hexagon, two of its sides, which meet in a corner, being the axes of coordinates.

8. From each corner of a parallelogram a perpendicular is drawn upon the diagonal which does not pass through that corner and these are produced to form another parallelogram; shew that its diagonals are perpendicular to the sides of the first parallelogram and that they both have the same centre.

9. If the straight lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$ make equal angles with the axis of x and be not parallel to one another, prove that $m_1 + m_2 + 2m_1 m_2 \cos \omega = 0$.

10. The axes being inclined at an angle of 30°, find the equation to the straight line which passes through the point $(-2, 3)$ and is perpendicular to the straight line $y + 3x = 6$.

11. Find the length of the perpendicular drawn from the point $(4, -3)$ upon the straight line $6x + 3y - 10 = 0$, the angle between the axes being 60°.

12. Find the equation to, and the length of, the perpendicular drawn from the point $(1, 1)$ upon the straight line $3x + 4y + 5 = 0$, the angle between the axes being 120°.

13. The coordinates of a point P referred to axes meeting at an angle ω are (h, k); prove that the length of the straight line joining the feet of the perpendiculars from P upon the axes is

$$\sin\omega\sqrt{h^2+k^2+2hk\cos\omega}.$$

14. From a given point (h, k) perpendiculars are drawn to the axes, whose inclination is ω, and their feet are joined. Prove that the length of the perpendicular drawn from (h, k) upon this line is

$$\frac{hk\sin^2\omega}{\sqrt{h^2+k^2+2hk\cos\omega}},$$

and that its equation is $hx-ky=h^2-k^2$.

ANSWERS

1. (1) $\tan^{-1}\frac{\sqrt{3}}{2}$; (2) $15°$. **2.** $\tan^{-1}\frac{30\sqrt{3}}{37}$.

3. $\tan^{-1}\left(\frac{m^2+1}{m^2-1}\tan\omega\right)$.

7. $y=0$, $y=x-a$, $x=2a$, $y=2a$, $y=x+a$, $x=0$, $y=x$, $x=a$, and $y=a$, where a is the length of a side.

10. $y(6-\sqrt{3})+x(3\sqrt{3}-2)=22-9\sqrt{3}$. **11.** $\frac{5}{8}$.

12. $10y-11x+1=0$; $\frac{6}{37}\sqrt{111}$.

SOLUTIONS/HINTS

1. (1) The angle $=\tan^{-1}\frac{2.\sin 60°}{1+2\cos 60°}$, [Art. 91]

$$=\tan^{-1}\frac{\sqrt{3}}{2}.$$

(2) The angle $=\tan^{-1}\dfrac{\frac{\sqrt{3}-1}{2}.\sin 60°}{1+\frac{\sqrt{3}-1}{2}.\cos 60°}$

$$=\tan^{-1}(2-\sqrt{3})=15°.$$

2. The angle $=\tan^{-1}\dfrac{(\frac{28}{73}+\frac{8}{7})\sin 120^\circ}{1+(\frac{28}{73}-\frac{8}{7})\cos 120^\circ-\frac{28}{73}\cdot\frac{8}{7}}$

$$=\tan^{-1}\frac{30\sqrt{3}}{37}.$$

3. The angle $=\tan^{-1}\dfrac{\left(m+\frac{1}{m}\right)\sin\omega}{1+\left(m-\frac{1}{m}\right)\cos\omega-1}$

$$=\tan^{-1}\left(\frac{m^2+1}{m^2-1}\cdot\tan\omega\right).$$

4. $1+\left(\tan\dfrac{11\pi}{24}+\tan\dfrac{19\pi}{24}\right)\cos\omega+\tan\dfrac{11\pi}{24}\cdot\tan\dfrac{19\pi}{24}=0.$

[Art. 93, Cor. 2]

$$\therefore\ \cos\omega=-\frac{1+\tan\frac{11\pi}{24}\cdot\tan\frac{19\pi}{24}}{\tan\frac{11\pi}{24}+\tan\frac{19\pi}{24}}=\frac{\cos\frac{19\pi-11\pi}{24}}{\sin\frac{19\pi+11\pi}{24}}=\frac{\frac{1}{2}}{-\frac{1}{\sqrt{2}}}$$

$$=-\frac{1}{\sqrt{2}}.\quad\therefore\ \omega=135^\circ.$$

5. $1+(1-1)\cos\omega-1=0$, whatever be the value of ω.

6. The lines $y=0$, and $y-k=m(x-h)$ are perpendicular if $1+m\cos\omega=0$. [Art. 92, Cor. 2.]

$$\therefore\ m=-\sec\omega.$$

Substituting, the equation becomes

$$x+y\cos\omega=h+k\cos\omega.$$

7. Let *OABCDE* be the hexagon, *OA* and *OE* being the axes of x and y.

The equations of *OA* and *OE* are $y=0$, $x=0$.

The intercepts cut from the axes by *AB* are a, $-a$.

$$\therefore\ \text{its equation is } x-y=a.$$

Those cut off by ED are $-a$, a.

$$\therefore \text{ its equation is } y - x = a.$$

CD and BE are parallel to the axis of x and their ordinates are $2a$, a. $\therefore$ their equations are $y = 2a$, $y = a$.

CB and AD are parallel to the axis of y and their abscissae are $2a$, a. $\therefore$ their equations are $x = 2a$, $x = a$.

OC bisects the angle between the axes.

$$\therefore \text{ its equation is } x = y.$$

8. Let $(a, 0)$; $(0, b)$; $(-a, 0)$; $(0, -b)$ be the coordinates of the angular points, the diagonals being the axes.

The equations of the sides of the 2nd parallelogram will be (as in Ex. 6)

$$x + y \sec \omega = a, \quad \text{..................} \quad \text{(i)}$$

$$x \sec \omega + y = b, \quad \text{..................} \quad \text{(ii)}$$

$$x + y \sec \omega = -a, \quad \text{..................} \quad \text{(iii)}$$

and

$$x \sec \omega + y = -b. \quad \text{..................} \quad \text{(iv)}$$

(i) $\times b -$ (ii) $\times a$ gives $x(b - a \sec \omega) + y(b \sec \omega - a) = 0$, (v)

which is *some* line through the intersection of (i) and (ii), but (iii) $\times b -$ (iv) $\times a$ gives the same line.

Hence (v) is the equation of a diagonal.

Similarly, $x(b + a \sec \omega) + y(b \sec \omega + a) = 0$ ...... (vi),

is the other diagonal; and these diagonals both pass through the origin.

Also (v) is perpendicular to a side of the first parallelogram, viz. $\frac{x}{a} + \frac{y}{b} = 1$, since

$$1 + \left(\frac{b - a \sec \omega}{a - b \sec \omega} - \frac{b}{a}\right) \cos \omega - \frac{b^2 - ab \sec \omega}{a^2 - ab \sec \omega} = 0.$$

Similarly (vi) is perpendicular to $\frac{x}{a} - \frac{y}{b} = 1$.

9. The angles they make with the axis of x are supplementary.

$$\therefore\ \tan^{-1}\frac{m_2 \sin\omega}{1+m_1\sin\omega} = \pi - \tan^{-1}\frac{m_2\sin\omega}{1+m_2\cos\omega};$$

$$\therefore\ \frac{m_1}{1+m_1\cos\omega}+\frac{m_2}{1+m_2\cos\omega}=0;$$

$$\therefore\ m_1+m_2+2m_1m_2\cos\omega=0.$$

10. Any line through $(-2, 3)$ is $y-3=m(x+2)$.

If this is perpendicular to $y+3x=6$,

$$1+(m-3)\cos 30^\circ - 3m = 0;\ \therefore\ m=\frac{3\sqrt{3}-2}{\sqrt{3}-6}.$$

Substituting, the equation becomes

$$y(6-\sqrt{3})+x(3\sqrt{3}-2)=22-9\sqrt{3}.$$

11. Perpendicular

$$=\frac{6\,.\,4-3\,.\,3-10}{\sqrt{6^2+3^2-2\,.\,6\,.\,3\cos 60^\circ}}\,.\,\sin 60^\circ\ [\text{Art. } 96]=\tfrac{5}{6}.$$

12. Length

$$=\frac{3+4+5}{\sqrt{3^2+4^2-2\,.\,3\,.\,4\cos 120^\circ}}\,.\,\sin 120^\circ\ [\text{Art. } 96]=\frac{6\sqrt{111}}{37}.$$

Any line through $(1, 1)$ is $y-1=m(x-1)$;
if this is perpendicular to $3x+4y+5=0$,

$$1+(m-\tfrac{3}{4})\cos 120^\circ-\tfrac{3}{4}\,.\,m=0;\quad [\text{Art. } 92, \text{ Cor. } 2]$$

$\therefore\ m=\frac{11}{10}$, and the equation becomes $10y-11x+1=0$.

13. Draw PN, PM perpendicular to the axes of x and y. Then $ON=h+k\cos\omega$, and $OM=k+h\cos\omega$.

Hence $MN^2=(h+k\cos\omega)^2+(k+h\cos\omega)^2$

$$-2\cos\omega\,(h+k\cos\omega)(k+h\cos\omega)$$

$$=(h^2+k^2+2hk\cos\omega)\sin^2\omega.$$

14. See last Ex. Draw PK perpendicular to MN.

Then $PK \,.\, MN = 2\triangle PMN = PM \,.\, PN \sin M\hat{P}N$

$$= h \sin \omega \,.\, k \sin \omega \,.\, \sin \omega.$$

$$\therefore\ PK = \frac{hk\sin^2\omega}{\sqrt{h^2+k^2+2hk\cos\omega}}, \text{ by Ex. 13.}$$

The equation of MN is

$$\frac{x}{h+k\cos\omega} + \frac{y}{k+h\cos\omega} = 1. \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Any line through (h, k) is $y - k = m(x-h)$. ...... (ii)

If (i) and (ii) are perpendicular, then (by Art. 92, Cor. 2)

$$1 + \left(m - \frac{k+h\cos\omega}{h+k\cos\omega}\right)\cos\omega - \frac{m(k+h\cos\omega)}{h+k\cos\omega} = 0.$$

$\therefore\ m = \frac{h}{k}$. On substitution, the required equation is

$$ky - k^2 = hx - h^2.$$

Straight lines passing through fixed points.

97. *If the equation to a straight line be of the form*

$$ax + by + c + \lambda(a'x + b'y + c') = 0 \quad \text{.........(1)},$$

where λ *is any arbitrary constant, it always passes through one fixed point whatever be the value of* λ.

For the equation (1) is satisfied by the coordinates of the point which satisfies *both* of the equations

$$ax + by + c = 0,$$

and $$a'x + b'y + c' = 0.$$

This point is, by Art. 77,

$$\left(\frac{bc' - b'c}{ab' - a'b}, \ \frac{ca' - c'a}{ab' - a'b}\right),$$

and these coordinates are independent of λ.

Ex. *Given the vertical angle of a triangle in magnitude and position, and also the sum of the reciprocals of the sides which contain it; shew that the base always passes through a fixed point.*

Take the fixed angular point as origin and the directions of the sides containing it as axes; let the lengths of these sides in any such triangle be a and b, which are not therefore given.

We have $$\frac{1}{a} + \frac{1}{b} = \text{const.} = \frac{1}{k} \text{ (say)} \quad \text{.......................(1)}.$$

The equation to the base is

$$\frac{x}{a} + \frac{y}{b} = 1,$$

i.e., by (1), $$\frac{x}{a} + y\left(\frac{1}{k} - \frac{1}{a}\right) = 1,$$

i.e. $$\frac{1}{a}(x - y) + \frac{y}{k} - 1 = 0.$$

Whatever be the value of a this straight line always passes through the point given by

$$x - y = 0 \text{ and } \frac{y}{k} - 1 = 0,$$

i.e. through the *fixed* point (k, k).

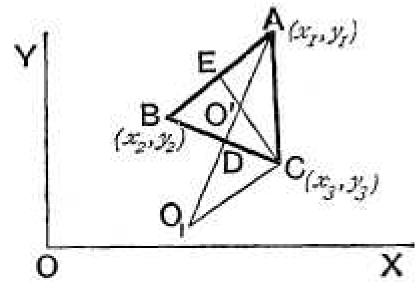
98. *Prove that the coordinates of the centre of the circle inscribed in the triangle, whose vertices are the points* (x_1, y_1), (x_2, y_2), *and* (x_3, y_3), *are*

$$\frac{ax_1 + bx_2 + cx_3}{a+b+c} \text{ and } \frac{ay_1 + by_2 + cy_3}{a+b+c},$$

where a, b, and c are the lengths of the sides of the triangle.

Find also the coordinates of the centres of the escribed circles.

Let ABC be the triangle and let AD and CE be the bisectors of the angles A and C and let them meet in O'.



Then O' is the required point.

Since AD bisects the angle BAC we have, by Euc. VI. 3,

$$\frac{BD}{BA} = \frac{DC}{AC} = \frac{BD+DC}{BA+AC} = \frac{a}{b+c},$$

so that

$$DC = \frac{ba}{b+c}.$$

Also, since CO' bisects the angle ACD, we have

$$\frac{AO'}{O'D} = \frac{AC}{CD} = \frac{b}{\frac{ba}{b+c}} = \frac{b+c}{a}.$$

The point D therefore divides BC in the ratio

$$BA : AC, \text{ i.e. } c : b.$$

Also O' divides AD in the ratio $b+c : a$.

Hence, by Art. 22, the coordinates of D are

$$\frac{cx_3 + bx_2}{c+b} \text{ and } \frac{cy_3 + by_2}{c+b}.$$

Also, by the same article, the coordinates of O' are

$$\frac{(b+c) \times \frac{cx_3 + bx_2}{c+b} + ax_1}{(b+c)+a} \text{ and } \frac{(b+c) \times \frac{cy_3 + by_2}{c+b} + ay_1}{(b+c)+a},$$

i.e.
$$\frac{ax_1 + bx_2 + cx_3}{a+b+c} \text{ and } \frac{ay_1 + by_2 + cy_3}{a+b+c}.$$

Again, if O_1 be the centre of the escribed circle opposite to the angle A, the line CO_1 bisects the exterior angle of ACB.

Hence (Euc. VI. A) we have

$$\frac{AO_1}{O_1D} = \frac{AC}{CD} = \frac{b+c}{a}.$$

Therefore O_1 is the point which divides AD *externally* in the ratio $b+c : a$.

Its coordinates (Art. 22) are therefore

$$\frac{(b+c)\dfrac{cx_3+bx_2}{c+b} - ax_1}{(b+c)-a} \quad \text{and} \quad \frac{(b+c)\dfrac{cy_3+by_2}{c+b} - ay_1}{(b+c)-a},$$

i.e.

$$\frac{-ax_1+bx_2+cx_3}{-a+b+c} \quad \text{and} \quad \frac{-ay_1+by_2+cy_3}{-a+b+c}.$$

Similarly, it may be shewn that the coordinates of the centres of the escribed circles opposite to B and C are respectively

$$\left(\frac{ax_1-bx_2+cx_3}{a-b+c},\quad \frac{ay_1-by_2+cy_3}{a-b+c}\right),$$

and

$$\left(\frac{ax_1+bx_2-cx_3}{a+b-c},\quad \frac{ay_1+by_2-cy_3}{a+b-c}\right).$$

99. As a numerical example consider the case of the triangle formed by the straight lines

$$3x+4y-7=0, \quad 12x+5y-17=0 \quad \text{and} \quad 5x+12y-34=0.$$

These three straight lines being BC, CA, and AB respectively we easily obtain, by solving, that the points A, B, and C are

$$\left(\frac{2}{7},\ \frac{19}{7}\right),\ \left(\frac{-52}{16},\ \frac{67}{16}\right) \text{ and } (1,\ 1).$$

Hence

$$a=\sqrt{\left(\frac{-52}{16}-1\right)^2+\left(\frac{67}{16}-1\right)^2}=\sqrt{\frac{68^2}{16^2}+\frac{51^2}{16^2}}$$

$$=\frac{17}{16}\sqrt{4^2+3^2}=\frac{85}{16},$$

$$b=\sqrt{\left(1-\frac{2}{7}\right)^2+\left(1-\frac{19}{7}\right)^2}=\sqrt{\frac{5^2}{7^2}+\frac{12^2}{7^2}}=\frac{13}{7},$$

and

$$c=\sqrt{\left(\frac{2}{7}+\frac{52}{16}\right)^2+\left(\frac{19}{7}-\frac{67}{16}\right)^2}=\sqrt{\frac{396^2+165^2}{112^2}}$$

$$=\frac{33}{112}\sqrt{169}=\frac{429}{112}.$$

Hence

$$ax_1=\frac{85}{16}\times\frac{2}{7}=\frac{170}{112};\quad ay_1=\frac{85}{16}\times\frac{19}{7}=\frac{1615}{112};$$

$$bx_2=\frac{13}{7}\times\frac{-52}{16}=-\frac{676}{112};\quad by_2=\frac{13}{7}\times\frac{67}{16}=\frac{871}{112};$$

$$cx_3=\frac{429}{112};\quad\text{and}\quad cy_3=\frac{429}{112}.$$

The coordinates of the centre of the incircle are therefore

$$\frac{\frac{170}{112}-\frac{676}{112}+\frac{429}{112}}{\frac{85}{16}+\frac{13}{7}+\frac{429}{112}}\quad\text{and}\quad\frac{\frac{1615}{112}+\frac{871}{112}+\frac{429}{112}}{\frac{85}{16}+\frac{13}{7}+\frac{429}{112}},$$

i.e.

$$\frac{-1}{16}\quad\text{and}\quad\frac{265}{112}.$$

The length of the radius of the incircle is the perpendicular from $\left(-\frac{1}{16},\frac{265}{112}\right)$ upon the straight line

$$3x+4y-7=0,$$

and therefore $$=\frac{\left(3\times-\frac{1}{16}\right)+\left(4\times\frac{265}{112}\right)-7}{\sqrt{3^2+4^2}}$$

$$=\frac{-21+1060-784}{5\times 112}=\frac{255}{5\times 112}=\frac{51}{112}.$$

The coordinates of the centre of the escribed circle which touches the side BC externally are

$$\frac{-\frac{170}{112}-\frac{676}{112}+\frac{429}{112}}{-\frac{85}{16}+\frac{13}{7}+\frac{429}{112}} \text{ and } \frac{-\frac{1615}{112}+\frac{871}{112}+\frac{429}{112}}{-\frac{85}{16}+\frac{13}{7}+\frac{429}{112}},$$

i.e. $$\frac{-417}{42} \text{ and } \frac{-315}{42}.$$

Similarly the coordinates of the centres of the other escribed circles can be written down.

100. Ex. *Find the radius, and the coordinates of the centre, of the circle circumscribing the triangle formed by the points*

$$(0, 1),\quad (2, 3),\ \textit{and}\ (3, 5).$$

Let (x_1, y_1) be the required centre and R the radius.

Since the distance of the centre from each of the three points is the same, we have

$$x_1^2+(y_1-1)^2=(x_1-2)^2+(y_1-3)^2=(x_1-3)^2+(y_1-5)^2=R^2 \ldots(1).$$

From the first two we have, on reduction,

$$x_1+y_1=3.$$

From the first and third equations we obtain

$$6x_1+8y_1=33.$$

Solving, we have $x_1=-\frac{9}{2}$ and $y_1=\frac{15}{2}$.

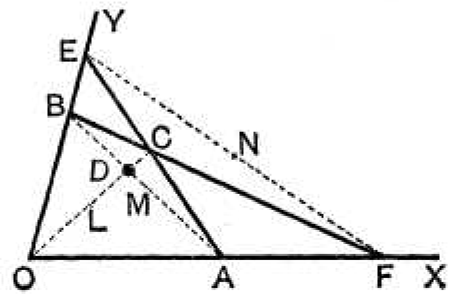
Substituting these values in (1) we get

$$R=\tfrac{5}{2}\sqrt{10}.$$

101. Ex. *Prove that the middle points of the diagonals of a complete quadrilateral lie on the same straight line.*

[**Complete quadrilateral. Def.** Let $OACB$ be any quadrilateral. Let AC and OB be produced to meet in E, and BC and OA to meet in F. Join AB, OC, and EF. The resulting figure is called a complete quadrilateral; the lines AB, OC, and EF are called its diagonals, and the points E, F, and D (the intersection of AB and OC) are called its vertices.]

Take the lines OAF and OBE as the axes of x and y.



Let $OA=2a$ and $OB=2b$, so that A is the point $(2a, 0)$ and B is the point $(0, 2b)$; also let C be the point $(2h, 2k)$.

Then L, the middle point of OC, is the point (h, k), and M, the middle point of AB, is (a, b).

The equation to LM is therefore $$y-b=\frac{k-b}{h-a}(x-a),$$

i.e. $$(h-a)y-(k-b)x=bh-ak \qquad \text{.....................(1).}$$

Again, the equation to BC is $y-2b=\frac{k-b}{h}x$.

Putting $y=0$, we have $x=\frac{-2bh}{k-b}$, so that F is the point

$$\left(\frac{-2bh}{k-b}, 0\right).$$

Similarly, E is the point $\left(0, -\frac{2ak}{h-a}\right)$.

Hence N, the middle point of EF, is $\left(\frac{-bh}{k-b}, \frac{-ak}{h-a}\right)$.

These coordinates clearly satisfy (1), *i.e.* N lies on the straight line LM.

EXAMPLES X

1. A straight line is such that the algebraic sum of the perpendiculars let fall upon it from any number of fixed points is zero; shew that it always passes through a fixed point.

2. Two fixed straight lines OX and OY are cut by a variable line in the points A and B respectively and P and Q are the feet of the perpendiculars drawn from A and B upon the lines OBY and OAX. Shew that, if AB pass through a fixed point, then PQ will also pass through a fixed point.

3. If the equal sides AB and AC of an isosceles triangle be produced to E and F so that $BE \cdot CF = AB^2$, shew that the line EF will always pass through a fixed point.

4. If a straight line move so that the sum of the perpendiculars let fall on it from the two fixed points (3, 4) and (7, 2) is equal to three times the perpendicular on it from a third fixed point (1, 3), prove that there is another fixed point through which this line always passes and find its coordinates.

Find the centre and radius of the circle which is inscribed in the triangle formed by the straight lines whose equations are

5. $3x+4y+2=0$, $3x-4y+12=0$, and $4x-3y=0$.

6. $2x+4y+3=0$, $4x+3y+3=0$, and $x+1=0$.

7. $y=0$, $12x-5y=0$, and $3x+4y-7=0$.

8. Prove that the coordinates of the centre of the circle inscribed in the triangle whose angular points are (1, 2), (2, 3), and (3, 1) are $\frac{8+\sqrt{10}}{6}$ and $\frac{16-\sqrt{10}}{6}$.

Find also the coordinates of the centres of the escribed circles.

9. Find the coordinates of the centres, and the radii, of the four circles which touch the sides of the triangle the coordinates of whose angular points are the points (6, 0), (0, 6), and (7, 7).

10. Find the position of the centre of the circle circumscribing the triangle whose vertices are the points (2, 3), (3, 4), and (6, 8).

Find the area of the triangle formed by the straight lines whose equations are

11. $y=x$, $y=2x$, and $y=3x+4$.

12. $y+x=0$, $y=x+6$, and $y=7x+5$.

13. $2y+x-5=0$, $y+2x-7=0$, and $x-y+1=0$.

14. $3x-4y+4a=0$, $2x-3y+4a=0$, and $5x-y+a=0$, proving also that the feet of the perpendiculars from the origin upon them are collinear.

15. $y=ax-bc$, $y=bx-ca$, and $y=cx-ab$.

16. $y=m_1x+\frac{a}{m_1}$, $y=m_2x+\frac{a}{m_2}$, and $y=m_3x+\frac{a}{m_3}$.

17. $y=m_1x+c_1$, $y=m_2x+c_2$, and the axis of y.

18. $y=m_1x+c_1$, $y=m_2x+c_2$, and $y=m_3x+c_3$.

19. Prove that the area of the triangle formed by the three straight lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$, and $a_3x+b_3y+c_3=0$ is

$$\frac{1}{2}\left\{\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}\right\}^2 \div (a_1b_2-a_2b_1)(a_2b_3-a_3b_2)(a_3b_1-a_1b_3).$$

20. Prove that the area of the triangle formed by the three straight lines

$$x \cos \alpha + y \sin \alpha - p_1 = 0, \quad x \cos \beta + y \sin \beta - p_2 = 0,$$

and
$$x \cos \gamma + y \sin \gamma - p_3 = 0,$$

is
$$\frac{1}{2} \frac{\{p_1 \sin(\gamma - \beta) + p_2 \sin(\alpha - \gamma) + p_3 \sin(\beta - \alpha)\}^2}{\sin(\gamma - \beta) \sin(\alpha - \gamma) \sin(\beta - \alpha)}.$$

21. Prove that the area of the parallelogram contained by the lines

$$4y - 3x - a = 0, \; 3y - 4x + a = 0, \; 4y - 3x - 3a = 0,$$

and
$$3y - 4x + 2a = 0 \text{ is } \tfrac{2}{7}a^2.$$

22. Prove that the area of the parallelogram whose sides are the straight lines

$$a_1x + b_1y + c_1 = 0, \; a_1x + b_1y + d_1 = 0, \; a_2x + b_2y + c_2 = 0,$$

and
$$a_2x + b_2y + d_2 = 0$$

is
$$\frac{(d_1 - c_1)(d_2 - c_2)}{a_1b_2 - a_2b_1}.$$

23. The vertices of a quadrilateral, taken in order, are the points (0, 0), (4, 0), (6, 7), and (0, 3); find the coordinates of the point of intersection of the two lines joining the middle points of opposite sides.

24. The lines $x + y + 1 = 0$, $x - y + 2 = 0$, $4x + 2y + 3 = 0$, and

$$x + 2y - 4 = 0$$

are the equations to the sides of a quadrilateral taken in order; find the equations to its three diagonals and the equation to the line on which their middle points lie.

25. Shew that the orthocentre of the triangle formed by the three straight lines

$$y = m_1x + \frac{a}{m_1}, \quad y = m_2x + \frac{a}{m_2}, \text{ and } y = m_3x + \frac{a}{m_3}$$

is the point

$$\left\{-a, \; a\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_1m_2m_3}\right)\right\}.$$

26. A and B are two fixed points whose coordinates are (3, 2) and (5, 1) respectively; ABP is an equilateral triangle on the side of AB remote from the origin. Find the coordinates of P and the orthocentre of the triangle ABP.

ANSWERS

4. $(-7, 3)$.

5. $(-\frac{13}{23}, \frac{5}{4})$; $\frac{157}{140}$.

6. $\left(\dfrac{-85-7\sqrt{5}}{120}, \dfrac{21\sqrt{5}-65}{120}\right)$; $\dfrac{35-7\sqrt{5}}{120}$.

7. $(\frac{7}{5}, \frac{14}{27})$; $\frac{14}{27}$.

8. $\left\{\dfrac{6+\sqrt{10}}{2}, \dfrac{2+\sqrt{10}}{2}\right\}$; $\left(\dfrac{6-\sqrt{10}}{2}, \dfrac{2-\sqrt{10}}{2}\right)$; $\left(\dfrac{8-\sqrt{10}}{6}, \dfrac{16+\sqrt{10}}{6}\right)$.

9. $(\frac{9}{2}, \frac{9}{2})$, $(2, 12)$, $(12, 2)$, and $(-3, -3)$; $\frac{3}{2}\sqrt{2}$, $4\sqrt{2}$, $4\sqrt{2}$, and $6\sqrt{2}$.

10. $(-13\frac{1}{2}, 19\frac{1}{2})$.

11. 4.

12. $7\frac{25}{48}$.

13. $\frac{3}{2}$.

14. $\dfrac{17a^2}{26}$.

15. $\frac{1}{2}(b-c)(c-a)(a-b)$.

16. $a^2(m_2-m_3)(m_3-m_1)(m_1-m_2) \div 2m_1^2 m_2^2 m_3^2$.

17. $\frac{1}{2}(c_1-c_2)^2 \div (m_1-m_2)$.

18. $\dfrac{1}{2}\left\{\dfrac{(c_2-c_3)^2}{m_2-m_3} + \dfrac{(c_3-c_1)^2}{m_3-m_1} + \dfrac{(c_1-c_2)^2}{m_1-m_2}\right\}$.

23. $(\frac{5}{2}, \frac{5}{2})$.

24. $10y+32x+43=0$; $25x+29y+5=0$; $y=5x+2$; $52x+80y=47$.

26. $(4+\frac{1}{2}\sqrt{3}, \frac{3}{2}+\sqrt{3})$; $(4+\frac{1}{6}\sqrt{3}, \frac{3}{2}+\frac{1}{3}\sqrt{3})$.

SOLUTIONS/HINTS

1. Let $$x\cos\alpha + y\sin\alpha = p \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$
be the equation to the straight line, and
$$(a_1, b_1), (a_2, b_2) \ldots (a_n, b_n)$$
the coordinates of the fixed points.

Then $$\Sigma a_1 . \cos\alpha + \Sigma b_1 . \sin\alpha - np = 0,$$
which is the condition that (i) should pass through the fixed point
$$\left(\frac{\Sigma a_1}{n}, \frac{\Sigma b_1}{n}\right).$$

2. Take OX, OY for axes and let $OA = a$, $OB = b$.

If the line AB goes through the fixed point (h, k),
$$\frac{h}{a} + \frac{k}{b} = 1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

The equation of PQ is $\dfrac{x}{b\cos\omega} + \dfrac{y}{a\cos\omega} = 1.$ …… (ii)

Now (i) is the condition that (ii) should pass through the fixed point $(k\cos\omega, h\cos\omega)$.

3. Take AB, AC for axes, and let $AB = a = AC$, and let $AE = h$, $AF = k$.

Then since $BE \cdot CF = AB^2$, $\therefore (h-a)(k-a) = a^2$, *i.e.* $\frac{a}{h} + \frac{a}{k} = 1$; which is the condition that $\frac{x}{h} + \frac{y}{k} = 1$ (the equation of EF) should pass through the fixed point (a, a).

4. Let $$x \cos \alpha + y \sin \alpha - p = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$
be the equation of the straight line. Then, by Art. 75,

$$3 \cos \alpha + 4 \sin \alpha - p + 7 \cos \alpha + 2 \sin \alpha - p$$
$$= 3(\cos \alpha + 3 \sin \alpha - p),$$

i.e. $-7 \cos \alpha + 3 \sin \alpha - p = 0$, which is the condition that (i) should pass through the point $(-7, 3)$.

5. By reference to a figure we see that two of the internal bisectors of the angles of the $\triangle$ are

$$\frac{3x + 4y + 2}{5} = + \frac{3x - 4y + 12}{5},$$

and $$\frac{3x - 4y + 12}{5} = - \frac{4x - 3y}{5}, \quad \text{[Art. 84]}$$

i.e. $$4y = 5,$$
and $$7x - 7y + 12 = 0.$$

Whence, solving, the coordinates of the incentre are $(-\frac{13}{28}, \frac{5}{4})$, and the radius = the perpendicular from this point upon $4x - 3y = 0$,

$$= \frac{4(-\frac{13}{28}) - 3(\frac{5}{4})}{5} = -\frac{157}{140}.$$

6. As in Ex. 5, two internal bisectors are

$$\frac{2x + 4y + 3}{2\sqrt{5}} = +(x+1), \text{ and } \frac{4x + 3y + 3}{5} = -(x+1), \text{[Art. 84]},$$

i.e. $x(2 - 2\sqrt{5}) + 4y + (3 - 2\sqrt{5}) = 0$, and $9x + 3y + 8 = 0$.

Whence, solving, the coordinates of the incentre are

$$\left(\frac{-85 - 7\sqrt{5}}{120}, \frac{21\sqrt{5} - 65}{120}\right).$$

and radius = perpendicular from this point upon $x+1=0$,

$$=\frac{-85-7\sqrt{5}}{120}+1=\frac{35-7\sqrt{5}}{120}.$$

7. As in Ex. 5, two internal bisectors are

$$\frac{12x-5y}{13}=+y, \text{ and } \frac{3x+4y-7}{5}=-y, \quad \text{[Art. 84]}$$

i.e. $$x+3y=\tfrac{7}{3}, \text{ and } 2x=3y.$$

Whence, solving, the coordinates of the incentre are $(\frac{7}{9}, \frac{14}{27})$, and $\therefore$ radius $=\frac{14}{27}$.

8. Using the formulae of Art. 98,

$$a=\sqrt{1^2+2^2}=\sqrt{5}, \quad b=\sqrt{2^2+1^2}=\sqrt{5}, \quad c=\sqrt{1^2+1^2}=\sqrt{2};$$

$\therefore$ the coordinates of the incentre are

$$\left(\frac{\sqrt{5}+2\sqrt{5}+3\sqrt{2}}{\sqrt{5}+\sqrt{5}+\sqrt{2}}, \frac{2\sqrt{5}+3\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{5}+\sqrt{2}}\right),$$

i.e. $$\left(\frac{8+\sqrt{10}}{6}, \frac{16-\sqrt{10}}{6}\right),$$

and the coordinates of the excentres are

$$\left(\frac{-\sqrt{5}+2\sqrt{5}+3\sqrt{2}}{-\sqrt{5}+\sqrt{5}+\sqrt{2}}, \frac{-2\sqrt{5}+3\sqrt{5}+\sqrt{2}}{-\sqrt{5}+\sqrt{5}+\sqrt{2}}\right),$$

i.e. $$\left(\frac{6+\sqrt{10}}{2}, \frac{2+\sqrt{10}}{2}\right);$$

$$\left(\frac{\sqrt{5}-2\sqrt{5}+3\sqrt{2}}{\sqrt{5}-\sqrt{5}+\sqrt{2}}, \frac{2\sqrt{5}-3\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{5}+\sqrt{2}}\right),$$

i.e. $$\left(\frac{6-\sqrt{10}}{2}, \frac{2-\sqrt{10}}{2}\right);$$

$$\left(\frac{\sqrt{5}+2\sqrt{5}-3\sqrt{2}}{\sqrt{5}+\sqrt{5}-\sqrt{2}}, \frac{2\sqrt{5}+3\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{5}-\sqrt{2}}\right),$$

i.e. $$\left(\frac{8-\sqrt{10}}{6}, \frac{16+\sqrt{10}}{6}\right).$$

9. Using the formulae of Art. 98,

$$a=\sqrt{7^2+1^2}=5\sqrt{2}, \quad b=\sqrt{1^2+7^2}=5\sqrt{2}, \quad c=\sqrt{6^2+6^2}=6\sqrt{2}.$$

The coordinates of the incentre are

$$\left(\frac{30\sqrt{2}+42\sqrt{2}}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}},\ \frac{30\sqrt{2}+42\sqrt{2}}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}\right),\ i.e.\left(\frac{9}{2},\ \frac{9}{2}\right).$$

and the radius (= perpendicular from incentre upon

$$x+y-6=0)=\frac{3}{\sqrt{2}}=\frac{3}{2}\sqrt{2},$$

and the coordinates of the excentres are

$$\left(\frac{-30\sqrt{2}+42\sqrt{2}}{-5\sqrt{2}+5\sqrt{2}+6\sqrt{2}},\ \frac{+30\sqrt{2}+42\sqrt{2}}{-5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}\right),\ i.e.\ (2, 12),$$

and radius $= 4\sqrt{2}$.

$$\left(\frac{30\sqrt{2}+42\sqrt{2}}{5\sqrt{2}-5\sqrt{2}+6\sqrt{2}},\ \frac{-30\sqrt{2}+42\sqrt{2}}{5\sqrt{2}-5\sqrt{2}+6\sqrt{2}}\right),\ i.e.\ (12,\ 2),$$

and radius $= 4\sqrt{2}$.

$$\left(\frac{30\sqrt{2}-42\sqrt{2}}{5\sqrt{2}+5\sqrt{2}-6\sqrt{2}},\ \frac{30\sqrt{2}-42\sqrt{2}}{5\sqrt{2}+5\sqrt{2}-6\sqrt{2}}\right),\ i.e.\ (-3,\ -3),$$

and radius = perpendicular upon the line joining the second pair of given points from $(-3, -3) = 6\sqrt{2}$.

10. Let (x, y) be the coordinates of circumcentre.

Then $(x-2)^2+(y-3)^2=(x-3)^2+(y-4)^2=(x-6)^2+(y-8)^2$.

Whence $\quad x+y=6, \quad 6x+8y=75.$

Solving $\quad x=-13\frac{1}{2}, \quad y=19\frac{1}{2}.$

11. Solving, the coordinates of the angular points are

$$(0, 0);\ (-2,\ -2);\ (-4,\ -8).$$

$\therefore$ By Art. 25,

$$\Delta=\tfrac{1}{2}\{0(-2+8)-2(-8-0)-4(0+2)\}=4.$$

12. Solving, the coordinates of the angular points are

$$(-3,\ 3);\ (-\tfrac{5}{8},\ \tfrac{5}{8});\ (\tfrac{1}{6},\ 6\tfrac{1}{6}).$$

Call them B, C, A respectively.

Length of $BC = \sqrt{(2\frac{3}{8})^2+(2\frac{3}{8})^2}=\frac{19}{8}\sqrt{2}.$

Perpendicular from A on $BC = \dfrac{6\frac{1}{3}}{\sqrt{2}} = \dfrac{19}{3\sqrt{2}}$.

$$\therefore \ \Delta = \frac{1}{2} \cdot \frac{19}{8} \cdot \sqrt{2} \cdot \frac{19}{3\sqrt{2}} = \frac{361}{48} = 7\tfrac{25}{48}.$$

13. Solving, the coordinates of the angular points are

$$(1, 2)\,;\ (2, 3)\,;\ (3, 1).$$

$\therefore$ by Art. 25, $\Delta = \frac{1}{2}\{1(3-1) + 2(1-2) + 3(2-3)\}$
$= \frac{3}{2}$.

14. Solving, the coordinates of the angular points are

$$(0, a)\,;\ \left(\frac{a}{13}, \frac{18a}{13}\right);\ (4a, 4a).$$

$\therefore$ by Art. 25,

$$\Delta = \tfrac{1}{2}\left\{0\left(\frac{18a}{13} - 4a\right) + \frac{a}{13}(4a - a) + 4a\left(a - \frac{18a}{13}\right)\right\}$$
$$= \frac{17a^2}{26}.$$

The foot of the perpendicular from the origin upon

$$x\cos\alpha + y\sin\alpha = p \ \text{ is } \ (p\cos\alpha,\ p\sin\alpha).$$

Writing the equations to the sides in this form, viz.

$$\frac{3x}{5} - \frac{4y}{5} = -\frac{4a}{5}\,;\ \frac{2x}{\sqrt{13}} - \frac{3y}{\sqrt{13}} = -\frac{4a}{\sqrt{13}}\,;\ \frac{5x}{\sqrt{26}} - \frac{y}{\sqrt{26}} = -\frac{a}{\sqrt{26}},$$

the coordinates of the feet of the perpendiculars from the origin upon the sides are seen to be

$$\left(-\frac{12a}{25}, \frac{16a}{25}\right);\ \left(-\frac{8a}{13}, \frac{12a}{13}\right);\ \left(-\frac{5a}{26}, \frac{a}{26}\right).$$

And the area of the Δ formed by these points

$$= \frac{a^2}{2}\left\{\frac{144}{325} - \frac{6}{325} + \frac{4}{169} - \frac{128}{325} + \frac{8}{65} - \frac{30}{169}\right\} = 0.$$

Hence the feet of these perpendiculars are collinear.

15. Solving, the coordinates of the angular points are

$$\{-c,\ -c(a+b)\},\ \text{etc.}$$

$\therefore$ By Art. 25, $\Delta = \frac{1}{2}\{c(a+b)(a-b) + \ldots + \ldots\}$,
$= \frac{1}{2}\Sigma c(a^2 - b^2) = \frac{1}{2}(b-c)(c-a)(a-b)$.

16. Solving, the coordinates of the angular points are

$$\left\{\frac{a}{m_1 m_2},\ \frac{a}{m_1}+\frac{a}{m_2}\right\},\ \text{etc.}$$

$\therefore$ by Art. 25,

$$\Delta = \tfrac{1}{2}a^2\Sigma\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\left(\frac{1}{m_2 m_3}-\frac{1}{m_3 m_1}\right)$$

$$=\tfrac{1}{2}a^2\Sigma\left\{\frac{1}{m_3}\left(\frac{1}{m_1^2}-\frac{1}{m_2^2}\right)\right\}=\frac{a^2}{2m_1^2m_2^2m_3^2}\Sigma\, m_3\left(m_1^2-m_2^2\right)$$

$$=\frac{a^2(m_1-m_2)(m_2-m_3)(m_3-m_1)}{2m_1^2m_2^2m_3^2}.$$

17. The coordinates of the angular points are,

$$(0,\ c_1);\ (0,\ c_2);\ \left(\frac{c_2-c_1}{m_1-m_2},\ \frac{m_1c_2-m_2c_1}{m_1-m_2}\right).$$

Hence by Art. 25, $\Delta=\frac{1}{2}\frac{c_1^2-c_2^2}{m_1-m_2}$.

18. If the three $\triangle$s be drawn that are formed by taking every pair of the lines with the axis of y, it will be seen that the required $\triangle$ is obtained by subtracting one of these $\triangle$s from the sum of the other two.

Hence (regard being had to sign) by Ex. 17,

$$\Delta=\tfrac{1}{2}\left\{\frac{(c_2-c_3)^2}{m_2-m_3}+\frac{(c_3-c_1)^2}{m_3-m_1}+\frac{(c_1-c_2)^2}{m_1-m_2}\right\}.$$

19. See Ex. 20. Put a_1, b_1, c_1 for $\cos\alpha$, $\sin\alpha$, $-p$ respectively, etc., and result follows at once. Or it may be deduced from the previous article, but the work is much longer.

20. Solving, the angular points are

$$\left(\frac{p_2\sin\alpha-p_1\sin\beta}{\sin(\alpha-\beta)},\ \frac{p_1\cos\beta-p_2\cos\alpha}{\sin(\alpha-\beta)}\right),\ \text{etc.}$$

Hence by Art. 25

$$2\Delta=\Sigma\left[\frac{p_3\sin\beta-p_2\sin\gamma}{\sin(\beta-\gamma)}\left(\frac{p_3\cos\alpha-p_1\cos\gamma}{\sin(\gamma-\alpha)}-\frac{p_1\cos\beta-p_2\cos\alpha}{\sin(\alpha-\beta)}\right)\right].$$

$$\therefore\ 2\Delta\,.\sin(\alpha-\beta)\,.\sin(\beta-\gamma)\,.\sin(\gamma-\alpha)$$

$$=\Sigma(p_3\sin\beta-p_2\sin\gamma)\left[p_3\cos\alpha\,.\sin(\alpha-\beta)+p_2\cos\alpha\sin(\gamma-\alpha)\right.$$
$$\left.-p_1\{\cos\gamma\sin(\alpha-\beta)+\cos\beta\sin(\gamma-\alpha)\}\right]$$

$$=\Sigma(p_3\sin\beta-p_2\sin\gamma)\{p_3\cos\alpha\,.\sin(\alpha-\beta)$$
$$+p_2\cos\alpha\,.\sin(\gamma-\alpha)+p_1\cos\alpha\,.\sin(\beta-\gamma)\},$$

since $$\Sigma\,.\cos\alpha\,.\sin(\beta-\gamma)=0,$$

$$=\{p_1\sin(\beta-\gamma)+p_2\sin(\gamma-\alpha)$$
$$+p_3\sin(\alpha-\beta)\}\,.\,\Sigma(p_3\sin\beta\,.\cos\alpha-p_2\sin\gamma\cos\alpha)$$
$$=-\{p_1\sin(\beta-\gamma)+p_2\sin(\gamma-\alpha)+p_3\sin(\alpha-\beta)\}^2.$$

$$\therefore\ \Delta=\tfrac{1}{2}\,\frac{\{p_1\sin(\beta-\gamma)+p_2\sin(\gamma-\alpha)+p_3\sin(\alpha-\beta)\}^2}{\sin(\gamma-\beta)\,.\sin(\alpha-\gamma)\,.\sin(\beta-\alpha)}.$$

21. Solving (i) and (ii); (i) and (iv); (ii) and (iii),

we obtain $$(a,a);\ \left(\frac{11a}{7},\ \frac{10a}{7}\right);\ \left(\frac{13a}{7},\ \frac{15a}{7}\right).$$

Let Δ be the triangle formed by these points;

$$\therefore\ \text{parallelogram}=2\Delta=a^2\{-\tfrac{5}{7}+\tfrac{11}{7}\,.\,\tfrac{8}{7}-\tfrac{13}{7}\,.\,\tfrac{3}{7}\}=\tfrac{2}{7}a^2.$$

22. Let $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+d_1=0$ be inclined at α to and intercept a length l_1 on the axis of x, and take β, l_2 similarly for the other pair of lines.

If p, q be the sides of the parallelogram, it will be seen from a figure that

$$\frac{p}{l_1}=\frac{\sin\alpha}{\sin(\beta-\alpha)},\quad \frac{q}{l_2}=\frac{\sin\beta}{\sin(\beta-\alpha)}.$$

Area of parallelogram

$$=p\,.\,q\,.\sin(\beta-\alpha)=\frac{\sin\alpha\,.\sin\beta}{\sin(\beta-\alpha)}\,.\,l_1l_2$$
$$=\frac{1}{\cot\alpha-\cot\beta}\,.\,\frac{(d_1-c_1)(d_2-c_2)}{a_1a_2}$$
$$=\frac{1}{-\frac{b_1}{a_1}+\frac{b_2}{a_2}}\,.\,\frac{(d_1-c_1)(d_2-c_2)}{a_1a_2}=\frac{(d_1-c_1)(d_2-c_2)}{a_2b_1\sim a_1b_2}.$$

23. The middle point of the line joining (0, 0) and (4, 0) is (2, 0).

The middle point of the line joining (0, 3) and (6, 7) is (3, 5), and the middle point of the line joining (2, 0) and (3, 5) is $(\frac{5}{2}, \frac{5}{2})$.

Similarly it can be shewn that the line joining the middle points of the other pair of sides also goes through this point.

24. By solving, the diagonals are the lines joining

$$(-\tfrac{7}{3}, \tfrac{19}{6}) \text{ and } (-\tfrac{3}{2}, \tfrac{1}{2});\ (-\tfrac{7}{6}, \tfrac{5}{6}) \text{ and } (-6, 5);$$

and $$(-\tfrac{1}{2}, -\tfrac{1}{2}) \text{ and } (0, 2).$$

Hence by Art. 62 the equations of the diagonals can be written down.

Also the line on which the middle points lie is that joining $$(-\tfrac{43}{12}, \tfrac{35}{12}) \text{ and } (-\tfrac{1}{4}, \tfrac{3}{4}).$$

25. See page 202.

26. Draw AH, PN, BK perpendicular to the axis of x and BL parallel to the axis of x meeting AH in L.

Let $A\hat{B}L = \theta$. Then $AB^2 = 1^2 + 2^2$. $\therefore\ AB = \sqrt{5}$.

Hence $$\sin\theta = \frac{1}{\sqrt{5}},\ \cos\theta = \frac{2}{\sqrt{5}},$$

$$ON = OK - NK = OK - PB\cos(\theta + 60°)$$

$$= 5 - \sqrt{5}\left(\frac{2}{\sqrt{5}}\cdot\frac{1}{2} - \frac{1}{\sqrt{5}}\cdot\frac{\sqrt{3}}{2}\right) = 4 + \frac{\sqrt{3}}{2}.$$

$$PN = BK + PB\,.\sin(\theta + 60°)$$

$$= 1 + \sqrt{5}\left\{\frac{1}{\sqrt{5}}\cdot\frac{1}{2} + \frac{2}{\sqrt{5}}\cdot\frac{\sqrt{3}}{2}\right\} = \tfrac{3}{2} + \sqrt{3}.$$

If $\bar{x}$ and $\bar{y}$ are the coordinates of the orthocentre, by Ex. 2, page 13

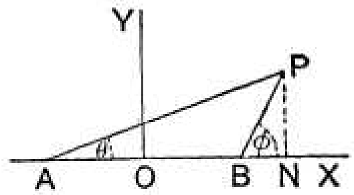
$$3\bar{x} = 3 + 5 + 4 + \frac{\sqrt{3}}{2}, \quad i.e.\ \bar{x} = 4 + \frac{\sqrt{3}}{6},$$

and $$3\bar{y} = 2 + 1 + \tfrac{3}{2} + \sqrt{3}, \quad i.e.\ \bar{y} = \tfrac{3}{2} + \frac{\sqrt{3}}{3},$$

since the centroid and the orthocentre coincide in an equilateral triangle.

102. Ex. *The base of a triangle is fixed; find the locus of the vertex when one base angle is double of the other.*

Let AB be the fixed base of the triangle; take its middle point O as origin, the direction of OB as the axis of x and a perpendicular line as the axis of y.



Let $AO = OB = a$.

If P be one position of the vertex, the condition of the problem then gives

$\angle PBA = 2\angle PAB$, *i.e.* $\pi - \phi = 2\theta$, *i.e.* $-\tan\phi = \tan 2\theta$..(1).

Let P be the point (h, k). We then have

$$\frac{k}{h+a} = \tan\theta \text{ and } \frac{k}{h-a} = \tan\phi.$$

Substituting these values in (1), we have

$$-\frac{k}{h-a} = \frac{2\dfrac{k}{h+a}}{1-\left(\dfrac{k}{h+a}\right)^2} = \frac{2(h+a)k}{(h+a)^2-k^2},$$

i.e. $-(h+a)^2 + k^2 = 2(h^2 - a^2)$, *i.e.* $k^2 - 3h^2 - 2ah + a^2 = 0$.

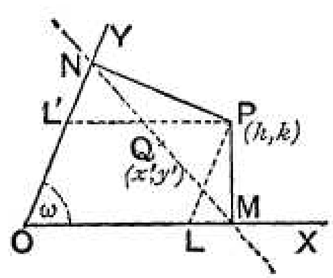
But this is the condition that the point (h, k) should lie on the curve

$$y^2 - 3x^2 - 2ax + a^2 = 0.$$

This is therefore the equation to the required locus.

103. Ex. *From a point P perpendiculars PM and PN are drawn upon two fixed lines which are inclined at an angle ω and meet in a fixed point O; if P move on a fixed straight line, find the locus of the middle point of MN.*

Let the two fixed lines be taken as the axes. Let the coordinates of P, any position of the moving point, be (h, k).



Let the equation of the straight line on which P lies be

$$Ax + By + C = 0,$$

so that we have

$$Ah + Bk + C = 0 \text{ ...... (1).}$$

Draw PL and PL' parallel to the axes.

We then have

$$OM = OL + LM = OL + LP\cos\omega = h + k\cos\omega,$$

and $$ON = OL' + L'N = LP + L'P\cos\omega = k + h\cos\omega.$$

M is therefore the point $(h + k\cos\omega,\ 0)$ and N is the point $(0,\ k + h\cos\omega)$.

Hence, if (x', y') be the coordinates of the middle point of MN, we have

$$2x' = h + k\cos\omega \ \ldots\ldots\ldots\ldots\ldots\ldots \ (2),$$

and $$2y' = k + h\cos\omega \ \ldots\ldots\ldots\ldots\ldots\ldots \ (3).$$

Equations (1), (2), and (3) express analytically all the relations which hold between x', y', h, and k.

Also h and k are the quantities which by their variation cause Q to take up different positions. If therefore between (1), (2), and (3) we eliminate h and k we shall obtain a relation between x' and y' which is true for *all* values of h and k, *i.e.* a relation which is true whatever be the position that P takes on the given straight line.

From (2) and (3), by solving, we have

$$h = \frac{2(x' - y'\cos\omega)}{\sin^2\omega} \text{ and } k = \frac{2(y' - x'\cos\omega)}{\sin^2\omega}.$$

Substituting these values in (1), we obtain

$$2A(x' - y'\cos\omega) + 2B(y' - x'\cos\omega) + C\sin^2\omega = 0.$$

But this is the condition that the point (x', y') shall always lie on the straight line

$$2A(x - y\cos\omega) + 2B(y - x\cos\omega) + C\sin^2\omega = 0,$$

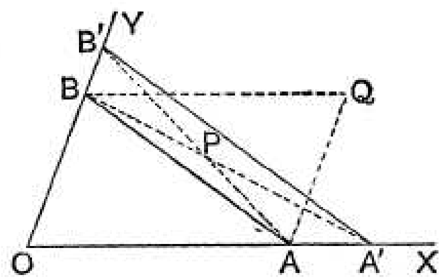
i.e. on the straight line

$$x(A - B\cos\omega) + y(B - A\cos\omega) + \tfrac{1}{2}C\sin^2\omega = 0,$$

which is therefore the equation to the locus of Q.

104. Ex. *A straight line is drawn parallel to the base of a given triangle and its extremities are joined transversely to those of the base; find the locus of the point of intersection of the joining lines.*

Let the triangle be OAB and take O as the origin and the directions of OA and OB as the axes of x and y.



Let $OA = a$ and $OB = b$, so that a and b are given quantities.

Let $A'B'$ be the straight line which is parallel to the base AB, so that

$$\frac{OA'}{OA} = \frac{OB'}{OB} = \lambda \text{ (say)},$$

and hence $OA' = \lambda a$ and $OB' = \lambda b$.

For different values of λ we therefore have different positions of $A'B'$.

The equation to AB' is

$$\frac{x}{a} + \frac{y}{\lambda b} = 1 \quad \text{.......................(1)},$$

and that to $A'B$ is

$$\frac{x}{\lambda a} + \frac{y}{b} = 1 \quad \text{.......................(2)}.$$

Since P is the intersection of AB' and $A'B$ its coordinates satisfy both (1) and (2). Whatever equation we derive from them must therefore denote a locus going through P. Also if we derive from (1) and (2) an equation which does not contain λ, it must represent a locus which passes through P whatever be the value of λ; in other words it must go through all the different positions of the point P.

Subtracting (2) from (1), we have

$$\frac{x}{a}\left(1 - \frac{1}{\lambda}\right) + \frac{y}{b}\left(\frac{1}{\lambda} - 1\right) = 0,$$

i.e.

$$\frac{x}{a} = \frac{y}{b}.$$

This then is the equation to the locus of P. Hence P always lies on the straight line

$$y = \frac{b}{a}x,$$

which is the straight line OQ where $OAQB$ is a parallelogram.

Aliter. By solving the equations (1) and (2) we easily see that they meet at the point

$$\left(\frac{\lambda}{\lambda+1}a, \ \frac{\lambda}{\lambda+1}b\right).$$

Hence, if P be the point (h, k), we have

$$h=\frac{\lambda}{\lambda+1}a \ \text{ and } \ k=\frac{\lambda}{\lambda+1}b.$$

Hence for all values of λ, *i.e.* for all positions of the straight line $A'B'$, we have

$$\frac{h}{a}=\frac{k}{b}.$$

But this is the condition that the point (h, k), *i.e.* P, should lie on the straight line

$$\frac{x}{a}=\frac{y}{b}.$$

The straight line is therefore the required locus.

105. Ex. *A variable straight line is drawn through a given point O to cut two fixed straight lines in R and S; on it is taken a point P such that*

$$\frac{2}{OP}=\frac{1}{OR}+\frac{1}{OS};$$

shew that the locus of P is a third fixed straight line.

Take any two fixed straight lines, at right angles and passing through O, as the axes and let the equation to the two given fixed straight lines be

$$Ax+By+C=0,$$

and

$$A'x+B'y+C'=0.$$

Transforming to polar coordinates these equations are

$$\frac{1}{r}=-\frac{A\cos\theta+B\sin\theta}{C} \ \text{ and } \ \frac{1}{r}=-\frac{A'\cos\theta+B'\sin\theta}{C'}.$$

If the angle XOR be θ the values of $\frac{1}{OR}$ and $\frac{1}{OS}$ are therefore

$$-\frac{A\cos\theta + B\sin\theta}{C} \text{ and } -\frac{A'\cos\theta + B'\sin\theta}{C'}.$$

We therefore have

$$\frac{2}{OP} = -\frac{A\cos\theta + B\sin\theta}{C} - \frac{A'\cos\theta + B'\sin\theta}{C'}$$

$$= -\left(\frac{A}{C} + \frac{A'}{C'}\right)\cos\theta - \left(\frac{B}{C} + \frac{B'}{C'}\right)\sin\theta.$$

The equation to the locus of P is therefore, on again transforming to Cartesian coordinates,

$$2 = -x\left(\frac{A}{C} + \frac{A'}{C'}\right) - y\left(\frac{B}{C} + \frac{B'}{C'}\right),$$

and this is a fixed straight line.

EXAMPLES XI

The base BC $(=2a)$ of a triangle ABC is fixed; the axes being BC and a perpendicular to it through its middle point, find the locus of the vertex A, when

1. the difference of the base angles is given $(=\alpha)$.

2. the product of the tangents of the base angles is given $(=\lambda)$.

3. the tangent of one base angle is m times the tangent of the other.

4. m times the square of one side added to n times the square of the other side is equal to a constant quantity c^2.

From a point P perpendiculars PM and PN are drawn upon two fixed lines which are inclined at an angle ω, and which are taken as the axes of coordinates and meet in O; find the locus of P

5. if $OM + ON$ be equal to $2c$.

6. if $OM - ON$ be equal to $2d$.

7. if $PM + PN$ be equal to $2c$.

8. if $PM - PN$ be equal to $2c$.

9. if MN be equal to $2c$.

10. if MN pass through the fixed point (a, b).

11. if MN be parallel to the given line $y = mx$.

12. Two fixed points A and B are taken on the axes such that $OA=a$ and $OB=b$; two variable points A' and B' are taken on the same axes; find the locus of the intersection of AB' and $A'B$

(1) when $$OA'+OB'=OA+OB,$$

and (2) when $$\frac{1}{OA'}-\frac{1}{OB'}=\frac{1}{OA}-\frac{1}{OB}.$$

13. Through a fixed point P are drawn any two straight lines to cut one fixed straight line OX in A and B and another fixed straight line OY in C and D; prove that the locus of the intersection of the straight lines AC and BD is a straight line passing through O.

14. OX and OY are two straight lines at right angles to one another; on OY is taken a fixed point A and on OX any point B; on AB an equilateral triangle is described, its vertex C being on the side of AB away from O. Shew that the locus of C is a straight line.

15. If a straight line pass through a fixed point, find the locus of the middle point of the portion of it which is intercepted between two given straight lines.

16. A and B are two fixed points; if PA and PB intersect a constant distance $2c$ from a given straight line, find the locus of P.

17. Through a fixed point O are drawn two straight lines at right angles to meet two fixed straight lines, which are also at right angles, in the points P and Q. Shew that the locus of the foot of the perpendicular from O on PQ is a straight line.

18. Find the locus of a point at which two given portions of the same straight line subtend equal angles.

19. Find the locus of a point which moves so that the difference of its distances from two fixed straight lines at right angles is equal to its distance from a fixed straight line.

20. A straight line AB, whose length is c, slides between two given oblique axes which meet at O; find the locus of the orthocentre of the triangle OAB.

21. Having given the bases and the sum of the areas of a number of triangles which have a common vertex, shew that the locus of this vertex is a straight line.

22. Through a given point O a straight line is drawn to cut two given straight lines in R and S; find the locus of a point P on this variable straight line, which is such that

(1) $2OP=OR+OS$,

and (2) $OP^2=OR.OS$.

23. Given n straight lines and a fixed point O; through O is drawn a straight line meeting these lines in the points R_1, R_2, R_3, $\ldots R_n$, and on it is taken a point R such that

$$\frac{n}{OR} = \frac{1}{OR_1} + \frac{1}{OR_2} + \frac{1}{OR_3} + \ldots + \frac{1}{OR_n};$$

shew that the locus of R is a straight line.

24. A variable straight line cuts off from n given concurrent straight lines intercepts the sum of the reciprocals of which is constant. Shew that it always passes through a fixed point.

25. If a triangle ABC remain always similar to a given triangle, and if the point A be fixed and the point B always move along a given straight line, find the locus of the point C.

26. A right-angled triangle ABC, having C a right angle, is of given magnitude, and the angular points A and B slide along two given perpendicular axes; shew that the locus of C is the pair of straight lines whose equations are $y = \pm \frac{b}{c} x$.

27. Two given straight lines meet in O, and through a given point P is drawn a straight line to meet them in Q and R; if the parallelogram $OQSR$ be completed find the equation to the locus of S.

28. Through a given point O is drawn a straight line to meet two given parallel straight lines in P and Q; through P and Q are drawn straight lines in given directions to meet in R; prove that the locus of R is a straight line.

ANSWERS

1. $x^2 + 2xy \cot \alpha - y^2 = a^2$.

2. $y^2 + \lambda x^2 = \lambda a^2$.

3. $(m+1)\, x = (m-1)\, a$.

4. $(m+n)\,(x^2 + y^2 + a^2) - 2ax\,(m-n) = c^2$.

5. $x + y = c \sec^2 \frac{\omega}{2}$.

6. $x - y = d \operatorname{cosec}^2 \frac{\omega}{2}$.

7. $x + y = 2c \operatorname{cosec} \omega$.

8. $y - x = 2c \operatorname{cosec} \omega$.

9. $x^2 + 2xy \cos \omega + y^2 = 4c^2 \operatorname{cosec}^2 \omega$.

10. $(x^2 + y^2) \cos \omega + xy\,(1 + \cos^2 \omega) = x\,(a \cos \omega + b) + y\,(b \cos \omega + a)$.

11. $x\,(m + \cos \omega) + y\,(1 + m \cos \omega) = 0$.

12. (i) $x + y - a - b = 0$;
(ii) $y = x$.

19. A straight line.

20. A circle, centre O.

25. A straight line.

27. If P be the point (h, k), the equation to the locus of S is

$$\frac{h}{x}+\frac{k}{y}=1.$$

SOLUTIONS/HINTS

1. We have $$\frac{\frac{y}{a-x}-\frac{y}{a+x}}{1+\frac{y^2}{a^2-x^2}}=\tan\alpha,$$

whence $$x^2+2xy\cot\alpha-y^2=a^2.$$

2. $\frac{y}{a-x}\cdot\frac{y}{a+x}=\lambda$; $\therefore$ $y^2+\lambda x^2=\lambda a^2$.

3. $\frac{y}{a-x}=\frac{my}{a+x}$. $\therefore$ $(m+1)x=(m-1)a$.

4. $n\{y^2+(x+a)^2\}+m\{y^2+(a-x)^2\}=c^2$,

i.e. $$(m+n)(x^2+y^2+a^2)-2ax(m-n)=c^2.$$

5. $x+y\cos\omega+y+x\cos\omega=2c$; $\therefore$ $x+y=c\sec^2\frac{\omega}{2}$.

6. $x+y\cos\omega-y-x\cos\omega=2d$; $x-y=d\operatorname{cosec}^2\frac{\omega}{2}$.

7. $x\sin\omega+y\sin\omega=2c$; $\therefore$ $x+y=2c\operatorname{cosec}\omega$.

8. $y\sin\omega-x\sin\omega=2c$; $\therefore$ $y-x=2c\operatorname{cosec}\omega$.

9. $(x+y\cos\omega)^2+(y+x\cos\omega)^2$

$$-2(x+y\cos\omega)(y+x\cos\omega)\cos\omega=4c^2;$$

$$\therefore\ x^2+y^2+2xy\cos\omega=4c^2\operatorname{cosec}^2\omega.$$

10. If (x', y') are the coordinates of P, the equation of MN is

$$\frac{x}{x'+y'\cos\omega}+\frac{y}{y'+x'\cos\omega}=1.$$

This goes through (a, b) if $\frac{a}{x'+y'\cos\omega}+\frac{b}{y'+x'\cos\omega}=1.$

Hence the locus of P is $\dfrac{a}{x+y\cos\omega}+\dfrac{b}{y+x\cos\omega}=1$,

i.e. $x(a\cos\omega+b)+y(a+b\cos\omega)$
$$=(x^2+y^2)\cos\omega+xy(1+\cos^2\omega).$$

11. Here similarly $-\dfrac{y+x\cos\omega}{x+y\cos\omega}=m.$

$$\therefore\ x(m+\cos\omega)+y(1+m\cos\omega)=0.$$

12. (1) Since $a+b=a'+b'$,

$$\therefore\ a-a'=b'-b. \quad \text{..................} \text{(i)}$$

Equations of AB' and $A'B$ are

$$\frac{x}{a}+\frac{y}{b'}=1, \quad \text{....................(ii)}$$

and
$$\frac{x}{a'}+\frac{y}{b}=1. \quad \text{....................(iii)}$$

Subtract; $\therefore\ \dfrac{x}{aa'}+\dfrac{y}{bb'}=0$, by (i). ....(iv)

Multiplying (ii) by $\dfrac{1}{b}$ and (iii) by $\dfrac{1}{a}$, and adding, we have

$$\frac{x}{ab}+\frac{y}{ab}+\frac{x}{aa'}+\frac{y}{bb'}=\frac{1}{a}+\frac{1}{b};$$

$$\therefore\ x+y=a+b, \text{ by (iv),}$$

(2) Subtract (ii) and (iii),

$$x\left\{\frac{1}{a}-\frac{1}{a'}\right\}-y\left\{\frac{1}{b}-\frac{1}{b'}\right\}=0.$$

$$\therefore\ x=y.$$

Aliter. If we eliminate a' and b' by merely substituting from (ii) and (iii) in (i) we get

$$\frac{y}{1-\dfrac{x}{a}}+\frac{x}{1-\dfrac{y}{b}}=a+b,$$

i.e., on reduction,

$ay^2+(a+b)xy+bx^2-bx(2a+b)-ay(a+2b)+ab(a+b)=0,$

i.e. $[x+y-a-b][bx+ay-ab]=0.$

The second factor gives $\frac{x}{a}+\frac{y}{b}=1$, *i.e.* the straight line AB itself. This part of the locus arises from the case when A' coincides with A, and B' with B, and then the line $A'B'$ entirely coincides with the line AB.

Neglecting this factor, the ordinary locus is given by

$$x+y=a+b.$$

13. Let P be the point (h, k), $OA=a$ etc.

Since AD and BC pass through P,

$$\therefore \frac{h}{a}+\frac{k}{d}=1, \text{ and } \frac{h}{b}+\frac{k}{c}=1,$$

and hence $$h\left(\frac{1}{a}-\frac{1}{b}\right)+k\left(\frac{1}{d}-\frac{1}{c}\right)=0. \quad \text{..........(i)}$$

The equations of AC and BD are

$$\frac{x}{a}+\frac{y}{c}=1, \quad \frac{x}{b}+\frac{y}{d}=1.$$

On subtraction, we have $x\left(\frac{1}{a}-\frac{1}{b}\right)+y\left(\frac{1}{c}-\frac{1}{d}\right)=0.$

Hence, from (i)

$$\frac{x}{h}+\frac{y}{k}=0,$$

which is a straight line through O.

14. Let the coordinates of C be (x, y), $O\hat{B}A=\theta$, $OA=a$.

Then $$a \operatorname{cosec}\theta = AB = BC = CA.$$

It is easily seen from a figure that

$$x=a\operatorname{cosec}\theta\,.\cos(60^\circ-\theta)=\frac{a}{2}(\sqrt{3}+\cot\theta),$$

and $$y=a\operatorname{cosec}\theta\,.\sin(60^\circ+\theta)=\frac{a}{2}(\sqrt{3}\cot\theta+1).$$

$\therefore\ \sqrt{3}\,.\,x-y=a$, which is a straight line.

15. Let h, k be the intercepts on the axes, and (a, b) the fixed point.

Then $$\frac{a}{h}+\frac{b}{k}=1,$$

is the condition that the straight line passes through (a, b).

The mid pt. of the intercept is given by $2x=h$, $2y=k$.

Hence the required locus is $\frac{a}{2x}+\frac{b}{2y}=1$.

16. Take the fixed line as the axis of x, and AB as the axis of y. Let them meet in O; let $OA=a$, $OB=b$ and let P be the point (h, k). The equation of AP is

$$y-a=\frac{k-a}{h}x.$$

The intercept of this on OX is $\frac{ah}{a-k}$. So for BP.

Hence the condition gives $\frac{ah}{a-k}-\frac{bh}{b-k}=2c$.

The locus of P is thus, on reduction,

$$2c(y-a)(y-b)+(a-b)xy=0.$$

17. Take for axes the lines through O parallel to the two *fixed* lines $O'P$, $O'Q$.

Let O' be (h, k), and $x\cos\alpha+y\sin\alpha=p$ ....(i)

the equation of PQ.

The equation of $O'P$, $O'Q$ is $(x-h)(y-k)=0$,

i.e. $$xy-hy-kx+hk=0. \quad \text{..............(ii)}$$

The equation of the lines joining the origin to the common points of (i) and (ii) is

$$p^2xy-p(hy+kx)(x\cos\alpha+y\sin\alpha)$$
$$+hk(x\cos\alpha+y\sin\alpha)^2=0. \quad \text{[Art. 122.]}$$

Since these are at right angles,

$$-p\cos\alpha k-p\sin\alpha h+hk=0.$$

But $(p\cos\alpha,\ p\sin\alpha)$ is the point whose locus is required. Hence the locus is the straight line $kx + hy = hk$.

18. Take the straight line for axis of x and any origin, let $OA = a$, $OB = b$, etc., and let $P(x, y)$ be any point on the locus.

$$\text{Then } \tan A\hat{P}B = \frac{\dfrac{y}{x-a} - \dfrac{y}{x-b}}{1 + \dfrac{y^2}{(x-a)(x-b)}} = \frac{(a-b)\,y}{(x-a)(x-b)+y^2}.$$

Hence the equation of the locus is

$$\frac{a-b}{(x-a)(x-b)+y^2} = \frac{c-d}{(x-c)(x-d)+y^2}.$$

19. Take the fixed lines for axes, and

$$x\cos\alpha + y\sin\alpha - p = 0$$

as the equation of the other fixed line.

Then $$x - y = x\cos\alpha + y\sin\alpha - p,$$

which is the equation to a straight line.

20. Let $$OA = a, \quad OB = b.$$

The equation of the line through A perpendicular to OB is

$$x\cos\omega + y = a\cos\omega,$$

and of the line through B perpendicular to OA is

$$x + y\cos\omega = b\cos\omega.$$

Also $$c^2 = a^2 + b^2 - 2ab\cos\omega.$$

Eliminating a, b, we have for the required locus

$$c^2 = (x + y\sec\omega)^2 + (y + x\sec\omega)^2 - 2\cos\omega\,(x + y\sec\omega)(y + x\sec\omega),$$

or $$c^2\cot^2\omega = x^2 + y^2 + 2xy\cos\omega.$$

21. Let $x\cos\alpha_1 + y\sin\alpha_1 = p_1$, $x\cos\alpha_2 + y\sin\alpha_2 = p_2$, etc. be the equations of the bases; c_1, c_2 etc. their lengths. Then we have

$$\Sigma c_1 (x \cos \alpha_1 + y \sin \alpha_1 - p_1) = \text{a constant},$$

which is the equation to a straight line.

22. As in Art. 105, we have

(1) $$-2r = \frac{c}{a\cos\theta + b\sin\theta} + \frac{c'}{a'\cos\theta + b'\sin\theta};$$

$$\therefore\ c(a'x + b'y) + c'(ax + by) + 2(ax + by)(a'x + b'y) = 0.$$

(2) $$r^2 = \frac{c}{a\cos\theta + b\sin\theta} \cdot \frac{c'}{a'\cos\theta + b'\sin\theta};$$

$$\therefore\ (ax + by)(a'x + b'y) = cc'.$$

23. Let $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, etc. be the equations of the n fixed lines.

As in Art. 105, we have

$$\frac{n}{r} = -\cos\theta\,\Sigma\frac{a_1}{c_1} - \sin\theta\,\Sigma\frac{b_1}{c_1};\quad \therefore\ x\,.\,\Sigma\frac{a_1}{c_1} + y\Sigma\frac{b_1}{c_1} + n = 0,$$

a straight line.

24. Let $r\cos\theta\,.\,a + r\sin\theta\,.\,b = 1$ be the equation of the variable line, and α_1, α_2, α_3 etc. the inclinations of the n given concurrent lines, the point of concurrence being the origin.

Then, as in Art. 105, $a\,.\,\Sigma\cos\alpha_1 + b\,.\,\Sigma\sin\alpha_1 = \lambda$, where λ is a constant.

But this is the condition that the given line $ax + by = 1$ should pass through the fixed point

$$\left(\frac{\Sigma\cos\alpha_1}{\lambda},\ \frac{\Sigma\sin\alpha_1}{\lambda}\right).$$

25. Let (r, θ) be coordinates of C referred to A as origin and perpendicular from A to locus of B (length c) as initial line.

Since the triangle ABC is always similar to a given one, $\therefore\ \dfrac{AB}{AC} = \text{constant} = \mu$ (say).

Then $\quad c = AB\,.\,\cos(\theta - A) = \mu r\cos(\theta - A)$.

Hence locus of C is a straight line.

26. Let $(h, 0)$, $(0, k)$, (x, y) be the coordinates of A, B, C respectively.

Then $$(x-h)^2+y^2=b^2, \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$
and $$(y-k)^2+x^2=a^2. \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Also, since CA, CB are at right angles,

$$\therefore \frac{y}{x-h}\cdot\frac{y-k}{x}+1=0, \text{ whence } \frac{y^2}{x^2}=\frac{(x-h)^2}{(y-k)^2}.$$

Also, from (i) and (ii), $\dfrac{(x-h)^2}{(y-k)^2}=\dfrac{b^2-y^2}{a^2-x^2}$.

$$\therefore \frac{b^2-y^2}{a^2-x^2}=\frac{y^2}{x^2}=\frac{b^2}{a^2}. \quad \therefore y=\pm\frac{b}{a}.x.$$

Aliter. If the axes meet in O, then $CBOA$ is a cyclic quadrilateral. $\therefore \angle COA=\angle CBA$.

$\therefore$ locus of C is $\dfrac{y}{x}=\tan COA=\tan CBA=\dfrac{b}{a}$.

So, if B lie on the negative axis of y the equation is

$$\frac{y}{x}=-\frac{b}{a}.$$

27. Let OQ, $OR=a, b$ respectively.

Then (a, b) is the point whose locus is required.
Since QR passes through a fixed point (h, k),

$$\therefore \frac{h}{a}+\frac{k}{b}=1.$$

Hence the required locus is $\dfrac{h}{x}+\dfrac{k}{y}=1$.

28. Let O be the origin and a perpendicular to the given parallel straight lines the axis of x, and let their distances from O be a and b. Let $\angle POX$ be θ; then P is the point $(a, a\tan\theta)$ and Q is $(b, b\tan\theta)$. The equations to PR, QR are thus

$$\left.\begin{aligned} y - a\tan\theta &= (x-a)\tan\alpha, \\ \text{and} \qquad y - b\tan\theta &= (x-b)\tan\beta, \end{aligned}\right\}$$

where α and β are given.

Eliminating $\tan\theta$, we have, as the required locus,

$$(a-b)\,y = x\,[a\tan\beta - b\tan\alpha] + ab\,(\tan\alpha - \tan\beta).$$