Straight Lines

Slope of a Line

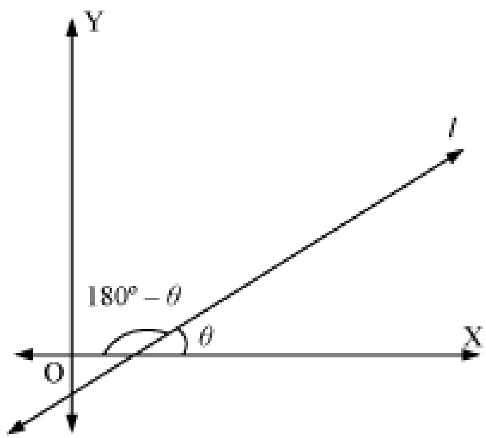
Have you ever wondered why it is difficult to climb a mountain while it is easy to walk down a straight road?

In such cases, we generally use the term ‘slope’ and say that the slope of the mountain is steep.

But do we actually know what slope is and how it is calculated?

Here, we will study about the slopes of straight lines. To understand what we mean by slope, let us first understand what we mean by inclination of a line.

Consider a straight line *l*, as shown in the figure.



Observe that the line *l* makes an angle θ with the positive direction of *x*-axis when measured in the anticlockwise direction. We say that this angle θ is the **inclination** of the line *l*.

The angle which a straight line makes with the positive direction of *x*-axis measured in the anticlockwise direction is called the inclination (or angle of inclination) of the line.

Now, from this definition, we can observe the following points:

1. Inclination of a line parallel to *y*-axis or the *y*-axis itself is 90°.
2. Inclination of a line parallel to *x*-axis or the *x*-axis itself is 0°.

Now that we have understood what we mean by inclination, let us now understand the meaning of the slope of a line.

In the above figure, we have seen that the inclination of line *l* is θ. In this case, we say that tan θ is the slope of line *l*.

If θ is the inclination of a line *l* with the positive direction of *x*-axis, then tan θ is called the slope or gradient of line *l*. The slope of a line is denoted by *m*.

For example, the slope of the line which makes an inclination of 45° with the positive direction of *x*-axis is given by $m = \tan 45° = 1$

Note:

1. Since tan θ is not defined for $\theta = 90°$, we say that the slope of a vertical line is not defined. We also conclude that the slope of *y*-axis is not defined.

2. The slope of *x*-axis is 0.

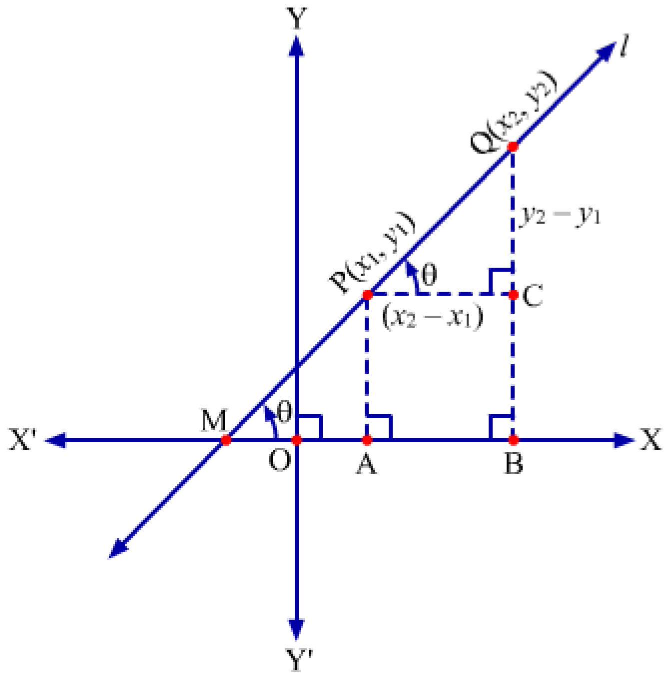
Now, if we have a line which passes through two given points, then can we find the slope of that line?

Yes, we can find the slope of that line using the formula given below.

If P(x_1, y_1) and Q(x_2, y_2) are two points on a non-vertical line *l* whose inclination is θ, then the slope of line *l* is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Let us prove this formula.

We have two points P(x_1, y_1) and Q(x_2, y_2) on a line *l* whose inclination is θ as shown in the following figure.



Let us draw perpendiculars from P and Q to X-axis which meet X-axis at A and B respectively.

Also, let us draw PC ⊥ QB.

∴ PC || AB

It can be seen that PQ is transversal with respect to X-axis and PC such that PC || X-axis.

Now,

∠QMB = θ (Given)

∠QPC = ∠QMB (Corresponding angles)

∴ ∠QPC = θ

Also, we have

OA = x_1 and OB = x_2 ∴ AB = $x_2 - x_1$

PA = y_1 and OB = y_2 ∴ QC = $y_2 - y_1$

Since AB = PC

$\therefore$ PC = $x_2 - x_1$

In right-angled triangle ΔPQC, we have

$\angle$QPC = θ

$$\tan\theta = \frac{\text{Side opposite to angle } \theta}{\text{Side adjacent to angle } \theta}$$
$$\Rightarrow \tan\theta = \frac{QC}{PC}$$
$$\Rightarrow \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of line PQ = Slope of line l = tan θ

$$\text{Slope of line PQ} = \text{Slope of line } l = \frac{y_2 - y_1}{x_2 - x_1}$$

Hence proved.

Using this formula, we can find the slope of any line passing through two distinct points.

For example, the slope of the line passing through the points (3, −7) and (5, 1) is $\frac{1-(-7)}{5-3} = \frac{8}{2} = 4$

Now, we know that if there are two lines in a coordinate plane, then they will be either parallel or perpendicular. In either of the two cases, a relation between the slopes of the two lines is exhibited. The relation is explained as follows:

- Two non-vertical lines l_1 and l_2 are parallel, if and only if their slopes are equal. In other words, if m_1 and m_2 are the slopes of lines l_1 and l_2 respectively, then the lines l_1 and l_2 are parallel to each other, if $m_1 = m_2$.

- Two non-vertical lines l_1 and l_2 are perpendicular to each other, if and only if their slopes are negative reciprocals of each other. In other words, if m_1 and m_2 are the slopes of lines l_1 and l_2 respectively, then the lines l_1 and l_2 are perpendicular to each other, if $m_1 m_2 = -1$.

Let us go through the following video to understand the proof of the above mentioned conditions for parallel and perpendicular lines in terms of their slopes.

Now, if we have three points A, B, and C, then we can conclude the following statement:

Three points A, B, and C will lie on a line i.e., they will be collinear, if and only if the slope of AB is the same as the slope of BC.

Let us now look at some examples to understand the concept of slope better.

Example 1:

A line l_1 passes through points (5, −3) and (4, −6). Another line, l_2, passes through points (8, 1) and (2, 3). Are lines l_1 and l_2 perpendicular, parallel or neither of the two?

Solution:

We will first find the slopes of the two lines.

We know that if a line passes through points (x_1, y_1) and (x_2, y_2), then the slope of that line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Thus,

Slope of line l_1 is given by

$$m_1 = \frac{(-6)-(-3)}{4-5} = \frac{-3}{-1} = 3$$

Slope of line l_2 is given by

$$m_2 = \frac{3-1}{2-8} = \frac{2}{-6} = -\frac{1}{3}$$

Here, we can observe that $m_1 m_2 = -1$. Hence, lines l_1 and l_2 are perpendicular to each other.

Example 2:

The line passing through points (0, 2) and (8, 4) is parallel to the line passing through points $\left(4, \frac{8}{5}\right)$ and (2, p). Find the value of p.

Solution:

We know that two lines are parallel if and only if their slopes are equal. The slope of a line passing through points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Therefore,

Slope of the line passing through points (0, 2) and (8, 4) is given by

$$m_1 = \frac{4-2}{8-0} = \frac{2}{8} = \frac{1}{4}$$

Slope of the line passing through points $\left(4, \frac{8}{5}\right)$ and (2, p) is given by

$$m_2 = \frac{p - \frac{8}{5}}{2-4} = \frac{5p-8}{-2\times 5} = \frac{-5p+8}{10}$$

Since the two lines are parallel,

$m_1 = m_2$

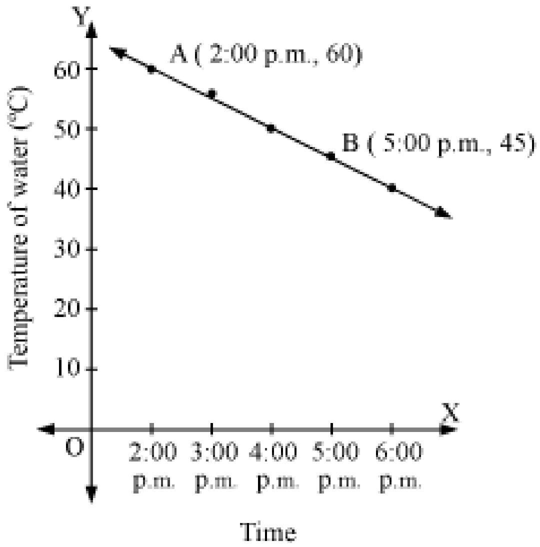
$$\Rightarrow \frac{1}{4} = \frac{-5p+8}{10}$$
$$\Rightarrow 10 = -20p + 32$$
$$\Rightarrow 5 = -10p + 16$$
$$\Rightarrow 10p = 16 - 5 = 11$$
$$\Rightarrow p = \frac{11}{10}$$

Thus, the value of p is $\frac{11}{10}$.

Example 3:

The given graph shows the temperature of water, which was kept on fire for some time, at different intervals of time.

What will be the temperature of water at 8 p.m. if it was kept in the same conditions from 2 p.m. to 9 p.m.?



Solution:

Since line AB passes through points A (2:00 p.m., 60°C) and B (5:00 p.m., 45°C), its slope is $\frac{45-60}{5-2}=\frac{-15}{3}=-5$.

Let y be the temperature of water at 8:00 p.m. Accordingly, on the basis of the given graph, line AB must pass through point C (8:00 p.m., y).

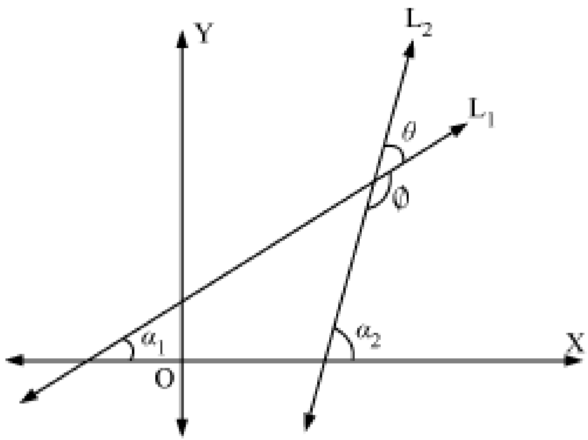
∴ Slope of AB = Slope of BC

$$\Rightarrow -5=\frac{y-45}{8-5}$$
$$\Rightarrow -5=\frac{y-45}{3}$$
$$\Rightarrow -15=y-45$$
$$\Rightarrow y=-15+45$$
$$\Rightarrow y=30$$

Thus, the temperature of water will be 30°C at 8:00 p.m.

Angle between Two Lines

- The acute angle θ between two lines L_1 and L_2 with slopes m_1 and m_2 respectively is given by
$\tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right|$, $1+m_1m_2\neq 0$. In other words, $\tan\theta=\left|\frac{\tan\alpha_2-\tan\alpha_1}{1+\tan\alpha_1\tan\alpha_2}\right|$.



- The obtuse angle Φ can be found by using Φ = 180° − θ.

Solved Examples

Example 1:

The coordinates of a quadrilateral ABCD are A(−4, −3), B(0, 5), C(10, 5), and D(4, −3). Find the measures of angles A, B, C, and D.

Solution:

The coordinates of the vertices of the quadrilateral ABCD are given as A(−4, −3), B(0, 5), C(10, 5) and D(4, −3).

We know that the angle between two lines with slopes m_1 and m_2 is given by $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

Slope of line AB = $\frac{5-(-3)}{0-(-4)} = \frac{8}{4} = 2$

Slope of line BC = $\frac{5-5}{10-0} = 0$

Slope of line CD = $\frac{(-3)-5}{4-10} = \frac{-8}{-6} = \frac{4}{3}$

Slope of line DA = $\frac{(-3)-(-3)}{(-4)-4}=0$

$$\tan \angle A = \left|\frac{2-0}{1+2\times 0}\right| = 2 \Rightarrow \angle A = \tan^{-1}(2)$$

$$\tan \angle B = \left|\frac{2-0}{1+2\times 0}\right| = 2 \Rightarrow \angle B = \tan^{-1}(2)$$

$$\tan\angle C = \left|\frac{\frac{4}{3}-0}{1+\frac{4}{3}\times 0}\right| = \frac{4}{3} \Rightarrow \angle C = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\tan\angle D = \left|\frac{\frac{4}{3}-0}{1+\frac{4}{3}\times 0}\right| = \frac{4}{3} \Rightarrow \angle D = \tan^{-1}\left(\frac{4}{3}\right)$$

Example 2:

The angle made by two line segments is $\tan^{-1}\left(\frac{1}{7}\right)$. If the slope of one line segment is $\frac{1}{2}$, then find the slope of the other line segment.

Solution:

We know that the angle between two lines with slopes m_1 and m_2 is given by $\tan\theta = \left|\frac{m_2-m_1}{1+m_1m_2}\right|$

Let the required slope of the other line be m.

According to the given information,

$$\frac{1}{7}=\left|\frac{\frac{1}{2}-m}{1+\frac{1}{2}\times m}\right|$$

$$\Rightarrow \frac{1}{7}=\left|\frac{\frac{1-2m}{2}}{\frac{2+m}{2}}\right|$$

$$\Rightarrow \frac{1}{7}=\pm\left(\frac{1-2m}{2+m}\right)$$

$$\Rightarrow \frac{1}{7}=\left(\frac{1-2m}{2+m}\right) \text{ or } \frac{1}{7}=-\left(\frac{1-2m}{2+m}\right)$$

$$\Rightarrow 2+m=7-14m \text{ or } 2+m=14m-7$$

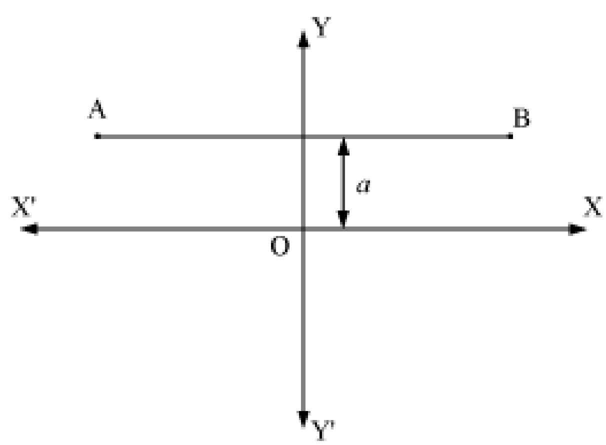
$$\Rightarrow 15m=5 \text{ or } 13m=9$$

$$\Rightarrow m=\frac{5}{15}=\frac{1}{3} \text{ or } m=\frac{9}{13}$$

Thus, the required slope of the line is $\frac{1}{3}$ or $\frac{9}{13}$.

Equations of Horizontal and Vertical Lines

Suppose we have a line AB lying above the *x*-axis at a distance of *a* units and parallel to *x*-axis.



Can we find anything common between the points lying on this line AB?

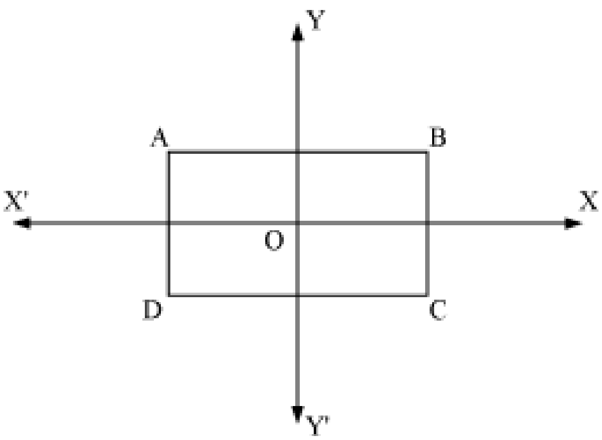
Observe that if we take any point on this line, then its *y*-coordinate will be *a*.

Therefore, we conclude that any point on this line AB will be of the form (x, a). Therefore, the equation of this line will be $y = a$.

Let us now look at some examples to understand the concept better.

Example 1:

In the given figure, a rectangle ABCD is drawn on a coordinate plane, and it is symmetric about the *x*-axis and the *y*-axis. The perimeter of the rectangle is 32 units. The equation of line AB is $y = 3$. Find the equations of the lines BC, CD, and DA.



Solution:

It is given that the rectangle ABCD is symmetric about the x and the y-axis. Hence, the distances of lines AB and CD from the x-axis will be the same.

Also, we know that the equation of the horizontal line that lies at distance a below the x-axis is $y = -a$.

It is given that the equation of line AB is $y = 3$. Thus, the equation of line CD is $y = -3$.

Thus, from the given figure it is clear that AD = BC = (3 + 3) units = 6 units.

It is given that the perimeter of rectangle ABCD is 32 units.

Hence,

2(AB + AD) = 32

⇒ AB + 6 = 16

$\Rightarrow$ AB = 10 units

Since the rectangle is symmetrical about the y-axis, lines AD and BC are at a distance of $\frac{10}{2}$ = 5 units from the y-axis.

We know that the equation of the vertical line that lies at a distance b to the right of the y-axis is $x = b$.

Thus, the equation of line BC is $x = 5$.

Also, we know that the equation of the vertical line that lies at a distance b to the left of the y-axis is $x = -b$.

Thus, the equation of line AD is $x = -5$.

Example 2:

Find the value(s) of p for which the equation $(p^2 + 2p - 3)y + (p^2 + 4p + 1)x + 5 = 0$ is parallel to the y-axis.

Solution:

We know that if a line is parallel to the y-axis, then the coefficient of y in its equation will be zero. Therefore,

$p^2 + 2p - 3 = 0$
$\Rightarrow p^2 + 3p - p - 3 = 0$

$\Rightarrow p(p + 3) - 1(p + 3) = 0$
$\Rightarrow (p + 3)(p - 1) = 0$

$\Rightarrow p = -3$ or $p = 1$, are the required values of p for which the given equation is parallel to the y-axis are −3 and 1.

Point-Slope and Two-Point Form of Straight Lines

Point-Slope Form

- The equation of a non-vertical line with slope m and passing through the point (x_1, y_1) is given by $(y - y_1) = m(x - x_1)$.

- In other words, the point (x, y) lies on the line with slope m through the fixed point (x_1, y_1) if and only if its coordinates satisfy the equation $(y - y_1) = m(x - x_1)$.

- The point-slope form of the equation is used when the information about the slope of the line and a point through which it passes is given.

- Let us go through the following video to understand how we arrived at the point-slope form of an equation and learn how to apply it to solve a problem.

Two-Point Form

- The equation of a non-vertical line passing through two given points (x_1, y_1) and (x_2, y_2) is given by $\frac{(y-y_1)}{(x-x_1)}=\frac{(y_2-y_1)}{(x_2-x_1)}$.

Solved Examples

Example 1:

Find the equation of the line that passes through the points $\left(\frac{1}{3},\frac{1}{2}\right)$ and (−5, 2).

Solution:

Using the two-point form of the equation of line, we know that the equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by $\frac{(y-y_1)}{(x-x_1)}=\frac{(y_2-y_1)}{(x_2-x_1)}$.

Thus, the required equation of the line passing through the points $\left(\frac{1}{3},\frac{1}{2}\right)$ and (−5, 2) is given by

$$\frac{\left(y-\frac{1}{2}\right)}{\left(x-\frac{1}{3}\right)}=\frac{\left(2-\frac{1}{2}\right)}{\left(-5-\frac{1}{3}\right)}$$

$$\Rightarrow \frac{\left(\frac{2y-1}{2}\right)}{\left(\frac{3x-1}{3}\right)}=\frac{\left(\frac{4-1}{2}\right)}{\left(\frac{-15-1}{3}\right)}$$

$$\Rightarrow \frac{2y-1}{3x-1}=\frac{3}{-16}$$

$$\Rightarrow -16(2y-1)=3(3x-1)$$

$$\Rightarrow -32y+16=9x-3$$

$$\Rightarrow 9x+32y-19=0$$

Example 2:

Find the equation of the line that passes through the point (8, 5) and makes an inclination of 210° with the *x*-axis.

Solution:

It is given that the line makes an inclination of 210° with the *x*-axis. Therefore, the slope of the line is given by

m = tan 210° = tan (180° + 30°) = tan 30° = $\frac{1}{\sqrt{3}}$

Using point-slope form of equation of line, we know that the equation of the line with slope m that passes through the point (x_1, y_1) is given by

$(y - y_1) = m(x - x_1)$.

Thus, the required equation of the line is given by

$$y-5=\frac{1}{\sqrt{3}}(x-8)$$

$$\Rightarrow \sqrt{3}(y-5)=(x-8)$$

$$\Rightarrow \sqrt{3}y-x-5\sqrt{3}+8=0$$

Example 3:

Find the equation of the line that passes through the intersection of lines $2x + y + 6 = 0$ and $x - y + 9 = 0$ and is perpendicular to the line that passes through points (−6, 3) and (4, 5).

Solution:

It is given that the required line passes through the intersection of lines $2x + y + 6 = 0$ and $x - y + 9 = 0$.

The points of intersection of lines $2x + y + 6 = 0$ and $x - y + 9 = 0$ can be found by adding the two equations. Hence,

$3x + 15 = 0$

$\Rightarrow x = -5$

$\therefore y = 4$

Thus, the required equation passes through the point (−5, 4).

It is also given that the required line is perpendicular to the line that passes through points (−6, 3) and (4, 5).

We know that the slope of a line that passes through points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Thus, the slope of the line that passes through points (−6, 3) and (4, 5) is given by

$m = \frac{5-3}{4+6} = \frac{2}{10} = \frac{1}{5}$.

We also know that if two lines are perpendicular to each other, then their slopes are negative reciprocals of each other.

Thus, the slope of the required line is −5.

Thus, we are required to find the equation of the line that passes through the point (−5, 4) having slope −5.

Using point-slope form, the equation of the required line is given by

$y - 4 = (-5)(x + 5)$

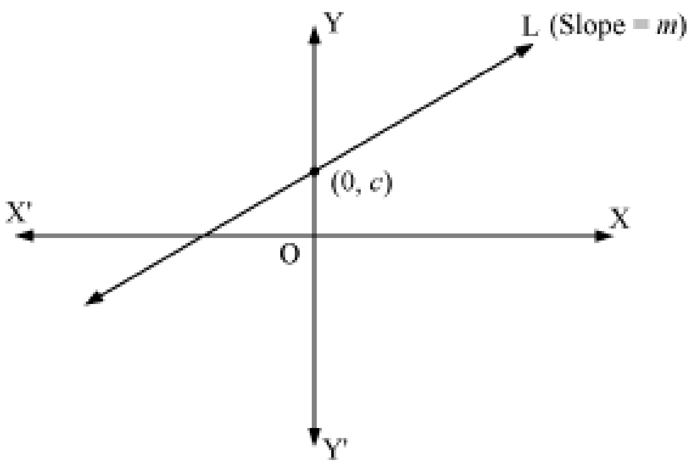
$\Rightarrow y - 4 = -5x - 25$

$\Rightarrow 5x + y + 21 = 0$

Slope-Intercept Form of Straight Lines

Slope-intercept Form

- If a line with slope m makes y-intercept as c, then the equation of the line is given by $y = mx + c$.
- In other words, we can say that point (x, y) on the line with slope m and y-intercept c lies on the line if and only if $y = mx + c$.



- If a line with slope m makes x-intercept as d, then the equation of the line is given by $y = m(x - d)$.
- Let us go through the following video to understand how we arrived at the above formulae of slope-intercept form of equation of a line.
- A general equation $Ax + By + C = 0$ can be written in slope-intercept form as follows:
- $y = -\frac{A}{B}x - \frac{C}{B}$, if B ≠ 0, where $m = -\frac{A}{B}$ and $c = -\frac{C}{B}$.
- $x = -\frac{C}{A}$, if B = 0, which is a vertical line whose slope is undefined and whose x-intercept is $-\frac{C}{A}$.

Solved Examples

Example 1:

The equation of a line is given by 12x + 8y − 9 = 0. Find the angle made by this line with the positive direction of the x-axis.

Solution:

The equation of the line is given by

$12x + 8y - 9 = 0$

$\Rightarrow 8y = 9 - 12x$

$$\Rightarrow y = \frac{9}{8} - \frac{12}{8}x$$

$$\Rightarrow y = \frac{9}{8} - \frac{3}{2}x$$

Comparing this equation with the general form $y = mx + c$, we obtain the slope of the line as

$m = -\frac{3}{2} = \tan\theta$

Thus, the angle made by the line with the positive direction of the x-axis is $\tan^{-1}\left(-\frac{3}{2}\right)$.

Example 2:

Find the equation of the line that makes x-intercept as 5 and is perpendicular to the line 16x + 4y = 5.

Solution:

It is given that the line is perpendicular to the line 16x + 4y = 5.

The slope of this line can be calculated as

$4y = -16x + 5$

$$\Rightarrow y = -4x + \frac{5}{4}$$

Thus, the slope of this line is −4. Therefore, the slope of the required line is $\frac{1}{4}$.

Also, it is given that the line makes *x*-intercept as 5.

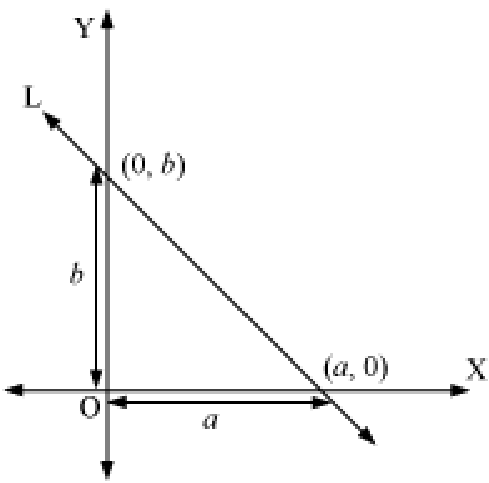
By using the slope-intercept form, we get the required equation of the line as

$$y = \frac{1}{4}(x-5)$$
$$\Rightarrow 4y = x-5$$
$$\Rightarrow x-4y-5=0$$

Intercept Form of Straight Lines

Intercept Form or Double Intercept Form

- The equation of the line that makes intercepts *a* and *b* on *x*-axis and *y*-axis respectively is given by $\frac{x}{a} + \frac{y}{b} = 1$.



- A general equation $Ax + By + C = 0$ can be written in double intercept form as follows:
- $\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1$, if C ≠ 0, where the *x* and *y*-intercepts are $-\frac{C}{A}$ and $-\frac{C}{B}$ respectively.

- $Ax + By = 0$, if $C = 0$, which is a line that passes through the origin making zero intercepts on the axes.

Solved Examples

Example 1:

The equation of a line is given by $11x - 8y + 12 = 0$. Find the x- and y-intercepts made by this line.

Solution:

The equation of the given line is

$11x - 8y + 12 = 0$

$$\Rightarrow \frac{11x}{12} - \frac{8y}{12} + 1 = 0$$

$$\Rightarrow \frac{11x}{12} - \frac{2y}{3} + 1 = 0$$

$$\Rightarrow \frac{x}{\frac{12}{11}} + \frac{y}{-\frac{3}{2}} = -1$$

$$\Rightarrow \frac{x}{-\frac{12}{11}} + \frac{y}{\frac{3}{2}} = 1$$

Thus, the given line makes the x-intercept as $-\frac{12}{11}$ and y-intercept as $\frac{3}{2}$.

Example 2:

The sum of x and y-intercepts made by a line is 25 and their product is 144. Find the equation of the line.

Solution:

We know that the equation of the line that makes x and y-intercepts as a and b is given by $\frac{x}{a} + \frac{y}{b} = 1$.

It is given that

$a + b = 25$ and $a \times b = 144$

$a \times b = 144$ gives $b = \frac{144}{a}$.

Therefore, on using $a + b = 25$, we obtain

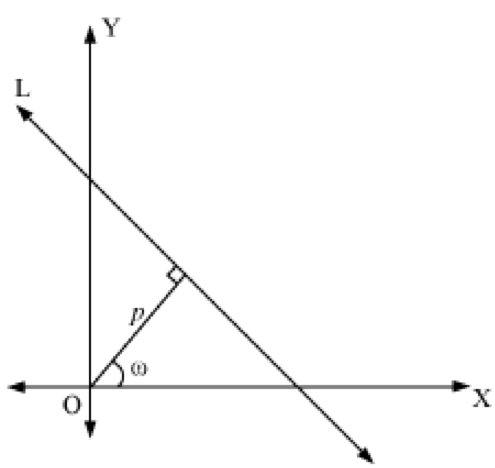
$$a + \frac{144}{a} = 25$$
$$\Rightarrow a^2 - 25a + 144 = 0$$
$$\Rightarrow a^2 - 9a - 16a + 144 = 0$$
$$\Rightarrow a(a-9) - 16(a-9) = 0$$
$$\Rightarrow (a-9)(a-16) = 0$$
$$\Rightarrow a = 9 \text{ or } a = 16$$

This gives $b = 16$ or $b = 9$.

Therefore, the required equation of the line is $\frac{x}{9} + \frac{y}{16} = 1$ or $\frac{x}{16} + \frac{y}{9} = 1$.

Normal Form of Straight Lines

- The equation of a line, which is at perpendicular distance p from the origin and the normal drawn from origin to this line makes angle ω with the positive direction of x-axis, is given by $x \cos \omega + y \sin \omega = p$.



- A general equation $Ax + By + C = 0$ can be written in its normal form as $x \cos \omega + y \sin \omega = p$, where $\cos \omega = \pm\frac{A}{\sqrt{A^2+B^2}}$, $\sin \omega = \pm\frac{B}{\sqrt{A^2+B^2}}$ and $p = \pm\frac{C}{\sqrt{A^2+B^2}}$.

Solved Examples

Example 1:

Find the equation of the line whose perpendicular distance from the origin is 10 units and the angle which the normal makes with the positive direction of *x*-axis is 240°.

Solution:

We know that the normal form of the equation is given by

$x \cos \omega + y \sin \omega = p$

Here, we are given that

$p = 10$ and $\omega = 240°$

Thus, the required equation of the line is given by

$x \cos 240° + y \sin 240° = 10$

$\Rightarrow x \cos (180° + 60°) + y \sin (180° + 60°) = 10$

$\Rightarrow -x \cos 60° - y \sin 60° = 10$

$$\Rightarrow -x\left(\frac{1}{2}\right) - y\left(\frac{\sqrt{3}}{2}\right) = 10$$
$$\Rightarrow x + y\sqrt{3} + 20 = 0$$

Example 2:

Find the equation of the line AB in normal form where the coordinates of points A and B are A(16, 4), and B(10, 10).

Solution:

Using the two-point form, the equation of line AB is given by

$$\frac{y-4}{x-16}=\frac{10-4}{10-16}$$
$$\Rightarrow \frac{y-4}{x-16}=\frac{6}{-6}=-1$$
$$\Rightarrow y-4=-x+16$$
$$\Rightarrow x+y-20=0$$

On dividing this equation by $\sqrt{(1)^2+(1)^2}=\sqrt{2}$, we obtain

$$\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}-\frac{20}{\sqrt{2}}=0$$
$$\Rightarrow \frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}-10\sqrt{2}=0$$

This is the required equation of line AB in the normal form.

Example 3:

The equation of a line is given as $-x + y + 7 = 0$. Find the normal distance of the line from the origin and the angle which the normal makes with the positive direction of *x*-axis.

Solution:

The given equation of the line is $-x + y + 7 = 0$.

On dividing this equation by $\sqrt{(-1)^2+(1)^2}=\sqrt{2}$, we obtain

$$-\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}+\frac{7}{\sqrt{2}}=0$$
$$\Rightarrow x\cos 45^\circ - y\sin 45^\circ=\frac{7}{\sqrt{2}}$$
$$\Rightarrow x\cos\left(360^\circ-45^\circ\right)+y\sin\left(360^\circ-45^\circ\right)=\frac{7}{\sqrt{2}}$$
$$\Rightarrow x\cos 315^\circ + y\sin 315^\circ=\frac{7}{\sqrt{2}}$$

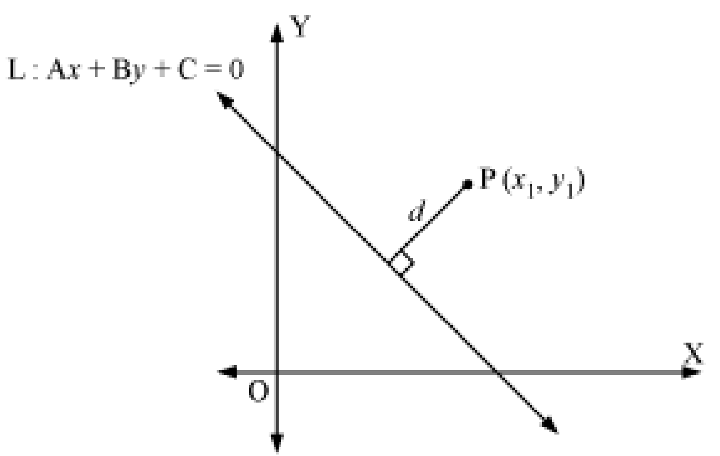
On comparing this equation with the standard normal form i.e., $x \cos \omega + y \sin \omega = p$, we obtain

$\omega = 315°$, $p = \frac{7}{\sqrt{2}}$

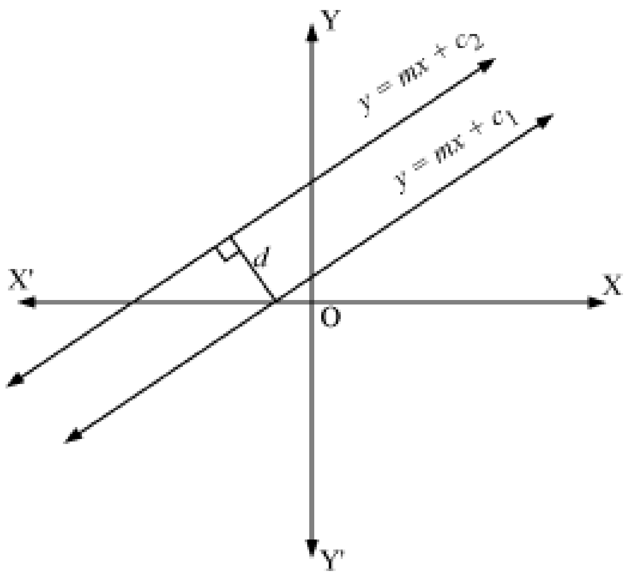
Thus, the normal distance of the line from the origin is $\frac{7}{\sqrt{2}}$ and the angle which the normal makes with the positive direction of the x-axis is 315°.

Distance of a Point from a Line and Distance between Two Parallel Lines

- The perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is given by $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.



- The distance between two parallel lines $y = mx + c_1$ and $y = mx + c_2$ is given by $d = \frac{|c_1 - c_2|}{\sqrt{1 + m^2}}$.



- In general, the distance between the parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by $d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$.

Solved Examples

Example 1:

The distance between two parallel lines is given as $\frac{12}{\sqrt{13}}$. If the equation of one of the parallel lines is $2x + 3y - 17 = 0$, then find the equation of the other line.

Solution:

We know that the distance between two parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by $d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$.

It is given that the required line is parallel to line $2x + 3y - 17 = 0$. Hence, the equation of the required line is $2x + 3y + C = 0$.

Therefore, using the given information, we obtain

$$\frac{12}{\sqrt{13}} = \frac{|-17-C|}{\sqrt{(2)^2+(3)^2}}$$
$$\Rightarrow \frac{12}{\sqrt{13}} = \frac{|-17-C|}{\sqrt{13}}$$
$$\Rightarrow 12 = |-17-C|$$
$$\Rightarrow -17-C = 12 \text{ or } -17-C = -12$$
$$\Rightarrow C = -29 \text{ or } C = -5$$

Thus, the required equation of the line is $2x + 3y - 29 = 0$ or $2x + 3y - 5 = 0$.

Example 2:

The distance of line $4x - 3y + 7 = 0$ from a point P that is lying on the x-axis is 3 units. Find the coordinates of point P.

Solution:

We know that the perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is given by $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2+B^2}}$.

Let the required point P lying on the x-axis be $(x', 0)$.

Therefore, on using the given information, we obtain

$$3 = \frac{|4 \times x' + (-3) \times 0 + 7|}{\sqrt{(4)^2 + (-3)^2}}$$
$$\Rightarrow 3 = \frac{|4x'+7|}{\sqrt{16+9}}$$
$$\Rightarrow 3 = \frac{|4x'+7|}{\sqrt{25}}$$
$$\Rightarrow |4x'+7| = 3 \times 5 = 15$$
$$\Rightarrow 4x'+7 = -15 \text{ or } 4x'+7 = 15$$
$$\Rightarrow 4x' = -22 \text{ or } 4x' = 8$$
$$\Rightarrow x' = -\frac{11}{2} \text{ or } x' = 2$$

Thus, the required point P can be $\left(-\frac{11}{2}, 0\right)$ or (2, 0).

Example 3:

Find the distance between the lines $y = 7x + 11$ and $y = 7x - 5$.

Solution:

We know that the distance between the parallel lines $y = mx + c_1$ and $y = mx + c_2$ is given by $d = \frac{|c_1 - c_2|}{\sqrt{1+m^2}}$.

Thus, the distance between the given parallel lines is given by

$$d = \frac{|11-(-5)|}{\sqrt{1+(7)^2}} = \frac{16}{\sqrt{50}} = \frac{16}{5\sqrt{2}} = \frac{8\sqrt{2}}{5}.$$

Thus, the required distance between the given parallel lines is $\frac{8\sqrt{2}}{5}$.