* Matrix Algebra *

Determinant value of a square matrix :-

The sum of the products of elements of a raw column with their corresponding cofactors is known as determinant value of a square, matrix.

2x2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3x3

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 3 & 2 \end{vmatrix} = 1\begin{vmatrix} 5 & 1 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 5 \\ 1 & 3 \end{vmatrix}$$

$$= 1(10-3) - (2-1) + (3-5)$$

$$= 7 - 1 - 2 = 4$$

4x4

$$\begin{vmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix}$$

$C_1 + C_2 + C_3 + C_4$ $\qquad$ $R_2 - R_1,\ R_3 - R_1,\ R_4 - R_1$

$$= \begin{vmatrix} 10+x & 2 & 3 & 4 \\ 10+x & 2+x & 3 & 4 \\ 10+x & 2 & 3+x & 4 \\ 10+x & 2 & 3 & 4+x \end{vmatrix} = \begin{vmatrix} 10+x & 2 & 3 & 4 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix}$$

$$= (10+x)\begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} = (10+x)(x^3)$$

Trick it works only when all diagonal element are same and all other elemsts are same.

$$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x+3a)(x-a)(x-a)(x-a)$$

$$= (x+3a)(x-a)^3$$

* (sum of all element of 1^{st} row) x (substract each element of 1^{st} row ~~to~~ from 1^{st} element)

eg

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = (2+1+1)(2-1)(2-1) = 4$$

$$\hookrightarrow = 2(4-1) - 1(2-1) + 1(1-2) = 6-1-1 = 4$$

* A square matrix A is said to be

(i) symmetric if $A^T = A$ i.e.) $a_{ij} = a_{ji} \; \forall \; i,j$

(ii) skew-symmetric if $A^T = -A$ i.e.) $a_{ij} = -a_{ji} \; \forall \; i,j$

(iii) orthogonal matrix if $AA^T = A^TA = I$

* Every square matrix matrix can be expressed as the sum of symmetric and skew-symmetric

i.e.

$$A = \underbrace{\left(\frac{A+A^T}{2}\right)}_{\text{symmetric}} + \underbrace{\left(\frac{A-A^T}{2}\right)}_{\text{skew-symetrix.}}$$

Eg: $A = \begin{bmatrix} 2 & 5 \\ 8 & 9 \end{bmatrix}$ $A^T = \begin{bmatrix} 2 & 8 \\ 5 & 9 \end{bmatrix}$

$$\frac{A+A^T}{2} = \frac{1}{2}\begin{bmatrix} 4 & 13 \\ 13 & 18 \end{bmatrix} = \begin{bmatrix} 2 & 13/2 \\ 13/2 & 9 \end{bmatrix}$$

$$\frac{A-A^T}{2} = \frac{1}{2}\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3/2 \\ 3/2 & 0 \end{bmatrix}$$

Properties of Determinant :-

(i) $|A^T| = |A|$, $|AB| = |A||B|$, $|A+B| \neq |A| + |B|$

(ii) The determinant value of a Triangular/a Diagonal matrix is the product of its leading diagonal element.

Eg:- $A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 8 \end{bmatrix} = 2(32-1) - 3(0-0) + 5(0-0) = 64$

or $= 2 \times 4 \times 8 = 64$

(iii) In a square matrix of each element of a row (column) is zero then the value of its determinant is zero.

eg. $A = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 2 & 7 \\ 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow |A| = 2(0-0) - 3(0-0) + 5(0-0)$

$|A| = 0$

(iv) In a square matrix if two rows(column) are identical / proportional then the value of its determinant is zero.

eg. $A = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 9 & 8 \\ 6 & 9 & 8 \end{bmatrix}$ $|A| = 2(72-72) - 3(48-48) + 5(54-54)$

$|A| = 0$

(v) The determinant value of a skew-symmetric of odd order is always zero.

eg: 3x3

$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$ $|A| = 0 - 1(0+6) + 2(3-0)$

$|A| = 0$

(vi) The determinant value of a non-zero skew-symmetric matrix even order is always a perfect square.

$A = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$ $|A| = 0 + 9$

$|A| = 9$

(vii) The determinant value of an orthogonal matrix is always either 1 (or) -1

Proof. $AA^T = I$

$|AA^T| = |I|$

$|A||A^T| = 1$

$|A||A| = 1$

$|A|^2 = 1$

$|A| = \pm 1$

(viii) If A is a square matrix of order n and k is any scalar the $|KA| = K^n|A|$

Eg.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad KA = \begin{bmatrix} ka_{11} & Ka_{12} \\ ka_{21} & Ka_{22} \end{bmatrix}$$

$$|KA| = \begin{bmatrix} k a_{11} & k a_{12} \\ k a_{21} & k a_{22} \end{bmatrix} = k^2 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = k^2|A|$$

(ix) If A is a non-~~sy~~singular matrix of order 'n' then → $|A| \neq 0$

det A $\neq 0$

singular matrix $|A| = 0$

(a) $A(\text{Adj } A) = |A| I$

(b) $A^{-1} = \dfrac{\text{Adj } A}{|A|}$

(c) $|\text{Adj } A| = |A|^{n-1}$

(d) $|\text{Adj Adj } A| = |A|^{(n-1)^2}$

(e) $|A^{-1}| = \dfrac{1}{|A|}$

Eg: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Cofactor matrix $= \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ $\text{Adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

transpose of co-factor

$|A| = ad - bc$

So $A^{-1} = \dfrac{\text{Adj}}{|A|} = \dfrac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

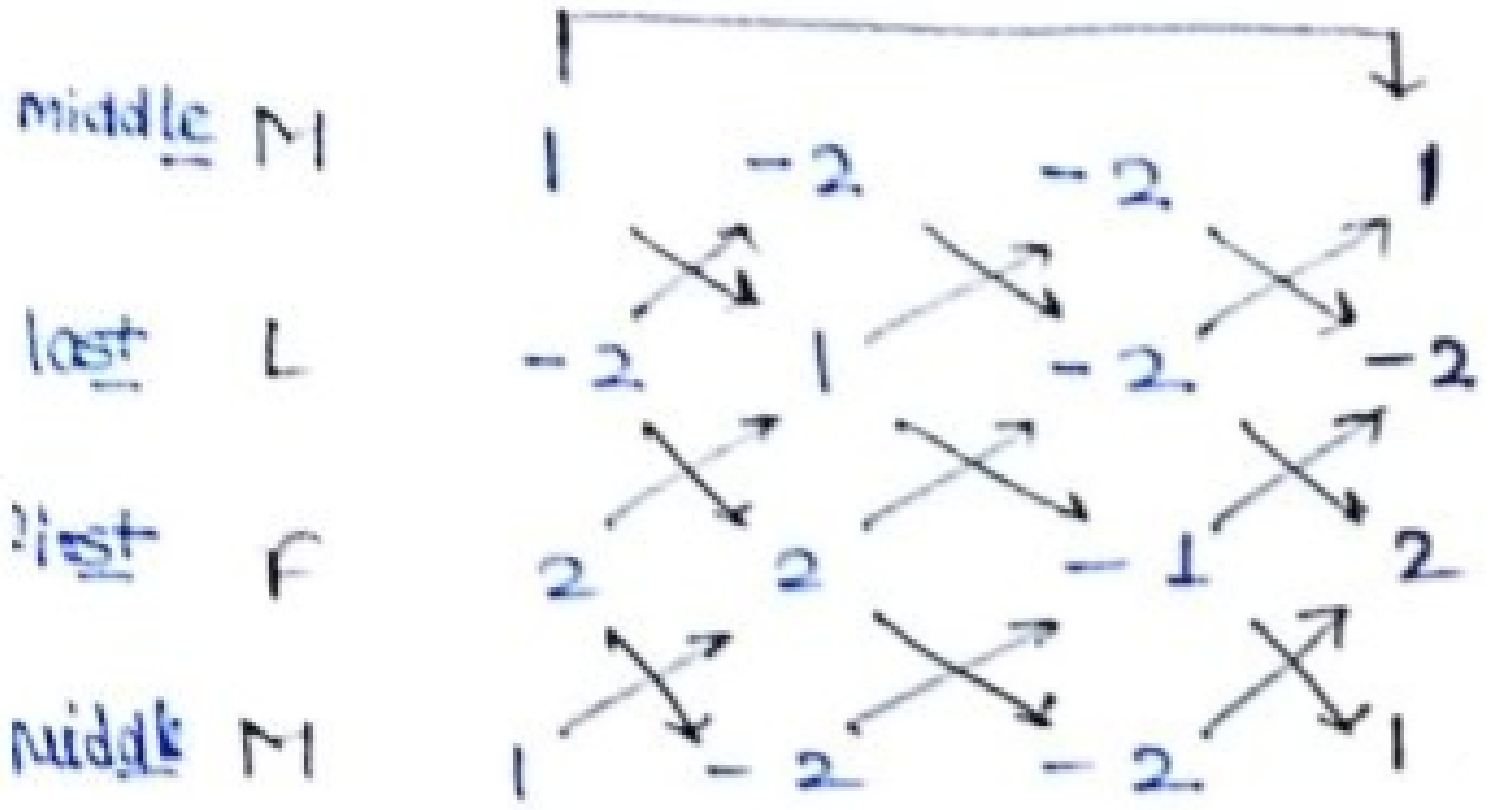
Eg:-

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$|A| = -1(1-4)+2(2+4)-2(-4-2)$

$|A| = 27$

work only for 3×3



Now

$$\text{Adj}A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

Question

If $A_{m\times n}$ and $B_{n\times p}$ are multiplied then the number of multiplicative and additive operation are needed to get matrix AB

(a) mpn, mpn

✓(b) mpn, mp(n-1)

(c) mp(n-1), mpn

(d) mpn, mpn - 1

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{\text{[illegible]}} & a & & a_{m\times n} \end{bmatrix}_{m\times n} \qquad B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ \vdots \\ b_{n1} \end{bmatrix}_{n\times p}$$

$$AB = \begin{bmatrix} \underset{1}{a_{11}\cdot b_{11}} \overset{1}{+} \underset{2}{a_{12}\cdot b_{21}} \cdot \overset{(n-1)}{+} \underset{n}{a_{1n}\cdot b_{n1}} \\ \\ \end{bmatrix}_{m\times n}$$

	(×)	(+)
for one lement	n	n-1
for (mA) element	mpn	mp(n-1)

Ques If $A_{3\times4}$ $B_{4\times5}$ $C_{5\times3}$ are multiplied then the minimum number of multiplication operations are needed to get matrix ABC.

[✓ (a) 96]   (b) 95   (c) 105   (d) 106

$A(BC)$

$4\times5 \quad 5\;3 \rightarrow 4\times3\times5 = 60$

$3\times4 \quad 4\times3 \rightarrow 3\times3\times4 = 36$

(3×3)

Total = 96 → minimum.

$(AB)\,C$

$3\times4 \quad 4\times5 \rightarrow 3\times5\times4 = 60$

$3\times5 \quad 5\times3 \rightarrow 3\times3\times5 = 45$

3×3

Total = 105

Question Which of the following is correct for 3×3 matrices P, Q, R

(a) $P(Q+R) = PQ + RP \neq PQ + PR$

(b) $(P-Q)^2 = P^2 - 2PQ + Q^2 \neq P^2 - PQ - QP + Q^2$

[✓ (c)] $(P+Q)^2 = P^2 + PQ + QP + Q^2$

(d) $|P+Q| = |P| + |Q|$ ✗

In matrix theory $\boxed{AB \neq BA}$

Question: If $A=(a_{ij})_{3\times 3}$ where $a_{ij}=i^2-j^2 \ \forall\ i,j$,

then $|A| = \underline{\ 0\ }$?

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{bmatrix}}_{\hookrightarrow \text{ skew-Symmetric of odd order}} = 0$$

$|A| = 0$

Question:- if $A=\begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix}$ is expressed as $(P+Q)$

wher P is symmetric

then $|Q| = $ — ?

Q is skew symmetric

Sol^n

$$P + Q$$

$$\swarrow \qquad \searrow$$

$$\frac{A+A^T}{2} \qquad \frac{A-A^T}{2}$$

$$Q = \frac{A-A^T}{2} = \frac{1}{2}\left\{\begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}\right\}$$

$$Q = \frac{1}{2}\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3/2 \\ 3/2 & 0 \end{bmatrix}$$

$$|Q| = 0 + 9/4 = 9/4$$

Question :- The number of different $(n\times n)$ symmetric matrix with each element being 0 (or) 1 is ——— ?

(a) 2^n (b) 2^{n^2} (c) $2^{\frac{n^2+n}{2}}$ (d) $2^{\frac{n^2+n}{2}}$

Sol^n 2×2

$$\begin{bmatrix} a & c \\ c & d \end{bmatrix} \qquad \underset{(0,1)}{\overset{a_{11}}{\square}} \qquad \underset{(0,1)}{\overset{a_{12} \& a_{21}}{\square}} \qquad \underset{(0,1)}{\overset{a_{22}}{\square}}$$

$$2 \times 2 \times 2 = 8$$

Check option

eg

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

3×3

$$\begin{bmatrix} \textcircled{a_{11}} & a_{12} & a_{13} \\ a_{21} & \textcircled{a_{22}} & a_{23} \\ a_{31} & a_{32} & \textcircled{a_{33}} \end{bmatrix} \quad \overset{a_{11}}{\square} \quad \overset{a_{22}}{\square} \quad \overset{a_{33}}{\square} \quad \overset{a_{12} \& a_{21}}{\square} \quad \overset{a_{13} \& a_{31}}{\square} \quad \overset{a_{23} \& a_{32}}{\square}$$

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$$

Question :- If A, B, C, D are non-singular matrix of same order such that DABEC = I then B^{-1} = ___

(a) DAEC (b) CEAD (c) ECDA (d) ADCE

An. TRICK

$$\underset{ACW}{DABEC} = I$$

B^{-1} = ECDA

A^{-1}. = BECD

D^{-1} = ABEC

$$ABC = I$$

$$A^{-1} \rightarrow A^{-1}ABC = A^{-1} I$$

$$BC = A^{-1}$$

$$C^{-1} \quad BCC^{-1} = A^{-1}C^{-1}$$

$$B = A^{-1}C^{-1}$$

$$B^{-1} = (A^{-1}C^{-1})^{-1} = (C^{-1})^{-1}(A^{-1})^{-1}$$

$$B = CA$$

Question:- Let $M^4 = I$ $(M \neq I, M^2 \neq I, M^3 \neq I)$ for any positive integer k, $M^{-1} = ----$?

(a) M^{4k} (b) M^{4k+1} (c) M^{4k+2} (d) M^{4k+3}

$M^8 = I \Rightarrow M^8 M^{-1} = M^{-1} \Rightarrow M^7 = I$

$M^{12} = I \Rightarrow M^{12} M^{-1} = M^{-1} \Rightarrow M^{11} = I$

$M^{16} = I \Rightarrow M^{16} M^{-1} = M^{-1} \Rightarrow M^{15} = I$

$M^{20} = I \Rightarrow M^{20} M^{-1} = M^{-1} \Rightarrow M^{19} = I$

$M^4 = I$

$M^4 M^4 = M^4$

$M^8 = I$

!

$M^{-1} = M^7 = M^{11} = M^{15} = M^{19} = \;---\; = M^?$

7, 11, 15, 19 - -

$a = 7, d = 4$ $\quad t_n = a + (n-1)d = 7 + (n-1)4$

$t_n = 7 + 4n - 4$

$$\boxed{t_n = 4n + 3}$$

Question:- If x & y are two singular matrix such that $XY = Y$, $YX = X$, then $X^2 + Y^2 = \_\_$?

Sol^n $X^2 + Y^2$

Singular $|X| = |Y| = 0$

$XX + YY$

$\downarrow \;\downarrow \quad\quad \downarrow$

$X\underbrace{YX}_{} + Y\underbrace{XY}_{}$

$\underbrace{\;}_{I} \downarrow \quad\quad \downarrow \;\downarrow$

$\underbrace{YX}_{} + \underbrace{XY}_{}$

$\downarrow \quad\quad \downarrow$

$X + Y$

= Ans

Rank of a Matrix :- The order of highest ordered non-zero minor is called rank of the matrix.

$$\underbrace{\text{minor}}_{\downarrow}$$

The determinant
Value of Square sub matrix.

Eg

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 5 \\ 1 & -1 & 3 & 7 \end{bmatrix}_{3\times4}$$

Take $\boxed{3\times3}$ Atlest on highest $\overset{\text{det}}{\text{sq.}} \neq 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & -1 & 3 \end{bmatrix} = 1(9+2)-1(3-2)+1(-1-3)$$

$$= 11-1-4$$

$$= 6 \neq 0$$

3x3 Matrix is non-zero

$\therefore \rho(A) = 3$

Eg

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}$$

Take (3x3)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} = 1(14-12)-1(7-3)+1(4-2)$$

$$= 2-4+2 = 0$$

(3x3) minor is zero

Take any (2x2)

$$\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2-1 = 1 \neq 0$$

a (2x2) minor is non zero

$\therefore \rho(A) = 2$

Properties of Rank

- The rank of Null matrix is taken as zero.
- The rank of Non-Singular matrix is its order.
- The rank of Singular matrix is less than its order
- If A is $(m \times n)$ matrix then $\rho(A) \leq \min(m, n)$

- $\rho(A) = \rho(A)^T$

- $\rho(AB) \leq \min\{\rho(A), \rho(B))\}$

- In a matrix if all rows are identical/proportional then it's rank is always one.

eg $A = (a_{ij})_{m \times n}$ where $a_{ij} = 5 \; \forall \; i, j$ then $\rho(A) = \underline{?}$

$$\begin{bmatrix} 5 & 5 & \cdots & 5 \\ 5 & 5 & \cdots & 5 \\ \vdots & \vdots & & \vdots \\ 5 & 5 & \cdots & 5 \end{bmatrix}_{n \times n} \left\} \text{All rows are identical} \right. \quad \therefore \rho(A) = 1$$

eg If $A = (a_{ij})_{m \times n}$ where $a_{ij} = i \cdot j$ then $\rho(A)$ ___ ?

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 2 & \cdots & n \\ 2 & 4 & \cdots & 2n \\ \vdots & & & \\ n & 2n & \cdots & n^2 \end{bmatrix} \left\} \text{All row's are proportional} \right. \quad \therefore \rho(A) = 1$$

- If A and B are two matrices of same order the

$$\rho(A + B) \leq \rho(A) + \rho(B)$$

eg

$$A + B = \underset{\substack{\downarrow \\ 2}}{\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}} \quad A = \underset{\substack{\downarrow \\ 1}}{\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}} \quad B = \underset{\substack{\downarrow \\ 1}}{\begin{bmatrix} 0 & 0 \\ 3 & 7 \end{bmatrix}}$$

$$2 = 1 + 1$$

$$A + B = \underset{\substack{\downarrow \\ 2}}{\begin{bmatrix} 3 & 7 \\ 8 & 15 \end{bmatrix}} \quad A = \underset{\substack{\downarrow \\ 2}}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \quad B = \underset{\substack{\downarrow \\ 2}}{\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}}$$

$$2 < 2 + 2$$

* If A is $n \times n$ matrix with rank $\underline{n}$ then $\underline{\rho(\text{Adj } A) = n}$

* If A is $n \times n$ matrix with rank $\underline{(n-1)}$ then $\underline{\rho(\text{Adj } A) = 1}$

* If A is $n \times n$ matrix with rank $\underline{(n-2)}$ then $\underline{\rho(\text{Adj } A) = 0}$

eg $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ 3x3 $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 1(8-9)-1(4-6)+1(3-4)$

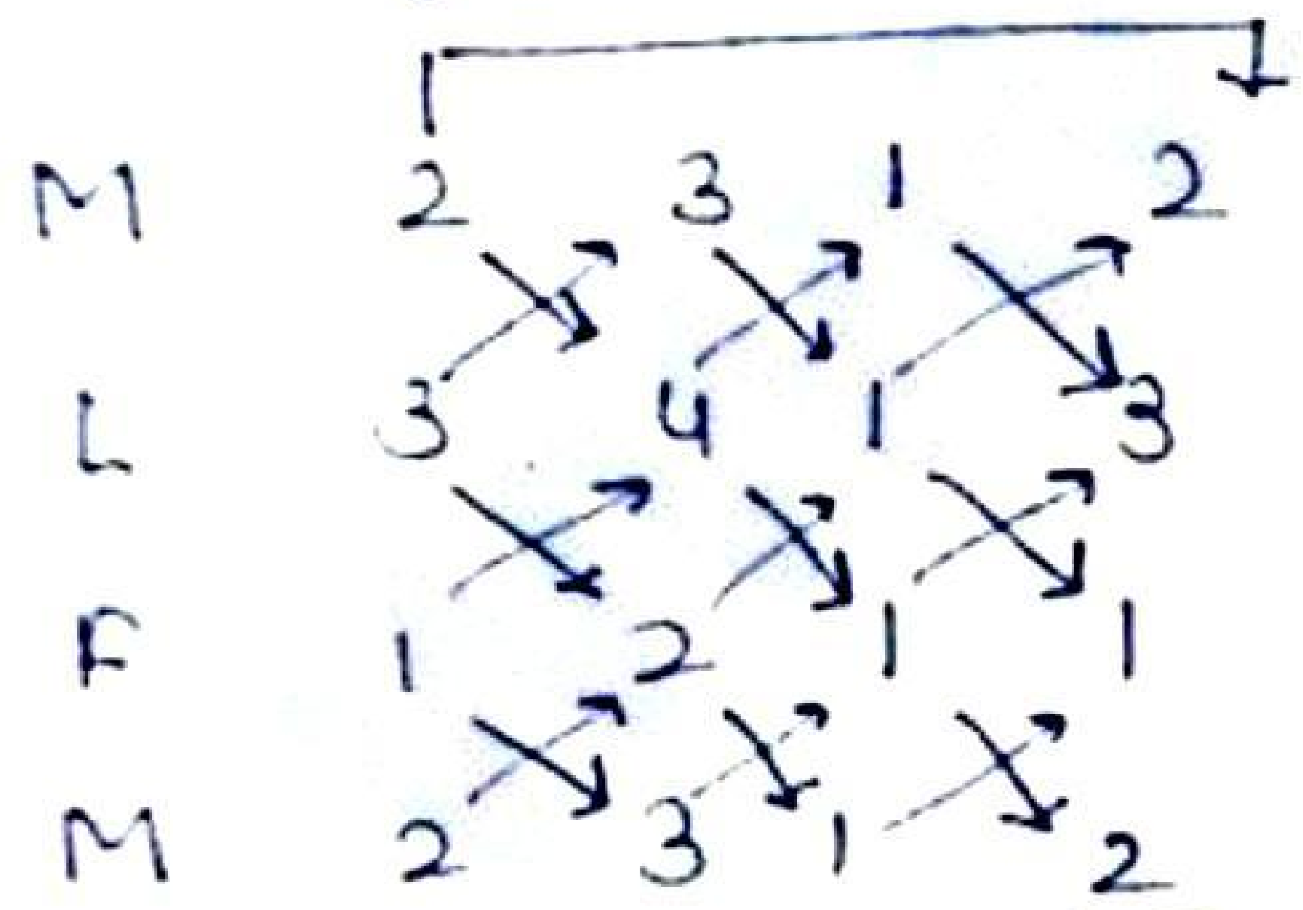
$= 0$

a (3x3) minor is zero.

take (2x2)

$$\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2-1 = 1 \neq 0 \quad \therefore \rho(A) = 2$$

For Adj A → trick



$$\begin{bmatrix} (8-9) & (3-4) & (3-2) \\ (6-4) & (4-2) & (1-3) \\ (3-4) & (2-3) & (2-1 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 2 & -2 \\ -1 & -1 & +1 \end{bmatrix}$$ All row are proportional

$\therefore \rho(\text{Adj } A) = 1$

Echelon form:-

* The number of zeros before non-zero element in a row are less than such number of zeros in the next row.

* zero rows (if any) must follow non-zero rows.

⇒ The number of non-zero rows is called rank of the matrix when it is in echelon form.

don't count these type of zero

$$A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

increasing order

zero row should be last

$e(A) = 3$

$$B = \begin{bmatrix} 1 & -1 & 3 & 0 & 6 \\ 0 & 5 & 7 & 2 & 3 \\ 0 & 0 & 0 & 7 & 2 \end{bmatrix}$$

$e(B) = 3$

$$C = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$e(C) = 3$

$$D = \begin{bmatrix} 3 & 7 & 0 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

last

$e(D) = 3$

* All are in echelon for

eg

non zero → start operation

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 2 & 1 & 3 \\ 2 & 4 & 3 & 2 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

$R_2 - 3R_1$

$R_3 - 2R_1$

$R_4 - (R_1 + R_2 + R_3$

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Bigg\} ③$$

last

$e(A) = 3$

eg

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$e(A) = 3$

eg

interchang

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 3 & 7 \\ 1 & 5 & 11 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 7 \\ 1 & 5 & 11 \end{bmatrix}$$

$R_3 - R_1, R_4 - R_1$

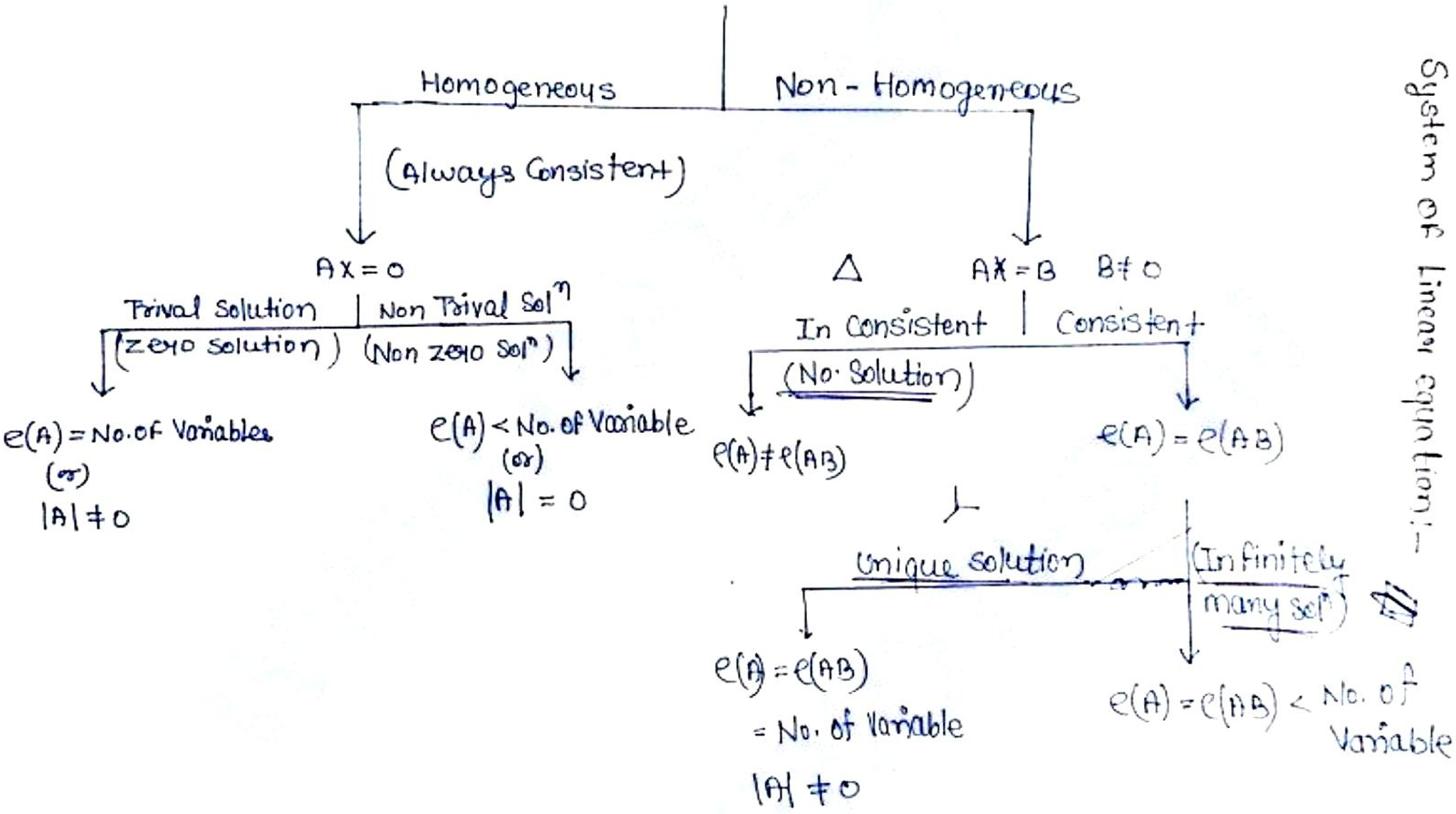
$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \end{bmatrix}$$

$R_3 - 2R_2, R_4 - 4R_2$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$e(A) = \underline{\underline{2}}$

System of Linear equation:-



Homogeneous system.

eg. $x_1 + x_2 + 2x_3 + x_4 = 0$

$2x_1 + 2x_2 + 4x_3 + 2x_4 = 0$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 1 <$ No. of Variables $= 4$

$x_1 + x_2 + 2x_3 + x_4 = 0$

Free variables = Total No. of Variables − Rank

$= 4 - 1 = 3$

let $x_1 = k_1$, $x_2 = k_2$, $x_3 = k_3$

$k_1 + k_2 + 2k_3 + x_4 = 0$

$x_4 = -k_1 - k_2 - 2k_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ -k_1 - k_2 - 2k_3 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

* Different values of k_1, k_2, k_3 we can Generate infinitely many number of solution to the given solution

Solve:-

$$x + y + z = 4$$
$$x + 2y + 3z = 6$$
$$2x + 3y + 4z = 10$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \end{bmatrix}$$

$$AX = B$$

Consider augmented matrix

$$(AB) = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 6 \\ 2 & 3 & 4 & 10 \end{bmatrix}$$

No. of Variable $= 3$

$R_2 - R_1$

$R_3 - (R_2 + R_1)$

$$\overset{\rho(A)}{\begin{bmatrix} 1 & 1 & 1 & : 4 \\ 0 & 1 & 2 & : 2 \\ 0 & 0 & 0 & : 0 \end{bmatrix}}$$

$\rho(A) = 2$

$\rho(AB) = 2$

$\rho(A) = 2 = \rho(AB) < $ No. of variable (3)

Infinitely many solutions

The eq^{n}s are

$$x + y + z = 4 \quad -①$$

$$y + 2z = 2 \quad -②$$

free variable $= 3 - 2 = 1$

let $z = k$

$$y = 2 - 2k$$

$$x = 2 + k$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+k \\ 2-2k \\ k \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

different values of k we can Generate infinitely many solution.

Remember :- free Variables
(or)
Number of independent Variables
(or)
Nullity
(or)
Dimension of Null space (or) Dimension of space of solution.

$$= \begin{pmatrix} \text{Total Number} \\ \text{of Variables} \end{pmatrix} - (\text{Rank})$$

$$= \begin{pmatrix} \text{Total Number} \\ \text{of Columns} \end{pmatrix} - (\text{Rank})$$

Question The Nullity of a matrix $A = \begin{bmatrix} 4 & 7 & 2 \\ 3 & -1 & 5 \\ 7 & 6 & x \end{bmatrix}$ is 1

then the value of $x = \_$?

Solⁿ Nullity $= \begin{pmatrix} \text{Total No.} \\ \text{Of Colums} \end{pmatrix} - (\text{Rank})$

$$1 = 3 - (\text{Rank})$$

Rank $\rho(A) = 2$

$A_{3\times3}$ matrix is with rank 2

$$\therefore |A| = 0$$

$$|A| = 4(-x-30) - 7(3x-35) + 2(18+7) = 0$$

$$\Rightarrow -4x - 120 - 21x + 245 + 50 = 0$$

$$25x = 175$$

$$x = 7$$

or

$\Rightarrow$ $\rho(A) = 2$ so least row of (3×3) echelon matrix should be zero

$R_2 - \frac{3}{4}R_1$, $R_3 - (R_1 + R_2)$

$$\rho(A)=2 \begin{bmatrix} 4 & 7 & 2 \\ 0 & 25/4 & 7/2 \\ 0 & 0 & x-7 \end{bmatrix} \quad \Rightarrow x - 7 = 0$$

$$x = \underline{7}$$

Q.6
WB

$A = (a_{ij})_{n\times n}$ where $a_{ij} = 3 \;\forall\; i,j$

$$A = \begin{bmatrix} 3 & 3 & \cdots & 3 \\ 3 & 3 & \cdots & 3 \\ \vdots & & & \vdots \\ 3 & \cdots & \cdots & 3 \end{bmatrix}$$

all row are identical

$\rho(A) = 1$

Nullity $= n - 1$

Q.7

$$\begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has infinity many solution.

$AX = 0$ has non trival sol^n

$|A| = 0$

$$\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$$

$$(a+2)(a-1)(a-1) = 0$$

$$a = -2, 1, 1$$

Q.8 $A_{m\times n}$, ~~ρ~~ $\rho(A) = r$

dimension of space of sol^n = (No. of Columns) − Rank

or Nullity. $= n - r$

Q.9

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 6 \\ 1 & 4 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ k \end{bmatrix}$$

Consider augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & k \end{array}\right]$$

$R_2 - R_1$
$R_3 - R_1$ $\qquad\qquad\qquad\qquad$ $R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 3 & \lambda-1 & k-6 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & \lambda-6 & k-20 \end{array}\right]$$

System has no solution

$$\rho(A) \neq \rho(A|B)$$

$$\lambda = 6, \quad k \neq 20$$

(i) $\left.\begin{array}{l}\lambda = 6 \\ k = 20\end{array}\right\}$ $\rho(A) = 2$, $\rho(A|B) = 2$, $n = 3$

Infinity many sol^n

(ii) $\lambda \neq 6$ $\quad$ $\rho(A) = 3$, $\rho(A|B) = 3$, $n = 3$

$k = 20$

(iii) $\lambda \neq 6$ $\quad$ $\rho(A) = 3$, $\rho(A|B) = 3$, $n = 3$

$k \neq 20$

} unique sol^n

(iv) $\lambda = 6$ $\quad$ $\rho(A) = 2$, $\rho(A|B) = 3$ $\quad$ $n = 3$

$k \neq 20$

$\rho(A) \neq \rho(A|B)$ $\quad$ No Solution

Q.10 A is n×n matrix such that (consider angmented matrix)

$A^2 = I$ & B is n×1 real

Vector then $AX = B$

$|A^2| = |I|$

$|A.A| = 1$ $\qquad |A| = \pm 1$

$|A||A| = 1$ $\qquad |A| \neq 0$

$|A| \neq 0$ matrix is inversible so

$$AX = B$$

$$X = \underset{n\times n}{A^{-1}} \underset{n\times 1}{B}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \cdot \\ \\ \\ \end{bmatrix}_{n\times 1}$$ Unique solution

Q.38 $$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 3 \\ 5 & 9 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Consider augmented matrix $R_2 - 2R_1,\ R_3 - 5R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 3 & 3 & b \\ 5 & 9 & -6 & c \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & -1 & 9 & c-5a \end{array}\right]$$

$R_3 - R_2$ $$\left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & 0 & 0 & c-5a-b+2a \end{array}\right]$$ so

$\rho(A) = 2$ so $\rho(A|B) = 2$

because system is Consistance

$(-5a - b + 2a = 0$

$3a + b - c = 0$

Linearly dependent Vectors and independent Vectors:-

Two vectors X_1 & X_2 are said to be linerly dependent if one can be expressed as scalar multiplication of the other. otherwise they are linearly independent.

Eg. $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$X_2 = 2X_1$

X_1 & X_2 are L.D.

eg. $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$X_2 \neq kX_1$

X_1 & X_2 are L. I.

Note:- * If $e(A) = n$ (or) $|A| \neq 0$ then the set of Vectors are linearly Independent

* If $e(A) < n$ (or) $|A| = 0$ then the set of vectors are Linearly dependent.

Eg $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$

$|A| = 0$

$\therefore$ X_2 & X_1 are L. D.

Eg $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = 3 - 2$

$|A| \neq 0$

X_1 & X_2 are L.I.

Orthogonal ~~Matrix~~ Vectors:-

~~The~~ Two vectors X_1 & X_2 are said to be orthogonal vector if $X_1^T . X_2 = 0$

vector n rows

$$\boxed{X_1^T X_2 = 0}$$

Eg.

$$X_1 = \begin{bmatrix}-1\\1\end{bmatrix} \quad X_2 = \begin{bmatrix}1\\1\end{bmatrix} \quad X_1^T . X_2 = \left(-1 \quad 1\right)_{1\times2} \begin{bmatrix}1\\1\end{bmatrix}_{2\times1}$$

$$= -1 + 1 = 0$$

$\therefore X_1$ & X_2 are orthogonal vection

~~Orthogonal~~ Orthonormal vector:-

Two vectors X_1 & X_2 are said to be orthonormal vectors if (i) $X_1^T . X_2 = 0$

(ii) $\|X_1\| = \|X_2\| = 1$

Eg:

$$X_1 = \begin{bmatrix}1\\0\end{bmatrix} \quad X_2 = \begin{bmatrix}0\\1\end{bmatrix}$$

$$X_1^T . X_2 = [1 \ 0]\begin{bmatrix}0\\1\end{bmatrix} = 0 + 0 = 0$$

$$\|X_1\| = \sqrt{1^2+0} = 1$$

$$\|X_2\| = \sqrt{0^2+1^2} = 1$$

So Vector are orthonormal Vector.

Normalised Vector :-

The normalised vector of x is given by $\frac{x}{||x||}$

Eg $x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ $\frac{x}{||x||} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$||x|| = \sqrt{1+4+1}$

$||x|| = \sqrt{6}$ $\frac{x}{||x||} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

Note!- orthonormal Vectors are always orthogonal but reverse need not be true.

Basis:- A set $S = \{x_1, x_2 \cdots x_n\}$ is said to be a Basis of vector space V if it satisfies the following two Conditions

(i) $X_1, X_2, X_3, \cdots X_n$ are linearly Independent.

(ii) set S spans Vector space V.

Example:- $V = \underset{\text{it means 2-D}}{R^2} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} / a, b \in R \right\}$ $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

Check 1st Condition $\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3-2 = 1 \neq 0$

so. $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are linear independent.

check second condition

take any $a, b \in R$

$a = 5$
$b = 9$

$$\begin{bmatrix} 5 \\ 9 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ 2c_1 + 3c_2 \end{bmatrix}$$

$$\Rightarrow \quad c_1 + c_2 = 5$$

$$2c_1 + 3c_2 = 9$$

$$c_1 = 6, \quad c_2 = -1$$

$$\therefore \begin{bmatrix} 5 \\ 9 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\therefore$ set S spans vector space $V = R^2$

$\therefore$ Set $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ is a basis of Vector space R^2

Example $V = R^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} / a, b \in R \right\}$ $\quad S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$ so L.I. $\quad \begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Set S spans vector space R^2

$\therefore$ Set $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is Basis of Vector space R^2

Eigen Values & Eigen Vector :-

- The roots of characteristic eqⁿ $|A-\lambda I| = 0$ are known as eigen value of matrix A.

- To each eigen value λ, if there exists a non-zero Vector X such that $\boxed{AX = \lambda X}$ then X is called Eigen Vector of matrix A corresponding to the eigen value λ.

Example :- Matrix A 2x2 $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

characteristic matrix $[A-\lambda I] = \begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$

ch. eqⁿ is $|A-\lambda I| = 0$

$$(4-\lambda)^2 - 4 = 0$$

$$\lambda^2 - 8\lambda + 12 = 0$$

$$\lambda = 2, 6$$

$\underline{\lambda = 2}$ consider $(A-2I)X = 0$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

rank = 1 No. of Variable = 2

free variable = 2-1 = 1

The eq^n is

$$2x_1 + 2x_2 = 0$$

let $x_1 = k$

$x_2 = -k$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Eigen vector $= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\underline{\lambda = 6}$ Consider $(A - 6I)x = 0$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_1 + R_2$ $\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

rank = 1, No. of variables = 2

free variables = 2 - 1 = 1

let $x_2 = k$ the eq^n is

$$-2x_1 + 2x_2 = 0$$

$$x_1 = k$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigen vector is. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

* The Eigen Vector Corresponding to the distinct Eign~~vector~~ Values of a real symmetric matrix are always orthogonal.

Eg. $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ symmetric

$\lambda = 2, \quad \lambda = 6$

$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$X_1^T X_2 = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1 + 1 = 0$

$\therefore X_1$ & X_2 are othogonal.

* If all the leading minors of a real symmetric matrix are positive then all its eigen values are positive.

$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

leading minor

$1 \times 1 \quad a$

$2 \times 2 \quad \begin{vmatrix} a & b \\ b & d \end{vmatrix}$

$3 \times 3 \quad \begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}$

Eg $A = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$

$1 \times 1 \quad 2 > 0$

$2 \times 2 \quad \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} = 10 - 1 = 9 > 0$

eigen values

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 5-\lambda \end{vmatrix} = 0$

$10 - 2\lambda - 5\lambda + \lambda^2 = 0$

$\lambda = 5.3, 1.69$

* The eigen vectors corresponding to the distinct eigen values of any square matrix are alway linear independent.

Eg: $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

$\lambda = 2 \;\&\; \lambda = 6$

$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \;\&\; X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1-1 = -2 \neq 0$

$\therefore X_1 \;\&\; X_2$ L. I

* The eigen vectors corresponding to the repeated eigen values of any square matrix may be L.I (or) may be L.D.

eg $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Ch. eq^n $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^3 = 0$$

$$\lambda = 2, 2, 2$$

$\boxed{\lambda = 2}$

consider $(A - 2I)X = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rank = 0 No. of variables = 3

free variable = 3 - 0 = 3

let $x_1 = k_1$, $x_2 = k_2$, $x_3 = k_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

eigen vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

eg: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Cha.eqn $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^3 = 0$$

$$\lambda = 1, 1, 1$$

$\boxed{\lambda = 1}$

Consider $(A-\lambda I)X = 0$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank = 1 No. of Variable = 3

free variable = 3 - 1 = 2.

let $x_1 = k_1$, $x_2 = k_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 0 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

eigen vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

The third eigen vector is depending on two independent eigen vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore$ The set of eigen Vectors are L.D.

Properties of Eigen Values: -

* The sum of eigen values = Trace of the matrix

* The product of eigen values = Det. Value of the matrix.

* The eigen values of A = Eigen Values of A^T

* The eigen value of a triangular matrix / a diagonal matrix are its leading diagonal element.

* If λ is the eigen value of a square matrix A then eigen value of matrix

(i) A^2 is λ^2

(ii) A^m is λ^m, $m \in N$

(iii) KA is $K\lambda$, k is a scalar

(iv) $A + KI$ is $\lambda + K$, K is a scalar

(v) $A - KI$ is $\lambda - K$, K is a scaler

(vi) A^{-1} is $1/\lambda$

(vii) Adj A is $\frac{|A|}{\lambda}$

(viii) $A^2 + C_1A + C_2I$ is $\lambda^2 + C_1\lambda + C_2$

* The eigen vector of A, A^{-1}, Adj A, A^2, A^3 - --are always same.

$$\begin{array}{cc} A_{3\times3} & A^2_{3\times3} \\ \lambda = 2,\ 3,\ 6 & 4,\ 9,\ 36 \\ \downarrow\ \ \downarrow\ \ \downarrow & \downarrow\ \ \downarrow\ \ \downarrow \\ x_1\ \ x_2\ \ x_3 & x_1\ \ x_2\ \ x_3 \end{array}$$

* In a square matrix if n rows are identical then (n-1) eigen vector will be zero.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \text{Ch. eq}^n \quad |A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left((1-\lambda)^2-1\right) - 1\left(1-\lambda-1\right) + 1\left(1-1+\lambda\right) = 0$$

$$\lambda^2(\lambda-3) = 0$$

$$\lambda = 0, 0, 3$$

3 rows are identical, 2 eigen vector will be zero.

Ques 13 wb

Sum of eigen values = trace

$$\text{trace} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} \quad \cdots \quad + \frac{1}{n(n+1)}$$

$$= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} \quad - - - \cdot + \frac{1}{n} - \frac{1}{n+1}$$

$$\text{trace} = 1 - \frac{1}{(n+1)}$$

Q.14 triangular matrix ~~of eigen~~ so $|A|$ = product of diagonal ele.

$$|A| = 1 \cdot \frac{1}{2} \cdots \frac{1}{n} = \frac{1}{n!}$$

Q.15

$$A = \begin{bmatrix} 40 & -29 & -11 \\ -18 & 30 & -12 \\ 26 & 24 & -50 \end{bmatrix}$$

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_2 = |A|$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{trace}(A)$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 20$$

det A $\quad C_1 + C_2 + C_3$

$$\begin{vmatrix} 0 & -29 & -11 \\ 0 & 30 & -12 \\ 0 & 24 & -50 \end{vmatrix} = 0$$

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 0$$

alteast one of eighen value = 0

$$0 + \lambda + \lambda_3 = 20$$

$$\lambda_3 = 20 - \lambda$$

Q.16

$$A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Ch. eqⁿ

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 0 & 0 \\ 2 & 1-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$(4-\lambda) \begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda)\left((2-\lambda)(-\lambda) - 4\right)$$

$$= (4-\lambda)(3-\lambda)\left(\qquad \right)$$

Trick

$$\left[\begin{array}{cc|cc} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Trick works only 4×4, 6×6,

$$\begin{bmatrix} \textcircled{B} & 0 \\ 0 & \textcircled{C} \end{bmatrix}, \begin{bmatrix} \textcircled{B} & 0 \\ C & \textcircled{D} \end{bmatrix}, \begin{bmatrix} \textcircled{B} & C \\ 0 & \textcircled{D} \end{bmatrix}$$

find eigen values

by B & C

B & D

B & D

$$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$$

$$B = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0 \qquad C = \begin{vmatrix} 3-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix}$$

$$3\lambda^2 - 3\lambda - 2 = 0 \qquad \lambda = 3, 4$$

$$\lambda = \frac{3 \pm \sqrt{17}}{2}$$

$$\lambda = 3, 4, \frac{3+\sqrt{17}}{2}, \frac{3-\sqrt{17}}{2}$$

4 distinct λ so 4 L.I. eigen vector.

* By applying operations eigen value changes.

Q.18

$$A_{3\times 3} = \longrightarrow \quad 3 \quad 2 \quad -1$$

$$A^2 = \longrightarrow \quad 9 \quad 4 \quad 1$$

$$A^2 - A \longrightarrow \quad 6 \quad 2 \quad 2$$

$$|B| = 6 \times 2 \times 2 = 24$$

Q.19

$$A_{3\times 3} \rightarrow \quad 1 \quad -\frac{1}{2} + \frac{i\sqrt{3}}{2} \quad -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

$$1 \quad \omega \quad \omega^2$$

$$A^{102} \quad (1)^{102} \quad (\omega)^{102} \quad (\omega^2)^{102}$$

$$1 \quad (\omega^3)^{34} \quad ((\omega^2)^3)^{34}$$

$$1 \quad 1 \quad 1$$

trace of $A^{102} = 1 + 1 + 1 = 3$

$$\omega = e^{i2\pi/3} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$$

$$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\omega^3 = e^{i2\pi} = \cos(2\pi) + i\sin(2\pi)$$

$$\omega^3 = 1 + i(0)$$

Q.20

$A_{3\times3}$ skew symmetric

$$\therefore |A| = 0$$

$$\lambda_1 \lambda_2 \lambda_3 = 0$$

at least one eigen value = 0

Q.21 $\lambda_1 = 1$ $\lambda_2 = 4$

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$AX_1 = \lambda_1 X_1$ $AX_2 = \lambda_2 X_2$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$a - b = 1$ $c - d = 1$

$2a + b = 8$ $2c + d = 4$

$a = 3, b = 2$ $c = 1$ $d = 2$

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

Q.22

$$\begin{array}{llll} \lambda & 1 & -1 & \longrightarrow \quad A \\ 3\lambda & 3 & -3 & \quad\quad 3A \\ 9\lambda^2 & 19 & 19 & \quad\quad A^2 \\ \hline \lambda^2+3\lambda & 4 & -2 & A^2+3A \end{array}$$

$$\lambda^2+3\lambda = 18$$

$\lambda \neq 0$ $\qquad |A^2+3A| = (18)(4)(-2)$

$|A| \neq 0$ $\qquad\qquad\qquad \doteq -144 \neq 0$

Q23 $\quad A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\qquad$ Ch eqⁿ $|A - \lambda I| = 0$

$$(1-\lambda)\left((2-\lambda)^2-1\right) = 0$$

$$\lambda_g = 1, 1, 3$$

$\underline{\lambda = 3}$ consider $(A - 3I)X = 0$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

// Trick

// select Non

// zero row

$$\begin{array}{cccc} M & L & F & M \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{array}$$

$$\frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Q.24

$$PQ = I$$

$$|PQ| = |I|$$

$$|P||Q| = 1$$

$$|P| \neq 0 \quad , \quad Q \neq 0$$

$$\lambda_1 , \lambda_2 \neq 0$$

each $\lambda \neq 0$

Tranjugate of a matrix :-

The transpose of conjugate of a matrix is known as tranjugato of a matrix.

$$A = \begin{bmatrix} 2 & 1+i \\ 3i & 1-3i \end{bmatrix}$$

Conjugate of A i.e $\bar{A} = \begin{bmatrix} 2 & 1-i \\ -3i & 1+3i \end{bmatrix}$

Tranjugate of A i.e. $A^{\theta} = \bar{A}^{T} = \begin{bmatrix} 2 & -3i \\ 1-i & 1+3i \end{bmatrix}$

A Square matrix A is said to be

(i) Hermition matrix if $A^\theta = A$

i.e. $a_{ij} = \overline{a_{ji}} \ \forall \ i,j$

(ii) skew - Hermition matrix if $A^\theta = -A$

i.e. $a_{ij} = -\bar{a}_{ji} \ \forall \ i,j$

(iii) Unitary Matrix if $AA^\theta = A^\theta A = I$

* The eigen values of a Hermition matrix (or) a symmetric matrix are always real.

* The eigen values of a skew Hermition matrix (or) a skew - symetric matrix are either zero (or) purely imaginary

* Eigen values of a unitory matrix (or) an othogonal matrix have absolute value 1

i.e $|\lambda| = 1$

Cauchy - Hamilton Matrix :-

Every square matrix satisfies it characteristic equation

Q.27

$$\begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & 6i & 3 \end{bmatrix}$$

Hermition → eigen are real (B)

Q.28

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad a+d = 1$$

$$ad - bc = 1$$

Ch. eqn $|A - \lambda I| = 0$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\lambda^2 - \lambda + 1 = 0 \qquad -①$$

By C-H eqn

$$A^2 - A + I = 0$$

$$A^2 = A - I$$

Multiple A

$$A^3 = A^2 - A = A - I + A$$

$$A^3 = -I$$

Q.29

$$f(t) = t^n + c_{n-1}\, t^{n-1} \;.\;.\;.\; + c_0$$

$\underline{n \times n}$

$$|A| = \lambda_1 \lambda_2 \;..\; \lambda_n = (-1)^n \frac{\text{Constant term}}{\text{coef. of } t^n}$$

$$= (-1)^n \frac{c_0}{1}$$

$$= (-1)^n c_0$$

Q30

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Ch. eq^n $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda^2 - 1) = 0$$

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

By Ch. $\underline{eq^n}$ $A^3 - A^2 - A + I = 0$

$$A^3 = A^2 + A - I$$

$$A^4 = A^3 + A^2 \bullet A$$

$$= A^2 + A - I + A^2 - A$$

$$A^4 = 2A^2 - I$$

$$A^8 = A^4 . A^4$$

$$A^8 = (2A^2 - I)(2A^2 - I)$$

$$A^8 = 4A^4 - 2A^2 - 2A^2 + I$$

$$A^8 = 4A^2 - 3I$$

$$A^4 = 2A^2 - I$$

$$A^{50} = 25A^2 - 24I$$

$$A^{50} = 25\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{bmatrix} - \begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

Q.31 let A be a 3x3 matrix such that

$$|A - I| = 0$$

$$|A - \lambda I| = 0 \qquad \lambda_3 = 1$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 13 \qquad \lambda_1 \lambda_2 = 32$$

$$\lambda_1 + \lambda_2 = 12 \qquad \lambda_2 \lambda_3 = 32$$

$$\lambda_1 = 8 \quad \lambda_2 = 4 \qquad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 64 + 16 + 1 = 81$$

Q.32 A is Hermition

$a_{ji} = \bar{a}_{ij}$

$a_{12} = 5+i$

$a_{21} = 5-i$

Q.32

$$\begin{bmatrix} p & q & r \\ q & r & p \\ r & p & q \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Ax = 0$ has no soln

$|A| = 0$

$$\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$$

$|A| = 0$ only for $p = q = r$.

Q.34

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & 2 \\ 0 & 1-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{bmatrix}$$

$\lambda = 1$

$$A - I = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

apply trick selectt 2 ~~bet~~ non zero row

M	L	F	M
-1	2	0	-1
2	0	-1	2

$$\frac{x_1}{-4} = \frac{x_2}{2} = \frac{x_3}{1} = k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$$

Q.39

$$P^3 = P$$

$$P^3 - P = 0$$

$$\lambda^3 - \lambda = 0$$

$$\lambda(\lambda^2 - 1) = 0$$

$$\lambda = 0, 1, -1$$