5. Logarithms

How do Logarithms Work?

Logarithms

"The Logarithm of a given number to a given base is the index of the power to which the base must be raised in order to equal the given number."

Logarithm of a number is the exponent by which another fixed number (the base) must be raised to produce that number.

If $x = b^y$, then $y = \log_b(x)$

(y is the logarithm of x to base b)

$1000 = 10 \times 10 \times 10 = 10^3$, its written in logarithm as $3 = \log_{10}(1000)$

The Logarithmic Function

The logarithmic function of x is defined as $f(x) = \log_a x$ where $a > 0$, $a \neq 1$

Laws of Logarithms

Logarithm of a product (Product Law): $\log_a xy = \log_a x + \log_a y$

Logarithm of a quotient (Quotient Law): $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

Logarithm of a power (Power Law): $\log_a x^m = m\log_a x$

Note:

$\log_a 1 = 0$

$\log_a a = 1$

If a > 0 and ≠ 1 then logarithm of a positive number N is defined as the index x of that power of 'a' which equals N i.e.,

$$\log_a N = x \text{ iff } a^x = N \Rightarrow a^{\log_a N} = N, a > 0, a \neq 1 \text{ and } N > 0$$

It is also known as fundamental logarithmic identity.

Its domain is (0, ∞) and range is R. a is called the base of the logarithmic function.

When base is 'e' then the logarithmic function is called **natural** or **Napierian logarithmic function** and when base is 10, then it is called common logarithmic function.

Characteristic and mantissa

1. The integral part of a logarithm is called the characteristic and the fractional part is called mantissa.

$$\log_{10} N = \underbrace{\text{integer}}_{\text{Characters tics}} + \underbrace{\text{fraction}(+ve)}_{\text{Mantissa}}$$

2. The mantissa part of log of a number is always kept positive.
3. If the characteristics of $\log_{10}$ N be n, then the number of digits in N is (n+1).
4. If the characteristics of $\log_{10}$ N be (– n) then there exists (n – 1) number of zeros after decimal part of N.

Properties of logarithms

Let *m* and *n* be arbitrary positive numbers such that a > 0, a ≠ 1, b > 0, b ≠ 1 then

$$(1)\ \log_a a = 1,\ \log_a 1 = 0$$

$$(2)\ \log_a b . \log_b a = 1 \Rightarrow \log_a b = \frac{1}{\log_b a}$$

$$(3)\ \log_c a = \log_b a.\ \log_c b \text{ or } \log_c a = \frac{\log_b a}{\log_b c}$$

$$(4)\ \log_a (mn) = \log_a m + \log_a n$$

$$(5)\ \log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$(6)\ \log_a m^n = n \log_a m \qquad (7)\ a^{\log_a m} = m$$

$$(8)\ \log_a \left(\frac{1}{n}\right) = -\log_a n \qquad (9)\ \log_{a^\beta} n = \frac{1}{\beta} \log_a n$$

$$(10)\ \log_{a^\beta} n^\alpha = \frac{\alpha}{\beta} \log_a n\ ,\ (\beta \neq 0)$$

$$(11)\ a^{\log_c b} = b^{\log_c a}\ ,\ (a, b, c > 0 \text{ and } c \neq 1)$$

Logarithmic inequalities

$$(1)\ \text{If } a > 1, p > 1 \Rightarrow \log_a p > 0$$

$$(2)\ \text{If } 0 < a < 1, p > 1 \Rightarrow \log_a p < 0$$

$$(3)\ \text{If } a > 1,\ 0 < p < 1 \Rightarrow \log_a p < 0$$

$$(4)\ \text{If } p > a > 1 \Rightarrow \log_a p > 1$$

$$(5)\ \text{If } a > p > 1 \Rightarrow 0 < \log_a p < 1$$

$$(6)\ \text{If } 0 < a < p < 1 \Rightarrow 0 < \log_a p < 1$$

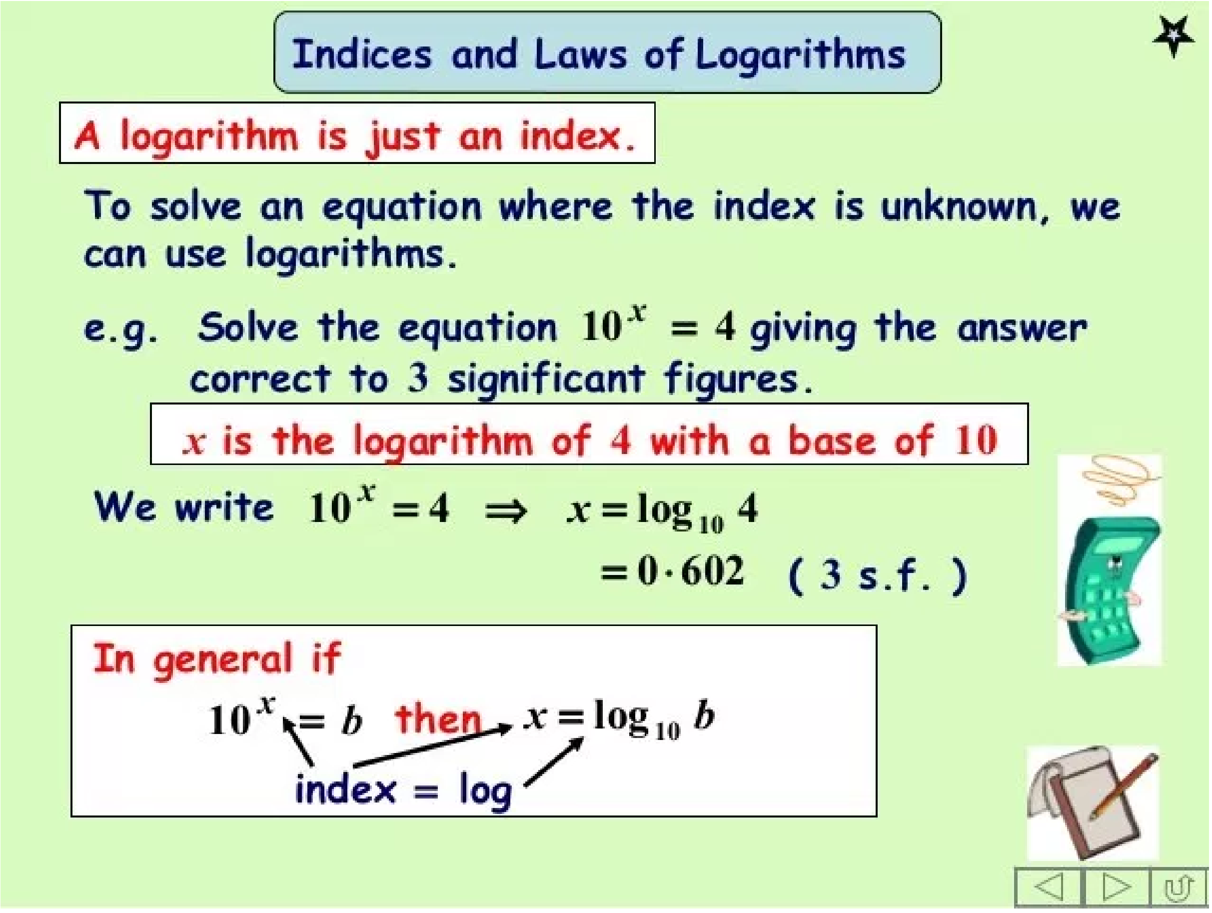
$$(7)\ \text{If } 0 < p < a < 1 \Rightarrow \log_a p > 1$$

$$(8)\ \text{If } \log_m a > b \Rightarrow \begin{cases} a > m^b, & \text{if } m > 1 \\ a < m^b, & \text{if } 0 < m < 1 \end{cases}$$

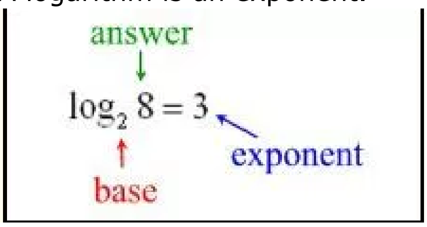
$$(9)\ \log_m a < b \Rightarrow \begin{cases} a < m^b, & \text{if } m > 1 \\ a > m^b, & \text{if } 0 < m < 1 \end{cases}$$

(10) $\log_p a > \log_p b \Rightarrow a \geq b$ if base *p* is positive and >1 or $a \leq b$ if base *p* is positive and < 1 *i.e.,* $0 < p < 1$.
In other words, if base is greater than 1 then inequality remains same and if base is positive but less than 1 then the sign of inequality is reversed.

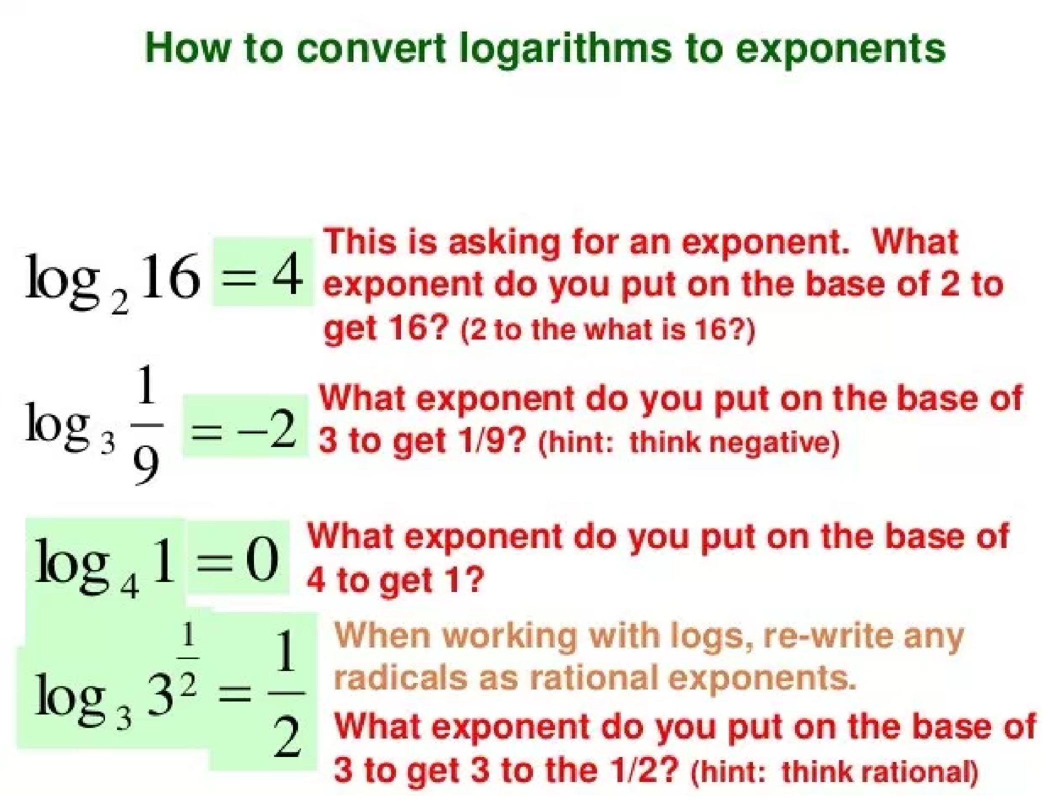
Logarithmic Expressions



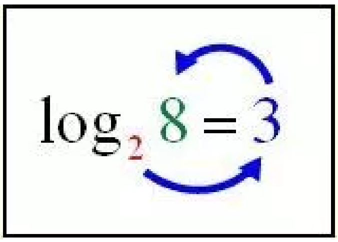
A logarithm is an exponent.



In the example shown above, 3 is the exponent to which the base 2 must be raised to create the answer of 8, or 2^3 = = 8. In this example, 8 is called the anitlogarithm base 2 of 3.



Try to remember the “spiral” relationship between the values as shown at the right. Follow the arrows starting with base 2 to get the equivalent exponential form.



Logarithms with base 10 are called common logarithms.
When the base is not indicated, base 10 is implied.

Logarithms with base e are called natural logarithms.
Natural logarithms are denoted by ln.

Operation	Laws of exponents	Laws of logs
Multiplication	$x^m \cdot x^n = x^{m+n}$	$\log(a \cdot b) = \log(a) + \log(b)$
Division	$\frac{x^m}{x^n} = x^{m-n}$	$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$
Exponentiation	$(x^m)^n = x^{mn}$	$\log(a^n) = n \cdot \log(a)$ One of the most useful properties of logs
Zero property	$x^0 = 1$	$\log(1) = 0$
Inverse	$x^{-1} = \frac{1}{x}$	$\log(x^{-1}) = \log\left(\frac{1}{x}\right) = -\log(x)$

On the graphing calculator:

Origins of Change of Base Formula:

$\log_b a = x$	Set $= x$.
$b^x = a$	Convert to exponential form.
$\log b^x = \log a$	Take common log of both sides.
$x \log b = \log a$	Use power rule.
$x = \frac{\log a}{\log b}$	Divide by $\log b$.

Change of Base Formula:

$$\log_b a = \frac{\log a}{\log b} = \frac{\ln a}{\ln b}$$

The base 10 logarithm is the log key.
The base e logarithm is the ln key.
To enter a logarithm with a different base,
use the Change of Base Formula:

$$\log_b x = \frac{\log x}{\log b}$$

Properties of Logs:

Using the properties of exponents, we can arrive at the properties of logarithms.

Properties of Exponents:

$$b^n \bullet b^m = b^{n+m}$$

$$\frac{b^n}{b^m} = b^{n-m}$$

$$\left(b^n\right)^m = b^{nm}$$

Let's see the connection:

Let $x = \log_b m$ and $y = \log_b n$

Using exponential form:

$b^x = m$ and $b^y = n$

Multiplication gives $mn = b^x b^y = b^{x+y}$

Returning to logarithmic form:

$\log_b(mn) = x + y$

$\log_b(mn) = \log_b m + \log_b n$

Similar investigations lead to the other logarithmic properties.

Properties of Logs:

$$\log_b(m \bullet n) = \log_b m + \log_b n$$

$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$$

$$\log_b m^r = r\log_b m$$

Also: $\log_b b = 1$ and $\log_b 1 = 0$

These log properties remain the same when working with the **natural log**:

$$\ln(ab) = \ln a + \ln b$$

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$\ln a^b = b \ln a$$

Remember: $\ln 1 = 0$ and $\ln e = 1$

Examples:

1.	Write in exponential form: $\log_2 64 = 6$	Answer: $2^6 = 64$
2.	Write in logarithmic form: $3^{-1} = \frac{1}{3}$	Answer: $\log_3\left(\frac{1}{3}\right) = -1$
3.	Evaluate: $\log_4 1$	Answer: $4^{?} = 1;\ ? = 0$ If using your calculator, remember to use the change of base formula and enter log 1 / log 4.
4.	What is the value of x? $\log_2 x = 5$	Answer: $2^5 = x$; $x = 32$
5.	Write in expanded form: $\log \frac{a\sqrt{b}}{c^6}$ (Apply the "properties of logs" rules.)	Answer: $\log\left(a\sqrt{b}\right) - \log c^6$ $\log a + \log\sqrt{b} - \log c^6$ $\log a + \log b^{\frac{1}{2}} - \log c^6$ $\log a + \frac{1}{2}\log b - 6\log c$
6.	Write in expanded form: $\ln\sqrt{\sin x \bullet \cos x}$	Answer: $\ln(\sin x \bullet \cos x)^{\frac{1}{2}}$ $\frac{1}{2}\ln(\sin x \bullet \cos x)$ $\frac{1}{2}\left(\ln(\sin x) + \ln(\cos x)\right)$
7.	Express as a single logarithm: $(\log x + 4\log y) - 5\log z$ (Apply the "properties of logs" rules in reverse.)	Answer: $\log\left(\frac{xy^4}{z^5}\right)$
8.	Express as a single logarithm: $\frac{1}{2}[(4\ln a + \ln b) - 4\ln c]$	Answer: $\ln\sqrt{\frac{a^4 b}{c^4}}$
9.	Using properties of logs, show that $\ln 4 = \ln\left(\frac{1}{4}\right)^{-1}$	Answer: $\ln\left(\frac{1}{4}\right)^{-1}$ $= -1(\ln 1 - \ln 4)$ $= -(0 - \ln 4)$ $= \ln 4$
10.	Using properties of logs, solve for x: $\log_3 x = \log_3 4 + \log_3 7$	Answer: $\log_3 4 + \log_3 7$ $= \log_3(4 \bullet 7) = \log_3 28$ $x = 28$

Logarithmic Equations

A logarithmic equation can be solved using the properties of logarithms along with the use of a common base.

Properties of Logs:

$$\log_b(m \cdot n) = \log_b m + \log_b n$$
$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$$
$$\log_b m^r = r\log_b m$$

$$\ln(ab) = \ln a + \ln b$$
$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$
$$\ln a^b = b\ln a$$

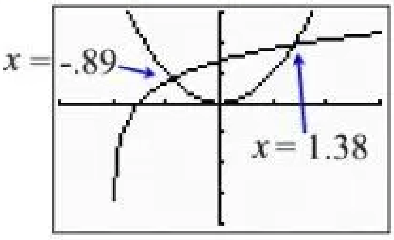
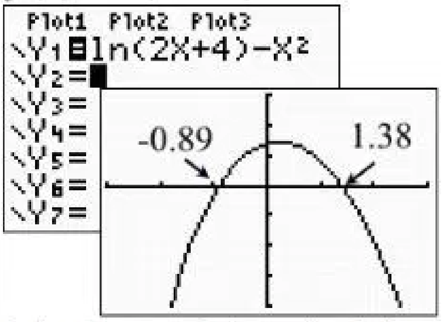
To solve most logarithmic equations:
1. Isolate the logarithmic expression.
(you may need to use the properties to create one logarithmic term)
2. Rewrite in exponential form
(with a common base)
3. Solve for the variable.

Things to remember about logs:

$\log_b 1 = 0$	$\ln 1 = 0$
$\log_b b = 1$	$\ln e = 1$
$\log_b b^x = x$	$\ln e^x = x$

Examples:

	Solve for x:	Answer:
1.	$3\log(x+4) = 6$	ANSWER: $3\log(x+4) = 6$ • Isolate the log expression $\log(x+4) = 2$ • Choose base 10 to correspond with log (base 10) $10^{\log(x+4)} = 10^2$ $x+4 = 100$ • Apply composition of inverses and solve. $x = 96$
2.	$\ln x = 4$	ANSWER: $\ln x = 4$ • Remember that e^x and $\ln x$ are inverse functions. $e^{\ln x} = e^4$ $x = e^4$
3.	$\log_5(x+1) = 2$	ANSWER: $\log_5(x+1) = 2$ $5^{\log_5(x+1)} = 5^2$ $x+1 = 25$ $x = 24$
4.	$2\ln(3x) = 18$	ANSWER: $2\ln(3x) = 18$ • Isolate the logarithmic expression first. $\ln(3x) = 9$ $e^{\ln(3x)} = e^9$ $3x = e^9$ $x = \frac{e^9}{3}$

5.	$\log_9 x + \log_9(x-8) = 1$	ANSWER: $\log_9 x + \log_9(x-8) = 1$ $\log_9 [x(x-8)] = 1$ $9^{\log_9[x(x-8)]} = 9^1$ $x(x-8) = 9$ $x^2 - 8x - 9 = 0$ $(x-9)(x+1) = 0$ $x - 9 = 0 \quad x + 1 = 0$ $x = 9 \quad$ ~~$x = -1$~~ • Use the log property to express the two terms on the left as a single term. • Remember that log of a negative value is not a real number and is not considered a solution.
6.	$\ln(2x-3) + \ln(x+4) = \ln(2x^2+11)$	ANSWER: $\ln(2x-3) + \ln(x+4) = \ln(2x^2+11)$ $\ln(2x-3)(x+4) = \ln(2x^2+11)$ $e^{\ln(2x-3)(x+4)} = e^{\ln(2x^2+11)}$ $(2x-3)(x+4) = 2x^2 + 11$ ~~$2x^2$~~ $+ 5x - 12 =$ ~~$2x^2$~~ $+ 11$ $5x - 12 = 11$ $5x = 23$ $x = \frac{23}{5}$
7.	Using your graphing calculator, solve for x to the *nearest hundredth.* $\ln(2x+4) = x^2$ **Method 2:** Place the left side of the equation into Y_1 and the right side into Y_2. Under the CALC menu, use #5 Intersect to find where the two graphs intersect. 	ANSWER: **Method 1:** Rewrite so the equation equals zero. $\ln(2x+4) - x^2 = 0$ Find the zeros of the function. $f(x) = \ln(2x+4) - x^2$  Both values are solutions, since both values allow for the ln of a positive value.