Binary Operations

1 Mark Questions

1. Let $* : R \times R \to R$ given by $(a,b) \to a + 4b^2$ be a binary operation. Compute $(-5) * (2 * 0)$.

All India 2014C

$$(-5) * (2 * 0) = (-5) * (2 + 4(0)^2)$$

$$[\because (ab) \to a + 4b^2]$$

$$= (-5) * (2) = -5 + 4(2)^2 = -5 + 16 = 11 \quad \textbf{(1)}$$

2. Let $*$ is a binary operation on the set of all non-zero real numbers, given by $a * b = \frac{ab}{5}$ for all $a, b \in R - \{0\}$. Find the value of x, given that $2 * (x * 5) = 10$.

Delhi 2014

$$\text{Given, } a * b = \frac{ab}{5}, \forall\, a, b \in R - \{0\} \quad \ldots\text{(i)}$$

$$\text{Also given, } \quad 2 * (x * 5) = 10$$

$$\Rightarrow \quad 2 * \left(\frac{x \cdot 5}{5}\right) = 10 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad 2 * x = 10 \Rightarrow \frac{2x}{5} = 10 \Rightarrow x = 25 \quad \textbf{(1)}$$

3. Let * is a binary operation on N given by $a*b =$ LCM (a, b) for all $a, b \in N$. Find $5*7$.

Delhi 2012; Foreign 2008

Given, $a * b = \text{LCM}(a, b), \forall\ a, b \in N$

$\therefore$ $5 * 7 = \text{LCM}(5, 7) = 35$

4. Let $*: R \times R \to R$ is defined as $a * b = 2a + b$. Find $(2 * 3) * 4$.

All India 2012

Given, $* : R \times R \to R$

such that $a * b = 2a + b$.

On putting $a = 2$ and $b = 3$, we get

$$(2 * 3) = 2(2) + 3 = 4 + 3 = 7$$

$\therefore$ $(2 * 3) * 4 = 7 * 4 = 2(7) + 4 = 14 + 4 = 18$ **(1)**

5. If the binary operation * on the set of integers Z, is defined by $a * b = a + 3b^2$, then find the value of $8 * 3$.

All India 2012C

Given, $a * b = a + 3b^2, \forall\, a, b \in z$

On putting $a = 8$ and $b = 3$, we get

$$8 * 3 = 8 + 3 \cdot 3^2 = 8 + 27 = 35$$

6. Let * is a binary operation on set of integers I defined by $a * b = 3a + 4b - 2$, then find the value of $4 * 5$. **All India 2011C**

Given, $a * b = 3a + 4b - 2$

On putting $a = 4$ and $b = 5$, we get

$$4 * 5 = 3(4) + 4(5) - 2$$
$$= 12 + 20 - 2 = 30$$

7. Let * is a binary operation on set of integers I, defined by $a * b = 2a + b - 3$. Find value of $3 * 4$. **Delhi 2011C; All India 2008**

Given, $a * b = 2a + b - 3$

On putting $a = 3$ and $b = 4$, we get

$$3 * 4 = 2(3) + 4 - 3$$
$$= 6 + 4 - 3 = 7$$

8. If the binary operation * on set of integers Z is defined by $a * b = a + 3b^2$, then find the value of $2 * 4$. **Delhi 2009**

Do same as Que 5. **[Ans. 50]**

9. Let * is the binary operation on N given by $a * b = \text{HCF}(a, b)$ where, $a, b \in N$. Write the value of $22 * 4$. **All India 2009**

Given, $a * b =$ HCF of a and b, where a and $b \in N$.

Now, $22 * 4 =$ HCF of 22 and 4

$= \text{HCF of } (2 \times 11) \text{ and } (2 \times 2) = 2$

$\therefore \quad 22 * 4 = 2$ **(1)**

10. If the binary operation *, defined on Q, is defined as $a * b = 2a + b - ab$, for all as $b \in Q$. Find the value of $3 * 4$. **Foreign 2009**

Given, $a * b = 2a + b - ab, \forall a, b \in Q$.

On putting $a = 3$ and $b = 4$, we get

$$3 * 4 = 2 \cdot 3 + 4 - 3 \cdot 4$$
$$= 6 + 4 - 12 = -2$$

11. If * is a binary operation on set Q of rational numbers defined as $a * b = \frac{ab}{5}$. Write the identity for *, if any. **All India 2009C; HOTS**

Given, binary operation is $a * b = \frac{ab}{5}$.

Let e be the identity element of * on Q.

Then, $a * e = a, \forall a \in Q$

[by definition of identity element]

12. If S is the set of all rational numbers except 1 and * be defined on S by $a * b = a + b - ab$, for all $a, b \in$ s.

Prove that

(i) * is a binary operation on S.

(ii) * is commutative as well as associative.

Delhi 2014C

(i) We know that, addition of two rational numbers is a rational number. Also, multiplication of two rational numbers is also a rational number.

Here, a and b are rational numbers other than 1. So, $a + b - ab$ is also a rational number [since difference of two rational numbers is rational number]. So, * is a binary operation on set S. **(1)**

(ii) **Commutative**

$$a * b = a + b - ab = b + a - ba$$

$\Rightarrow \quad a * b = b * a$

Hence, * is commutative. **(1)**

Associative $(a * b) * c$

$= (a + b - ab) * c$

$= a + b - ab + c - (a + b - ab)c$

$= a + b + c - ab - bc - ac + abc$ ...(i) **(1)**

and $a * (b * c) = a * (b + c - bc)$

$= a + b + c - bc - a(b + c - bc)$

$= a + b + c - ab - bc - (ac + abc)$ ...(ii)

From Eqs. (i) and (ii), we get

$$(a * b) * c = a * (a * c)$$

Hence, * is associative. **(1)**

13. Consider the binary operations $* : R \times R \rightarrow R$ and $o : R \times R \rightarrow R$ defined as $a * b = |a - b|$ and $a o b = a$. For all $a, b \in R$. Show that * is commutative but not associative, 'o' is associative but not commutative.

All India 2012

Given $* : R \times R \to R$ such that $a * b = |a - b|$ and $a \, o \, b = a, \forall \, a, b \in R$.

We have to show that, * is commutative but not associative.

(i) **Commutative**

$$a * b = |a - b|, \forall \, a, b \in R \qquad \text{[given]}$$

and $b * a = |b - a| \forall \, a, b \in R$

$= |-(a - b)|$

$= |a - b| \; [\because |-x| = |x|, \forall \, x \in R]$

Thus, $a * b = b * a, \forall \, a, b \in R$

Hence, * is commutative. **(1)**

14. Consider the binary operation * on the set $\{1, 2, 3, 4, 5\}$ defined by $a * b = \min\{a, b\}$. Write operation table of operation *.

Delhi 2011

Given, binary operation is $a * b = \min\{a, b\}$ defined on the set $\{1, 2, 3, 4, 5\}$. **(1/2)**

The operation table for operation * is given as follows:

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

$[\because 1 * 1 = \min\{1, 1\} = 1, 1 * 2 = \min\{1, 2\} = 1$

$\dots 5 * 4 = \min\{5, 4\} = 4,$

$5 * 5 = \min\{5, 5\} = 5]$ **(3½)**

15. A binary operation * on the set {0, 1, 2, 3, 4, 5} is defined as $a * b = \begin{cases} a+b, & \text{if } a+b<6 \\ a+b-6, & \text{if } a+b \geq 6 \end{cases}$.

Show that zero is the identity for this operation and each element 'a' of the set is invertible with $6-a$, being the inverse of 'a'.

All India 2011; HOTS

Given $a * b = \begin{cases} a+b, & \text{if } a+b<6 \\ a+b-6, & \text{if } a+b \geq 6 \end{cases}$

The operation table for * is as follows:

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

(2)

$[\because 0+0<6 \Rightarrow 0+0=0; 0+1<6 \Rightarrow 0+1=1; \ldots 1+4<6 \Rightarrow 1+4=5; 1+5 \geq 6 \Rightarrow 1+5-6=0; \ldots]$

From table, we note that

(i) $a * 0 = 0 * a = a$. Hence, 0 is the identity for an operation. **(1)**

(ii) $1 * 5 = 0, 2 * 4 = 0, 3 * 3 = 0, 4 * 2 = 0, 5 * 1 = 0$

Hence, inverse of 1 is 5, i.e.for element a, $6-a$ is its inverse. **(1)**

16. If * is a binary operation on Q, defined by $a*b = 3ab/5$. Show that * is commutative as well as associative. Also, find its identity, if it exists. **Delhi 2010**

Given, binary operation is

$$a * b = \frac{3ab}{5}, a, b \in Q$$

(i) **Commutative** $a * b = \frac{3ab}{5}$ [given]

and $b * a = \frac{3ba}{5}$

[by using definition of *]

$\because \quad \frac{3ab}{5} = \frac{3ba}{5}, \forall a, b \in Q$ is true

$\therefore \quad a * b = b * a, \forall a, b \in Q$

Therefore, * is commutative. **(1)**

(ii) **Associative**

$$a * (b * c) = a * \left(\frac{3bc}{5}\right) \left[\text{using } a * b = \frac{3ab}{5}\right]$$

$$= \frac{3a\left(\frac{3bc}{5}\right)}{5} = \frac{9abc}{25}$$

and $(a * b) * c = \left(\frac{3ab}{5}\right) * c$

$$\left[\because a * b = \frac{3ab}{5}, \forall a, b \in Q\right]$$

$$= \frac{3\left(\frac{3ab}{5}\right)(c)}{5} = \frac{9abc}{25}$$

Clearly, $a * (b * c) = (a * b) * c,$

$\forall\ a, b, c \in Q$

Therefore, * is associative. **(2)**

(iii) **Existence of identity** Let e be the identity element of * on Q. Then, by definition of identity element, we must have

$$a * e = e * a = a, \forall\ a \in Q$$

Let $a * e = a, \forall\ a \in Q$

$\Rightarrow \quad \frac{3ae}{5} = a \quad \left[\because a * b = \frac{3\,ab}{5}\right]$

$\Rightarrow \quad e = \frac{5}{3} \in Q$

$\therefore e = \frac{5}{3}$ is the identity element of * defined on Q. **(1)**

17. If $A = N \times N$ and * is a binary operation on A defined by $(a, b) * (c, d) = (a + c, b + d)$. Show that * is commutative and associative. Also, find identity element for * on A, if any.

Foreign 2010

The given binary operation is

$$(a, b) * (c, d) = (a + c, b + d)$$

defined on $A = N \times N$, we have to show that * is commutative and associative.

(i) **Commutative**

$(a, b) * (c, d) = (a + c, b + d),$

$\forall (a, b) (c, d) \in N \times N$ [given]...(i)

Also, $(c, d) * (a, b) = (c + a, d + b)$

$\forall (a, b), (c, d) \in N \times N$ ... (ii)

Since, $a + c = c + a, \forall a, c \in N$

and $b + d = d + b, \forall b, d \in N$

From Eqs. (i) and (ii), we get

$(a + c, b + d) = (c + a, b + d),$

$\forall a, b, c, d \in N$

$\Rightarrow (a, b) * (c, d) = (c, d) * (a, b),$

$\forall (a, b) (c, d) \in N \times N$

Therefore, * is commutative. **(1)**

(ii) **Associative** $(a, b) * [(c, d) * (e, f)]$

$= (a, b) * (c + e, d + f)$

[using given definition of *]

$= (a + c + e, b + d + f)$ ...(i)

Also, $[(a, b) * (c, d)] * (e, f)$

$= (a + c, b + d) * (e, f)$

[using definition of *]

$= (a + c + e, b + d + f)$ ...(ii)

From Eqs. (i) and (ii), we get

$(a, b) * [(c, d) * (e, f)] = [(a, b) * (c, d)] * (e, f)$

$\forall (a, b), (c, d), (e, f) \in N \times N$ **(2)**

Therefore * is associative.

(iii) Now, we check the existence of identity of the given operation *.

Let if possible (s, t) be the identity element of the operation *. Then, by definition of identity, we must have

$(a, b) * (s, t) = (a, b), \forall (a, b), (s, t) \in N \times N$

$\Rightarrow \quad (a + s, b + t) = (a, b)$

$[\because (a, b) * (c, d) = (a + c, b + d)$ is given$]$

On equating corresponding elements, we get

$a + s = a$ ...(i)

and $b + t = b$ ...(ii)

Eqs. (i) and (ii) are true, when $s = 0$ and $t = 0$

$\therefore \quad (s, t) = (0, 0)$

But $(0, 0) \notin N \times N$

$\Rightarrow$ Identity of the above operation * does not exist as there does not exist any $(s, t) \in N \times N$ such that

$(a, b) * (s, t) = (a, b), \forall (a, b) \in N \times N$ **(1)**

18. If * is the binary operation on N given by $a * b =$ LCM of a and b. Find 20 * 16. Is * (i) commutative and (ii) associative?

All India 2008C

Given $a * b =$ LCM of a and b

LCM of 20 and 16 = 80

$\therefore \quad 20 * 16 = 80$ **(1)**

(i) **Commutative**

$$a * b = \text{LCM of } a \text{ and } b \qquad \text{[given]}$$

and $\quad b * a = \text{LCM of } b \text{ and } a$

$$= \text{LCM of } (b \text{ and } a), \forall\ a, b \in Q$$

and $\quad a * b = b * a, \forall\ a, b \in Q$

Therefore, * is commutative. **(1½)**

(ii) **Associative** $a * (b * c) = a * (\text{LCM of } b \text{ and } c)$

$$[\because\ a * b = \text{LCM of } a \text{ and } b]$$

$\Rightarrow \quad a * (b * c) = \text{LCM of } (a, b \text{ and } c)$ ...(i)

and $\quad (a * b) * c = (\text{LCM of } a \text{ and } b) * c$

$\Rightarrow \quad (a * b) * c = \text{LCM of } (a, b \text{ and } c)$ ...(ii)

From Eqs. (i) and (ii), we get

$$a * (b * c) = (a * b) * c, \forall\ a, b, c, \in Q$$

Therefore, * is associative. **(1½)**

19. If * is a binary operation on set Q of rational numbers such that $a * b = (2a - b)^2, a, b \in Q$. Find 3 * 5, 5 * 3. Is 3 * 5 = 5 * 3? **Delhi 2008C**

The given binary operation is

$$a * b = (2a - b)^2,\ a, b \in Q$$

$$\therefore \quad 3 * 5 = [2(3) - 5]^2$$

$$[\text{put } a = 3 \text{ and } b = 5 \text{ in } a * b = (2a - b)^2]$$

$$= (6 - 5)^2 = (1)^2 = 1 \qquad \textbf{(1½)}$$

Also, $\quad 5 * 3 = [2(5) - 3]^2$

$$[\text{put } a = 5 \text{ and } b = 3 \text{ in } a * b = (2a - b)^2]$$

$$= (10 - 3)^2 = (7)^2 = 49 \qquad \textbf{(1½)}$$

Clearly, from above $3 * 5 \neq 5 * 3$ **(1)**