Long Answer Questions-I (PYQ)

[4 Mark]

Q.1. Consider the binary operation * on the set {1, 2, 3, 4, 5} defined by $a * b$ = min. {a, b}. Write the operation table of the operation *.

Ans. Required operation table of the operation * is given as

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Q.2. Show that the relation R in the set $N \times N$ defined by $(a, b)R(c, d)$ if $a^2 + d^2 = b^2 + c^2$ $\forall$ $a, b, c, d \in N$, is an equivalence relation.

Ans.

Given, R is a relation in $N \times N$ defined by $(a, b)\ R\ (c, d) \Rightarrow a^2 + d^2 = b^2 + c^2$

Reflexivity:

$\because \quad a^2 + b^2 = b^2 + a^2\ \forall\ a, b \in N$

$\Rightarrow \quad (a, b)\ R\ (a, b) \quad \Rightarrow \quad R$ is reflexive

Symmetry: Let $(a, b) R (c, d)$

$\Rightarrow \quad a^2 + d^2 = b^2 + c^2 \quad \Rightarrow \quad b^2 + c^2 = a^2 + d^2 \quad \Rightarrow \quad c^2 + b^2 = d^2 + a^2$

$\Rightarrow \quad (c, d)\ R\ (a, b) \quad \Rightarrow \quad R$ is symmetric

Transitivity: Let $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

$\Rightarrow \quad a^2 + d^2 = b^2 + c^2$ and $c^2 + f^2 = d^2 + e^2 \quad \Rightarrow \quad a^2 + d^2 + c^2 + f^2 = b^2 + c^2 + d^2 + e^2$

$\Rightarrow \quad a^2 + f^2 = b^2 + e^2 \quad \Rightarrow \quad (a, b)\ R\ (e, f)$

$\Rightarrow \quad R$ is transitive.

Hence, R is an equivalence relation.

[6 Mark]

Q.1. Consider $f : R_+ \to [-9, \infty]$ given by $f(x) = 5x^2 + 6x - 9$. Prove that f is invertible with $f^{-1}(y) = \left(\frac{\sqrt{54+5y} - 3}{5}\right)$.

Ans.

To prove f is invertible, it is sufficient to prove f is one–one onto

Here, $f(x) = 5x^2 + 6x - 9$

One-one: Let $x_1, x_2 \in R_+$, then

$$f(x_1) = f(x_2) \quad \Rightarrow \quad 5{x_1}^2 + 6x_1 - 9 = 5{x_2}^2 + 6x_2 - 9$$
$$\Rightarrow \quad 5{x_1}^2 + 6x_1 - 5{x_2}^2 - 6x_2 = 0 \quad \Rightarrow \quad 5({x_1}^2 - {x_2}^2) + 6(x_1 - x_2) = 0$$
$$\Rightarrow \quad 5(x_1 - x_2)(x_1 + x_2) + 6(x_1 - x_2) = 0 \quad \Rightarrow \quad (x_1 - x_2)(5x_1 + 5x_2 + 6) = 0$$
$$\Rightarrow \quad x_1 - x_2 = 0 \qquad [\because \quad 5x_1 + 5x_2 + 6 \neq 0]$$
$$\Rightarrow \quad x_1 = x_2$$

i.e., f is one-one function.

Onto: Let $f(x) = y$

$$\therefore \quad y = 5x^2 + 6x - 9 \quad \Rightarrow \quad 5x^2 + 6x - (9 + y) = 0$$
$$\Rightarrow \quad x = \frac{-6 \pm \sqrt{36 + 4 \times 5(9 + y)}}{10} \quad \Rightarrow \quad x = \frac{-6 \pm \sqrt{216 + 20y}}{10}$$
$$\Rightarrow \quad x = \frac{\pm\sqrt{54 + 5y} - 3}{5} \quad \Rightarrow \quad x = \frac{\sqrt{54 + 5y} - 3}{5} \qquad [\because \quad x \in R_+]$$

Obviously, $\forall\ y \in [-9, \infty]$ the value of $x \in R_+$

$\Rightarrow$ f is onto function.

Hence, f is one-one onto function, *i.e.*, invertible.

Also, f is invertible with

$$f^{-1}(y) = \frac{\sqrt{54 + 5y} - 3}{5}.$$

Q.2. Let *A* = *R* – {3} and *B* = *R* – {1}. Consider the function *f* : *A* → *B* defined by $f(x) = \left(\frac{x - 2}{x - 3}\right)$. Show that *f* is one-one and onto and hence find f^{-1}.

Ans.

One-one:

Let $x_1, x_2 \in A$

Now, $f(x_1) = f(x_2)$

$\Rightarrow \quad \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3} \quad \Rightarrow \quad (x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$

$\Rightarrow \quad x_1 x_2 - 3x_1 - 2x_2 + 6 = x_1 x_2 - 2x_1 - 3x_2 + 6 \quad \Rightarrow \quad -3x_1 - 2x_2 = -2x_1 - 3x_2$

$\Rightarrow \quad -x_1 = -x_2 \quad \Rightarrow \quad x_1 = x_2$

Hence, f is one-one function.

Onto:

Let $\quad y = \frac{x - 2}{x - 3} \quad \Rightarrow \quad xy - 3y = x - 2$

$\Rightarrow \quad xy - x = 3y - 2 \quad \Rightarrow \quad x(y - 1) = 3y - 2$

$\Rightarrow \quad x = \frac{3y - 2}{y - 1} \quad \ldots(i)$

From above it is obvious that $\forall\ y$ except 1, *i.e.*, $\forall\ y \in B = R - \{1\} \exists\ x \in A$

Hence, f is onto function.

Thus, f is one-one onto function.

If f^{-1} is inverse function of f then $f^{-1}(y) = \frac{3y - 2}{y - 1}$ [from (i)]

Q.3. Let $f : N \to N$ be defined by f(n)= $\begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$ for all $n \in N$. Find whether the function f is bijective.

Ans.

Given $f: N \rightarrow N$ defined such that $f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$

Let $x, y \in N$ and let they are odd then

$$f(x) = f(y) \quad \Rightarrow \quad \frac{x+1}{2} = \frac{y+1}{2} \quad \Rightarrow \quad x = y$$

If $x, y \in N$ are both even then also

$$f(x) = f(y) \Rightarrow \frac{x}{2} = \frac{y}{2} \Rightarrow x = y$$

If $x, y \in N$ are such that x is odd and y is even then

$$f(x) = \frac{x+1}{2} \quad \text{and} \quad f(y) = \frac{y}{2}$$

Thus, $x \neq y$ for $f(x) = f(y)$

Let $x = 6$ and $y = 5$

We get $f(6) = \frac{6}{2} = 3, f(5) = \frac{5+1}{2} = 3$

$\therefore \quad f(x) = f(y)$ but $x \neq y$

So, $f(x)$ is not one-one.

Hence, $f(x)$ is not bijective.

Q.4. Consider the binary operations * : *R* × *R* → *R* and *o* : *R* × *R* → *R* defined as *a* * *b* = |*a* – *b*| and *aob* = *a* for all *a*, *b* ∈ *R*. Show that '*' is commutative but not associative, '*o*' is associative but not commutative.

Ans.

For operation '*'

$$\text{'*'} : R \times R \rightarrow R \text{ such that } a * b = |a - b| \ \forall\ a, b \in R$$

Commutativity:

$\forall\ a, b \in R,\ a * b = |a - b| = |b - a| = b * a$

i.e., '*' is commutative

Associativity:

$\forall\ a, b, c \in R,\ (a * b) * c = |a - b| * c = ||a - b| - c|$

and $a * (b * c) = a * |b - c| = |a - |b - c||$

But $||a - b| - c| \neq |a - |b - c||$

$\Rightarrow$ $(a * b) * c \neq a * (b * c)$

$\Rightarrow$ * is not associative.

Hence, '*' is commutative but not associative.

For Operation 'o'

$$o : R \times R \rightarrow R \text{ such that } aob = a$$

Commutativity:

$\forall\ a, b \in R,\ aob = a$ and $boa = b$ $\qquad \because \quad a \neq b \ \Rightarrow\ aob \neq boa$

$\Rightarrow \qquad$ 'o' is not commutative.

Associativity:

$\forall\ a, b, c \in R,\ (aob)\ oc = aoc = a$

$\Rightarrow \qquad ao(boc) = aob = a \qquad \Rightarrow \quad (aob)\ oc = ao\ (boc)$

$\Rightarrow \qquad$ 'o' is associative

Hence 'o' is not commutative but associative.

Q.5. If $f, g : R \rightarrow R$ be two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x$, $\forall\ x \in R$. Then find fog and gof. Hence find $fog\ (-3)$, $fog(5)$ and $gof\ (-2)$.

Ans.

Here, $f(x) = |x| + x$ can be written as

$$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

And $g(x) = |x| - x$, can be written as

$$g(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$

Therefore, gof is defined as

For $x \geq 0$, $gof(x) = g(f(x))$ $\Rightarrow$ $gof(x) = g(2x) = 0$

and for $x < 0$, $gof(x) = g(f(x)) = g(0) = 0$

Hence, $gof(x) = 0 \ \forall \ x \in R$.

Again, fog is defined as

For $x \geq 0$, $fog(x) = f(g(x)) = f(0) = 0$

and for $x < 0$, $fog(x) = f(g(x)) = f(-2x) = 2(-2x) = -4x$

Hence,

2nd part

$fog(5) = 0$ $[\because \ 5 \geq 0]$

$fog(-3) = -4 \times (-3) = 12$ $[\because -3 < 0]$

$gof(-2) = 0$

Q.6. Show that the relation *R* on the set *A* = {*x* ∈ *Z* : 0 ≤ *x* ≤ 12}, given by *R* = {(*a*, *b*) : |*a* – *b*| is *a* multiple of 4} is an equivalence relation.

Ans.

We have the given relation

$R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$, where $a, b \in A$ and $A = \{x \in Z : 0 \leq x \leq 12\} = \{0, 1, 2, \ldots\ldots, 12\}$.

We discuss the following properties of relation R on set A.

Reflexivity: For any $a \in A$ we have

$|a - a| = 0$, which is multiple of 4

$(a, a) \in R$ for all $a \in R$.

So, R is reflexive.

Symmetry: Let $(a, b) \in R$.

$\Rightarrow \quad |a - b|$ is divisible by 4 $\quad \Rightarrow \quad |a - b| = 4k \quad [\text{Where } k \in Z]$

$\Rightarrow \quad a - b = \pm 4k \quad \Rightarrow \quad b - a = \mp 4k$

$\Rightarrow \quad |b - a| = 4k \quad \Rightarrow \quad |b - a|$ is divisible by 4

$\Rightarrow \quad (b - a) \in R$

So, R is Symmetric

Transitivity: Let $a, b, c \in A$ such that $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow \quad |a - b|$ is multiple of 4 $\quad$ and $\quad |b - c|$ is multiple of 4.

$\Rightarrow \quad |a - b| = 4m \quad$ and $\quad |b - c| = 4n,\ m, n \in N$

$\Rightarrow \quad a - b = \pm 4m \quad$ and $\quad |a - c| = \pm 4n$

$\therefore \quad (a - b) + (b - c) = \pm 4(m + n)$

$\Rightarrow \quad a - c = \pm 4(m + n) \quad \Rightarrow \quad |a - c| = 4(m + n)$

$\Rightarrow \quad |a - c|$ is a multiple of 4 $\quad \Rightarrow \quad (a, c) \in R$

Thus, $(a, b) \in R$ and $(b, c) \in R \quad \Rightarrow \quad (a, c) \in R$.

So, R is transitive.

Hence, R is an equivalence relation.

Q.7. Let *N* denote the set of all natural numbers and *R* be the relation on *N* × *N* defined by (*a*, *b*) *R* (*c*, *d*) if *ad*(*b* + *c*) = *bc*(*a* + *d*). Show that *R* is an equivalence relation.

Ans.

Here R is a relation defined as

$$R = \{[(a, b), (c, d)] : ad(b + c) = bc(a + d)\}$$

Reflexivity: By commutative law under addition and multiplication

$$b + a = a + b \ \forall \ a, b \in N$$

$$ab = ba \ \forall \ a, b \in N$$

$\therefore$ $$ab(b + a) = ba(a + b) \ \forall \ a, b \in N$$

$(a, b) \ R \ (a, b)$ Hence, R is reflexive

Symmetry: Let $(a, b) \ R \ (c, d)$

$$(a, b) \ R \ (c, d) \Rightarrow ad(b + c) = bc(a + d)$$

$$\Rightarrow bc(a + d) = ad(b + c)$$

$$\Rightarrow cb(d + a) = da(c + b)$$

[By commutative law under addition and multiplication]

$$\Rightarrow (c, d) \ R \ (a, b)$$

Hence, R is symmetric.

Transitivity: Let $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

Now, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

$\Rightarrow \quad ad(b + c) = bc(a + d)$ and $cf(d + e) = de(c + f)$

$\Rightarrow \quad \frac{b+c}{bc} = \frac{a+d}{ad}$ and $\frac{d+e}{de} = \frac{c+f}{cf}$

$\Rightarrow \quad \frac{1}{c} + \frac{1}{b} = \frac{1}{d} + \frac{1}{a}$ and $\frac{1}{e} + \frac{1}{d} = \frac{1}{f} + \frac{1}{c}$

Adding both, we get

$\Rightarrow \quad \frac{1}{c} + \frac{1}{b} + \frac{1}{e} + \frac{1}{d} = \frac{1}{d} + \frac{1}{a} + \frac{1}{f} + \frac{1}{c}$

$\Rightarrow \quad \frac{1}{b} + \frac{1}{e} = \frac{1}{a} + \frac{1}{f} \quad \Rightarrow \frac{e+b}{be} = \frac{f+a}{af}$

$\Rightarrow \quad af\,(b+e) = be\,(a+f) \quad \Rightarrow (a,b)R(e,f) \qquad [c,\ d \neq 0]$

Hence, R is transitive.

In this way, R is reflexive, symmetric and transitive.

Therefore, R is an equivalence relation.

Q.8. Consider $f : R_+ \to [4, \infty]$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse (f^{-1}) of f given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

Ans.

One-one: Let $x_1, x_2 \in R$ (Domain)

$$f(x_1) = f(x_2) \quad \Rightarrow \quad {x_1}^2 + 4 = {x_2}^2 + 4$$

$$\Rightarrow \quad {x_1}^2 = {x_2}^2$$

$$\Rightarrow \quad x_1 = x_2 \qquad [\therefore x_1, x_2 \text{ are +ve real number}]$$

Hence, f is one-one function.

Onto: Let $y \in [4, \infty)$ such that

$$y = f(x) \ \forall \ x \in R_+ \qquad \text{(set of non-negative reals)}$$

$$\Rightarrow \quad y = x^2 + 4$$

$$\Rightarrow \quad x = \sqrt{y - 4} \qquad [\therefore \text{x is + ve real number}]$$

Obviously, $\forall \ y \in [4, \infty)$, x is real number $\in R$ (domain)

i.e., all elements of codomain have pre image in domain.

$\Rightarrow$ f is onto.

Hence, f is invertible being one-one onto.
Inverse function: If f^{-1} is inverse of f, then

$$fof^{-1} = I \qquad \text{(Identity function)}$$

$$\Rightarrow \quad fof^{-1}(y) = y \ \forall \ y \in [4, \infty)$$

$$\Rightarrow \quad f(f^{-1}(y)) = y$$

$$\Rightarrow \quad (f^{-1}(y))^2 + 4 = y \qquad [\because f(x) = x^2 + 4]$$

$$\Rightarrow \quad f^{-1}(y) = \sqrt{y - 4}$$

Therefore, required inverse function is $f^{-1}[4, \infty) \to R$ defined by

$$f^{-1}(y) = \sqrt{y - 4} \quad \forall \ y \in [4, \infty)$$

Q.9. Determine whether the relation *R* defined on the set *R* of all real numbers as *R* = {(*a*, *b*) : *a*, *b* ∈ *R* and *a* – *b* + 3–√ ∈ *S*, where *S* is the set of all irrational numbers}, is reflexive, symmetric and transitive.

Ans.

Here, relation R defined on the set R is given as

$$R = \{(a, b) : a, b \in R \text{ and } a - b + \sqrt{3} \in S\}$$

Reflexivity: Let $a \in R$ (set of real numbers)

Now, $(a, a) \in R$ as $a - a + \sqrt{3} = \sqrt{3} \in S$

i.e., R is reflexive ... (i)

Symmetric: Let $a, b \in R$ (set of real numbers)

Let $a, b \in R \quad \Rightarrow \quad a - b + \sqrt{3} \in S$ (Set of irrational numbers)

$\Rightarrow \quad b - a + \sqrt{3} \in S$

$\Rightarrow \quad (b, a) \in R$

i.e., R is symmetric ... (ii)

Transitivity: Let $a, b, c \in R$

Now $(a, b) \in R$ and $(b, c) \in R \quad \Rightarrow \quad a - b + \sqrt{3} \in S$ and $b - c + \sqrt{3} \in S$

$\Rightarrow \quad a - b + \sqrt{3} + b - c + \sqrt{3} \in S$

$\Rightarrow \quad (a, c) \in R.$

i.e., R is transitive ... (iii)

(i), (ii) and (iii) $\Rightarrow$ R is reflexive, symmetric and transitive.

Q.10. Show that the function *f* in $A = |\mathrm{R} - \left\{\frac{2}{3}\right\}$ defined as $f(x) = \frac{4x+3}{6x-4}$ **is one-one and onto. Hence, find f^{-1}.**

Ans.

One-one: Let $x_1, x_2 \in A$

Now $, f(x_1) = f(x_2) \quad \Rightarrow \quad \frac{4x_1+3}{6x_1-4} = \frac{4x_2+3}{6x_2-4}$

$\Rightarrow 24x_1x_2 + 18x_2 - 16x_1 - 12 = 24x_1x_2 + 18x_1 - 16x_2 - 12$

$\Rightarrow -34x_1 = -34x_2 \quad \Rightarrow \quad x_1 = x_2$

Hence, f is one-one function.

Onto:

Let $\quad y = \frac{4x+3}{6x-4} \quad \Rightarrow \quad 6\,\text{xy} - 4y = 4x + 3$

$\Rightarrow \quad 6\,\text{xy} - 4x = 4y + 3 \quad \Rightarrow \quad x(6y - 4) = 4y + 3$

$\Rightarrow \quad x = \frac{4y+3}{6y-4}$

$\Rightarrow \quad \forall\, y \in$ codomain $\exists x \in$ domain $\left[\because \;\; x \neq \frac{2}{3}\right] \quad \Rightarrow \quad f$ is onto function.

Thus, f is one-one onto function.

Also, $f^{-1}(x) = \frac{4x+3}{6x-4}$

Q.11. Let T be the set of all triangles in a plane with R as relation in T given by $R = \{(T_1, T_2) : T_1 \cong T_2\}$. Show that R is an equivalence relation.

Ans. We have the relation, $R = \{(T_1, T_2) : T_1 \cong T_2\}$

Reflexivity: As Each triangle is congruent to itself,

i.e., $\quad T_1 \cong T_2 \quad \forall\, T_1 \in T$

Thus, R is reflexive.

Symmetry: Let $T_1, T_2 \in T$, such that

$$(T_1, T_2) \in R \quad \Rightarrow \quad T_1 \cong T_2$$

$$T_2 \cong T_1 \quad \Rightarrow \quad (T_2, T_1) \in R$$

i.e., R is symmetric.

Transitivity: Let $T_1, T_2, T_3 \in T$, such that $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$

$\Rightarrow \quad T_1 \cong T_2 \quad$ and $\quad T_2 \cong T_3$

$\Rightarrow \quad T_1 \cong T_3 \quad \Rightarrow \quad (T_1, T_3) \in R$

i.e., R is transitive.

Hence, R is an equivalence relation.

Q.12. Let $f : W \rightarrow W$, be defined as $f(x) = x - 1$, if x is odd and $f(x) = x + 1$, if x is even. Show that f is invertible. Find the inverse of f, where W is the set of all whole numbers.

Ans. One-one:

Case I When x_1, x_2 are even number

Now, $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$

i.e., f is one-one.

Case II When x_1, x_2 are odd number

Now, $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$

i.e., f is one-one.

Case III When x_1 is odd and, x_2 is even number

Then, $x_1 \neq x_2$. Also, in this case $f(x_1)$ is even and $f(x_2)$ is odd and so

$f(x_1) \neq f(x_2)$

i.e. $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

i.e., f is one-one.

Case IV When x_1 is even and, x_2 is odd number

Similar as Case III, we can prove f is one-one.

Onto:

Given, $f(x) = \begin{cases} x - 1, & \text{if} \, x is odd \\ x + 1, & \text{if} \, x is even \end{cases}$

$\Rightarrow$ For every even number 'y' of codomain $\exists$ odd number $y + 1$ in domain and for every odd number y of codomain there exists even number $y - 1$ in domain.

i.e. f is onto function. Hence, f is one-one onto *i.e.*, invertible function.

Inverse:

Let $f(x) = y$

Now, $y = x + 1 \Rightarrow x = y - 1$

And, $y = x - 1 \Rightarrow x = y + 1$

Therefore, required inverse function is given by

$$f^{-1}(x) = \begin{cases} x+1, & \text{if } x \text{ is odd} \\ x-1, & \text{if } x \text{ is even} \end{cases}$$

Q.13. If the function *f*: *R* → *R* be defined by $f(x) = 2x - 3$ and *g* : *R* → *R* by $g(x) = x^3 + 5$, then find the value of $(fog)^{-1}(x)$.

Ans. Here $f: R \to R$ and $g: R \to R$ be two functions such that

$f(x) = 2x - 3$ and $g(x) = x^3 + 5$

∵ *f* and *g* both are bijective (one-one onto) function.

⇒ *fog* is also bijective function.

⇒ *fog* is invertible function.

Now, $fog(x) = f\{(g(x)\}$ ⇒ $fog(x) = f(x^3 + 5)$

⇒ $fog(x) = 2(x^3 + 5) - 3$ ⇒ $fog(x) = 2x^3 + 10 - 3$

⇒ $fog(x) = 2x^3 + 7$ ...(*i*)

For inverse of fog (x)

Let $\text{fog}(x) = y \Rightarrow x = \text{fog}^{-1}(y)$

$(i) \Rightarrow y = 2x^3 + 7 \Rightarrow 2x^3 = y - 7$

$\Rightarrow x^3 = \frac{y-7}{2} \Rightarrow x = \left(\frac{y-7}{2}\right)^{\frac{1}{3}}$

$\Rightarrow \text{fog}^{-1}(y) = \left(\frac{y-7}{2}\right)^{\frac{1}{3}} \Rightarrow \text{fog}^{-1}(x) = \left(\frac{x-7}{2}\right)^{\frac{1}{3}}$

Q.14. Let $f : N \rightarrow R$ be a function defined as $f(x) = 4x^2 + 12x + 15$.

Show that $f : N \rightarrow S$ is invertible, where S is the range of f. Hence, find inverse of f.

Ans.

Let $y \in S$, then $y = 4x^2 + 12x + 15$, for some $x \in N$

$$\Rightarrow \quad y = (2x+3)^2 + 6 \quad \Rightarrow \quad x = \frac{(\sqrt{y-6})-3}{2}, \quad \text{as} \quad y > 6$$

Let $g : S \rightarrow N$ is defined by $g(y) = \frac{(\sqrt{y-6})-3}{2}$

$$\therefore \quad \text{gof}\,(x) = g(4x^2+12x+15) = g((2x+3)^2+6) = \frac{\sqrt{(2x+3)^2}-3}{2} = x$$

$$\text{and} \quad \text{fog}\,(y) = f\left(\frac{(\sqrt{y-6})-3}{2}\right) = \left[\frac{2[(\sqrt{y-6})-3]}{2}+3\right]^2 + 6 = y$$

Hence, $fog\,(y) = I_S$ and $gof\,(x) = I_N$

f is invertible, $f^{-1} = g$.

Q.15. Let Z be the set of all integers and R be relation on Z defined as $R = \{(a, b) : a, b \in Z$ and is divisible by 5$\}$. Prove that R is an equivalence relation.

Ans. Given $R = \{(a, b) : a, b \in Z$ and $(a - b)$ is divisible by 5$\}$

Reflexivity: $\forall\ a \in Z$

$a - a = 0$ is divisible by 5

$\Rightarrow \quad (a, a) \in R\ \forall\ a \in Z$

Hence, R is reflexive.

Symmetry: Let $(a, b) \in R \quad \Rightarrow \quad a - b$ is divisible by 5

$\Rightarrow \quad -(b - a)$ is divisible by 5

$\Rightarrow \quad (b - a)$ is divisible by 5

$\Rightarrow \quad (b, a) \in R$

Hence, R is symmetric.

Transitivity: Let $(a, b), (b, c) \in R$

$\Rightarrow$ $(a - b)$ and $(b - c)$ are divisible by 5

$\Rightarrow$ $(a - b + b - c)$ is divisible by 5

$\Rightarrow$ $a - c$ is divisible by 5

$\Rightarrow$ $(a, c) \in R$

Hence, R is transitive.

Thus, R is an equivalence relation.

Q.16. Let $\mathbf{f} : \mathbf{R} - \left\{-\frac{1}{3}\right\} \rightarrow \mathbf{R}$ be a function defined as $f(x) = \frac{4x}{3x+4}$. Show that, in $\mathbf{f} : \mathbf{R} - \left\{-\frac{4}{3}\right\} \rightarrow$ Range of *f*, *f* is one-one and onto. Hence find $\mathbf{f}^{-1}$:
Range $\mathbf{f} \rightarrow \mathbf{R} - \left\{-\frac{4}{3}\right\}$.

Ans.

Let $x_1, x_2 \in R - \left\{-\frac{4}{3}\right\}$

Now $f(x_1) = f(x_2) \quad \Rightarrow \quad \frac{4x_1}{3x_1+4} = \frac{4x_2}{3x_2+4}$

$\Rightarrow \quad 12\, x_1\, x_2 + 16\, x_1 = 12\, x_1\, x_2 + 16x_2$

$\Rightarrow \quad 16\, x_1 = 16\, x_2$

$\Rightarrow \quad x_1 = x_2$

Hence f is one-one function

Since, co-domain f is range of f

So, $f: \mathbb{R} - \left\{-\frac{4}{3}\right\} \rightarrow \mathbb{R}$ in one-one onto function.

For inverse function

Let $f(x) = y$

$\Rightarrow \quad \frac{4x}{3x+4} = y \quad \Rightarrow \quad 3xy + 4y = 4x$

$\Rightarrow \quad 4x - 3xy = 4y$

$\Rightarrow \quad x(4 - 3y) = 4y$

$\Rightarrow \quad x = \frac{4y}{4-3y}$

Therefore, f^{-1}: Range of $f \rightarrow R - \{- 4/3\}$ is $f^{-1}(y) = \frac{4y}{4-3y}$

Q.17. Let *A* = *R* × *R* and * be the binary operation on *A* defined by (*a*, *b*) * (*c*, *d*) = (*a* + *c*, *b* + *d*). Show that * is commutative and associative. Find the identity element for * on *A*, if any.

Ans. For Commutativity

Let $(a, b), (c, d) \in R \times R$

$(a, b) * (c, d) = (a + c, b + d)$ and $(c, d) * (a, b) = (c + a, d + b)$

$= (a + c, b + d)$ [∵ Commutative law holds for real number]

$\Rightarrow \quad (a, b) * (c, d) = (c, d) * (a, b)$

Hence, * is commutative

For Associativity

Let (a, b), (c, d) and $(e, f) \in R \times R$

$((a, b) * (c, d)) * (e, f) = (a + c, b + d) * (e, f) = (a + c + e, b + d + f)$

$(a, b) * ((c, d) * (e, f)) = (a, b) * (c + e, d + f) = (a + c + e, b + d + f)$

$((a, b) * (c, d)) * (e, f)) = (a, b) * ((c \, . \, d) * (e, f))$

$\therefore$ * is associative

Let (e_1, e_2) be identity

$\Rightarrow \quad (a, b) * (e_1, e_2) = (a, b) \quad \Rightarrow \quad (a + e_1, b + e_2) = (a, b)$

$\Rightarrow \quad a + e_1 = a$ and $b + e_2 = b \quad \Rightarrow \quad e_1 = 0, e_2 = 0$

$(0, 0) \in R \times R$ is the identity element.

Q.18. Let $A = Q \times Q$, where Q is the set of all rational numbers, and * be a binary operation on A defined by $(a, b) * (c, d) = (ac, b + ad)$ for $(a, b), (c, d) \in A$. Then find

Q. The identity element of * in A.

Ans. (*i*) Let (x, y) be the identity element in A.

Now, $(a, b) * (x, y) = (a, b) = (x, y) * (a, b) \ \forall \ (a, b) \in A$

$\Rightarrow \quad (ax, b + ay) = (a, b) = (xa, y + bx)$

Equating corresponding terms, we get

$\Rightarrow \quad ax = a, b + ay = b$ or $a = xa, b = y + bx$,

$\Rightarrow \quad x = 1$ and $y = 0$

Hence, $(1, 0)$ is the identity element in A.

Q. (*ii*) Invertible elements of A, and hence write the inverse of elements (5, 3) and $\left(\frac{1}{2}, 4\right)$

Ans.

(ii) Let (a, b) be an invertible element in A and let (c, d) be its inverse in A.

Now, $(a, b) * (c, d) = (1, 0) = (c, d) * (a, b)$

$\Rightarrow \quad (ac, b + ad) = (1, 0) = (ca, d + bc)$

$\Rightarrow \quad ac = 1, b + ad = 0$ or $1 = ca, 0 = d + bc$ [By equating coefficients]

$\Rightarrow \quad c = \frac{1}{a}$ and $d = -\frac{b}{a}$ where, $a \neq 0$

Therefore, all $(a, b) \in A$ is an invertible element of A if $a \neq 0$, and inverse of (a, b) is $\left(\frac{1}{a}, -\frac{b}{a}\right)$.

For inverse of (5, 3)

Inverse of $(5, 3) = \left(\frac{1}{5}, -\frac{3}{5}\right)$ $\left(\because \text{ Inverse of } (a, b) = \frac{1}{a}, \frac{-b}{a}\right)$

For inverse of $\left(\frac{1}{2}, 4\right)$

Inverse of $\left(\frac{1}{2}, 4\right) = (2, -8)$

Long Answer Questions-I (OIQ)

[4 Mark]

Q.1. Let $f: R \to \left[0, \frac{\pi}{2}\right)$ **defined by $f(x) = \tan^{-1}(x^2 + x + a)$, then find the value or set of values of 'a' for which f is onto.**

Ans.

Given function is $f: R \to \left[0, \frac{\pi}{2}\right)$.

Since, f is onto $\Rightarrow$ Range of f is $\left[0, \frac{\pi}{2}\right)$

$$\Rightarrow \quad 0 \le f(x) < \frac{\pi}{2} \qquad \Rightarrow \quad 0 \le \tan^{-1}(x^2 + x + a) < \frac{\pi}{2}$$

$$\Rightarrow \quad \tan 0 \le x^2 + x + a < \tan\frac{\pi}{2} \qquad \Rightarrow \quad 0 \le x^2 + x + a < \infty$$

It is possible only when $a = \frac{1}{4}$

As $\quad x^2 + x + \frac{1}{4} = x^2 + 2x \times \frac{1}{2} + \left(\frac{1}{2}\right)^2$

$$= \left(x + \frac{1}{2}\right)^2 \text{ and } 0 \le \left(x + \frac{1}{2}\right)^2 < \infty$$

Hence, the required value of $a = \frac{1}{4}$.

Q.2. Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$ and f, $g : A \to B$ be functions defined by $f(x) = x^2 - x$, $x \in A$ and, $g(x) = 2\left|x - \frac{1}{2}\right| - 1$ $x \in A$. Are f and g equal? Justify your answer.

Ans.

For two functions $f : A \to B$ and $g : A \to B$ to be equal, $f(a) = g(a)$ $\forall$ $a \in A$ and $R_f = R_g$.

Here, we have $f(x) = x^2 - x$

$$g(x) = 2\left|x - \frac{1}{2}\right| - 1 \qquad [x \in A = \{-1, 0, 1, 2\}]$$

We see that, $f(-1) = (-1)^2 - (-1) = 2$

$$g(-1) = 2\left|(-1) - \frac{1}{2}\right| - 1 = 2 \times \frac{3}{2} - 1 = 3 - 1 = 2$$

So, $\qquad f(-1) = g(-1)$

Again, we check that, $f(0) = g(0) = 0$, $f(1) = g(1) = 0$ and $f(2) = g(2) = 2$.

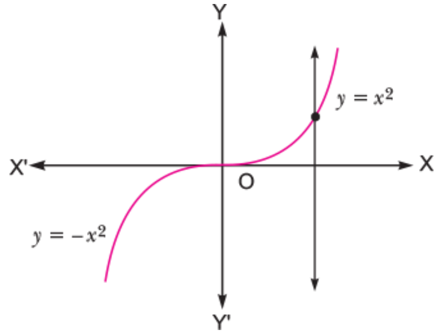
Hence, f and g are equal functions.

Q.3. Let $A = \{x \in R : -1 \le x \le 1\} = B$. Show that $f : A \to B$ given by $f(x) = x|x|$ is a bijection.

Ans. We have,

$$f(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

For $x \geq 0$, $f(x) = x^2$ represents a parabola opening upward and for $x < 0$, $f(x) = -x_2$ represents a parabola opening downward.



So, the graph of $f(x)$ is as shown in figure.

Since any line parallel to x-axis, will cut the graph at only one point, so f is one-one. Also, any line parallel to y-axis will cut the graph, so f is onto.

Thus, it is evident from the graph of $f(x)$ that f is one-one and onto.

Q.4. If $f(x) = \sqrt{x}\ (x \geq 0)$ and $g(x) = x^2 - 1$ are two real functions, then find *fog* and *gof* and check whether *fog* = *gof*.

Ans.

The given functions are $f(x) = \sqrt{x},\ x \geq 0$ and $g(x) = x^2 - 1$

We have, domain of $f = [0, \infty)$ and range of $f = [0, \infty)$

domain of $g = R$ and range of $g = [-1, \infty)$

Computation of *gof*. We observe that range of $f = [0, \infty) \subseteq$ domain of g

$\therefore$ gof exists and $gof : [0, \infty) \to R$

Also, $gof(x) = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 - 1 = x - 1$

Thus, $gof : [0, \infty) \to R$ is defined as $gof(x) = x - 1$

Computation of *fog*: We observe that range of $g = [-1, \infty) \subseteq$ domain of f.

$\therefore$ Domain of $fog = \{x : x \in$ domain of g and $g(x) \in$ domain of $f\}$

$\Rightarrow$ Domain of $fog = \{x : x \in R \text{ and } g(x) \in [0, \infty)\}$

$\Rightarrow$ Domain of $fog = \{x : x \in R \text{ and } x^2 - 1 \in [0, \infty)\}$

$\Rightarrow$ Domain of $fog = \{x : x \in R \text{ and } x^2 - 1 \geq 0\}$

Domain of $fog = \{x : x \in R \text{ and } x \leq -1 \text{ or } x \geq 1\}$

$\therefore$ Domain of $fog = x : x \in (-\infty, -1] \cup [1, \infty)$

Also, $\text{fog}\,(x) = f(g(x)) = f(x^2 - 1) = \sqrt{x^2 - 1}$

Thus, $fog : (-\infty, -1] \cup [1, \infty) \to R$ is defined as $\text{fog}\,(x) = \sqrt{x^2 - 1}$.

We find that *fog* and *gof* have distinct domains. Also, their formulae are not same.

Hence, $fog \neq gof$

Q.5. Let *X* be a non-empty set and * be a binary operation on *P*(*X*) (the power set of set *X*) defined by

$$A * B = A \cup B \text{ for all } A, B \in P(X)$$

Prove that '*' is both commutative and associative on *P*(*X*). Find the identity element with respect to '*' on *P*(*X*). Also, show that Φ∈ *P*(*X*) is the only invertible element of *P* (*X*).

Ans. As we studied in earlier class that for sets A, B, C

$$A \cup B = B \cup A \text{ and } (A \cup B) \cup C = A \cup (B \cup C)$$

Therefore, for any $A, B, C \in P(X)$, we have

$$A \cup B = B \cup A \text{ and } (A \cup B) \cup C = A \cup (B \cup C)$$

$\Rightarrow$ $A * B = B * A$ and $(A * B) * C = A * (B * C)$

Thus, '*' is both commutative and associative on $P(X)$

Now, $A \cup \Phi = A = \Phi \cup A$ for all $A \in P(X)$

$A * \Phi = A = \Phi * A$ for all $A \in P(X)$

So, Φ is the identity element for '*' on $P(X)$. Let $A \in P(X)$ be an invertible element. Then, there exists $S \in P(X)$ such that

$A * S = \Phi = S * A$

$\Rightarrow \quad A \cup S = \Phi = S \cup A \quad \Rightarrow \quad S = \Phi = A$

Hence, Φ is the only invertible element.

Q.6. Let $g(x) = 1 + x - [x]$ and f(x) = $\begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$ then for all x find fog (x).

Ans.

$fog(x) = f(g(x)) = f(1 + x - [x]) = f(1 + \{x\}) = 1$

Here, $\quad \{x\} = x - [x]$

Obviously, $\quad 0 \le x - [x] < 1$

$\Rightarrow \quad 0 \le \{x\} < 1$

$\Rightarrow \quad 1 + \{x\} \ge 1$

$\therefore \quad fog\ (x) = f(1 + \{x\}) = 1$

$\left[\begin{array}{ll} \text{Note:} & \text{Symbol } \{x\} \text{ denotes the partial part or decimal part of } x. \\ & \text{For example, } \{4.25\} = 0.25,\ \{4\} = 0,\ \{-3.45\} = 0.45 \\ & \text{In this way } x = x - [x] \quad \Rightarrow \quad 0 \le \{x\} < 1 \end{array}\right]$

[6 Mark]

Q.1. If the operation '*' on $Q - \{1\}$, defined by $a * b = a + b - ab$ for all $a, b \in Q - \{1\}$, then

(*i*) Is '*' commutative?

(*ii*) Is '*' associative?

(*iii*) Find the identity element.

(*iv*) Find the inverse of '*a*' for each $a \in Q - \{1\}$

Ans. We have, $a * b = a + b - ab \ \forall \ a, b \in Q - \{1\}$, then

(*i*) Commutative: Let $a, b \in Q - \{1\}$

Now, $a * b = a + b - ab$

$b * a = b + a - ba = a + b - ab$ [$\because$ Commutative law holds for + & ×]

Hence, $a * b = b * a$

i.e., '*' is commutative.

(*ii*) Associative: Let $a, b, c \in Q - \{1\}$

Now, $(a * b) * c = (a + b - ab) * c = a + b - ab + c - ac - bc + abc$

$a * (b * c) = a * (b + c - bc) = a + b + c - bc - ab - ac + abc$

i.e., $(a * b) * c = a * (b * c)$

Hence, '*' is associative.

(*iii*) Identity: Let e be the identity element.

Then, $\forall \ a \in Q - \{1\}$, we have

$a * e = a \quad \Rightarrow \quad a + e - ae = a$

$\Rightarrow \quad (1 - a) e = 0$

$\Rightarrow \quad e = 0 \in Q - \{1\}$ [$\because a \neq 1 \Rightarrow 1 - a \neq 0$]

Now, $a * 0 = a + 0 - a \times 0 = a$

$0 * a = 0 + a - 0 \times a = a$

Thus, 0 is the identity element in $Q - \{1\}$.

(*iv*) Inverse: Let b be the inverse element of a, for each $a \in Q - \{1\}$.

Then $a * b = e = 0 \Rightarrow a * b = 0$

$\Rightarrow a + b - ab = 0 \Rightarrow ab - b = a$

$\Rightarrow b(a - 1) = a$

$\Rightarrow b = \frac{a}{a-1} \in Q - \{1\}$

Therefore, for each a the corresponding inverse element is $\frac{a}{a-1} \in Q - \{1\}$.

Q.2. Show that the function $f : R \to R$ given by $f(x) = x^3 + x$ is a bijection.

Ans.

We have the function $f : R \to R$ given by $f(x) = x^3 + x$.

Injectivity: Let $x, y \in R$ such that $f(x) = f(y)$

$\Rightarrow x^3 + x = y^3 + y$

$\Rightarrow x^3 - y^3 + x - y = 0$

$\Rightarrow (x - y)(x^2 + xy + y^2) + (x - y) = 0$ $\left[\begin{array}{l} \because \ x^2 + xy + y^2 \geq 0 \text{ for all } x, y \in R \\ \because \ x^2 + xy + y^2 + 1 \geq 1 \text{ for all } x, y \in R \end{array}\right]$

$\Rightarrow (x - y)(x^2 + xy + y^2 + 1) = 0$

$\Rightarrow x - y = 0 \Rightarrow x = y$

Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in R$.

So, f is injective.

Surjectivity: Let y be an arbitrary element of R such that $f(x) = y$

$\Rightarrow \quad x^3 + x = y \quad \Rightarrow \quad x^3 + x - y = 0$

For every value of y, the equation $x^3 + x - y = 0$ has a real root a.

Therefore, $a^3 + a - y = 0$ [$\because$ An odd degree equation has at least one real root.]

$$a^3 + a = y \Rightarrow f(a) = y$$

Thus, for every $y \in R$ there exists $a \in R$ such that

$$f(a) = y$$

So, f is surjective.

Hence, $f: R \rightarrow R$ is a bijection.

Q.3. Let $A = R - \{3\}$ and $B = R - \left\{\frac{2}{3}\right\}$. If $f: A \rightarrow B: f(x) = \frac{2x-4}{3x-9}$, then prove that *f* is a bijective function.

Ans.

One-one: Let x_1, x_2 be any two elements of A, then

$$f(x_1) = f(x_2) \Rightarrow \frac{2x_1 - 4}{3x_1 - 9} = \frac{2x_2 - 4}{3x_2 - 9}$$

$$\Rightarrow 6x_1x_2 - 18x_1 - 12x_2 + 36 = 6x_1x_2 - 12x_1 - 18x_2 + 36$$

$$\Rightarrow -18x_1 - 12x_2 = -12x_1 - 18x_2$$

$$\Rightarrow -18x_1 + 12x_1 = -18x_2 + 12x_2$$

$$\Rightarrow -6x_1 = -6x_2 \Rightarrow x_1 = x_2$$

Hence, f is one-one function. ...(i)

Onto: Let $y = \frac{2x - 4}{3x - 9} \Rightarrow 3xy - 9y = 2x - 4$

$\Rightarrow 3xy - 2x = 9y - 4 \Rightarrow x(3y - 2) = 9y - 4$

$\Rightarrow x = \frac{9y - 4}{3y - 2}$

From above, it is obvious that $\forall\, y \neq \frac{2}{3}$ *i.e.* $\forall\, y \in B, \exists\, x \in A$

Hence, f is onto function ...(ii)

(i) and (ii) $\Rightarrow$ f is one-one onto *i.e*. bijective function.

Q.4. Given a non-empty set X. Let * : $P(X) \times P(X) \to P(X)$ defined as

$$A * B = (A - B) \cup (B - A)\ \forall\ A, B \in P(X).$$

Show that the empty set Φ is the identity for the operation * and all the elements A of $P(X)$ are invertible with $A^{-1} = A$.

Ans. Here operation '*' is defined as

$* : P(X) \times P(X) \to P(X)$ such that $A * B = (A - B) \cup (B - A)\ \forall\ A, B \in P(X)$

Existence of identity:

Let $E \in P(X)$ be identity for '*' in set $P(X)$

$$\Rightarrow A * E = A = E * A$$

$$\Rightarrow (A - E) \cup (E - A) = A = (E - A) \cup (A - E)$$

It is possible only when $E = \Phi$, Because

$$(A - \Phi) \cup (\Phi - A) = A \cup \Phi = A \quad \text{and} \quad (\Phi - A) \cup (A - \Phi) = \Phi \cup A = A$$

Hence, Φ is identity element.

Existence of inverse:

Let A^{-1} be the inverse of A for '*' on set $P(X)$.

$\therefore \quad A * A^{-1} = \Phi = A^{-1} * A \quad \Rightarrow \quad (A - A^{-1}) \cup (A^{-1} - A) = \Phi$

$\Rightarrow \quad A - A^{-1} = \Phi = A^{-1} - A = \Phi \quad \Rightarrow \quad A \subset A^{-1}$ and $A^{-1} \subset A$

$\Rightarrow \quad A = A^{-1}$

Hence, each element of $P(X)$ is inverse of itself.

Q.5. Show that the relation *R* on the set *A* of points in a plane, given by

***R* = {(*P*, *Q*) : Distance of the point *P* from the origin = Distance of point *Q* from origin} is an equivalence relation.**

Further, show that the set of all points related to a point *P* ≠ (0, 0) is the circle passing through *P* with origin as centre.

Ans. If O be the origin, then

$$R = \{(P, Q) : OP = OQ\}$$

Reflexivity: $\forall$ point $P \in A$

$$OP = OP \quad \Rightarrow \quad (P, P) \in R$$

i.e., R is reflexive.

Symmetry: Let $P, Q \in A$, such that $(P, Q) \in R$

$$OP = OQ \quad \Rightarrow \quad OQ = OP \quad \Rightarrow \quad (Q, P) \in R$$

i.e., R is symmetric.

Transitivity: Let $P, Q, S \in A$, such that $(P, Q) \in R$ and $(Q, S) \in R$

$$OP = OQ \quad \text{and} \quad OQ = OS$$

$$OP = OS \quad \Rightarrow \quad (P, S) \in R$$

i.e., R is transitive.

Now we have R is reflexive, symmetric and transitive.

Therefore, R is an equivalence relation.

Let P, Q, R... be points in the set A, such that

$$(P, Q), (P, R)... \in R$$

$\Rightarrow \quad OP = OQ; OP = OR; ...$ [where O is origin]

$\Rightarrow \quad OP = OQ = OR = ...$

i.e., All points P, Q, R ... $\in A$, which are related to P are equidistant from origin 'O'.

Hence, set of all points of A related to P is the circle passing through P, having origin as centre.