Chapter 7

Electrodynamics

7.1 Electromotive Force

7.1.1 Ohm's Law

To make a current flow, you have to *push* on the charges. How *fast* they move, in response to a given push, depends on the nature of the material. For most substances, the current density **J** is proportional to the *force per unit charge*, **f**:

$$\mathbf{J} = \sigma \mathbf{f}. \tag{7.1}$$

The proportionality factor σ (not to be confused with surface charge) is an empirical constant that varies from one material to another; it's called the **conductivity** of the medium. Actually, the handbooks usually list the *reciprocal* of σ, called the **resistivity**: $\rho = 1/\sigma$ (not to be confused with charge density—I'm sorry, but we're running out of Greek letters, and this is the standard notation). Some typical values are listed in Table 7.1. Notice that even *insulators* conduct slightly, though the conductivity of a metal is astronomically greater—by a factor of 10^{22} or so. In fact, for most purposes metals can be regarded as **perfect conductors**, with $\sigma = \infty$.

In principle, the force that drives the charges to produce the current could be anything—chemical, gravitational, or trained ants with tiny harnesses. For *our* purposes, though, it's usually an electromagnetic force that does the job. In this case Eq. 7.1 becomes

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{7.2}$$

Ordinarily, the velocity of the charges is sufficiently small that the second term can be ignored:

$$\boxed{\mathbf{J} = \sigma \mathbf{E}.} \tag{7.3}$$

(However, in plasmas, for instance, the magnetic contribution to **f** can be significant.) Equation 7.3 is called **Ohm's law**, though the physics behind it is really contained in Eq. 7.1, of which 7.3 is just a special case.

Material	Resistivity	Material	Resistivity
Conductors:		*Semiconductors:*	
Silver	1.59×10^{-8}	Salt water (saturated)	4.4×10^{-2}
Copper	1.68×10^{-8}	Germanium	4.6×10^{-1}
Gold	2.21×10^{-8}	Diamond	2.7
Aluminum	2.65×10^{-8}	Silicon	2.5×10^{3}
Iron	9.61×10^{-8}	*Insulators:*	
Mercury	9.58×10^{-7}	Water (pure)	2.5×10^{5}
Nichrome	1.00×10^{-6}	Wood	$10^{8} - 10^{11}$
Manganese	1.44×10^{-6}	Glass	$10^{10} - 10^{14}$
Graphite	1.4×10^{-5}	Quartz (fused)	$\sim 10^{16}$

Table 7.1 Resistivities, in ohm-meters (all values are for 1 atm, 20° C).
Source: Handbook of Chemistry and Physics, 78th ed.
(Boca Raton: CRC Press, Inc., 1997).

I know: you're confused because I said $\mathbf{E} = 0$ inside a conductor (Sect. 2.5.1). But that's for *stationary* charges ($\mathbf{J} = 0$). Moreover, for *perfect* conductors $\mathbf{E} = \mathbf{J}/\sigma = 0$ even if current *is* flowing. In practice, metals are such good conductors that the electric field required to drive current in them is negligible. Thus we routinely treat the connecting wires in electric circuits (for example) as equipotentials. **Resistors**, by contrast, are made from *poorly* conducting materials.

Example 7.1

A cylindrical resistor of cross-sectional area A and length L is made from material with conductivity σ. (See Fig. 7.1; as indicated, the cross section need not be circular, but I *do* assume it is the same all the way down.) If the potential is constant over each end, and the potential difference between the ends is V, what current flows?

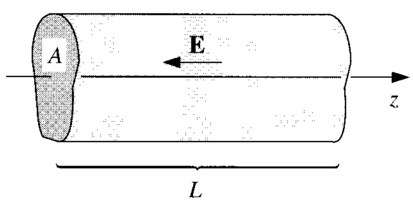

Figure 7.1

Solution: As it turns out, the electric field is *uniform* within the wire (I'll *prove* this in a moment). It follows from Eq. 7.3 that the current density is also uniform, so

$$I = JA = \sigma EA = \frac{\sigma A}{L} V.$$

Example 7.2

Two long cylinders (radii a and b) are separated by material of conductivity σ (Fig. 7.2). If they are maintained at a potential difference V, what current flows from one to the other, in a length L?

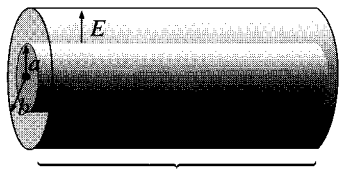

Figure 7.2

Solution: The field between the cylinders is

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}},$$

where λ is the charge per unit length on the inner cylinder. The current is therefore

$$I = \int \mathbf{J} \cdot d\mathbf{a} = \sigma \int \mathbf{E} \cdot d\mathbf{a} = \frac{\sigma}{\epsilon_0}\lambda L.$$

(The integral is over any surface enclosing the inner cylinder.) Meanwhile, the potential difference between the cylinders is

$$V = -\int_b^a \mathbf{E} \cdot d\mathbf{l} = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right),$$

so

$$I = \frac{2\pi\sigma L}{\ln(b/a)}V.$$

As these examples illustrate, the total current flowing from one **electrode** to the other is proportional to the potential difference between them:

$$\boxed{V = IR.} \tag{7.4}$$

This, of course, is the more familiar version of Ohm's law. The constant of proportionality R is called the **resistance**; it's a function of the geometry of the arrangement and the conductivity of the medium between the electrodes. (In Ex. 7.1, $R = (L/\sigma A)$; in Ex. 7.2, $R = \ln(b/a)/2\pi\sigma L$.) Resistance is measured in **ohms** (Ω): an ohm is a volt per ampere. Notice that the proportionality between V and I is a direct consequence of Eq. 7.3: if you want to double V, you simply double the charge everywhere—but that doubles $\mathbf{E}$, which doubles $\mathbf{J}$, which doubles I.

For *steady* currents and *uniform* conductivity,

$$\nabla \cdot \mathbf{E} = \frac{1}{\sigma} \nabla \cdot \mathbf{J} = 0, \tag{7.5}$$

(Eq. 5.31), and therefore the charge density is zero; any unbalanced charge resides on the *surface*. (We proved this long ago, for the case of *stationary* charges, using the fact that $\mathbf{E} = 0$; evidently, it is still true when the charges are allowed to move.) It follows, in particular, that Laplace's equation holds within a homogeneous ohmic material carrying a steady current, so all the tools and tricks of Chapter 3 are available for computing the potential.

Example 7.3

I asserted that the field in Ex. 7.1 is *uniform*. Let's *prove* it.

Solution: Within the cylinder V obeys Laplace's equation. What are the boundary conditions? At the left end the potential is constant—we may as well set it equal to zero. At the right end the potential is likewise constant—call it V_0. On the cylindrical surface, $\mathbf{J} \cdot \hat{\mathbf{n}} = 0$, or else charge would be leaking out into the surrounding space (which we take to be nonconducting). Therefore $\mathbf{E} \cdot \hat{\mathbf{n}} = 0$, and hence $\partial V/\partial n = 0$. With V or its normal derivative specified on all surfaces, the potential is uniquely determined (Prob. 3.4). But it's *easy* to guess *one* potential that obeys Laplace's equation and fits these boundary conditions:

$$V(z) = \frac{V_0 z}{L},$$

where z is measured along the axis. The uniqueness theorem guarantees that this is *the* solution. The corresponding field is

$$\mathbf{E} = -\nabla V = -\frac{V_0}{L}\hat{\mathbf{z}},$$

which is indeed uniform. qed

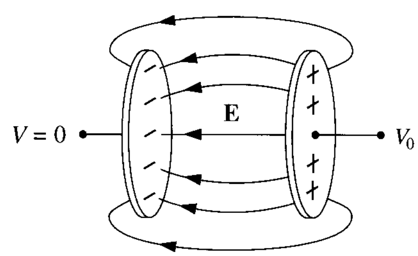

Figure 7.3

Contrast the *enormously* more difficult problem that arises if the conducting material is removed, leaving only a metal plate at either end (Fig. 7.3). Evidently in the present case charge arranges itself over the surface of the wire in just such a way as to produce a nice uniform field within.[1]

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[1] *Calculating* this surface charge is not easy. See, for example, J. D. Jackson, *Am. J. Phys.* **64**, 855 (1996). Nor is it a simple matter to determine the field *outside* the wire—see Prob. 7.57.

I don't suppose there is any formula in physics more widely known than Ohm's law, and yet it's not really a true law, in the sense of Gauss's law or Ampère's law; rather, it is a "rule of thumb" that applies pretty well to many substances. You're not going to win a Nobel prize for finding an exception. In fact, when you stop to think about it, it's a little surprising that Ohm's law *ever* holds. After all, a given field $\mathbf{E}$ produces a force $q\mathbf{E}$ (on a charge q), and according to Newton's second law the charge will accelerate. But if the charges are *accelerating*, why doesn't the current *increase* with time, growing larger and larger the longer you leave the field on? Ohm's law implies, on the contrary, that a constant field produces a constant *current*, which suggests a constant *velocity*. Isn't that a contradiction of Newton's law?

No, for we are forgetting the frequent collisions electrons make as they pass down the wire. It's a little like this: Suppose you're driving down a street with a stop sign at every intersection, so that, although you accelerate constantly in between, you are obliged to start all over again with each new block. Your *average* speed is then a constant, in spite of the fact that (save for the periodic abrupt stops) you are always accelerating. If the length of a block is λ and your acceleration is a, the time it takes to go a block is

$$t = \sqrt{\frac{2\lambda}{a}},$$

and hence the average velocity is

$$v_{\text{ave}} = \frac{1}{2}at = \sqrt{\frac{\lambda a}{2}}.$$

But wait! That's no good ***either!*** It says that the velocity is proportional to the *square root* of the acceleration, and therefore that the current should be proportional to the *square root* of the field! There's another twist to the story: The charges in practice are already moving quite fast because of their thermal energy. But the thermal velocities have random directions, and average to zero. The net **drift velocity** we're concerned with is a tiny extra bit (Prob. 5.19). So the time between collisions is actually much shorter than we supposed; in fact,

$$t = \frac{\lambda}{v_{\text{thermal}}},$$

and therefore

$$v_{\text{ave}} = \frac{1}{2}at = \frac{a\lambda}{2v_{\text{thermal}}}.$$

If there are n molecules per unit volume and f free electrons per molecule, each with charge q and mass m, the current density is

$$\mathbf{J} = nfq\mathbf{v}_{\text{ave}} = \frac{nfq\lambda}{2v_{\text{thermal}}}\frac{\mathbf{F}}{m} = \left(\frac{nf\lambda q^2}{2mv_{\text{thermal}}}\right)\mathbf{E}. \tag{7.6}$$

I don't claim that the term in parentheses is an accurate formula for the conductivity,[2] but it

[2]This classical model (due to Drude) bears little resemblance to the modern quantum theory of conductivity. See, for instance, D. Park's *Introduction to the Quantum Theory*, 3rd ed., Chap. 15 (New York: McGraw-Hill, 1992).

does indicate the basic ingredients, and it correctly predicts that conductivity is proportional to the density of the moving charges and (ordinarily) decreases with increasing temperature.

As a result of all the collisions, the work done by the electrical force is converted into heat in the resistor. Since the work done per unit charge is V and the charge flowing per unit time is I, the power delivered is

$$\boxed{P = VI = I^2 R.} \tag{7.7}$$

This is the **Joule heating law**. With I in amperes and R in ohms, P comes out in watts (joules per second).

Problem 7.1 Two concentric metal spherical shells, of radius a and b, respectively, are separated by weakly conducting material of conductivity σ (Fig. 7.4a).

(a) If they are maintained at a potential difference V, what current flows from one to the other?

(b) What is the resistance between the shells?

(c) Notice that if $b \gg a$ the outer radius (b) is irrelevant. How do you account for that? Exploit this observation to determine the current flowing between two metal spheres, each of radius a, immersed deep in the sea and held quite far apart (Fig. 7.4b), if the potential difference between them is V. (This arrangement can be used to measure the conductivity of sea water.)

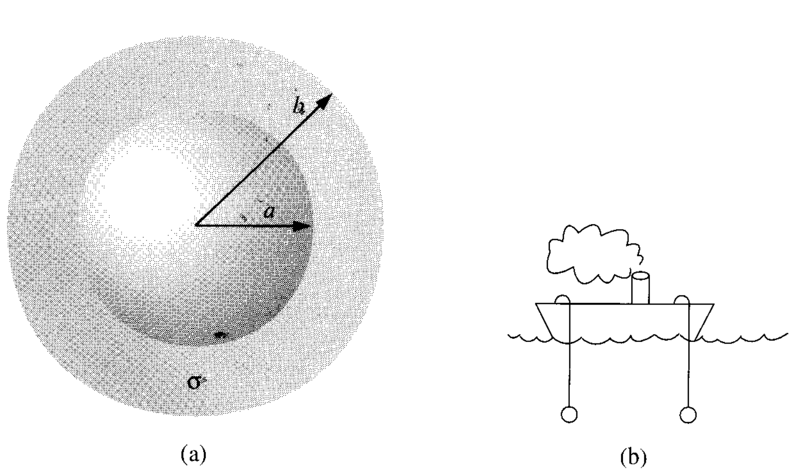

Figure 7.4

Problem 7.2 A capacitor C has been charged up to potential V_0; at time $t = 0$ it is connected to a resistor R, and begins to discharge (Fig. 7.5a).

(a) Determine the charge on the capacitor as a function of time, $Q(t)$. What is the current through the resistor, $I(t)$?

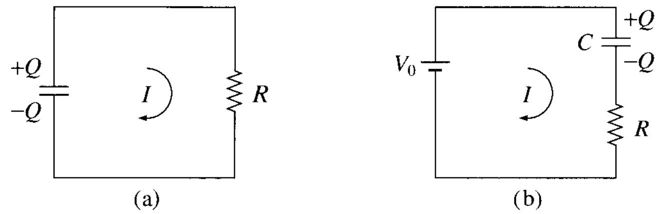

Figure 7.5

(b) What was the original energy stored in the capacitor (Eq. 2.55)? By integrating Eq. 7.7, confirm that the heat delivered to the resistor is equal to the energy lost by the capacitor.

Now imagine *charging up* the capacitor, by connecting it (and the resistor) to a battery of fixed voltage V_0, at time $t = 0$ (Fig. 7.5b).

(c) Again, determine $Q(t)$ and $I(t)$.

(d) Find the total energy output of the battery ($\int V_0 I\, dt$). Determine the heat delivered to the resistor. What is the final energy stored in the capacitor? What fraction of the work done by the battery shows up as energy in the capacitor? [Notice that the answer is independent of R!]

Problem 7.3

(a) Two metal objects are embedded in weakly conducting material of conductivity σ (Fig. 7.6). Show that the resistance between them is related to the capacitance of the arrangement by

$$R = \frac{\epsilon_0}{\sigma C}.$$

(b) Suppose you connected a battery between 1 and 2 and charged them up to a potential difference V_0. If you then disconnect the battery, the charge will gradually leak off. Show that $V(t) = V_0 e^{-t/\tau}$, and find the **time constant,** τ, in terms of ϵ_0 and σ.

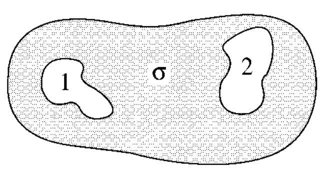

Figure 7.6

Problem 7.4 Suppose the conductivity of the material separating the cylinders in Ex. 7.2 is not uniform; specifically, $\sigma(s) = k/s$, for some constant k. Find the resistance between the cylinders. [*Hint:* Because σ is a function of position, Eq. 7.5 does not hold, the charge density is not zero in the resistive medium, and **E** does not go like $1/s$. But we *do* know that for steady currents I is the same across each cylindrical surface. Take it from there.]

7.1.2 Electromotive Force

If you think about a typical electric circuit (Fig. 7.7)—a battery hooked up to a light bulb, say—there arises a perplexing question: In practice, the *current is the same all the way around the loop*, at any given moment; *why* is this the case, when the only obvious driving force is inside the battery? Off hand, you might expect this to produce a large current in the battery and none at all in the lamp. Who's doing the pushing in the rest of the circuit, and how does it happen that this push is exactly right to produce the same current in each segment? What's more, given that the charges in a typical wire move (literally) at a *snail's* pace (see Prob. 5.19), why doesn't it take half an hour for the news to reach the light bulb? How do all the charges know to start moving at the same instant?

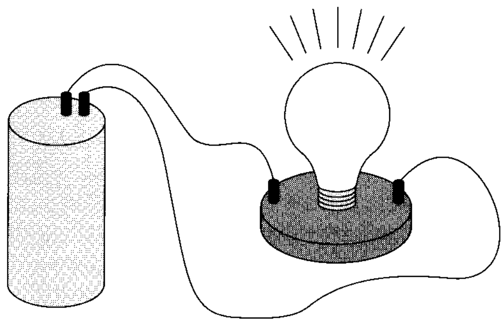

Figure 7.7

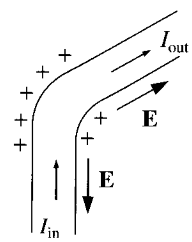

Figure 7.8

Answer: If the current is *not* the same all the way around (for instance, during the first split second after the switch is closed), then charge is piling up somewhere, and—here's the crucial point—the electric field of this accumulating charge is in such a direction as to even out the flow. Suppose, for instance, that the current *into* the bend in Fig. 7.8 is greater than the current *out*. Then charge piles up at the "knee," and this produces a field aiming *away* from the kink. This field *opposes* the current flowing in (slowing it down) and *promotes* the current flowing out (speeding it up) until these currents are equal, at which point there is no further accumulation of charge, and equilibrium is established. It's a beautiful system, automatically self-correcting to keep the current uniform, and it does it all so quickly that, in practice, you can safely assume the current is the same all around the circuit even in systems that oscillate at radio frequencies.

The upshot of all this is that there are really *two* forces involved in driving current around a circuit: the *source*, $\mathbf{f}_s$, which is ordinarily confined to one portion of the loop (a battery, say), and the *electrostatic* force, which serves to smooth out the flow and communicate the influence of the source to distant parts of the circuit:

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E}. \tag{7.8}$$

The physical agency responsible for $\mathbf{f}_s$ can be any one of many different things: in a battery it's a chemical force; in a piezoelectric crystal mechanical pressure is converted into an electrical impulse; in a thermocouple it's a temperature gradient that does the job; in a photoelectric cell it's light; and in a Van de Graaff generator the electrons are literally loaded onto a conveyer belt and swept along. Whatever the *mechanism*, its net effect is determined by the line integral of **f** around the circuit:

$$\boxed{\mathcal{E} \equiv \oint \mathbf{f} \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l}.} \tag{7.9}$$

(Because $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ for electrostatic fields, it doesn't matter whether you use **f** or $\mathbf{f}_s$.) $\mathcal{E}$ is called the **electromotive force**, or **emf**, of the circuit. It's a lousy term, since this is not a *force* at all—it's the *integral* of a *force per unit charge*. Some people prefer the word **electromotance**, but emf is so ingrained that I think we'd better stick with it.

Within an ideal source of emf (a resistanceless battery,[3] for instance), the *net* force on the charges is *zero* (Eq. 7.1 with $\sigma = \infty$), so $\mathbf{E} = -\mathbf{f}_s$. The potential difference between the terminals (a and b) is therefore

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b \mathbf{f}_s \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l} = \mathcal{E} \tag{7.10}$$

(we can extend the integral to the entire loop because $\mathbf{f}_s = 0$ outside the source). The function of a battery, then, is to establish and maintain a voltage difference equal to the electromotive force (a 6 V battery, for example, holds the positive terminal 6 V above the negative terminal). The resulting electrostatic field drives current around the rest of the circuit (notice, however, that *inside* the battery $\mathbf{f}_s$ drives current in the direction *opposite* to **E**).

Because it's the line integral of $\mathbf{f}_s$, $\mathcal{E}$ can be interpreted as the *work done, per unit charge*, by the source—indeed, in some books electromotive force is *defined* this way. However, as you'll see in the next section, there is some subtlety involved in this interpretation, so I prefer Eq. 7.9.

Problem 7.5 A battery of emf $\mathcal{E}$ and internal resistance r is hooked up to a variable "load" resistance, R. If you want to deliver the maximum possible power to the load, what resistance R should you choose? (You can't change $\mathcal{E}$ and r, of course.)

Problem 7.6 A rectangular loop of wire is situated so that one end (height h) is between the plates of a parallel-plate capacitor (Fig. 7.9), oriented parallel to the field **E**. The other end is way outside, where the field is essentially zero. What is the emf in this loop? If the total resistance is R, what current flows? Explain. [*Warning:* this is a trick question, so be careful; if you have invented a perpetual motion machine, there's probably something wrong with it.]

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[3] *Real* batteries have a certain **internal resistance**, r, and the potential difference between their terminals is $\mathcal{E} - Ir$, when a current I is flowing. For an illuminating discussion of how batteries work, see D. Roberts, *Am. J. Phys.* **51**, 829 (1983).

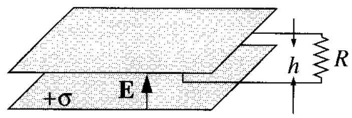


Figure 7.9

7.1.3 Motional emf

In the last section I listed several possible sources of electromotive force in a circuit, batteries being the most familiar. But I did not mention the most common one of all: the **generator**. Generators exploit **motional emf**'s, which arise when you *move a wire through a magnetic field*. Figure 7.10 shows a primitive model for a generator. In the shaded region there is a uniform magnetic field **B**, pointing into the page, and the resistor R represents whatever it is (maybe a light bulb or a toaster) we're trying to drive current through. If the entire loop is pulled to the right with speed v, the charges in segment ab experience a magnetic force whose vertical component qvB drives current around the loop, in the clockwise direction. The emf is

$$\mathcal{E} = \oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l} = vBh, \tag{7.11}$$

where h is the width of the loop. (The horizontal segments bc and ad contribute nothing, since the force here is perpendicular to the wire.)

Notice that the integral you perform to calculate $\mathcal{E}$ (Eq. 7.9 or 7.11) is carried out at *one instant of time*—take a "snapshot" of the loop, if you like, and work from that. Thus $d\mathbf{l}$, for the segment ab in Fig. 7.10, points straight up, even though the loop is moving to the right. You can't quarrel with this—it's simply the way emf is *defined*—but it *is* important to be *clear* about it.

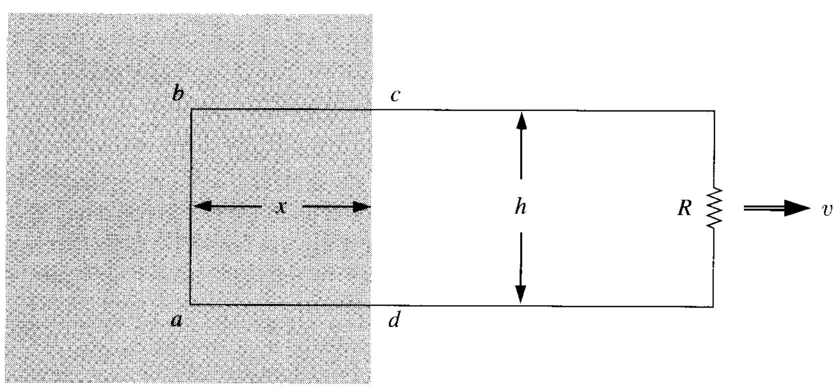


Figure 7.10

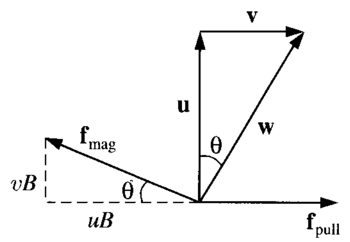


Figure 7.11

In particular, although the magnetic force is responsible for establishing the emf, it is certainly *not* doing any work—magnetic forces *never* do work. Who, then, *is* supplying the energy that heats the resistor? *Answer:* The person who's pulling on the loop! With the current flowing, charges in segment ab have a vertical velocity (call it $\mathbf{u}$) in addition to the horizontal velocity $\mathbf{v}$ they inherit from the motion of the loop. Accordingly, the magnetic force has a component quB to the left. To counteract this, the person pulling on the wire must exert a force per unit charge

$$f_{pull} = uB$$

to the *right* (Fig. 7.11). This force is transmitted to the charge by the structure of the wire. Meanwhile, the particle is actually *moving* in the direction of the resultant velocity $\mathbf{w}$, and the distance it goes is $(h/\cos\theta)$. The work done per unit charge is therefore

$$\int \mathbf{f}_{pull} \cdot d\mathbf{l} = (uB)\left(\frac{h}{\cos\theta}\right)\sin\theta = vBh = \mathcal{E}$$

($\sin\theta$ coming from the dot product). As it turns out, then, the *work done per unit charge is exactly equal to the emf,* though the integrals are taken along entirely different paths (Fig. 7.12) and completely different forces are involved. To calculate the emf you integrate around the loop at *one instant*, but to calculate the work done you follow a charge in its motion around the loop; $\mathbf{f}_{pull}$ contributes nothing to the emf, because it is perpendicular to the wire, whereas $\mathbf{f}_{mag}$ contributes nothing to work because it is perpendicular to the motion of the charge.[4]

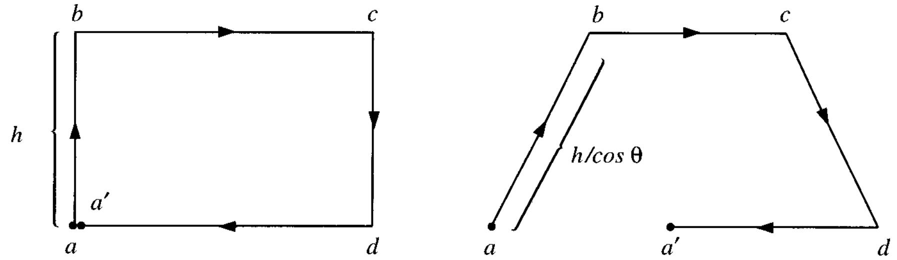
There is a particularly nice way of expressing the emf generated in a moving loop. Let Φ be the flux of $\mathbf{B}$ through the loop:

$$\Phi \equiv \int \mathbf{B} \cdot d\mathbf{a}. \tag{7.12}$$

For the rectangular loop in Fig. 7.10,

$$\Phi = Bhx.$$

[4]For further discussion, see E. P. Mosca, *Am. J. Phys.* **42**, 295 (1974).



(a) Integration for computing $\mathcal{E}$ (follow the wire at one instant of time).

(b) Integration path for calculating work done (follow the charge around the loop).

Figure 7.12

As the loop moves, the flux decreases:

$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv.$$

(The minus sign accounts for the fact that dx/dt is negative.) But this is precisely the emf (Eq. 7.11); evidently the emf generated in the loop is minus the rate of change of flux through the loop:

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}.} \tag{7.13}$$

This is the **flux rule** for motional emf. Apart from its delightful simplicity, it has the virtue of applying to *non*rectangular loops moving in *arbitrary* directions through *non*uniform magnetic fields; in fact, the loop need not even maintain a fixed shape.

Proof: Figure 7.13 shows a loop of wire at time t and also a short time dt later. Suppose we compute the flux at time t, using surface $\mathcal{S}$, and the flux at time $t + dt$, using the surface consisting of $\mathcal{S}$ plus the "ribbon" that connects the new position of the loop to the old. The *change* in flux, then, is

$$d\Phi = \Phi(t + dt) - \Phi(t) = \Phi_{\text{ribbon}} = \int_{\text{ribbon}} \mathbf{B} \cdot d\mathbf{a}.$$

Focus your attention on point P: in time dt it moves to P'. Let $\mathbf{v}$ be the velocity of the *wire*, and $\mathbf{u}$ the velocity of a charge *down* the wire; $\mathbf{w} = \mathbf{v} + \mathbf{u}$ is the resultant velocity of a charge at P. The infinitesimal element of area on the ribbon can be written as

$$d\mathbf{a} = (\mathbf{v} \times d\mathbf{l})\, dt$$

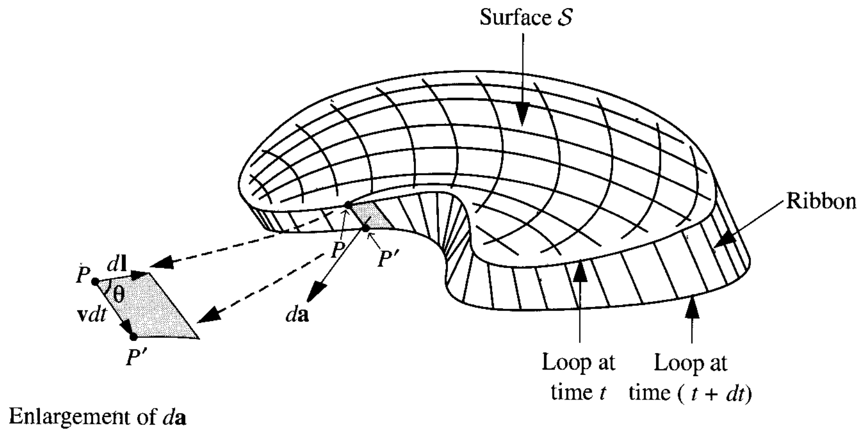


Figure 7.13

(see inset in Fig. 7.13). Therefore

$$\frac{d\Phi}{dt} = \oint \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l}).$$

Since $\mathbf{w} = (\mathbf{v} + \mathbf{u})$ and $\mathbf{u}$ is parallel to $d\mathbf{l}$, we can also write this as

$$\frac{d\Phi}{dt} = \oint \mathbf{B} \cdot (\mathbf{w} \times d\mathbf{l}).$$

Now, the scalar triple-product can be rewritten:

$$\mathbf{B} \cdot (\mathbf{w} \times d\mathbf{l}) = -(\mathbf{w} \times \mathbf{B}) \cdot d\mathbf{l},$$

so

$$\frac{d\Phi}{dt} = -\oint (\mathbf{w} \times \mathbf{B}) \cdot d\mathbf{l}.$$

But $(\mathbf{w} \times \mathbf{B})$ is the magnetic force per unit charge, $\mathbf{f}_{\text{mag}}$, so

$$\frac{d\Phi}{dt} = -\oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l},$$

and the integral of $\mathbf{f}_{\text{mag}}$ is the emf

$$\mathcal{E} = -\frac{d\Phi}{dt}. \qquad \text{qed}$$

There is a sign ambiguity in the definition of emf (Eq. 7.9): Which *way* around the loop are you supposed to integrate? There is a compensatory ambiguity in the definition of *flux* (Eq. 7.12): Which is the positive direction for $d\mathbf{a}$? In applying the flux rule, sign consistency is governed (as always) by your right hand: If your fingers define the positive direction around the loop, then your thumb indicates the direction of $d\mathbf{a}$. Should the emf come out negative, it means the current will flow in the negative direction around the circuit.

The flux rule is a nifty short-cut for calculating motional emf's. It does *not* contain any new physics. Occasionally you will run across problems that cannot be handled by the flux rule; for these one must go back to the Lorentz force law itself.

Example 7.4

A metal disk of radius a rotates with angular velocity ω about a vertical axis, through a uniform field $\mathbf{B}$, pointing up. A circuit is made by connecting one end of a resistor to the axle and the other end to a sliding contact, which touches the outer edge of the disk (Fig. 7.14). Find the current in the resistor.

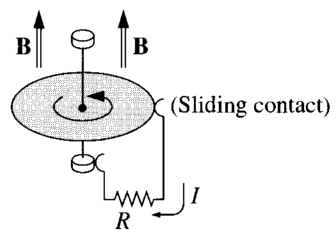


Figure 7.14

Solution: The speed of a point on the disk at a distance s from the axis is $v = \omega s$, so the force per unit charge is $\mathbf{f}_{\text{mag}} = \mathbf{v} \times \mathbf{B} = \omega s B \hat{\mathbf{s}}$. The emf is therefore

$$\mathcal{E} = \int_0^a f_{\text{mag}}\, ds = \omega B \int_0^a s\, ds = \frac{\omega B a^2}{2},$$

and the current is

$$I = \frac{\mathcal{E}}{R} = \frac{\omega B a^2}{2R}.$$

The trouble with the flux rule is that it assumes the current flows along a well-defined path, whereas in this example the current spreads out over the whole disk. It's not even clear what the "flux through the circuit" would *mean* in this context. Even more tricky is the case of **eddy currents**. Take a chunk of aluminum (say), and shake it around in a nonuniform magnetic field. Currents will be generated in the material, and you will feel a kind of "viscous drag"—as though you were pulling the block through molasses (this is the force I called $\mathbf{f}_{\text{pull}}$ in the discussion of motional emf). Eddy currents are notoriously difficult to calculate,[5] but easy and dramatic to demonstrate. You may have witnessed the classic experiment in which an

[5]See, for example, W. M. Saslow, *Am. J. Phys.*, **60**, 693 (1992).

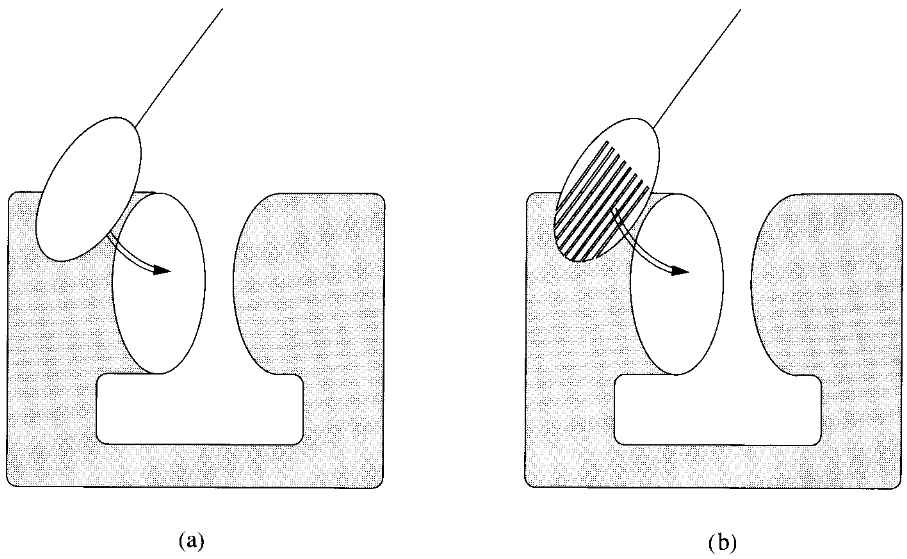


Figure 7.15

aluminum disk mounted as a pendulum on a horizontal axis swings down and passes between the poles of a magnet (Fig. 7.15a). When it enters the field region it suddenly slows way down. To confirm that eddy currents are responsible, one repeats the process using a disk that has many slots cut in it, to prevent the flow of large-scale currents (Fig. 7.15b). This time the disk swings freely, unimpeded by the field.

Problem 7.7 A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance l apart (Fig. 7.16). A resistor R is connected across the rails and a uniform magnetic field $\mathbf{B}$, pointing into the page, fills the entire region.

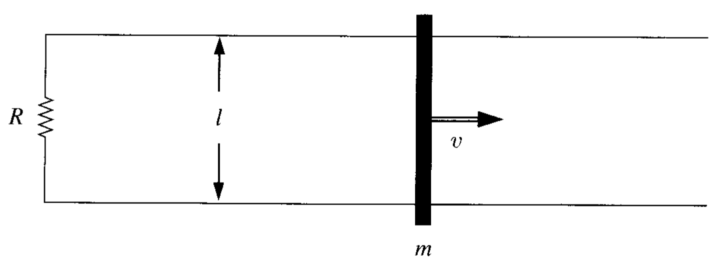


Figure 7.16

(a) If the bar moves to the right at speed v, what is the current in the resistor? In what direction does it flow?

(b) What is the magnetic force on the bar? In what direction?

(c) If the bar starts out with speed v_0 at time $t = 0$, and is left to slide, what is its speed at a later time t?

(d) The initial kinetic energy of the bar was, of course, $\frac{1}{2}mv_0{}^2$. Check that the energy delivered to the resistor is exactly $\frac{1}{2}mv_0{}^2$.

Problem 7.8 A square loop of wire (side a) lies on a table, a distance s from a very long straight wire, which carries a current I, as shown in Fig. 7.17.

(a) Find the flux of **B** through the loop.

(b) If someone now pulls the loop directly away from the wire, at speed v, what emf is generated? In what direction (clockwise or counterclockwise) does the current flow?

(c) What if the loop is pulled to the *right* at speed v, instead of away?

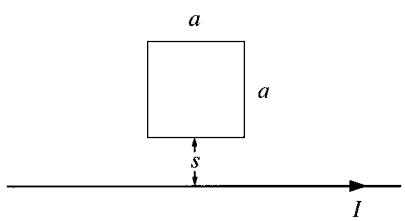


Figure 7.17

Problem 7.9 An infinite number of different surfaces can be fit to a given boundary line, and yet, in defining the magnetic flux through a loop, $\Phi = \int \mathbf{B} \cdot d\mathbf{a}$, I never specified the particular surface to be used. Justify this apparent oversight.

Problem 7.10 A square loop (side a) is mounted on a vertical shaft and rotated at angular velocity ω (Fig. 7.18). A uniform magnetic field **B** points to the right. Find the $\mathcal{E}(t)$ for this **alternating current** generator.

Problem 7.11 A square loop is cut out of a thick sheet of aluminum. It is then placed so that the top portion is in a uniform magnetic field **B**, and allowed to fall under gravity (Fig. 7.19). (In the diagram, shading indicates the field region; **B** points into the page.) If the magnetic field is 1 T (a pretty standard laboratory field), find the terminal velocity of the loop (in m/s). Find the velocity of the loop as a function of time. How long does it take (in seconds) to reach, say, 90% of the terminal velocity? What would happen if you cut a tiny slit in the ring, breaking the circuit? [*Note:* The dimensions of the loop cancel out; determine the actual *numbers*, in the units indicated.]

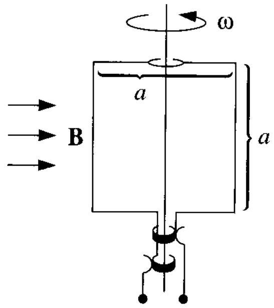


Figure 7.18

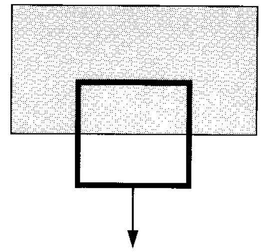


Figure 7.19

7.2 Electromagnetic Induction

7.2.1 Faraday's Law

In 1831 Michael Faraday reported on a series of experiments, including three that (with some violence to history) can be characterized as follows:

Experiment 1. He pulled a loop of wire to the right through a magnetic field (Fig. 7.20a). A current flowed in the loop.

Experiment 2. He moved the *magnet* to the *left*, holding the loop still (Fig. 7.20b). Again, a current flowed in the loop.

Experiment 3. With both the loop and the magnet at rest (Fig. 7.20c), he changed the *strength* of the field (he used an electromagnet, and varied the current in the coil). Once again, current flowed in the loop.

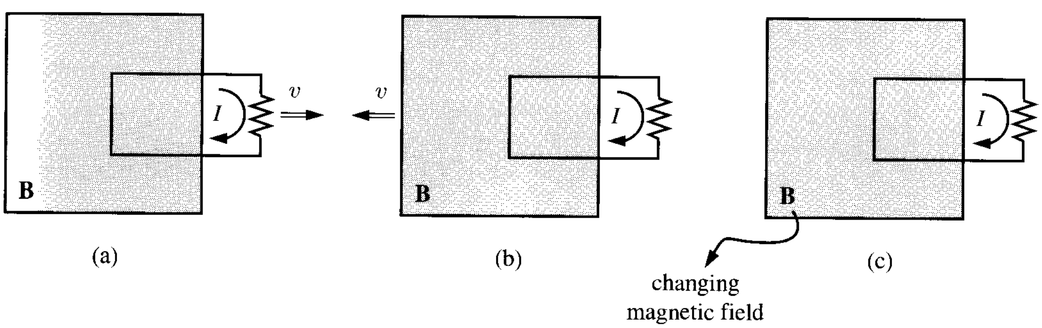


Figure 7.20

The first experiment, of course, is an example of motional emf, conveniently expressed by the flux rule:

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

I don't think it will surprise you to learn that exactly the same emf arises in Experiment 2—all that really matters is the *relative* motion of the magnet and the loop. Indeed, in the light of special relativity is *has* to be so. But Faraday knew nothing of relativity, and in classical electrodynamics this simple reciprocity is a coincidence, with remarkable implications. For if the *loop* moves, it's a *magnetic* force that sets up the emf, but if the loop is *stationary*, the force *cannot* be magnetic—stationary charges experience no magnetic forces. In that case, what *is* responsible? What sort of field exerts a force on charges at rest? Well, *electric* fields do, of course, but in this case there doesn't seem to be any electric field in sight.

Faraday had an ingenious inspiration:

A changing magnetic field induces an electric field.

It is this "induced" electric field that accounts for the emf in Experiment 2.[6] Indeed, if (as Faraday found empirically) the emf is again equal to the rate of change of the flux,

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}, \tag{7.14}$$

then $\mathbf{E}$ is related to the change in $\mathbf{B}$ by the equation

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}. \tag{7.15}$$

This is **Faraday's law**, in integral form. We can convert it to differential form by applying Stokes' theorem:

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.} \tag{7.16}$$

Note that Faraday's law reduces to the old rule $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ (or, in differential form, $\nabla \times \mathbf{E} = 0$) in the static case (constant $\mathbf{B}$) as, of course, it should.

In Experiment 3 the magnetic field changes for entirely different reasons, but according to Faraday's law an electric field will again be induced, giving rise to an emf $-d\Phi/dt$. Indeed, one can subsume all three cases (and for that matter any combination of them) into a kind of **universal flux rule**:

> Whenever (and for whatever reason) the magnetic flux through a loop changes, an emf
>
> $$\mathcal{E} = -\frac{d\Phi}{dt} \tag{7.17}$$
>
> will appear in the loop.

[6] You might argue that the magnetic field in Experiment 2 is not really *changing*—just *moving*. What I mean is that if you sit at a *fixed location*, the field *does* change, as the magnet passes by.

Many people call *this* "Faraday's law." Maybe I'm overly fastidious, but I find this confusing. There are really *two* totally different mechanisms underlying Eq. 7.17, and to identify them both as "Faraday's law" is a little like saying that because identical twins look alike we ought to call them by the same name. In Faraday's first experiment it's the Lorentz force law at work; the emf is *magnetic*. But in the other two it's an *electric* field (induced by the changing magnetic field) that does the job. Viewed in this light, it is quite astonishing that all three processes yield the same formula for the emf. In fact, it was precisely this "coincidence" that led Einstein to the special theory of relativity—he sought a deeper understanding of what is, in classical electrodynamics, a peculiar accident. But that's a story for Chapter 12. In the meantime I shall reserve the term "Faraday's law" for electric fields induced by changing magnetic fields, and I do *not* regard Experiment 1 as an instance of Faraday's law.

Example 7.5

A long cylindrical magnet of length L and radius a carries a uniform magnetization $\mathbf{M}$ parallel to its axis. It passes at constant velocity v through a circular wire ring of slightly larger diameter (Fig. 7.21). Graph the emf induced in the ring, as a function of time.

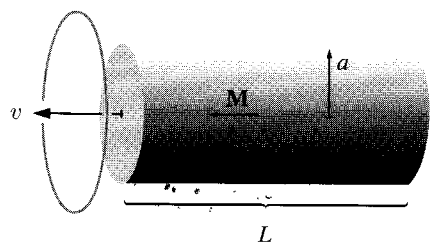


Figure 7.21

Solution: The magnetic field is the same as that of a long solenoid with surface current $\mathbf{K}_b = M\hat{\boldsymbol{\phi}}$. So the field inside is $\mathbf{B} = \mu_0\mathbf{M}$, except near the ends, where it starts to spread out. The flux through the ring is zero when the magnet is far away; it builds up to a maximum of $\mu_0 M\pi a^2$ as the leading end passes through; and it drops back to zero as the trailing end emerges (Fig. 7.22a). The emf is (minus) the derivative of Φ with respect to time, so it consists of two spikes, as shown in Fig. 7.22b.

Keeping track of the *signs* in Faraday's law can be a real headache. For instance, in Ex. 7.5 we would like to know which *way* around the ring the induced current flows. In principle, the right-hand rule does the job (we called Φ positive to the left, in Fig. 7.22a, so the positive direction for current in the ring is counterclockwise, as viewed from the left; since the first spike in Fig. 7.22b is *negative*, the first current pulse flows *clockwise*, and the second counterclockwise). But there's a handy rule, called **Lenz's law**, whose sole purpose is to help you get the directions right:[7]

[7]Lenz's law applies to *motional* emf's, too, but for them it is usually easier to get the direction of the current from the Lorentz force law.

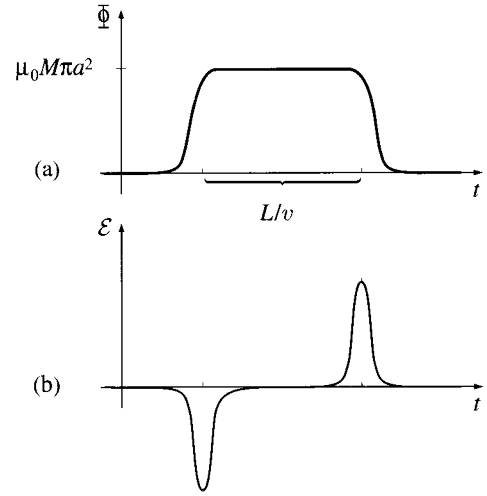


Figure 7.22

Nature abhors a change in flux.

The induced current will flow in such a direction that the flux *it* produces tends to cancel the change. (As the front end of the magnet in Ex. 7.5 enters the ring, the flux increases, so the current in the ring must generate a field to the *right*—it therefore flows *clockwise*.) Notice that it is the *change* in flux, not the flux itself, that nature abhors (when the tail end of the magnet exits the ring, the flux *drops*, so the induced current flows *counterclockwise*, in an effort to restore it). Faraday induction is a kind of "inertial" phenomenon: A conducting loop "likes" to maintain a constant flux through it; if you try to *change* the flux, the loop responds by sending a current around in such a direction as to frustrate your efforts. (It doesn't *succeed* completely; the flux produced by the induced current is typically only a tiny fraction of the original. All Lenz's law tells you is the *direction* of the flow.)

Example 7.6

The "jumping ring" demonstration. If you wind a solenoidal coil around an iron core (the iron is there to beef up the magnetic field), place a metal ring on top, and plug it in, the ring will jump several feet in the air (Fig. 7.23). Why?

Solution: *Before* you turned on the current, the flux through the ring was *zero*. *Afterward* a flux appeared (upward, in the diagram), and the emf generated in the ring led to a current (in the ring) which, according to Lenz's law, was in such a direction that *its* field tended to cancel this new flux. This means that the current in the loop is *opposite* to the current in the solenoid. And opposite currents repel, so the ring flies off.[8]

[8]For further discussion of the jumping ring (and the related "floating ring"), see C. S. Schneider and J. P. Ertel. *Am. J. Phys.* **66**, 686 (1998).

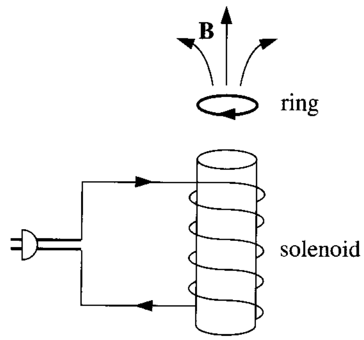


Figure 7.23

Problem 7.12 A long solenoid, of radius a, is driven by an alternating current, so that the field inside is sinusoidal: $\mathbf{B}(t) = B_0 \cos(\omega t)\,\hat{\mathbf{z}}$. A circular loop of wire, of radius $a/2$ and resistance R, is placed inside the solenoid, and coaxial with it. Find the current induced in the loop, as a function of time.

Problem 7.13 A square loop of wire, with sides of length a, lies in the first quadrant of the xy plane, with one corner at the origin. In this region there is a nonuniform time-dependent magnetic field $\mathbf{B}(y, t) = ky^3t^2\,\hat{\mathbf{z}}$ (where k is a constant). Find the emf induced in the loop.

Problem 7.14 As a lecture demonstration a short cylindrical bar magnet is dropped down a vertical aluminum pipe of slightly larger diameter, about 2 meters long. It takes several seconds to emerge at the bottom, whereas an otherwise identical piece of *unmagnetized* iron makes the trip in a fraction of a second. Explain why the magnet falls more slowly.

7.2.2 The Induced Electric Field

What Faraday's discovery tells us is that there are really two distinct kinds of electric fields: those attributable directly to electric charges, and those associated with changing magnetic fields.[9] The former can be calculated (in the static case) using Coulomb's law; the latter can be found by exploiting the analogy between Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

[9]You could, I suppose, introduce an entirely new word to denote the field generated by a changing $\mathbf{B}$. Electrodynamics would then involve *three* fields: $\mathbf{E}$-fields, produced by electric charges $[\nabla \cdot \mathbf{E} = (1/\epsilon_0)\rho,\ \nabla \times \mathbf{E} = 0]$; $\mathbf{B}$-fields, produced by electric currents $[\nabla \cdot \mathbf{B} = 0,\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}]$; and $\mathbf{G}$-fields, produced by changing magnetic fields $[\nabla \cdot \mathbf{G} = 0,\ \nabla \times \mathbf{G} = -\partial \mathbf{B}/\partial t]$. Because $\mathbf{E}$ and $\mathbf{G}$ exert *forces* in the same way $[\mathbf{F} = q(\mathbf{E} + \mathbf{G})]$, it is tidier to regard their sum as a *single* entity and call the whole thing "the electric field."

and Ampère's law,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Of course, the curl alone is not enough to determine a field—you must also specify the divergence. But as long as **E** is a *pure* Faraday field, due exclusively to a changing **B** (with $\rho = 0$), Gauss's law says

$$\nabla \cdot \mathbf{E} = 0,$$

while for magnetic fields, of course,

$$\nabla \cdot \mathbf{B} = 0$$

always. So the parallel is complete, and I conclude that *Faraday-induced electric fields are determined by* $-(\partial \mathbf{B}/\partial t)$ *in exactly the same way as magnetostatic fields are determined by* $\mu_0 \mathbf{J}$.

In particular, if symmetry permits, we can use all the tricks associated with Ampère's law in integral form,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}},$$

only this time it's *Faraday's* law in integral form:

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}. \tag{7.18}$$

The rate of change of (magnetic) flux through the Amperian loop plays the role formerly assigned to $\mu_0 I_{\text{enc}}$.

Example 7.7

A uniform magnetic field $\mathbf{B}(t)$, pointing straight up, fills the shaded circular region of Fig. 7.24. If **B** is changing with time, what is the induced electric field?

Solution: **E** points in the circumferential direction, just like the *magnetic* field inside a long straight wire carrying a uniform *current* density. Draw an Amperian loop of radius s, and apply Faraday's law:

$$\oint \mathbf{E} \cdot d\mathbf{l} = E(2\pi s) = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\pi s^2 B(t)\right) = -\pi s^2 \frac{dB}{dt}.$$

Therefore

$$\mathbf{E} = -\frac{s}{2}\frac{dB}{dt}\,\hat{\boldsymbol{\phi}}.$$

If **B** is *increasing*, **E** runs *clockwise*, as viewed from above.

Example 7.8

A line charge λ is glued onto the rim of a wheel of radius b, which is then suspended horizontally, as shown in Fig. 7.25, so that it is free to rotate (the spokes are made of some nonconducting material—wood, maybe). In the central region, out to radius a, there is a uniform magnetic field $\mathbf{B}_0$, pointing up. Now someone turns the field off. What happens?

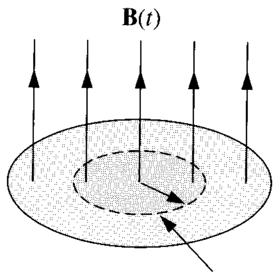


Figure 7.24

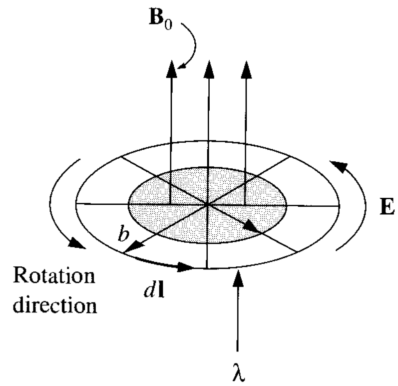


Figure 7.25

Solution: The changing magnetic field will induce an electric field, curling around the axis of the wheel. This electric field exerts a force on the charges at the rim, and the wheel starts to turn. According to Lenz's law, it will rotate in such a direction that *its* field tends to restore the upward flux. The motion, then, is counterclockwise, as viewed from above.

Quantitatively, Faraday's law says

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\pi a^2 \frac{dB}{dt}.$$

Now, the torque on a segment of length $d\mathbf{l}$ is $(\mathbf{r} \times \mathbf{F})$, or $b\lambda E\, dl$. The total torque on the wheel is therefore

$$N = b\lambda \oint E\, dl = -b\lambda\pi a^2 \frac{dB}{dt},$$

and the angular momentum imparted to the wheel is

$$\int N dt = -\lambda\pi a^2 b \int_{B_0}^{0} dB = \lambda\pi a^2 b B_0.$$

It doesn't matter how fast or slow you turn off the field; the ultimate angular velocity of the wheel is the same regardless. (If you find yourself wondering where this angular momentum *came* from, you're getting ahead of the story! Wait for the next chapter.)

A final word on this example: It's the *electric* field that did the rotating. To convince you of this I deliberately set things up so that the *magnetic* field is always *zero* at the location of the charge (on the rim). The experimenter may tell you she never put in any electric fields—all she did was switch off the magnetic field. But when she did that, an electric field automatically appeared, and it's this electric field that turned the wheel.

I must warn you, now, of a small fraud that tarnishes many applications of Faraday's law: Electromagnetic induction, of course, occurs only when the magnetic fields are *changing*, and yet we would like to use the apparatus of magneto*statics* (Ampère's law, the Biot-Savart law, and the rest) to *calculate* those magnetic fields. Technically, any result derived in this way is only approximately correct. But in practice the error is usually negligible unless the field fluctuates extremely rapidly, or you are interested in points very far from the source. Even the case of a wire snipped by a pair of scissors (Prob. 7.18) is *static enough* for Ampère's law to apply. This régime, in which magnetostatic rules can be used to calculate the magnetic field on the right hand side of Faraday's law, is called **quasistatic**. Generally speaking, it is only when we come to electromagnetic waves and radiation that we must worry seriously about the breakdown of magnetostatics itself.

Example 7.9

An infinitely long straight wire carries a slowly varying current $I(t)$. Determine the induced electric field, as a function of the distance s from the wire.[10]

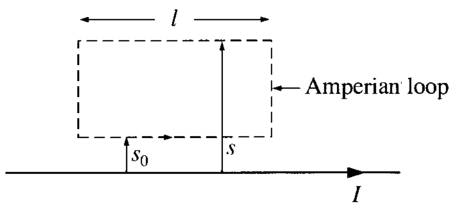


Figure 7.26

Solution: In the quasistatic approximation, the magnetic field is $(\mu_0 I/2\pi s)$, and it circles around the wire. Like the **B**-field of a solenoid, **E** here runs parallel to the axis. For the rectangular "Amperian loop" in Fig. 7.26, Faraday's law gives:

$$\begin{aligned}\oint \mathbf{E}\cdot d\mathbf{l} &= E(s_0)l - E(s)l = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a} \\ &= -\frac{\mu_0 l}{2\pi}\frac{dI}{dt}\int_{s_0}^{s}\frac{1}{s'}\,ds' = -\frac{\mu_0 l}{2\pi}\frac{dI}{dt}(\ln s - \ln s_0).\end{aligned}$$

Thus

$$\mathbf{E}(s) = \left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln s + K\right]\hat{\mathbf{z}}, \tag{7.19}$$

where K is a constant (that is to say, it is independent of s—it might still be a function of t). The actual *value* of K depends on the whole history of the function $I(t)$—we'll see some examples in Chapter 10.

[10]This example is artificial, and not just in the usual sense of involving infinite wires, but in a more subtle respect. It assumes that the current is the same (at any given instant) all the way down the line. This is a safe assumption for the *short* wires in typical electric circuits, but not (in practice) for *long* wires (transmission lines), unless you supply a distributed and synchronized driving mechanism. But never mind—the problem doesn't inquire how you would *produce* such a current; it only asks what *fields* would result if you *did*. (Variations on this problem are discussed in M. A. Heald, *Am. J. Phys.* **54**, 1142 (1986), and references cited therein.)

Equation 7.19 has the peculiar implication that E blows up as s goes to infinity. *That* can't be true ... What's gone wrong? *Answer:* We have overstepped the limits of the quasistatic approximation. As we shall see in Chapter 9, electromagnetic "news" travels at the speed of light, and at large distances **B** depends not on the current *now*, but on the current *as it was* at some earlier time (indeed, a whole *range* of earlier times, since different points on the wire are different distances away). If τ is the time it takes I to change substantially, then the quasistatic approximation should hold only for

$$s \ll c\tau, \tag{7.20}$$

and hence Eq. 7.19 simply does not apply, at extremely large s.

Problem 7.15 A long solenoid with radius a and n turns per unit length carries a time-dependent current $I(t)$ in the $\hat{\boldsymbol{\phi}}$ direction. Find the electric field (magnitude and direction) at a distance s from the axis (both inside and outside the solenoid), in the quasistatic approximation.

Problem 7.16 An alternating current $I = I_0 \cos(\omega t)$ flows down a long straight wire, and returns along a coaxial conducting tube of radius a.

(a) In what *direction* does the induced electric field point (radial, circumferential, or longitudinal)?

(b) Assuming that the field goes to zero as $s \to \infty$, find $\mathbf{E}(s, t)$. [Incidentally, this is not at all the way electric fields *actually* behave in coaxial cables, for reasons suggested in footnote 10. See Sect. 9.5.3, or J. G. Cherveniak, *Am. J. Phys.*, **54**, 946 (1986), for a more realistic treatment.]

Problem 7.17 A long solenoid of radius a, carrying n turns per unit length, is looped by a wire with resistance R, as shown in Fig. 7.27.

(a) If the current in the solenoid is increasing at a constant rate ($dI/dt = k$), what current flows in the loop, and which way (left or right) does it pass through the resistor?

(b) If the current I in the solenoid is constant but the solenoid is pulled out of the loop and reinserted in the opposite direction, what total charge passes through the resistor?

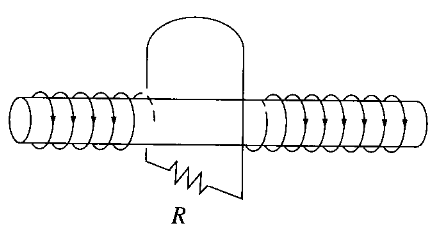


Figure 7.27

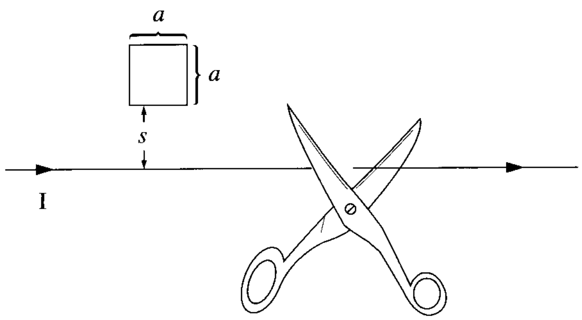


Figure 7.28

Problem 7.18 A square loop, side a, resistance R, lies a distance s from an infinite straight wire that carries current I (Fig. 7.28). Now someone cuts the wire, so that I drops to zero. In what direction does the induced current in the square loop flow, and what total charge passes a given point in the loop during the time this current flows? If you don't like the scissors model, turn the current down *gradually*:

$$I(t) = \begin{cases} (1-\alpha t)I, & \text{for } 0 \leq t \leq 1/\alpha, \\ 0, & \text{for } t > 1/\alpha. \end{cases}$$

Problem 7.19 A toroidal coil has a rectangular cross section, with inner radius a, outer radius $a + w$, and height h. It carries a total of N tightly wound turns, and the current is increasing at a constant rate ($dI/dt = k$). If w and h are both much less than a, find the electric field at a point z above the center of the toroid. [*Hint:* exploit the analogy between Faraday fields and magnetostatic fields, and refer to Ex. 5.6.]

7.2.3 Inductance

Suppose you have two loops of wire, at rest (Fig. 7.29). If you run a steady current I_1 around loop 1, it produces a magnetic field $\mathbf{B}_1$. Some of the field lines pass through loop 2; let Φ_2 be the flux of $\mathbf{B}_1$ through 2. You might have a tough time actually *calculating* $\mathbf{B}_1$, but a glance at the Biot-Savart law,

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\mathbf{l}_1 \times \hat{\boldsymbol{\imath}}}{\imath^2},$$

reveals one significant fact about this field: *It is proportional to the current* I_1. Therefore, so too is the flux through loop 2:

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2.$$

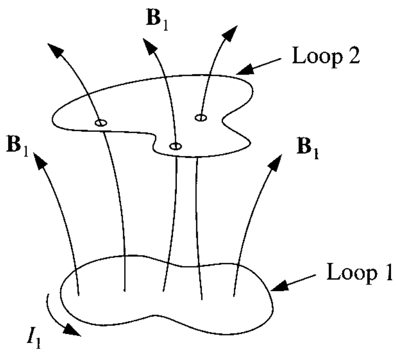


Figure 7.29

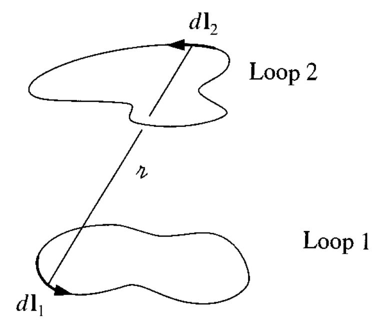


Figure 7.30

Thus

$$\Phi_2 = M_{21} I_1, \tag{7.21}$$

where M_{21} is the constant of proportionality; it is known as the **mutual inductance** of the two loops.

There is a cute formula for the mutual inductance, which you can derive by expressing the flux in terms of the vector potential and invoking Stokes' theorem:

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2 = \int (\nabla \times \mathbf{A}_1) \cdot d\mathbf{a}_2 = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2.$$

Now, according to Eq. 5.63,

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{\imath},$$

and hence

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\mathbf{l}_1}{\imath} \right) \cdot d\mathbf{l}_2.$$

Evidently

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\imath}. \tag{7.22}$$

This is the **Neumann formula**; it involves a double line integral—one integration around loop 1, the other around loop 2 (Fig. 7.30). It's not very useful for practical calculations, but it does reveal two important things about mutual inductance:

1. M_{21} is a purely geometrical quantity, having to do with the sizes, shapes, and relative positions of the two loops.

2. The integral in Eq. 7.22 is unchanged if we switch the roles of loops 1 and 2; it follows that

$$M_{21} = M_{12}. \tag{7.23}$$

This is an astonishing conclusion: *Whatever the shapes and positions of the loops, the flux through 2 when we run a current I around 1 is identical to the flux through 1 when we send the same current I around 2.* We may as well drop the subscripts and call them both M.

Example 7.10

A short solenoid (length l and radius a, with n_1 turns per unit length) lies on the axis of a very long solenoid (radius b, n_2 turns per unit length) as shown in Fig. 7.31. Current I flows in the short solenoid. What is the flux through the long solenoid?

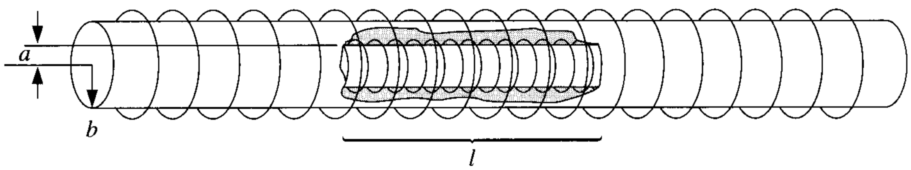


Figure 7.31

Solution: Since the inner solenoid is short, it has a very complicated field; moreover, it puts a different amount of flux through each turn of the outer solenoid. It would be a *miserable* task to compute the total flux this way. However, if we exploit the equality of the mutual inductances, the problem becomes very easy. Just look at the reverse situation: run the current I through the *outer* solenoid, and calculate the flux through the *inner* one. The field inside the long solenoid is constant:

$$B = \mu_0 n_2 I$$

(Eq. 5.57), so the flux through a single loop of the short solenoid is

$$B\pi a^2 = \mu_0 n_2 I \pi a^2.$$

There are $n_1 l$ turns in all, so the total flux through the inner solenoid is

$$\Phi = \mu_0 \pi a^2 n_1 n_2 l I.$$

This is also the flux a current I in the *short* solenoid would put through the *long* one, which is what we set out to find. Incidentally, the mutual inductance, in this case, is

$$M = \mu_0 \pi a^2 n_1 n_2 l.$$

Suppose now that you *vary* the current in loop 1. The flux through loop 2 will vary accordingly, and Faraday's law says this changing flux will induce an emf in loop 2:

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}. \tag{7.24}$$

(In quoting Eq. 7.21—which was based on the Biot-Savart law—I am tacitly assuming that the currents change slowly enough for the configuration to be considered quasistatic.) What

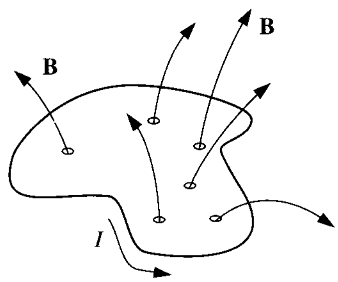


Figure 7.32

a remarkable thing: Every time you change the current in loop 1, an induced current flows in loop 2—even though there are no wires connecting them!

Come to think of it, a changing current not only induces an emf in any nearby loops, it also induces an emf in the source loop *itself* (Fig 7.32). Once again, the field (and therefore also the flux) is proportional to the current:

$$\Phi = LI. \tag{7.25}$$

The constant of proportionality L is called the **self-inductance** (or simply the **inductance**) of the loop. As with M, it depends on the geometry (size and shape) of the loop. If the current changes, the emf induced in the loop is

$$\mathcal{E} = -L\frac{dI}{dt}. \tag{7.26}$$

Inductance is measured in **henries** (H); a henry is a volt-second per ampere.

Example 7.11

Find the self-inductance of a toroidal coil with rectangular cross section (inner radius a, outer radius b, height h), which carries a total of N turns.

Solution: The magnetic field inside the toroid is (Eq. 5.58)

$$B = \frac{\mu_0 N I}{2\pi s}.$$

The flux through a single turn (Fig. 7.33) is

$$\int \mathbf{B} \cdot d\mathbf{a} = \frac{\mu_0 N I}{2\pi} h \int_a^b \frac{1}{s}\, ds = \frac{\mu_0 N I h}{2\pi} \ln\left(\frac{b}{a}\right).$$

The *total* flux is N times this, so the self-inductance (Eq. 7.25) is

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right). \tag{7.27}$$

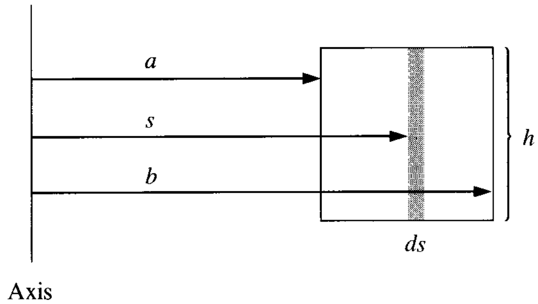


Figure 7.33

Inductance (like capacitance) is an intrinsically *positive* quantity. Lenz's law, which is enforced by the minus sign in Eq. 7.26, dictates that the emf is in such a direction as to *oppose* any *change in current*. For this reason, it is called a **back emf**. Whenever you try to alter the current in a wire, you must fight against this back emf. Thus inductance plays somewhat the same role in electric circuits that *mass* plays in mechanical systems: The greater L is, the harder it is to change the current, just as the larger the mass, the harder it is to change an object's velocity.

Example 7.12

Suppose a current I is flowing around a loop when someone suddenly cuts the wire. The current drops "instantaneously" to zero. This generates a whopping back emf, for although I may be small, dI/dt is enormous. That's why you often draw a spark when you unplug an iron or toaster—electromagnetic induction is desperately trying to keep the current going, even if it has to jump the gap in the circuit.

Nothing so dramatic occurs when you plug *in* a toaster or iron. In this case induction opposes the sudden *increase* in current, prescribing instead a smooth and continuous buildup. Suppose, for instance, that a battery (which supplies a constant emf $\mathcal{E}_0$) is connected to a circuit of resistance R and inductance L (Fig. 7.34). What current flows?

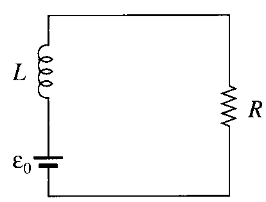


Figure 7.34

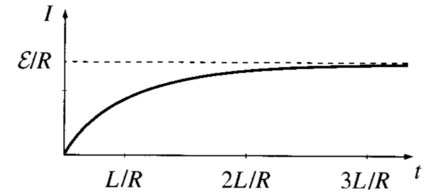


Figure 7.35

Solution: The total emf in this circuit is that provided by the battery plus that resulting from the self-inductance. Ohm's law, then, says[11]

$$\mathcal{E}_0 - L\frac{dI}{dt} = IR.$$

This is a first-order differential equation for I as a function of time. The general solution, as you can easily derive for yourself, is

$$I(t) = \frac{\mathcal{E}_0}{R} + ke^{-(R/L)t},$$

where k is a constant to be determined by the initial conditions. In particular, if the circuit is "plugged in" at time $t = 0$ (so $I(0) = 0$), then k has the value $-\mathcal{E}_0/R$, and

$$I(t) = \frac{\mathcal{E}_0}{R}\left[1 - e^{-(R/L)t}\right]. \tag{7.28}$$

This function is plotted in Fig. 7.35. Had there been no inductance in the circuit, the current would have jumped immediately to $\mathcal{E}_0/R$. In practice, *every* circuit has *some* self-inductance, and the current approaches $\mathcal{E}_0/R$ asymptotically. The quantity $\tau \equiv L/R$ is called the **time constant**; it tells you how long the current takes to reach a substantial fraction (roughly two-thirds) of its final value.

Problem 7.20 A small loop of wire (radius a) lies a distance z above the center of a large loop (radius b), as shown in Fig. 7.36. The planes of the two loops are parallel, and perpendicular to the common axis.

(a) Suppose current I flows in the big loop. Find the flux through the little loop. (The little loop is so small that you may consider the field of the big loop to be essentially constant.)

(b) Suppose current I flows in the little loop. Find the flux through the big loop. (The little loop is so small that you may treat it as a magnetic dipole.)

(c) Find the mutual inductances, and confirm that $M_{12} = M_{21}$.

[11]Notice that $-L(dI/dt)$ goes on the *left* side of the equation—it is part of the emf that (together with $\mathcal{E}_0$) establishes the voltage across the resistor (Eq. 7.10).

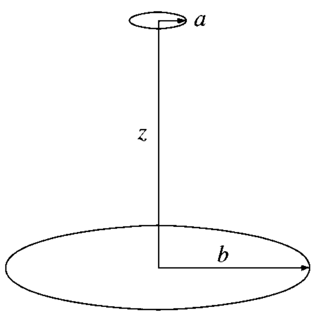


Figure 7.36

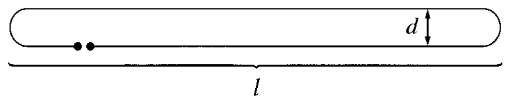


Figure 7.37

Problem 7.21 A square loop of wire, of side a, lies midway between two long wires, $3a$ apart, and in the same plane. (Actually, the long wires are sides of a large rectangular loop, but the short ends are so far away that they can be neglected.) A clockwise current I in the square loop is gradually increasing: $dI/dt = k$ (a constant). Find the emf induced in the big loop. Which way will the induced current flow?

Problem 7.22 Find the self-inductance per unit length of a long solenoid, of radius R, carrying n turns per unit length.

Problem 7.23 Try to compute the self-inductance of the "hairpin" loop shown in Fig. 7.37. (Neglect the contribution from the ends; most of the flux comes from the long straight section.) You'll run into a snag that is characteristic of many self-inductance calculations. To get a definite answer, assume the wire has a tiny radius ϵ, and ignore any flux through the wire itself.

Problem 7.24 An alternating current $I_0 \cos(\omega t)$ (amplitude 0.5 A, frequency 60 Hz) flows down a straight wire, which runs along the axis of a toroidal coil with rectangular cross section (inner radius 1 cm, outer radius 2 cm, height 1 cm, 1000 turns). The coil is connected to a 500 Ω resistor.

(a) In the quasistatic approximation, what emf is induced in the toroid? Find the current, $I_r(t)$, in the resistor.

(b) Calculate the back emf in the coil, due to the current $I_r(t)$. What is the ratio of the amplitudes of this back emf and the "direct" emf in (a)?

Problem 7.25 A capacitor C is charged up to a potential V and connected to an inductor L, as shown schematically in Fig. 7.38. At time $t = 0$ the switch S is closed. Find the current in the circuit as a function of time. How does your answer change if a resistor R is included in series with C and L?

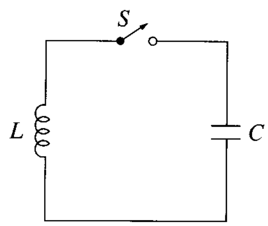


Figure 7.38

7.2.4 Energy in Magnetic Fields

It takes a certain amount of *energy* to start a current flowing in a circuit. I'm not talking about the energy delivered to the resistors and converted into heat—that is irretrievably lost as far as the circuit is concerned and can be large or small, depending on how long you let the current run. What I am concerned with, rather, is the work you must do *against the back emf* to get the current going. This is a *fixed* amount, and it is *recoverable*: you get it back when the current is turned off. In the meantime it represents energy latent in the circuit; as we'll see in a moment, it can be regarded as energy stored in the magnetic field.

The work done on a unit charge, against the back emf, in one trip around the circuit is $-\mathcal{E}$ (the minus sign records the fact that this is the work done *by you against* the emf, not the work done by the emf). The amount of charge per unit time passing down the wire is I. So the total work done per unit time is

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}.$$

If we start with zero current and build it up to a final value I, the work done (integrating the last equation over time) is

$$\boxed{W = \frac{1}{2}LI^2.} \tag{7.29}$$

It does not depend on how *long* we take to crank up the current, only on the geometry of the loop (in the form of L) and the final current I.

There is a nicer way to write W, which has the advantage that it is readily generalized to surface and volume currents. Remember that the flux Φ through the loop is equal to LI (Eq. 7.25). On the other hand,

$$\Phi = \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = \int_{\mathcal{S}} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{A} \cdot d\mathbf{l},$$

where $\mathcal{P}$ is the perimeter of the loop and $\mathcal{S}$ is any surface bounded by $\mathcal{P}$. Thus,

$$LI = \oint \mathbf{A} \cdot d\mathbf{l},$$

and therefore

$$W = \frac{1}{2} I \oint \mathbf{A} \cdot d\mathbf{l}.$$

The vector sign might as well go on the I:

$$W = \frac{1}{2} \oint (\mathbf{A} \cdot \mathbf{I})\, dl. \tag{7.30}$$

In this form, the generalization to volume currents is obvious:

$$W = \frac{1}{2} \int_{\mathcal{V}} (\mathbf{A} \cdot \mathbf{J})\, d\tau. \tag{7.31}$$

But we can do even better, and express W entirely in terms of the magnetic field: Ampère's law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, lets us eliminate $\mathbf{J}$:

$$W = \frac{1}{2\mu_0} \int \mathbf{A} \cdot (\nabla \times \mathbf{B})\, d\tau. \tag{7.32}$$

Integration by parts enables us to move the derivative from $\mathbf{B}$ to $\mathbf{A}$; specifically, product rule 6 states that

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}),$$

so

$$\mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{B}).$$

Consequently,

$$\begin{aligned} W &= \frac{1}{2\mu_0} \left[\int B^2\, d\tau - \int \nabla \cdot (\mathbf{A} \times \mathbf{B})\, d\tau \right] \\ &= \frac{1}{2\mu_0} \left[\int_{\mathcal{V}} B^2\, d\tau - \oint_{\mathcal{S}} (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a} \right], \end{aligned} \tag{7.33}$$

where $\mathcal{S}$ is the surface bounding the volume $\mathcal{V}$.

Now, the integration in Eq. 7.31 is to be taken over the *entire volume occupied by the current*. But any region *larger* than this will do just as well, for $\mathbf{J}$ is zero out there anyway. In Eq. 7.33 the larger the region we pick the greater is the contribution from the volume integral, and therefore the smaller is that of the surface integral (this makes sense: as the surface gets farther from the current, both $\mathbf{A}$ and $\mathbf{B}$ decrease). In particular, if we agree to integrate over *all* space, then the surface integral goes to zero, and we are left with

$$\boxed{W = \frac{1}{2\mu_0} \int_{\text{all space}} B^2\, d\tau.} \tag{7.34}$$

In view of this result, we say the energy is "stored in the magnetic field," in the amount $(B^2/2\mu_0)$ per unit volume. This is a nice way to think of it, though someone looking at Eq. 7.31 might prefer to say that the energy is stored in the *current distribution*, in the amount $\frac{1}{2}(\mathbf{A} \cdot \mathbf{J})$ per unit volume. The distinction is one of bookkeeping; the important quantity is the total energy W, and we shall not worry about where (if anywhere) the energy is "located."

You might find it strange that is takes energy to set up a magnetic field—after all, magnetic fields *themselves* do no work. The point is that producing a magnetic field, where previously there was none, requires *changing* the field, and a changing **B**-field, according to Faraday, induces an *electric* field. The latter, of course, *can* do work. In the beginning there is no **E**, and at the end there is no **E**; but in between, while **B** is building up, there *is* an **E**, and it is against *this* that the work is done. (You see why I could not calculate the energy stored in a magnetostatic field back in Chapter 5.) In the light of this, it is extraordinary how similar the magnetic energy formulas are to their electrostatic counterparts:

$$W_{\text{elec}} = \frac{1}{2}\int (V\rho)\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau, \qquad \text{(2.43 and 2.45)}$$

$$W_{\text{mag}} = \frac{1}{2}\int (\mathbf{A} \cdot \mathbf{J})\,d\tau = \frac{1}{2\mu_0}\int B^2\,d\tau. \qquad \text{(7.31 and 7.34)}$$

Example 7.13

A long coaxial cable carries current I (the current flows down the surface of the inner cylinder, radius a, and back along the outer cylinder, radius b) as shown in Fig. 7.39. Find the magnetic energy stored in a section of length l.

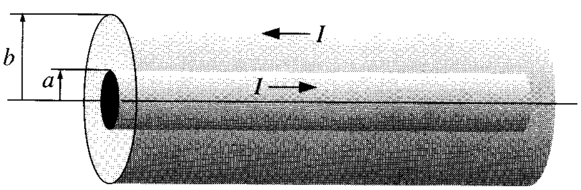


Figure 7.39

Solution: According to Ampère's law, the field between the cylinders is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}.$$

Elsewhere, the field is zero. Thus, the energy per unit volume is

$$\frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi s}\right)^2 = \frac{\mu_0 I^2}{8\pi^2 s^2}.$$

The energy in a cylindrical shell of length l, radius s, and thickness ds, then, is

$$\left(\frac{\mu_0 I^2}{8\pi^2 s^2}\right) 2\pi l s\,ds = \frac{\mu_0 I^2 l}{4\pi}\left(\frac{ds}{s}\right).$$

Integrating from a to b, we have:

$$W = \frac{\mu_0 I^2 l}{4\pi} \ln\left(\frac{b}{a}\right).$$

By the way, this suggests a very simple way to calculate the self-inductance of the cable. According to Eq. 7.29, the energy can also be written as $\frac{1}{2}LI^2$. Comparing the two expressions,[12]

$$L = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right).$$

This method of calculating self-inductance is especially useful when the current is not confined to a single path, but spreads over some surface or volume. In such cases different parts of the current may circle different amounts of flux, and it can be very tricky to get L directly from Eq. 7.25.

Problem 7.26 Find the energy stored in a section of length l of a long solenoid (radius R, current I, n turns per unit length), (a) using Eq. 7.29 (you found L in Prob. 7.22); (b) using Eq. 7.30 (we worked out **A** in Ex. 5.12); (c) using Eq. 7.34; (d) using Eq. 7.33 (take as your volume the cylindrical tube from radius $a < R$ out to radius $b > R$).

Problem 7.27 Calculate the energy stored in the toroidal coil of Ex. 7.11, by applying Eq. 7.34. Use the answer to check Eq. 7.27.

Problem 7.28 A long cable carries current in one direction uniformly distributed over its (circular) cross section. The current returns along the surface (there is a very thin insulating sheath separating the currents). Find the self-inductance per unit length.

Problem 7.29 Suppose the circuit in Fig. 7.40 has been connected for a long time when suddenly, at time $t = 0$, switch S is thrown, bypassing the battery.

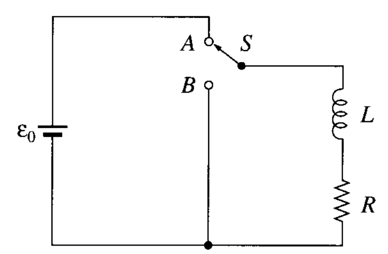


Figure 7.40

[12]Notice the similarity to Eq. 7.27—in a sense, the rectangular toroid *is* a short coaxial cable, turned on its side.

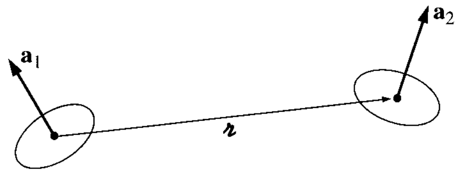


Figure 7.41

(a) What is the current at any subsequent time t?

(b) What is the total energy delivered to the resistor?

(c) Show that this is equal to the energy originally stored in the inductor.

Problem 7.30 Two tiny wire loops, with areas $\mathbf{a}_1$ and $\mathbf{a}_2$, are situated a displacement $\boldsymbol{\mathscr{r}}$ apart (Fig. 7.41).

(a) Find their mutual inductance. [*Hint:* Treat them as magnetic dipoles, and use Eq. 5.87.] Is your formula consistent with Eq. 7.23?

(b) Suppose a current I_1 is flowing in loop 1, and we propose to turn on a current I_2 in loop 2. How much work must be done, against the mutually induced emf, to keep the current I_1 flowing in loop 1? In light of this result, comment on Eq. 6.35.

7.3 Maxwell's Equations

7.3.1 Electrodynamics Before Maxwell

So far, we have encountered the following laws, specifying the divergence and curl of electric and magnetic fields:

$$\text{(i)} \quad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \quad \text{(Gauss's law)},$$

$$\text{(ii)} \quad \nabla \cdot \mathbf{B} = 0 \quad \text{(no name)},$$

$$\text{(iii)} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday's law)},$$

$$\text{(iv)} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \text{(Ampère's law)}.$$

These equations represent the state of electromagnetic theory over a century ago, when Maxwell began his work. They were not written in so compact a form in those days, but their physical content was familiar. Now, it happens there is a fatal inconsistency in these formulas. It has to do with the old rule that divergence of curl is always zero. If you apply the divergence to number (iii), everything works out:

$$\nabla \cdot (\nabla \times \mathbf{E}) = \nabla \cdot \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}).$$

The left side is zero because divergence of curl is zero; the right side is zero by virtue of equation (ii). But when you do the same thing to number (iv), you get into trouble:

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0(\nabla \cdot \mathbf{J}); \tag{7.35}$$

the left side must be zero, but the right side, in general, is *not*. For *steady* currents, the divergence of **J** is zero, but evidently when we go beyond magnetostatics Ampère's law cannot be right.

There's another way to see that Ampère's law is bound to fail for nonsteady currents. Suppose we're in the process of charging up a capacitor (Fig. 7.42). In integral form. Ampère's law reads

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}.$$

I want to apply it to the Amperian loop shown in the diagram. How do I determine I_{enc}? Well, it's the total current passing through the loop, or, more precisely, the current piercing a surface that has the loop for its boundary. In this case, the *simplest* surface lies in the plane of the loop—the wire punctures this surface, so $I_{enc} = I$. Fine—but what if I draw instead the balloon-shaped surface in Fig. 7.42? *No* current passes through *this* surface, and I conclude that $I_{enc} = 0$! We never had this problem in magnetostatics because the conflict arises only when charge is piling up somewhere (in this case, on the capacitor plates). But *for nonsteady currents* (such as this one) *"the current enclosed by a loop" is an ill-defined notion,* since it depends entirely on what surface you use. (If this seems pedantic to you—"obviously one should use the planar surface"—remember that the Amperian loop could be some contorted shape that doesn't even lie in a plane.)

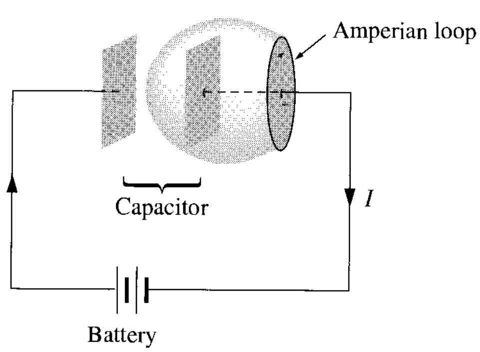


Figure 7.42

Of course, we had no right to *expect* Ampère's law to hold outside of magnetostatics; after all, we derived it from the Biot-Savart law. However, in Maxwell's time there was no *experimental* reason to doubt that Ampère's law was of wider validity. The flaw was a purely theoretical one, and Maxwell fixed it by purely theoretical arguments.

7.3.2 How Maxwell Fixed Ampère's Law

The problem is on the right side of Eq. 7.35, which *should be* zero, but *isn't*. Applying the continuity equation (5.29) and Gauss's law, the offending term can be rewritten:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right).$$

It might occur to you that if we were to combine $\epsilon_0(\partial \mathbf{E}/\partial t)$ with $\mathbf{J}$, in Ampère's law, it would be just right to kill off the extra divergence:

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.} \tag{7.36}$$

(Maxwell himself had other reasons for wanting to add this quantity to Ampère's law. To him the rescue of the continuity equation was a happy dividend rather than a primary motive. But today we recognize this argument as a far more compelling one than Maxwell's, which was based on a now-discredited model of the ether.)[13]

Such a modification changes nothing, as far as magneto*statics* is concerned: when $\mathbf{E}$ is constant, we still have $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. In fact, Maxwell's term is hard to detect in ordinary electromagnetic experiments, where it must compete for recognition with $\mathbf{J}$; that's why Faraday and the others never discovered it in the laboratory. However, it plays a crucial role in the propagation of electromagnetic waves, as we'll see in Chapter 9.

Apart from curing the defect in Ampère's law, Maxwell's term has a certain aesthetic appeal: Just as a changing *magnetic* field induces an *electric* field (Faraday's law), so

A changing electric field induces a magnetic field.

Of course, theoretical convenience and aesthetic consistency are only *suggestive*—there might, after all, be other ways to doctor up Ampère's law. The real confirmation of Maxwell's theory came in 1888 with Hertz's experiments on electromagnetic waves.

Maxwell called his extra term the **displacement current**:

$$\mathbf{J}_d \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{7.37}$$

It's a misleading name, since $\epsilon_0(\partial \mathbf{E}/\partial t)$ has nothing to do with current, except that it adds to $\mathbf{J}$ in Ampère's law. Let's see now how the displacement current resolves the paradox of the charging capacitor (Fig. 7.42). If the capacitor plates are very close together (I didn't

[13]For the history of this subject, see A. M. Bork, *Am. J. Phys.* **31**, 854 (1963).

draw them that way, but the calculation is simpler if you assume this), then the electric field between them is

$$E = \frac{1}{\epsilon_0}\sigma = \frac{1}{\epsilon_0}\frac{Q}{A},$$

where Q is the charge on the plate and A is its area. Thus, between the plates

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt} = \frac{1}{\epsilon_0 A}I.$$

Now, Eq. 7.36 reads, in integral form,

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \mu_0\epsilon_0 \int \left(\frac{\partial \mathbf{E}}{\partial t}\right)\cdot d\mathbf{a}. \qquad (7.38)$$

If we choose the *flat* surface, then $E = 0$ and $I_{\text{enc}} = I$. If, on the other hand, we use the balloon-shaped surface, then $I_{\text{enc}} = 0$, but $\int(\partial\mathbf{E}/\partial t)\cdot d\mathbf{a} = I/\epsilon_0$. So we get the same answer for either surface, though in the first case it comes from the genuine current and in the second from the displacement current.

Problem 7.31 A fat wire, radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor, as shown in Fig. 7.43. Find the magnetic field in the gap, at a distance $s < a$ from the axis.

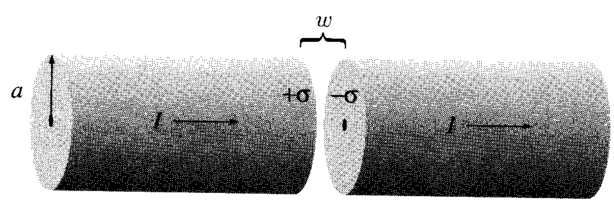


Figure 7.43

Problem 7.32 The preceding problem was an artificial model for the charging capacitor, designed to avoid complications associated with the current spreading out over the surface of the plates. For a more realistic model, imagine *thin* wires that connect to the centers of the plates (Fig. 7.44a). Again, the current I is constant, the radius of the capacitor is a, and the separation of the plates is $w \ll a$. Assume that the current flows out over the plates in such a way that the surface charge is uniform, at any given time, and is zero at $t = 0$.

(a) Find the electric field between the plates, as a function of t.

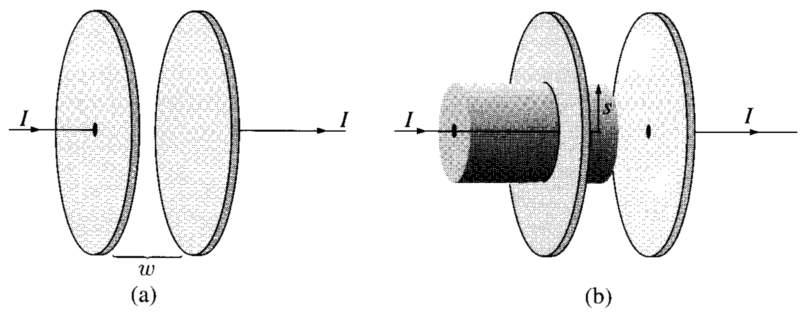


Figure 7.44

(b) Find the displacement current through a circle of radius s in the plane midway between the plates. Using this circle as your "Amperian loop," and the flat surface that spans it, find the magnetic field at a distance s from the axis.

(c) Repeat part (b), but this time use the cylindrical surface in Fig. 7.44b, which extends to the left through the plate and terminates outside the capacitor. Notice that the displacement current through this surface is zero, and there are two contributions to I_{enc}.[14]

Problem 7.33 Refer to Prob. 7.16, to which the correct answer was

$$\mathbf{E}(s, t) = \frac{\mu_0 I_0 \omega}{2\pi} \sin(\omega t) \ln\left(\frac{a}{s}\right) \hat{\mathbf{z}}.$$

(a) Find the displacement current density $\mathbf{J}_d$.

(b) Integrate it to get the total displacement current,

$$I_d = \int \mathbf{J}_d \cdot d\mathbf{a}.$$

(c) Compare I_d and I. (What's their ratio?) If the outer cylinder were, say, 2 mm in diameter, how high would the frequency have to be, for I_d to be 1% of I? [This problem is designed to indicate why Faraday never discovered displacement currents, and why it is ordinarily safe to ignore them unless the frequency is extremely high.]

[14]This problem raises an interesting quasi-philosophical question: If you measure **B** in the laboratory, have you detected the effects of displacement current (as (b) would suggest), or merely confirmed the effects of ordinary currents (as (c) implies)? See D. F. Bartlett, *Am. J. Phys.* **58**, 1168 (1990).

7.3.3 Maxwell's Equations

In the last section we put the finishing touches on Maxwell's equations:

$$
\boxed{
\begin{array}{lll}
\text{(i)} & \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho & \text{(Gauss's law),} \\[2ex]
\text{(ii)} & \nabla \cdot \mathbf{B} = 0 & \text{(no name),} \\[2ex]
\text{(iii)} & \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \text{(Faraday's law),} \\[2ex]
\text{(iv)} & \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} & \text{(Ampère's law with Maxwell's correction).}
\end{array}
}
\tag{7.39}
$$

Together with the force law,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{7.40}$$

they summarize the entire theoretical content of classical electrodynamics[15] (save for some special properties of matter, which we encountered in Chapters 4 and 6). Even the continuity equation,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}, \tag{7.41}$$

which is the mathematical expression of conservation of charge, can be derived from Maxwell's equations by applying the divergence to number (iv).

I have written Maxwell's equations in the traditional way, which emphasizes that they specify the divergence and curl of **E** and **B**. In this form they reinforce the notion that electric fields can be produced *either* by charges (ρ) *or* by changing magnetic fields ($\partial \mathbf{B}/\partial t$), and magnetic fields can be produced *either* by currents (**J**) *or* by changing electric fields ($\partial \mathbf{E}/\partial t$). Actually, this is somewhat misleading, because when you come right down to it $\partial \mathbf{B}/\partial t$ and $\partial \mathbf{E}/\partial t$ are *themselves* due to charges and currents. I think it is logically preferable to write

$$
\left.
\begin{array}{ll}
\text{(i)}\ \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(iii)}\ \nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0, \\[2ex]
\text{(ii)}\ \nabla \cdot \mathbf{B} = 0, & \text{(iv)}\ \nabla \times \mathbf{B} - \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J},
\end{array}
\right\}
\tag{7.42}
$$

with the fields (**E** and **B**) on the left and the sources (ρ and **J**) on the right. This notation emphasizes that all electromagnetic fields are ultimately attributable to charges and currents. Maxwell's equations tell you how *charges* produce *fields*; reciprocally, the force law tells you how *fields* affect *charges*.

[15] Like any differential equations, Maxwell's must be supplemented by suitable *boundary conditions*. Because these are typically "obvious" from the context (e.g. **E** and **B** go to zero at large distances from a localized charge distribution), it is easy to forget that they play an essential role.

Problem 7.34 Suppose

$$\mathbf{E}(\mathbf{r}, t) = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt - r)\hat{\mathbf{r}}; \quad \mathbf{B}(\mathbf{r}, t) = 0$$

(the theta function is defined in Prob. 1.45b). Show that these fields satisfy all of Maxwell's equations, and determine ρ and $\mathbf{J}$. Describe the physical situation that gives rise to these fields.

7.3.4 Magnetic Charge

There is a pleasing symmetry about Maxwell's equations; it is particularly striking in free space, where ρ and $\mathbf{J}$ vanish:

$$\left.\begin{array}{ll} \nabla \cdot \mathbf{E} = 0, & \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} = 0, & \nabla \times \mathbf{B} = \mu_0\epsilon_0\dfrac{\partial \mathbf{E}}{\partial t}. \end{array}\right\}$$

If you replace $\mathbf{E}$ by $\mathbf{B}$ and $\mathbf{B}$ by $-\mu_0\epsilon_0\mathbf{E}$, the first pair of equations turns into the second, and vice versa. This symmetry[16] between $\mathbf{E}$ and $\mathbf{B}$ is spoiled, though, by the charge term in Gauss's law and the current term in Ampère's law. You can't help wondering why the corresponding quantities are "missing" from $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$. What if we had

$$\left.\begin{array}{ll} \text{(i)}\ \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho_e, & \text{(iii)}\ \nabla \times \mathbf{E} = -\mu_0\mathbf{J}_m - \dfrac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)}\ \nabla \cdot \mathbf{B} = \mu_0\rho_m, & \text{(iv)}\ \nabla \times \mathbf{B} = \mu_0\mathbf{J}_e + \mu_0\epsilon_0\dfrac{\partial \mathbf{E}}{\partial t}. \end{array}\right\} \quad (7.43)$$

Then ρ_m would represent the density of magnetic "charge," and ρ_e the density of electric charge; $\mathbf{J}_m$ would be the current of magnetic charge, and $\mathbf{J}_e$ the current of electric charge. Both charges would be conserved:

$$\nabla \cdot \mathbf{J}_m = -\frac{\partial \rho_m}{\partial t}, \quad \text{and} \quad \nabla \cdot \mathbf{J}_e = -\frac{\partial \rho_e}{\partial t}. \quad (7.44)$$

The former follows by application of the divergence to (iii), the latter by taking the divergence of (iv).

In a sense, Maxwell's equations *beg* for magnetic charge to exist—it would fit in so nicely. And yet, in spite of a diligent search, no one has ever found any.[17] As far as we know, ρ_m is zero everywhere, and so is $\mathbf{J}_m$; $\mathbf{B}$ is *not* on equal footing with $\mathbf{E}$: there exist

[16]Don't be distracted by the pesky constants μ_0 and ϵ_0: these are present only because the SI system measures $\mathbf{E}$ and $\mathbf{B}$ in different units, and would not occur, for instance, in the Gaussian system.

[17]For an extensive bibliography, see A. S. Goldhaber and W. P. Trower, *Am. J. Phys.* **58**, 429 (1990).

stationary sources for **E** (electric charges) but none for **B**. (This is reflected in the fact that magnetic multipole expansions have no monopole term, and magnetic dipoles consist of current loops, not separated north and south "poles.") Apparently God just didn't *make* any magnetic charge. (In the quantum theory of electrodynamics, by the way, it's a more than merely aesthetic shame that magnetic charge does not seem to exist: Dirac showed that the *existence* of *magnetic* charge would explain why *electric* charge is *quantized*. See Prob. 8.12.)

Problem 7.35 Assuming that "Coulomb's law" for magnetic charges (q_m) reads

$$\mathbf{F} = \frac{\mu_0}{4\pi}\frac{q_{m_1}q_{m_2}}{\mathscr{r}^2}\hat{\mathscr{r}}, \tag{7.45}$$

work out the force law for a monopole q_m moving with velocity **v** through electric and magnetic fields **E** and **B**. [For interesting commentary, see W. Rindler, *Am. J. Phys.* **57**, 993 (1989).]

Problem 7.36 Suppose a magnetic monopole q_m passes through a resistanceless loop of wire with self-inductance L. What current is induced in the loop? [This is one of the methods used to search for monopoles in the laboratory; see B. Cabrera, *Phys. Rev. Lett.* **48**, 1378 (1982).]

7.3.5 Maxwell's Equations in Matter

Maxwell's equations in the form 7.39 are complete and correct as they stand. However, when you are working with materials that are subject to electric and magnetic polarization there is a more convenient way to *write* them. For inside polarized matter there will be accumulations of "bound" charge and current over which you exert no direct control. It would be nice to reformulate Maxwell's equations in such a way as to make explicit reference only to those sources we control directly: the "free" charges and currents.

We have already learned, from the static case, that an electric polarization **P** produces a bound charge density

$$\rho_b = -\nabla \cdot \mathbf{P} \tag{7.46}$$

(Eq. 4.12). Likewise, a magnetic polarization (or "magnetization") **M** results in a bound current

$$\mathbf{J}_b = \nabla \times \mathbf{M} \tag{7.47}$$

(Eq. 6.13). There's just one new feature to consider in the *non*static case: Any *change* in the electric polarization involves a flow of (bound) charge (call it $\mathbf{J}_p$), which must be included in the total current. For suppose we examine a tiny chunk of polarized material (Fig. 7.45.) The polarization introduces a charge density $\sigma_b = P$ at one end and $-\sigma_b$ at the other (Eq. 4.11). If P now *increases* a bit, the charge on each end increases accordingly, giving a net current

$$dI = \frac{\partial \sigma_b}{\partial t}da_\perp = \frac{\partial P}{\partial t}da_\perp.$$

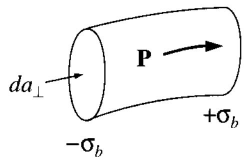


Figure 7.45

The current density, therefore, is

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}. \tag{7.48}$$

This **polarization current** has nothing whatever to do with the *bound* current $\mathbf{J}_b$. The latter is associated with *magnetization* of the material and involves the spin and orbital motion of electrons; $\mathbf{J}_p$, by contrast, is the result of the linear motion of charge when the electric polarization changes. If $\mathbf{P}$ points to the right and is increasing, then each plus charge moves a bit to the right and each minus charge to the left; the cumulative effect is the polarization current $\mathbf{J}_p$. In this connection, we ought to check that Eq. 7.48 is consistent with the continuity equation:

$$\nabla \cdot \mathbf{J}_p = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{P}) = -\frac{\partial \rho_b}{\partial t}.$$

Yes: The continuity equation *is* satisfied; in fact, $\mathbf{J}_p$ is essential to account for the conservation of bound charge. (Incidentally, a changing *magnetization* does *not* lead to any analogous accumulation of charge or current. The bound current $\mathbf{J}_b = \nabla \times \mathbf{M}$ varies in response to changes in $\mathbf{M}$, to be sure, but that's about it.)

In view of all this, the total charge density can be separated into two parts:

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P}, \tag{7.49}$$

and the current density into *three* parts:

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}. \tag{7.50}$$

Gauss's law can now be written as

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho_f - \nabla \cdot \mathbf{P}),$$

or

$$\nabla \cdot \mathbf{D} = \rho_f, \tag{7.51}$$

where $\mathbf{D}$, as in the static case, is given by

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}. \tag{7.52}$$

Meanwhile, Ampère's law (with Maxwell's term) becomes

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

or

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \tag{7.53}$$

where, as before,

$$\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}. \tag{7.54}$$

Faraday's law and $\nabla \cdot \mathbf{B} = 0$ are not affected by our separation of charge and current into free and bound parts, since they do not involve ρ or $\mathbf{J}$.

In terms of *free* charges and currents, then, Maxwell's equations read

$$\begin{array}{ll} \text{(i)}\ \nabla \cdot \mathbf{D} = \rho_f, & \text{(iii)}\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)}\ \nabla \cdot \mathbf{B} = 0, & \text{(iv)}\ \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t}. \end{array} \tag{7.55}$$

Some people regard these as the "true" Maxwell's equations, but please understand that they are in *no way* more "general" than 7.39; they simply reflect a convenient division of charge and current into free and nonfree parts. And they have the disadvantage of hybrid notation, since they contain both **E** and **D**, both **B** and **H**. They must be supplemented, therefore, by appropriate **constitutive relations**, giving **D** and **H** in terms of **E** and **B**. These depend on the nature of the material; for linear media

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \text{and} \quad \mathbf{M} = \chi_m \mathbf{H}, \tag{7.56}$$

so

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B}, \tag{7.57}$$

where $\epsilon \equiv \epsilon_0(1 + \chi_e)$ and $\mu \equiv \mu_0(1 + \chi_m)$. Incidentally, you'll remember that **D** is called the electric "displacement"; that's why the second term in the Ampère/Maxwell equation (iv) is called the **displacement current**, generalizing Eq. 7.37:

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}. \tag{7.58}$$

Problem 7.37 Sea water at frequency $\nu = 4 \times 10^8$ Hz has permittivity $\epsilon = 81\epsilon_0$, permeability $\mu = \mu_0$, and resistivity $\rho = 0.23\ \Omega\cdot\text{m}$. What is the ratio of conduction current to displacement current? [*Hint:* consider a parallel-plate capacitor immersed in sea water and driven by a voltage $V_0 \cos(2\pi\nu t)$.]

7.3.6 Boundary Conditions

In general, the fields **E, B, D,** and **H** will be discontinuous at a boundary between two different media, or at a surface that carries charge density σ or current density **K**. The explicit form of these discontinuities can be deduced from Maxwell's equations (7.55), in their integral form

$$\left.\begin{aligned}&\text{(i)}\quad \oint_{\mathcal{S}} \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}}\\ &\text{(ii)}\quad \oint_{\mathcal{S}} \mathbf{B}\cdot d\mathbf{a} = 0\end{aligned}\right\}\ \text{over any closed surface } \mathcal{S}.$$

$$\left.\begin{aligned}&\text{(iii)}\quad \oint_{\mathcal{P}} \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B}\cdot d\mathbf{a}\\ &\text{(iv)}\quad \oint_{\mathcal{P}} \mathbf{H}\cdot d\mathbf{l} = I_{f_{\text{enc}}} + \frac{d}{dt}\int_{\mathcal{S}} \mathbf{D}\cdot d\mathbf{a}\end{aligned}\right\}\ \begin{array}{l}\text{for any surface } \mathcal{S}\\ \text{bounded by the}\\ \text{closed loop } \mathcal{P}.\end{array}$$

Applying (i) to a tiny, wafer-thin Gaussian pillbox extending just slightly into the material on either side of the boundary, we obtain (Fig. 7.46):

$$\mathbf{D}_1\cdot\mathbf{a} - \mathbf{D}_2\cdot\mathbf{a} = \sigma_f\, a.$$

(The positive direction for **a** is *from* 2 *toward* 1. The edge of the wafer contributes nothing in the limit as the thickness goes to zero, nor does any *volume* change density.) Thus, the component of **D** that is perpendicular to the interface is discontinuous in the amount

$$\boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f.} \tag{7.59}$$

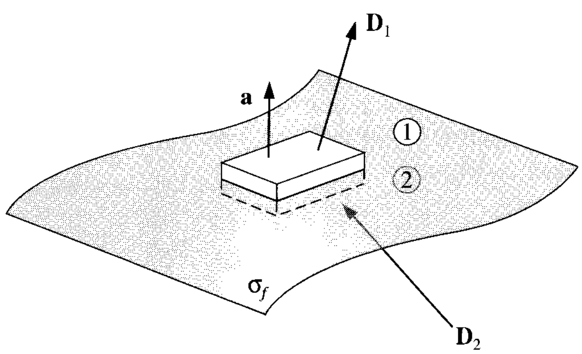


Figure 7.46

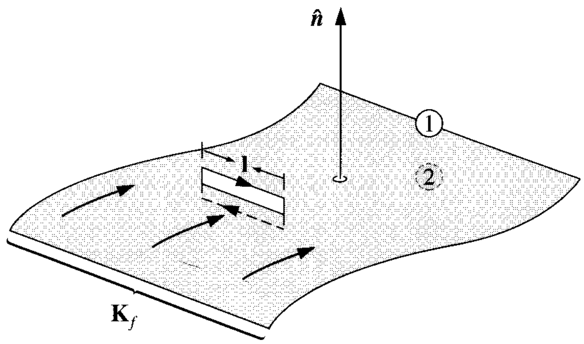


Figure 7.47

Identical reasoning, applied to equation (ii), yields

$$\boxed{B_1^\perp - B_2^\perp = 0.} \tag{7.60}$$

Turning to (iii), a very thin Amperian loop straddling the surface (Fig. 7.47) gives

$$\mathbf{E}_1 \cdot \mathbf{l} - \mathbf{E}_2 \cdot \mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{a}.$$

But in the limit as the width of the loop goes to zero, the flux vanishes. (I have already dropped the contribution of the two ends to $\oint \mathbf{E} \cdot d\mathbf{l}$, on the same grounds.) Therefore,

$$\boxed{\mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0.} \tag{7.61}$$

That is, the components of $\mathbf{E}$ *parallel* to the interface are continuous across the boundary. By the same token, (iv) implies

$$\mathbf{H}_1 \cdot \mathbf{l} - \mathbf{H}_2 \cdot \mathbf{l} = I_{f_{\text{enc}}},$$

where $I_{f_{\text{enc}}}$ is the free current passing through the Amperian loop. No *volume* current density will contribute (in the limit of infinitesimal width) but a *surface* current can. In fact. if $\hat{\mathbf{n}}$ is a unit vector perpendicular to the interface (pointing from 2 toward 1), so that $(\hat{\mathbf{n}} \times \mathbf{l})$ is normal to the Amperian loop, then

$$I_{f_{\text{enc}}} = \mathbf{K}_f \cdot (\hat{\mathbf{n}} \times \mathbf{l}) = (\mathbf{K}_f \times \hat{\mathbf{n}}) \cdot \mathbf{l},$$

and hence

$$\boxed{\mathbf{H}_1^\parallel - \mathbf{H}_2^\parallel = \mathbf{K}_f \times \hat{\mathbf{n}}.} \tag{7.62}$$

So the *parallel* components of $\mathbf{H}$ are discontinuous by an amount proportional to the free surface current density.

Equations 7.59-62 are the general boundary conditions for electrodynamics. In the case of *linear* media, they can be expressed in terms of **E** and **B** alone:

$$\left.\begin{array}{ll}\text{(i) } \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, & \text{(iii) } \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0, \\ \text{(ii) } B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv) } \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}. \end{array}\right\} \quad (7.63)$$

In particular, if there is no free charge or free current at the interface, then

$$\boxed{\begin{array}{ll}\text{(i) } \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0, & \text{(iii) } \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0, \\ \text{(ii) } B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv) } \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = 0. \end{array}} \quad (7.64)$$

As we shall see in Chapter 9, these equations are the basis for the theory of reflection and refraction.

More Problems on Chapter 7

Problem 7.38 Two very large metal plates are held a distance d apart, one at potential zero, the other at potential V_0 (Fig. 7.48). A metal sphere of radius a ($a \ll d$) is sliced in two, and one hemisphere placed on the grounded plate, so that its potential is likewise zero. If the region between the plates is filled with weakly conducting material of uniform conductivity σ, what current flows to the hemisphere? [*Answer:* $(3\pi a^2\sigma/d)V_0$. *Hint:* study Ex. 3.8.]

! **Problem 7.39** Two long, straight copper pipes, each of radius a, are held a distance $2d$ apart (see Fig. 7.49). One is at potential V_0, the other at $-V_0$. The space surrounding the pipes is filled with weakly conducting material of conductivity σ. Find the current, per unit length, which flows from one pipe to the other. [*Hint:* refer to Prob. 3.11.]

Problem 7.40 A common textbook problem asks you to calculate the resistance of a cone-shaped object, of resistivity ρ, with length L, radius a at one end, and radius b at the other (Fig. 7.50). The two ends are flat, and are taken to be equipotentials. The suggested method is to slice it into circular disks of width dz, find the resistance of each disk, and integrate to get the total.

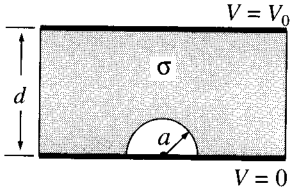


Figure 7.48

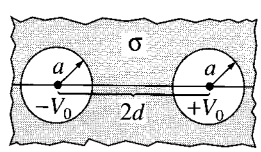


Figure 7.49

(c) The induced current on the surface of the superconductor (the xy plane) can be determined from the boundary condition on the *tangential* component of **B** (Eq. 5.74): $\mathbf{B} = \mu_0(\mathbf{K} \times \hat{\mathbf{z}})$. Using the field you get from the image configuration, show that

$$\mathbf{K} = -\frac{3mrh}{2\pi(r^2 + h^2)^{5/2}}\,\hat{\boldsymbol{\phi}},$$

where r is the distance from the origin.

! **Problem 7.44** If a magnetic dipole levitating above an infinite superconducting plane (Prob. 7.43) is free to rotate, what orientation will it adopt, and how high above the surface will it float?

Problem 7.45 A perfectly conducting spherical shell of radius a rotates about the z axis with angular velocity ω, in a uniform magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{z}}$. Calculate the emf developed between the "north pole" and the equator. [*Answer:* $\frac{1}{2}B_0\omega a^2$]

! **Problem 7.46** Refer to Prob. 7.11 (and use the result of Prob. 5.40, if it helps):

(a) Does the square ring fall faster in the orientation shown (Fig. 7.19), or when rotated 45° about an axis coming out of the page? Find the ratio of the two terminal velocities. If you dropped the loop, which orientation would it assume in falling? [*Answer:* $(\sqrt{2} - 2y/l)^2$, where l is the length of a side, and y is the height of the center above the edge of the magnetic field, in the rotated configuration.]

(b) How long does is take a *circular* ring to cross the bottom of the magnetic field, at its (changing) terminal velocity?

Problem 7.47

(a) Use the analogy between Faraday's law and Ampère's law, together with the Biot-Savart law, to show that

$$\mathbf{E}(\mathbf{r}, t) = -\frac{1}{4\pi}\frac{\partial}{\partial t}\int \frac{\mathbf{B}(\mathbf{r}', t) \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau', \tag{7.65}$$

for Faraday-induced electric fields.

(b) Referring to Prob. 5.50a, show that

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \tag{7.66}$$

where **A** is the vector potential. Check this result by taking the curl of both sides.

(c) A spherical shell of radius R carries a uniform surface charge σ. It spins about a fixed axis at an angular velocity $\omega(t)$ that changes slowly with time. Find the electric field inside and outside the sphere. [*Hint:* There are *two* contributions here: the Coulomb field due to the charge, and the Faraday field due to the changing **B**. Refer to Ex. 5.11, and use Eq. 7.66.]

Problem 7.48 Electrons undergoing cyclotron motion can be speeded up by increasing the magnetic field; the accompanying electric field will impart tangential acceleration. This is the principle of the **betatron**. One would like to keep the radius of the orbit constant during the process. Show that this can be achieved by designing a magnet such that the average field over the area of the orbit is twice the field at the circumference (Fig. 7.52). Assume the electrons start from rest in zero field, and that the apparatus is symmetric about the center of the orbit. (Assume also that the electron velocity remains well below the speed of light, so that nonrelativistic mechanics applies.) [*Hint:* differentiate Eq. 5.3 with respect to time, and use $F = ma = qE$.]

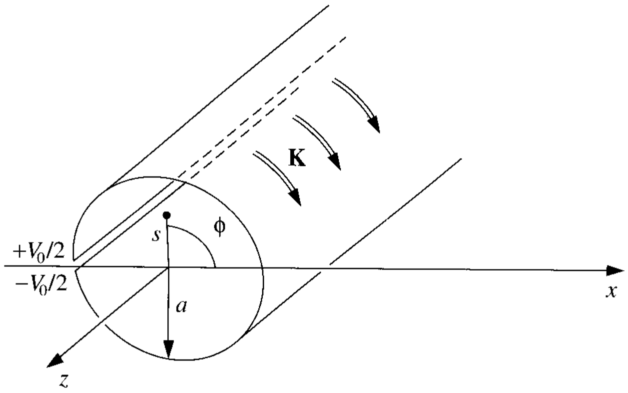


Figure 7.51

A **superconductor** is a perfect conductor with the *additional* property that the (constant) **B** inside is in fact *zero.* (This "flux exclusion" is known as the **Meissner effect.**[18])

(c) Show that the current in a superconductor is confined to the surface.

(d) Superconductivity is lost above a certain critical temperature (T_c), which varies from one material to another. Suppose you had a sphere (radius a) above its critical temperature, and you held it in a uniform magnetic field $B_0\hat{\mathbf{z}}$ while cooling it below T_c. Find the induced surface current density **K**, as a function of the polar angle θ.

Problem 7.43 A familiar demonstration of superconductivity (Prob. 7.42) is the levitation of a magnet over a piece of superconducting material. This phenomenon can be analyzed using the method of images.[19] Treat the magnet as a perfect dipole **m**, a height z above the origin (and constrained to point in the z direction), and pretend that the superconductor occupies the entire half-space below the xy plane. Because of the Meissner effect, $\mathbf{B} = 0$ for $z \leq 0$, and since **B** is divergenceless, the normal (z) component is continuous, so $B_z = 0$ just *above* the surface. This boundary condition is met by the image configuration in which an identical dipole is placed at $-z$, as a stand-in for the superconductor; the two arrangements therefore produce the same magnetic field in the region $z > 0$.

(a) Which way should the image dipole point ($+z$ or $-z$)?

(b) Find the force on the magnet due to the induced currents in the superconductor (which is to say, the force due to the image dipole). Set it equal to Mg (where M is the mass of the magnet) to determine the height h at which the magnet will "float." [*Hint:* refer to Prob. 6.3.]

[18]The Meissner effect is sometimes referred to as "perfect diamagnetism," in the sense that the field inside is not merely *reduced,* but canceled entirely. However, the surface currents responsible for this are *free,* not bound, so the actual *mechanism* is quite different.

[19]W. M. Saslow, *Am. J. Phys.* **59**, 16 (1991).

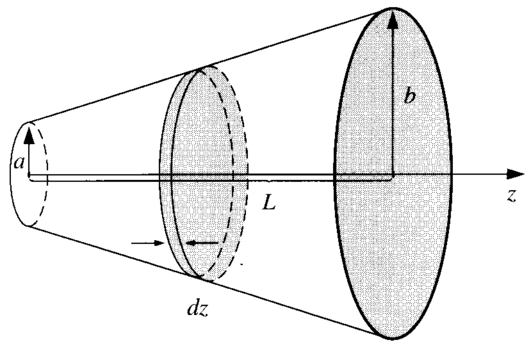


Figure 7.50

(a) Calculate R this way.

(b) Explain why this method is fundamentally flawed. [See J. D. Romano and R. H. Price, *Am. J. Phys.* **64**, 1150 (1996).]

(c) Suppose the ends are, instead, spherical surfaces, centered at the apex of the cone. Calculate the resistance in that case. (Let L be the distance between the centers of the circular perimeters of the end caps.) [*Answer:* $(\rho/2\pi ab)(b-a)^2/(\sqrt{L^2+(b-a)^2}-L)$]

! **Problem 7.41** A rare case in which the electrostatic field **E** for a circuit can actually be *calculated* is the following [M. A. Heald, *Am. J. Phys.* **52**, 522 (1984)]: Imagine an infinitely long cylindrical sheet, of uniform resistivity and radius a. A slot (corresponding to the battery) is maintained at $\pm V_0/2$, at $\phi = \pm\pi$, and a steady current flows over the surface, as indicated in Fig. 7.51. According to Ohm's law, then,

$$V(a,\phi) = \frac{V_0\phi}{2\pi}, \quad (-\pi < \phi < +\pi).$$

(a) Use separation of variables in cylindrical coordinates to determine $V(s,\phi)$ inside and outside the cylinder. [*Answer:* $(V_0/\pi)\tan^{-1}[(s\sin\phi)/(a+s\cos\phi)]$, $(s<a)$; $(V_0/\pi)\tan^{-1}[(a\sin\phi)/(s+a\cos\phi)]$, $(s>a)$]

(b) Find the surface charge density on the cylinder. [*Answer:* $(\epsilon_0 V_0/\pi a)\tan(\phi/2)$]

Problem 7.42 In a **perfect conductor**, the conductivity is infinite, so $\mathbf{E} = 0$ (Eq. 7.3), and any net charge resides on the surface (just as it does for an *im*perfect conductor, in electro*statics*).

(a) Show that the magnetic field is constant ($\partial\mathbf{B}/\partial t = 0$), inside a perfect conductor.

(b) Show that the magnetic flux through a perfectly conducting loop is constant.

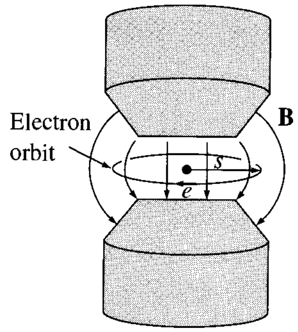


Figure 7.52

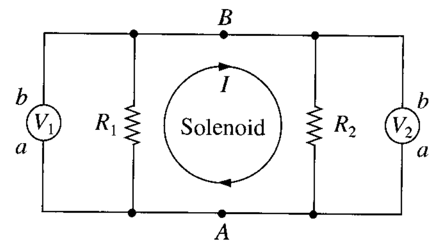


Figure 7.53

Problem 7.49 An atomic electron (charge q) circles about the nucleus (charge Q) in an orbit of radius r; the centripetal acceleration is provided, of course, by the Coulomb attraction of opposite charges. Now a small magnetic field dB is slowly turned on, perpendicular to the plane of the orbit. Show that the increase in kinetic energy, dT, imparted by the induced electric field, is just right to sustain circular motion *at the same radius r.* (That's why, in my discussion of diamagnetism, I assumed the radius is fixed. See Sect. 6.1.3 and the references cited there.)

Problem 7.50 The current in a long solenoid is increasing linearly with time, so that the flux is proportional to t: $\Phi = \alpha t$. Two voltmeters are connected to diametrically opposite points (A and B), together with resistors (R_1 and R_2), as shown in Fig. 7.53. What is the reading on each voltmeter? Assume that these are *ideal* voltmeters that draw negligible current (they have huge internal resistance), and that a voltmeter registers $\int_a^b \mathbf{E} \cdot d\mathbf{l}$ between the terminals and through the meter. [*Answer:* $V_1 = \alpha R_1/(R_1 + R_2)$; $V_2 = -\alpha R_2/(R_1 + R_2)$. Notice that $V_1 \neq V_2$, even though they are connected to the same points! See R. H. Romer, *Am. J. Phys.* **50**, 1089 (1982).]

Problem 7.51 In the discussion of motional emf (Sect. 7.1.3) I assumed that the wire loop (Fig. 7.10) has a resistance R; the current generated is then $I = vBh/R$. But what if the wire is made out of perfectly conducting material, so that R is *zero*? In that case the current is limited only by the back emf associated with the self-inductance L of the loop (which would ordinarily be negligible in comparison with IR). Show that in this régime the loop (mass m) executes simple harmonic motion, and find its frequency.[20] [*Answer:* $\omega = Bh/\sqrt{mL}$]

Problem 7.52

(a) Use the Neumann formula (Eq. 7.22) to calculate the mutual inductance of the configuration in Fig. 7.36, assuming a is very small ($a \ll b$, $a \ll z$). Compare your answer to Prob. 7.20.

(b) For the general case (*not* assuming a is small) show that

$$M = \frac{\mu_0 \pi \beta}{2} \sqrt{ab\beta} \left(1 + \frac{15}{8}\beta^2 + \ldots\right),$$

[20] For a collection of related problems, see W. M. Saslow, *Am. J. Phys.* **55**, 986 (1987), and R. H. Romer, *Eur. J. Phys.* **11**, 103 (1990).

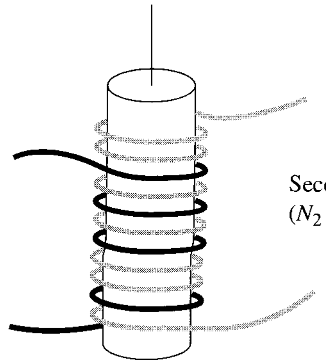


Figure 7.54

where

$$\beta \equiv \frac{ab}{z^2 + a^2 + b^2}.$$

Problem 7.53 Two coils are wrapped around a cylindrical form in such a way that the *same flux passes through every turn of both coils.* (In practice this is achieved by inserting an iron core through the cylinder; this has the effect of concentrating the flux.) The "primary" coil has N_1 turns and the secondary has N_2 (Fig. 7.54). If the current I in the primary is changing, show that the emf in the secondary is given by

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}, \tag{7.67}$$

where $\mathcal{E}_1$ is the (back) emf of the primary. [This is a primitive **transformer**—a device for raising or lowering the emf of an alternating current source. By choosing the appropriate number of turns, any desired secondary emf can be obtained. If you think this violates the conservation of energy, check out Prob. 7.54.]

Problem 7.54 A transformer (Prob. 7.53) takes an input AC voltage of amplitude V_1, and delivers an output voltage of amplitude V_2, which is determined by the turns ratio ($V_2/V_1 = N_2/N_1$). If $N_2 > N_1$ the output voltage is greater than the input voltage. Why doesn't this violate conservation of energy? *Answer:* Power is the product of voltage and current; evidently if the voltage goes *up*, the current must come *down*. The purpose of this problem is to see exactly how this works out, in a simplified model.

(a) In an ideal transformer the same flux passes through all turns of the primary and of the secondary. Show that in this case $M^2 = L_1 L_2$, where M is the mutual inductance of the coils, and L_1, L_2 are their individual self-inductances.

(b) Suppose the primary is driven with AC voltage $V_{in} = V_1 \cos(\omega t)$, and the secondary is connected to a resistor, R. Show that the two currents satisfy the relations

$$L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} = V_1 \cos(\omega t); \quad L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} = -I_2 R.$$

(c) Using the result in (a), solve these equations for $I_1(t)$ and $I_2(t)$. (Assume I_1 has no DC component.)

(d) Show that the output voltage ($V_{\text{out}} = I_2 R$) divided by the input voltage (V_{in}) is equal to the turns ratio: $V_{\text{out}}/V_{\text{in}} = N_2/N_1$.

(e) Calculate the input power ($P_{\text{in}} = V_{\text{in}} I_1$) and the output power ($P_{\text{out}} = V_{\text{out}} I_2$), and show that their averages over a full cycle are equal.

Problem 7.55 Suppose $\mathbf{J}(\mathbf{r})$ is constant in time but $\rho(\mathbf{r}, t)$ is *not*—conditions that might prevail, for instance, during the charging of a capacitor.

(a) Show that the charge density at any particular point is a linear function of time:

$$\rho(\mathbf{r}, t) = \rho(\mathbf{r}, 0) + \dot{\rho}(\mathbf{r}, 0)t,$$

where $\dot{\rho}(\mathbf{r}, 0)$ is the time derivative of ρ at $t = 0$.

This is *not* an electrostatic or magnetostatic configuration;[21] nevertheless—rather surprisingly—both Coulomb's law (in the form of Eq. 2.8) and the Biot-Savart law (Eq. 5.39) hold, as you can confirm by showing that they satisfy Maxwell's equations. In particular:

(b) Show that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, d\tau'$$

obeys Ampère's law *with Maxwell's displacement current term.*

Problem 7.56 The magnetic field of an infinite straight wire carrying a steady current I can be obtained from the *displacement* current term in the Ampère/Maxwell law, as follows: Picture the current as consisting of a uniform line charge λ moving along the z axis at speed v (so that $I = \lambda v$), with a tiny gap of length ϵ, which reaches the origin at time $t = 0$. In the next instant (up to $t = \epsilon/v$) there is no *real* current passing through a circular Amperian loop in the xy plane, but there *is* a *displacement* current, due to the "missing" charge in the gap.

(a) Use Coulomb's law to calculate the z component of the electric field, for points in the xy plane a distance s from the origin, due to a segment of wire with uniform density $-\lambda$ extending from $z_1 = vt - \epsilon$ to $z_2 = vt$.

(b) Determine the flux of this electric field through a circle of radius a in the xy plane.

(c) Find the displacement current through this circle. Show that I_d is equal to I, in the limit as the gap width (ϵ) goes to zero. [For a slightly different approach to the same problem, see W. K. Terry, *Am. J. Phys.* **50**, 742 (1982).]

Problem 7.57 The *magnetic* field outside a long straight wire carrying a steady current I is (of course)

$$\mathbf{B} = \frac{\mu_0}{2\pi} \frac{I}{s} \hat{\boldsymbol{\phi}}.$$

[21] Some authors *would* regard this as magnetostatic, since $\mathbf{B}$ is independent of t. For them, the Biot-Savart law is a general rule of magnetostatics, but $\nabla \cdot \mathbf{J} = 0$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ apply only under the *additional* assumption that ρ is constant. In such a formulation Maxwell's displacement term can (in this very special case) be *derived* from the Biot-Savart law, by the method of part (b). See D. F. Bartlett, *Am. J. Phys.* **58**, 1168 (1990); D. J. Griffiths and M. A. Heald, *Am. J. Phys.* **59**, 111 (1991).

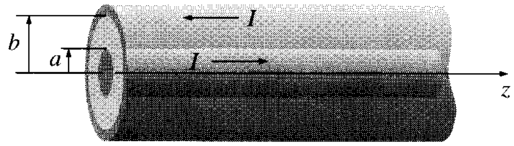


Figure 7.55

The *electric field inside* the wire is uniform:

$$\mathbf{E} = \frac{I\rho}{\pi a^2}\,\hat{\mathbf{z}},$$

where ρ is the resistivity and a is the radius (see Exs. 7.1 and 7.3). *Question:* What is the electric field *outside* the wire? This is a famous problem, first analyzed by Sommerfeld, and known in its most recent incarnation as "Merzbacher's puzzle."[22] The answer depends on how you complete the circuit. Suppose the current returns along a perfectly conducting grounded coaxial cylinder of radius b (Fig. 7.55). In the region $a < s < b$, the potential $V(s, z)$ satisfies Laplace's equation, with the boundary conditions

$$\text{(i)}\ \ V(a, z) = -\frac{I\rho z}{\pi a^2}; \quad \text{(ii)}\ \ V(b, z) = 0.$$

Unfortunately, this does not suffice to determine the answer—we still need to specify boundary conditions at the two ends. In the literature it is customary to sweep this ambiguity under the rug by simply *asserting* (in so many words) that $V(s, z)$ is proportional to z: $V(s, z) = zf(s)$. On this assumption:

(a) Determine $V(s, z)$.

(b) Find $\mathbf{E}(s, z)$.

(c) Calculate the surface charge density $\sigma(z)$ on the wire.

[*Answer:* $V = (-Iz\rho/\pi a^2)[\ln(s/b)/\ln(a/b)]$ This is a *peculiar* result, since E_s and $\sigma(z)$ are *not* independent of z—as one would certainly expect for a truly *infinite* wire.]

Problem 7.58 A certain transmission line is constructed from two thin metal "ribbons," of width w, a very small distance $h \ll w$ apart. The current travels down one strip and back along the other. In each case it spreads out uniformly over the surface of the ribbon.

(a) Find the capacitance per unit length, $\mathcal{C}$.

(b) Find the inductance per unit length, $\mathcal{L}$.

(c) What is the product $\mathcal{LC}$, numerically? [$\mathcal{L}$ and $\mathcal{C}$ will, of course, vary from one kind of transmission line to another, but their *product* is a universal constant—check, for example, the cable in Ex. 7.13—provided the space between the conductors is a vacuum. In the theory of transmission lines, this product is related to the speed with which a pulse propagates down the line: $v = 1/\sqrt{\mathcal{LC}}$.]

[22] A. Sommerfeld, *Electrodynamics*, p. 125 (New York: Academic Press, 1952); E. Merzbacher, *Am. J. Phys.* **48**, 104 (1980); further references in M. A. Heald, *Am. J. Phys.* **52**, 522 (1984).

(d) If the strips are insulated from one another by a nonconducting material of permittivity ϵ and permeablility μ, what then is the product $\mathcal{LC}$? What is the propagation speed? [*Hint:* see Ex. 4.6; by what factor does L change when an inductor is immersed in linear material of permeability μ?]

Problem 7.59 Prove **Alfven's theorem**: In a perfectly conducting fluid (say, a gas of free electrons), the magnetic flux through any closed loop moving with the fluid is constant in time. (The magnetic field lines are, as it were, "frozen" into the fluid.)

(a) Use Ohm's law, in the form of Eq. 7.2, together with Faraday's law, to prove that if $\sigma = \infty$ and $\mathbf{J}$ is finite, then

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}).$$

(b) Let $\mathcal{S}$ be the surface bounded by the loop ($\mathcal{P}$) at time t, and $\mathcal{S}'$ a surface bounded by the loop in its new position ($\mathcal{P}'$) at time $t + dt$ (see Fig. 7.56). The change in flux is

$$d\Phi = \int_{\mathcal{S}'} \mathbf{B}(t + dt) \cdot d\mathbf{a} - \int_{\mathcal{S}} \mathbf{B}(t) \cdot d\mathbf{a}.$$

Show that

$$\int_{\mathcal{S}'} \mathbf{B}(t + dt) \cdot d\mathbf{a} + \int_{\mathcal{R}} \mathbf{B}(t + dt) \cdot d\mathbf{a} = \int_{\mathcal{S}} \mathbf{B}(t + dt) \cdot d\mathbf{a}$$

(where $\mathcal{R}$ is the "ribbon" joining $\mathcal{P}$ and $\mathcal{P}'$), and hence that

$$d\Phi = dt \int_{\mathcal{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} - \int_{\mathcal{R}} \mathbf{B}(t + dt) \cdot d\mathbf{a}$$

(for infinitesimal dt). Use the method of Sect. 7.1.3 to rewrite the second integral as

$$dt \oint_{\mathcal{P}} (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{l},$$

and invoke Stokes' theorem to conclude that

$$\frac{d\Phi}{dt} = \int_{\mathcal{S}} \left(\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) \right) \cdot d\mathbf{a}.$$

Together with the result in (a), this proves the theorem.

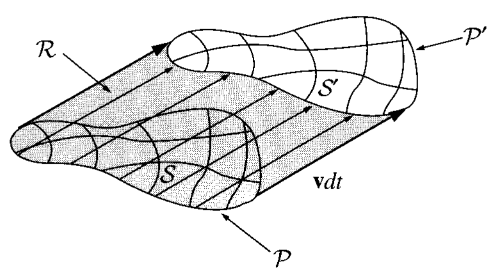


Figure 7.56

Problem 7.60

(a) Show that Maxwell's equations with magnetic charge (Eq. 7.43) are invariant under the **duality transformation**

$$\left.\begin{array}{rcl} \mathbf{E}' &=& \mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha, \\ c\mathbf{B}' &=& c\mathbf{B}\cos\alpha - \mathbf{E}\sin\alpha, \\ cq_e' &=& cq_e\cos\alpha + q_m\sin\alpha, \\ q_m' &=& q_m\cos\alpha - cq_e\sin\alpha, \end{array}\right\} \tag{7.68}$$

where $c \equiv 1/\sqrt{\epsilon_0\mu_0}$ and α is an arbitrary rotation angle in "**E**/**B**-space." Charge and current densities transform in the same way as q_e and q_m. [This means, in particular, that if you know the fields produced by a configuration of *electric* charge, you can immediately (using $\alpha = 90°$) write down the fields produced by the corresponding arrangement of *magnetic* charge.]

(b) Show that the force law (Prob. 7.35)

$$\mathbf{F} = q_e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + q_m(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}) \tag{7.69}$$

is also invariant under the duality transformation.

Intermission

All of our cards are now on the table, and in a sense my job is done. In the first seven chapters we assembled electrodynamics piece by piece, and now, with Maxwell's equations in their final form, the theory is complete. There are no more laws to be learned, no further generalizations to be considered, and (with perhaps one exception) no lurking inconsistencies to be resolved. If yours is a one-semester course, this would be a reasonable place to stop.

But in another sense we have just arrived at the starting point. We are at last in possession of a full deck, and we know the rules of the game—it's time to deal. This is the fun part, in which one comes to appreciate the extraordinary power and richness of electrodynamics. In a full-year course there should be plenty of time to cover the remaining chapters, and perhaps to supplement them with a unit on plasma physics, say, or AC circuit theory, or even a little General Relativity. But if you have room only for one topic, I'd recommend Chapter 9, on Electromagnetic Waves (you'll probably want to skim Chapter 8 as preparation). This is the segue to Optics, and is historically the most important application of Maxwell's theory.