Exercise 15.9

Chapter 15 Multiple Integrals 15.9 1E

(a)

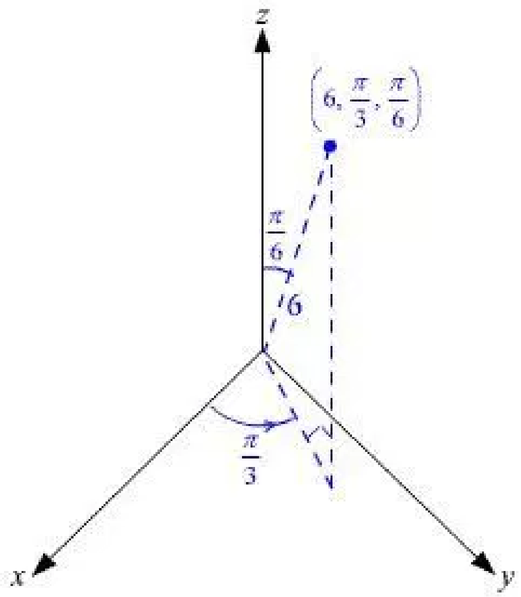
Consider the following spherical coordinates of the point:

$$\left(6, \frac{\pi}{3}, \frac{\pi}{6}\right)$$

Recall that spherical coordinates (ρ, θ, ϕ) is given by

$$x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi.$$

Plot the point with the given spherical coordinates as shown below:



Find the rectangular coordinates of the given point.

Find $x-$coordinate.

Substitute 6 for ρ, $\frac{\pi}{3}$ for θ, and $\frac{\pi}{6}$ for ϕ in $x = \rho \sin\phi \cos\theta$.

$$\begin{aligned} x &= 6 \sin\frac{\pi}{6} \cos\frac{\pi}{3} \\ &= 6\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{3}{2} \end{aligned}$$

Find $y-$coordinate.

Substitute 6 for ρ, $\frac{\pi}{3}$ for θ, and $\frac{\pi}{6}$ for ϕ in $y=\rho\sin\phi\sin\theta$.

$$\begin{aligned}y &= 6\sin\frac{\pi}{6}\sin\frac{\pi}{3}\\ &= 6\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{3\sqrt{3}}{2}\end{aligned}$$

Find $z-$coordinate.

Substitute 6 for ρ, $\frac{\pi}{3}$ for θ, and $\frac{\pi}{6}$ for ϕ in $z=\rho\cos\phi$.

$$\begin{aligned}z &= 6\cos\frac{\pi}{6}\\ &= 6\left(\frac{\sqrt{3}}{2}\right)\\ &= 3\sqrt{3}\end{aligned}$$

Therefore, the rectangular coordinates of the point $\left(6,\frac{\pi}{3},\frac{\pi}{6}\right)$ is,

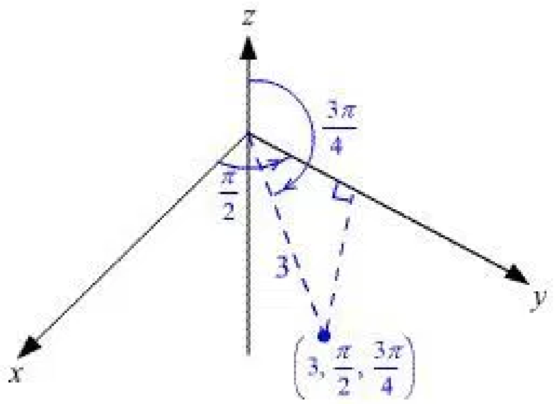
$$\boxed{\left(\frac{3}{2},\frac{3\sqrt{3}}{2},3\sqrt{3}\right)}.$$

(b)

Consider the following spherical coordinates of the point:

$$\left(3,\frac{\pi}{2},\frac{3\pi}{4}\right)$$

Plot the point with the given spherical coordinates as shown below:



Find the rectangular coordinates of the given point.

Find $x-$coordinate.

Substitute 3 for ρ, $\frac{\pi}{2}$ for θ, and $\frac{3\pi}{4}$ for ϕ in $x=\rho\sin\phi\cos\theta.$

$$\begin{aligned} x &= 3\sin\frac{3\pi}{4}\cos\frac{\pi}{2} \\ &= 3\sin\frac{\pi}{4}\cos\frac{\pi}{2} \\ &= 3\left(\frac{\sqrt{2}}{2}\right)(0) \\ &= 0 \end{aligned}$$

Find $y-$coordinate.

Substitute 3 for ρ, $\frac{\pi}{2}$ for θ, and $\frac{3\pi}{4}$ for ϕ in $y=\rho\sin\phi\sin\theta.$

$$\begin{aligned} y &= 3\sin\frac{3\pi}{4}\sin\frac{\pi}{2} \\ &= 3\sin\frac{\pi}{4}\sin\frac{\pi}{2} \\ &= 3\left(\frac{\sqrt{2}}{2}\right)(1) \\ &= \frac{3\sqrt{2}}{2} \end{aligned}$$

Find $z-$coordinate.

Substitute 3 for ρ, $\frac{\pi}{2}$ for θ, and $\frac{3\pi}{4}$ for ϕ in $z=\rho\cos\phi.$

$$\begin{aligned} z &= 3\cos\frac{3\pi}{4} \\ &= -3\cos\frac{\pi}{4} \\ &= -3\left(\frac{\sqrt{2}}{2}\right) \\ &= -\frac{3\sqrt{2}}{2} \end{aligned}$$

Therefore, the rectangular coordinates of the point $\left(3,\frac{\pi}{2},\frac{3\pi}{4}\right)$ is,

$$\boxed{\left(0,\frac{3\sqrt{2}}{2},-\frac{3\sqrt{2}}{2}\right)}.$$

Chapter 15 Multiple Integrals 15.9 2E

Write *x*, *y* and *z* in the form of spherical coordinates

$x = \rho \sin\phi \cos\theta$ ......(1)

$y = \rho \sin\phi \sin\theta$ ......(2)

$z = \rho \cos\phi$. ......(3)

Substitute the known values in equation (1) and find *x*.

$$x = 2\sin\frac{\pi}{2}\cos\frac{\pi}{2}$$
$$= 0$$

Replace ρ with 2, θ with $\frac{\pi}{2}$, and ϕ with $\frac{\pi}{2}$ equation (2).

$$y = 2\sin\frac{\pi}{2}\sin\frac{\pi}{2}$$
$$= 2$$

Substitute the known values in equation (3) and find *z*.

$$z = 2\cos\frac{\pi}{2}$$
$$= 0$$

Therefore, we get the corresponding rectangular coordinates as $(0, 2, 0)$.

(b) Plot the point with the given spherical coordinates $\left(4, \frac{-\pi}{4}, \frac{\pi}{3}\right)$.

Substitute the known values in equation (1) and find *x*.

$$x = 4\sin\frac{\pi}{3}\cos\left(-\frac{\pi}{4}\right)$$
$$= \sqrt{6}$$

Replace ρ with 4, θ with $-\frac{\pi}{4}$, and ϕ with $\frac{\pi}{3}$ equation (2).

$$y = 4\sin\frac{\pi}{3}\sin\left(-\frac{\pi}{4}\right)$$
$$= -\sqrt{6}$$

Substitute the known values in equation (3) and find *z*.

$$z = 4\cos\frac{\pi}{3}$$
$$= 2$$

Therefore, we get the corresponding rectangular coordinates as $\left(\sqrt{6}, -\sqrt{6}, 2\right)$.

Chapter 15 Multiple Integrals 15.9 3E

(a)

The relationship between rectangular coordinates (x, y, z) and the spherical coordinates (ρ, θ, ϕ) are

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta \quad \cdots\cdots(1)$$
$$z = \rho \cos\phi$$

and $\rho = \sqrt{x^2 + y^2 + z^2} \quad \cdots\cdots(2)$

Let $(x, y, z) = (0, -2, 0)$.

From (2), we have

$$\rho = \sqrt{0^2 + (-2)^2 + 0^2}$$
$$= 2$$

From (2), we have

$$\cos\phi = \frac{z}{\rho}$$
$$= \frac{0}{2}$$
$$= 0$$
$$\phi = \boxed{\frac{\pi}{2}}$$

And

$$\cos\theta = \frac{x}{\rho \sin\phi}$$
$$= \frac{0}{\rho \sin\phi}$$
$$= 0$$
$$\theta = \frac{3\pi}{2} \qquad [\text{since } y = -2 < 0]$$

Therefore, the suitable spherical coordinates are $\boxed{\left(2, \frac{3\pi}{2}, \frac{\pi}{2}\right)}$.

(b)

Let $(x,y,z)=\left(-1,1,-\sqrt{2}\right)$.

From (2), we have

$$\rho=\sqrt{(-1)^2+1^2+\left(-\sqrt{2}\right)^2}$$
$$=\sqrt{1+1+2}$$
$$=\boxed{2}$$

From (2), we have

$$\cos\phi=\frac{z}{\rho}$$
$$=\frac{-\sqrt{2}}{2}$$
$$=-\frac{1}{\sqrt{2}}$$
$$\phi=\boxed{\frac{3\pi}{4}} \qquad [\text{since } z=-\sqrt{2}<0]$$

And

$$\cos\theta=\frac{x}{\rho\sin\phi}$$
$$=\frac{-1}{2\cdot\left(\frac{1}{\sqrt{2}}\right)}$$
$$=-\frac{1}{\sqrt{2}}$$
$$\theta=\boxed{\frac{3\pi}{4}} \qquad [\text{since } x=-1<0, y=1>0]$$

Therefore the suitable spherical coordinates are $\boxed{\left(2,\frac{3\pi}{4},\frac{3\pi}{4}\right)}$.

Chapter 15 Multiple Integrals 15.9 4E

a)

The relation between rectangular coordinates (x, y, z) and the spherical coordinates (ρ, θ, ϕ) are

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta \quad \cdots\cdots(1)$$
$$z = \rho \cos\phi$$

And $\rho = \sqrt{x^2 + y^2 + z^2} \quad \cdots\cdots(2)$

Let $(x, y, z) = (1, 0, \sqrt{3})$

From (2), we have

$$\rho = \sqrt{1^2 + 0^2 + (\sqrt{3})^2} = \boxed{2}$$

From (2), we have

$$\cos\phi = \frac{z}{\rho} = \frac{\sqrt{3}}{2}$$
$$\phi = \boxed{\frac{\pi}{6}}$$

And

$$\cos\theta = \frac{x}{\rho \sin\phi} = \frac{1}{2 \cdot \frac{1}{2}} = 1$$
$$\theta = \boxed{0}$$

Therefore, the suitable spherical coordinates are $\boxed{\left(2, 0, \frac{\pi}{6}\right)}$.

(b)

Let $(x, y, z) = (\sqrt{3}, -1, 2\sqrt{3})$

From (2), we have

$$\rho = \sqrt{(\sqrt{3})^2 + (-1)^2 + (2\sqrt{3})^2}$$
$$= \sqrt{3+1+12}$$
$$= \boxed{4}$$

From (2), we have

$$\cos\phi = \frac{z}{\rho}$$
$$= \frac{2\sqrt{3}}{4}$$
$$= \frac{\sqrt{3}}{2}$$
$$\phi = \boxed{\frac{\pi}{6}}$$

And

$$\cos\theta = \frac{x}{\rho\sin\phi}$$
$$= \frac{\sqrt{3}}{4\cdot\frac{1}{2}}$$
$$= \frac{\sqrt{3}}{2}$$
$$\theta = \boxed{\frac{11\pi}{6}} \qquad [\text{since } y = -1 < 0]$$

Therefore, the suitable spherical coordinates are $\boxed{\left(4, \frac{11\pi}{6}, \frac{\pi}{6}\right)}$.

Chapter 15 Multiple Integrals 15.9 5E

Consider the surface $\phi = \frac{\pi}{3}$.

Need to describe the given surface in words.

Since we know that the surface $\phi = c$ represents a half-cone with the *z*-axis as its axis.

If $0 < c < \frac{\pi}{2}$, then the cone lies above the *xy*-plane.

If $\frac{\pi}{2} < c < \pi$, then the cone lies below the *xy*-plane.

For the given surface $c = \frac{\pi}{3}$, and $0 < \frac{\pi}{3} < \frac{\pi}{2}$.

Therefore the given surface represents a half-cone and it is lies above the *xy*-plane with the *z*-axis as its axis.

Graph the given surface.

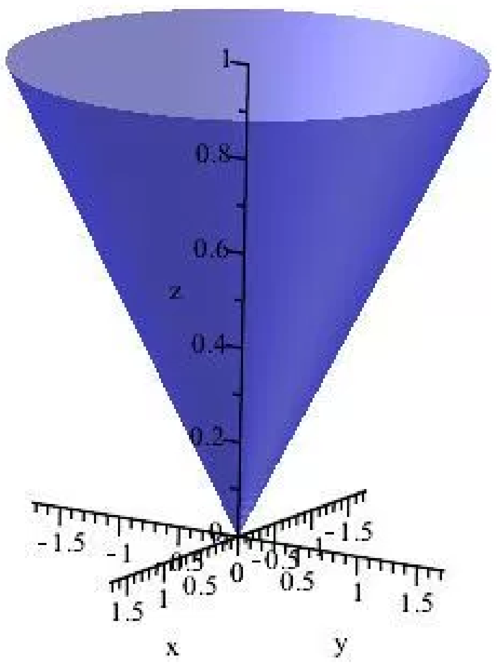
Using Maple command we draw the given surface.

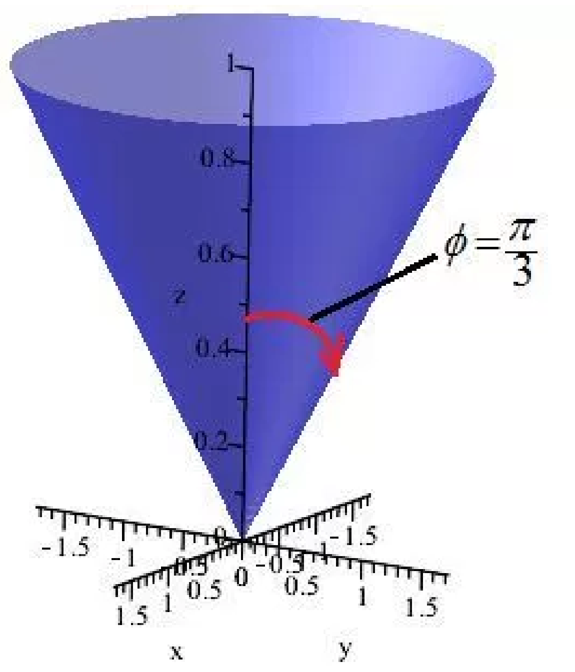
Keystrokes:

plot3d([r, theta, (1/3)*Pi], r = 0 .. 2, theta = 0 .. 2*Pi, coords = spherical, axes = normal, labels = ["x", "y", "z"], style = surface, transparency = .5, color = blue)

Maple result:

$$plot3d\left(\left[r, \text{theta}, \frac{\text{Pi}}{3}\right], r = 0..2, \text{theta} = 0..2\cdot\text{Pi}, coords = spherical, axes = normal, labels = ["x", "y", "z"], style = surface, transparency = 0.5, color = blue\right);$$





Chapter 15 Multiple Integrals 15.9 6E

Given surface is ρ = 3

A sphere centered at the origin with radius 3.

Chapter 15 Multiple Integrals 15.9 7E

ρ= sinθsinφ

ρ2= ρsinθsinφ

or x2+y2+z2= y

x2+y2- y+ 1/4 + z2= 1/4 (by completing the square)

x2+(y-1/2)2 + z2= 1/4

Thus, it is a sphere centered at (0,1/2,0) with radius 1/2.

Chapter 15 Multiple Integrals 15.9 8E

The surface is $\rho^2\left(\sin^2\phi\sin^2\theta+\cos^2\phi\right)=9$

This is an equation in spherical coordinates.

So, to convert this to the rectangular coordinates, we use

$x=\rho\sin\phi\cos\theta$

$y=\rho\sin\phi\sin\theta$

$z=\rho\cos\phi$

So, $\rho^2\left(\sin^2\phi\sin^2\theta+\cos^2\phi\right)=9$ can be written as $\left(\rho\sin\phi\sin\theta\right)^2+\left(\rho\cos\phi\right)^2=9$

In other words, $y^2+z^2=9$

Observe that the reference of x is not given in this equation.

So, we follow that $-\infty < x < \infty$

Thus, the given spherical equation can be understood to be $y^2 + z^2 = 3^2, -\infty < x < \infty$

This is a cylinder with base perpendicular to $yz-$plane with radius 3 units and height is infinite.

Chapter 15 Multiple Integrals 15.9 9E

(a) We have

$$z^2 = x^2 + y^2$$

To find the equation in spherical coordinates we use the equations

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$

Putting these values of x, y and z in $z^2 = x^2 + y^2$, we get

$$(\rho\cos\phi)^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$$

$$\Rightarrow \rho^2\cos^2\phi = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta$$

$$\Rightarrow \rho^2\cos^2\phi = \rho^2\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right)$$

$$\Rightarrow \rho^2\cos^2\phi = \rho^2\sin^2\phi\cdot\left(1\right)$$

$$\Rightarrow \cos^2\phi = \sin^2\phi$$

The required equation in the spherical coordinates is $\cos^2\phi = \sin^2\phi$

(b)

We have

$$x^2 + z^2 = 9$$

Put $x = \rho\sin\phi\cos\theta$ and $z = \rho\cos\phi$, we get

$$(\rho\sin\phi\cos\theta)^2 + (\rho\cos\phi)^2 = 9$$

$$\Rightarrow \rho^2\sin^2\phi\cos^2\theta + \rho^2\cos^2\phi = 9$$

$$\Rightarrow \rho^2\left(\sin^2\phi\cos^2\theta + \cos^2\phi\right) = 9$$

The required equation in the spherical coordinates is

$$\rho^2\left(\sin^2\phi\cos^2\theta + \cos^2\phi\right) = 9$$

Chapter 15 Multiple Integrals 15.9 10E

(a)

We have

$$x^2 - 2x + y^2 + z^2 = 0$$

To find the equation in spherical coordinates we use the equations

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$

Putting these values of $x, y \text{ and } z$ in the equation $x^2 - 2x + y^2 + z^2 = 0$, we get

$$(\rho\sin\phi\cos\theta)^2 - 2\left(\rho\sin\phi\cos\theta\right) + (\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi)^2 = 0$$

$$\Rightarrow \rho^2\sin^2\phi\cos^2\theta - 2\rho\sin\phi\cos\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi = 0$$
$$\Rightarrow \rho^2\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right) - 2\rho\sin\phi\cos\theta + \rho^2\cos^2\phi = 0$$
$$\Rightarrow \rho^2\sin^2\phi - 2\rho\sin\phi\cos\theta + \rho^2\cos^2\phi = 0$$
$$\Rightarrow \rho^2\left(\sin^2\phi + \cos^2\phi\right) - 2\rho\sin\phi\cos\theta = 0$$

$$\Rightarrow \rho^2 - 2\rho\sin\phi\cos\theta = 0$$

$$\Rightarrow \rho(\rho - 2\sin\phi\cos\theta) = 0$$

The required equation in the spherical coordinates is $\rho(\rho - 2\sin\phi\cos\theta) = 0$

(b)

We have

$$x + 2y + 3z = 1$$

Put $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$ in the equation $x + 2y + 3z = 1$, we get

$$\rho\sin\phi\cos\theta + 2(\rho\sin\phi\sin\theta) + 3\rho\cos\phi = 1$$

$$\Rightarrow \rho\sin\phi\cos\theta + 2\rho\sin\phi\sin\theta + 3\rho\cos\phi = 1$$

The required equation in the spherical coordinates is
$\rho\sin\phi\cos\theta + 2\rho\sin\phi\sin\theta + 3\rho\cos\phi = 1$

Chapter 15 Multiple Integrals 15.9 13E

Consider the solid formed by the following parameters:

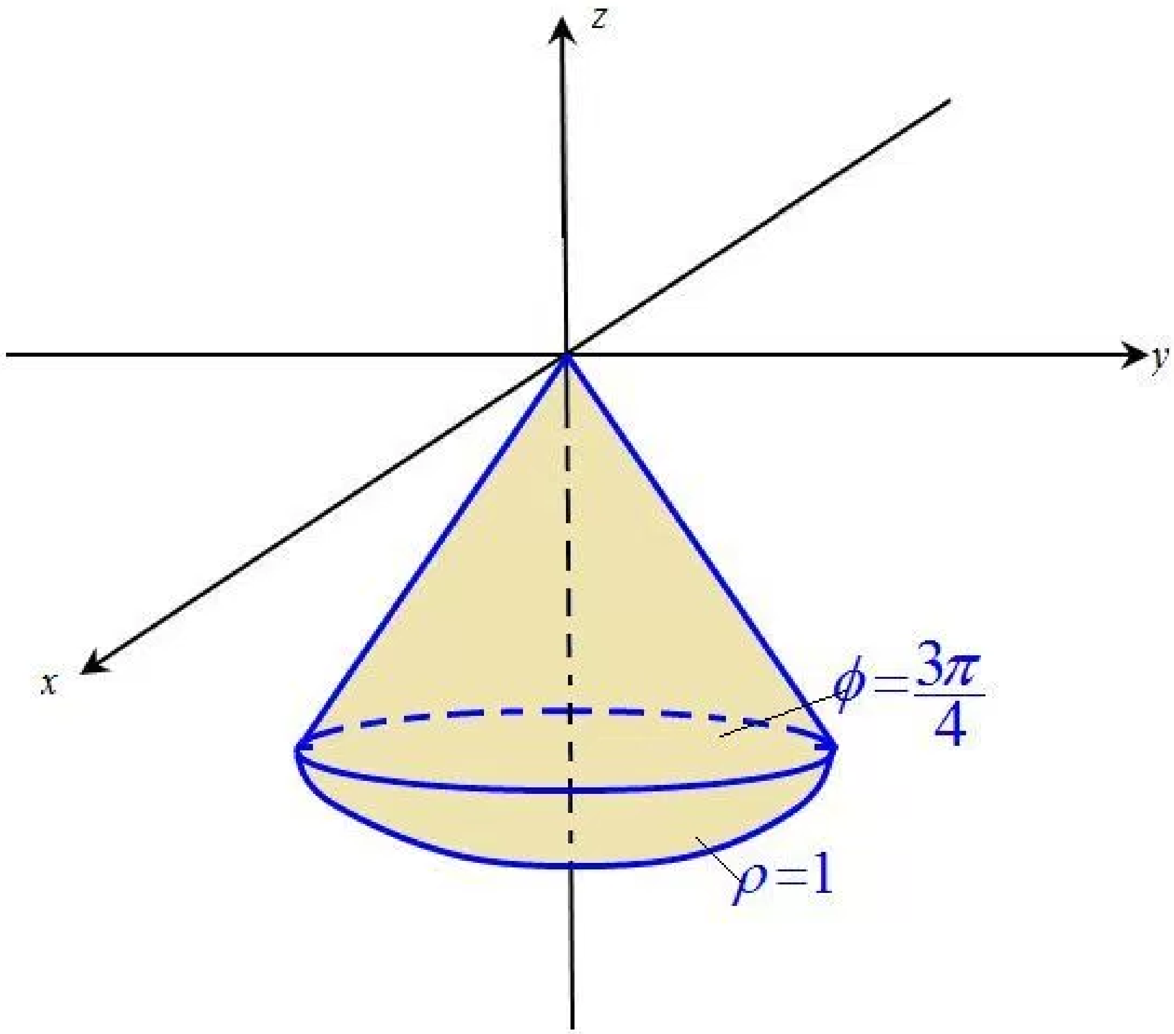
$$\rho \le 1, \frac{3\pi}{4} \le \phi \le \pi$$

In general, θ varies from 0 to 2π, and ρ is positive. To plot the region, use the inequalities $0 \le \rho \le 1, 3\pi/4 \le \phi \le \pi, 0 \le \theta \le 2\pi$.

The inequality $\rho \le 1$ is the solid, inside a sphere of radius 1 centered at the origin.

The set $\frac{3\pi}{4} \le \phi \le \pi$ describes a half-cone. It opens in the downward direction. The combination of the two sets $\rho \le 1, \frac{3\pi}{4} \le \phi \le \pi$ represents the plate with shape sphere of radius 1 which is parallel to the $xy-$plane and the center is located at $(0,0,1)$.

Sketch solid of the inequalities $\rho \le 1, \frac{3\pi}{4} \le \phi \le \pi$ as follows:



Chapter 15 Multiple Integrals 15.9 14E

Find the shapes described by the boundaries and then find their intersection in order to sketch the solid.

The inequality $\rho \le 2$ is the solid inside a sphere of radius 2 centered at the origin.

Rewrite the inequality $\rho \le \csc\phi$:

$$\rho \le \frac{1}{\sin\phi}$$

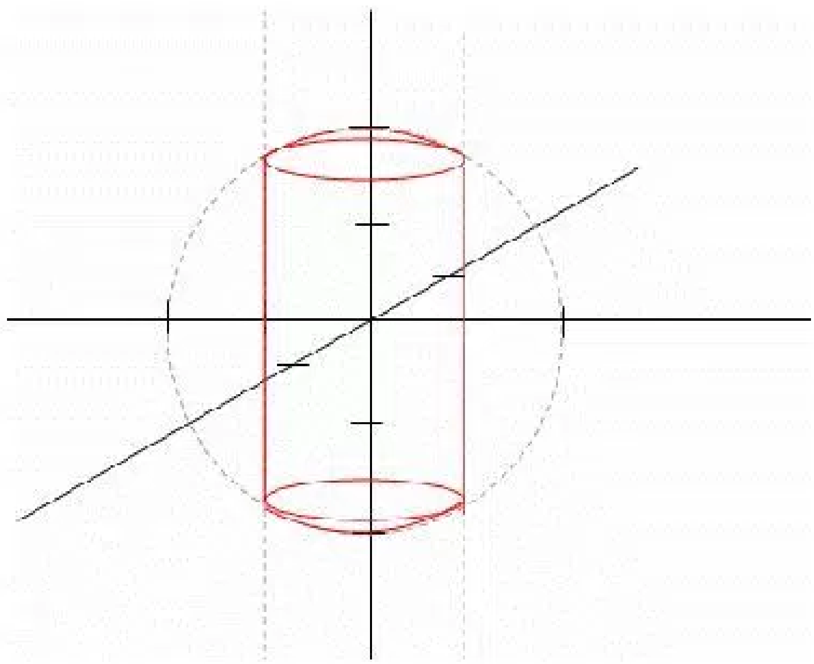
$$\rho\sin\phi \le 1$$

The expression $\rho\sin\phi$ gives the r distance from the z-axis to a point (ρ, θ, ϕ). Switching briefly to the cylindrical coordinate system, we can therefore rewrite this inequality as

$r \le 1$

Which is the solid inside a circular cylinder with axis the z-axis and radius 1.

The intersection of the two given inequalities is therefore the solid cut out of a sphere of radius 2 by a cylinder of radius 1:



The sphere and cylinder are shown with gray dashed lines and the solid of intersection shown with a red outline. The solid is something akin to a circular cylinder of radius 1 and height 2 with axis along the z-axis, but with rounded "caps" on the top and bottom where it is bounded by the sphere.

Chapter 15 Multiple Integrals 15.9 15E

Consider the solid which is lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

Need to describe the solid in terms of inequalities involving spherical coordinates.

First sketch the solid:

Use Maple to sketch the solid:

Keystrokes:

with(plots);

a := implicitplot3d(x^2+y^2+z^2-z = 0, x = -.5 .. .5, y = -.5 .. .5, z = 0 .. 1, style = surface, color = blue)

b := implicitplot3d(sqrt(x^2+y^2) = z, x = -.5 .. .5, y = -.5 .. .5, z = 0 .. .5, style = surface, color = red); display(a, b, axes = normal);

Maple result:

> $with(plots):$

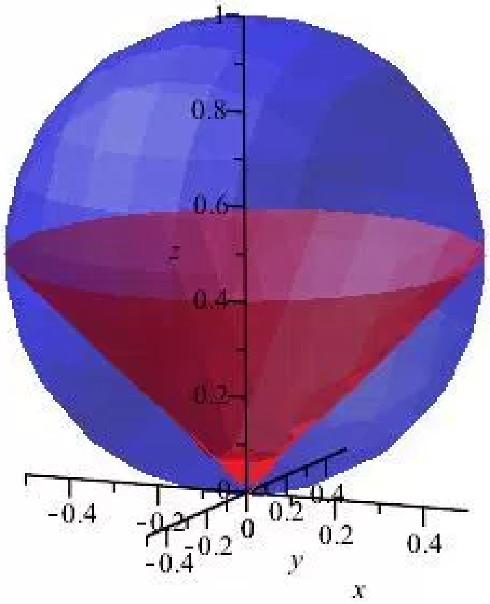
> $a := implicitplot3d(x^2 + y^2 + z^2 - z = 0, x = -0.5..0.5, y = -0.5..0.5, z = 0..1, style = surface, color = blue);$

$a := PLOT3D(...)$

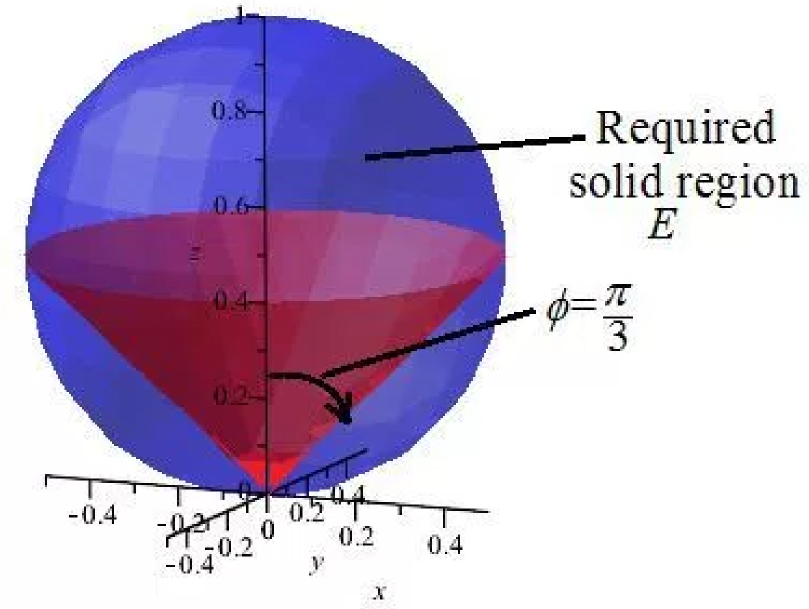
> $b := implicitplot3d(\sqrt{x^2 + y^2} = z, x = -0.5..0.5, y = -0.5..0.5, z = 0..0.5, style = surface, color = red);$

$b := PLOT3D(...)$

> $display(a, b, axes = normal);$



Observe the below graph:



Since we know that

$$\rho^2 = x^2 + y^2 + z^2$$
$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

Use these formulas convert the given surfaces from rectangular to spherical coordinates.

Rewrite $z = \sqrt{x^2 + y^2}$ as

$$\begin{aligned}\rho\cos\phi &= \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2} \\ &= \rho\sqrt{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta} \\ &= \rho\sqrt{\sin^2\phi(\cos^2\theta + \sin^2\theta)} \\ &= \rho\sin\phi\end{aligned}$$

Then

$$\rho\cos\phi = \rho\sin\phi$$
$$\tan\phi = 1$$
$$\phi = \frac{\pi}{4}$$

Rewrite $x^2+y^2+z^2=z$ as

$$\rho^2=\rho\cos\phi$$
$$\rho=\cos\phi$$

Therefore, the description of the solid is given by:

$$E=\left\{(\rho,\theta,\phi)\middle|0\le\rho\le\cos\phi, 0\le\phi\le\frac{\pi}{4}, 0\le\theta\le 2\pi\right\}.$$

Chapter 15 Multiple Integrals 15.9 16E

The inequality consists of the equations which may or May not be equal but are assigned with the symbols $\le, \ge, <, >$

Consider a hollow sphere of diameter $30\ cm$ and thickness $0.5\ cm$.

Let the ball is centered at the origin of the x, y, z axes.

The radius of the sphere is,

$$r=\frac{30}{2}\ cm$$
$$=15\ cm$$

Clearly it is a sphere of radius $15\ cm$

By the formula,

$x^2+y^2+z^2=r^2$ Where r is the radius

This implies,

$$x^2+y^2+z^2=(15)^2$$
$$=225$$

As the thickness is $0.5\ cm$ thus the new radius formed is,

$$r_n=15\ cm-0.5\ cm$$
$$=14.5\text{ cm}$$

Where r_n is the new radius

This implies

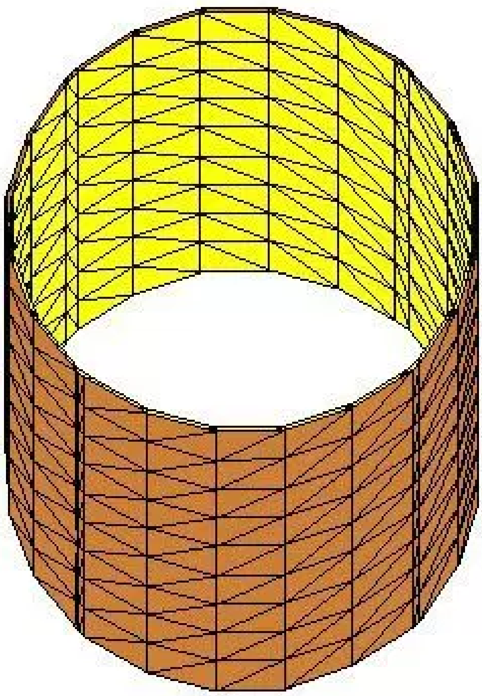
The new equation is,

$$x^2+y^2+z^2=(14.5)^2$$
$$=210.25$$

Thus, the equation of the data becomes an inequality,

$$14.5\le x^2+y^2+z^2$$
$$\le 15$$

The graph of the inequality is,



Therefore the required solution is

$$\boxed{14.5 \le x^2 + y^2 + z^2 \le 15}$$

(b)

Suppose that the sphere is cut in half. If this sphere is cut in half then this means that may be $z \ge 0$ or $z \le 0$.

This implies the upper half is,

$$14.5 \le x^2 + y^2 + z^2 \le 15$$

And,

$$z \ge 0$$

Also the lower half is,

$$14.5 \le x^2 + y^2 + z^2 \le 15$$

And,

$$z \le 0$$

Therefore the required solution is $\boxed{14.5 \le x^2 + y^2 + z^2 \le 15 \quad z \ge 0}$ and

$$\boxed{14.5 \le x^2 + y^2 + z^2 \le 15 \quad z \le 0}$$

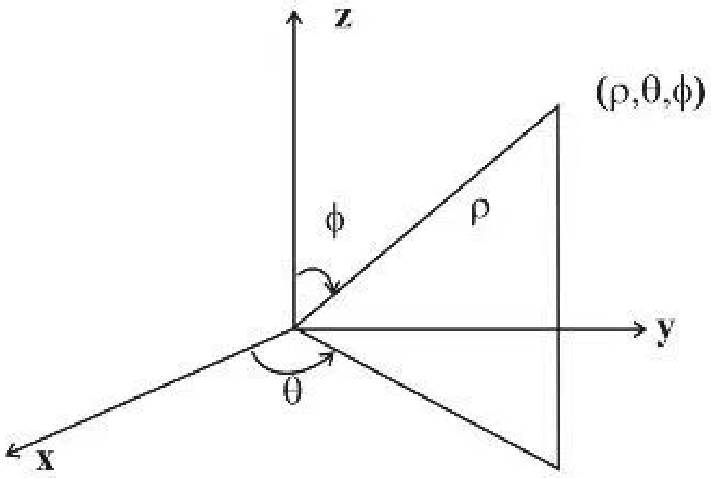
Chapter 15 Multiple Integrals 15.9 17E

$$\int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

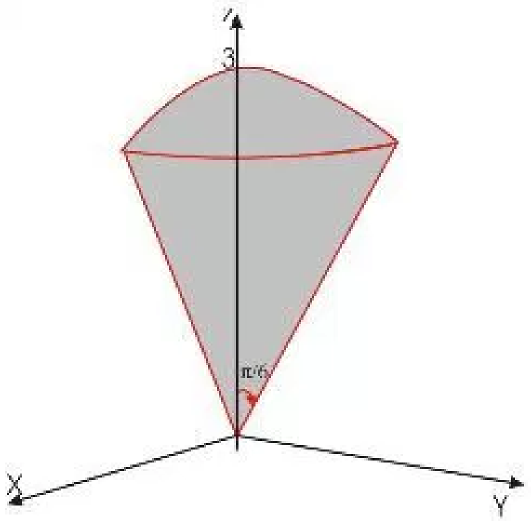
The region of integration is given in spherical co – ordinates

$$E = \left\{(\rho,\theta,\phi) : \ 0 \le \phi \le \frac{\pi}{6}, \ 0 \le \theta \le \frac{\pi}{2}, \ 0 \le \rho \le 3\right\}$$

As we know the spherical co - ordinates (ρ,θ,ϕ) are given as



Therefore the given solid can be given as



$$= \int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi/6} \sin\phi \, d\phi \int_0^{\pi/2} d\theta \int_0^3 \rho^2 \, d\rho$$

$$= [-\cos\phi]_0^{\pi/6} [\theta]_0^{\pi/2} \left[\frac{1}{3}\rho^3\right]_0^3$$

$$= \left(1 - \frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{2}\right)(9)$$

$$= \boxed{\frac{9\pi}{4}\left(2-\sqrt{3}\right)}$$

Chapter 15 Multiple Integrals 15.9 18E

Consider the triple integral,

$$\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}\int_1^2 \rho^2 \sin(\phi) d\rho d\phi d\theta.$$

The region of the integration can be written as,

$$E=\left\{(\rho,\theta,\phi)\,|\,0\le\theta\le 2\pi,\frac{\pi}{2}\le\phi\le\pi,1\le\rho\le 2\right\}.$$

Since ρ varies from 1 to 2.

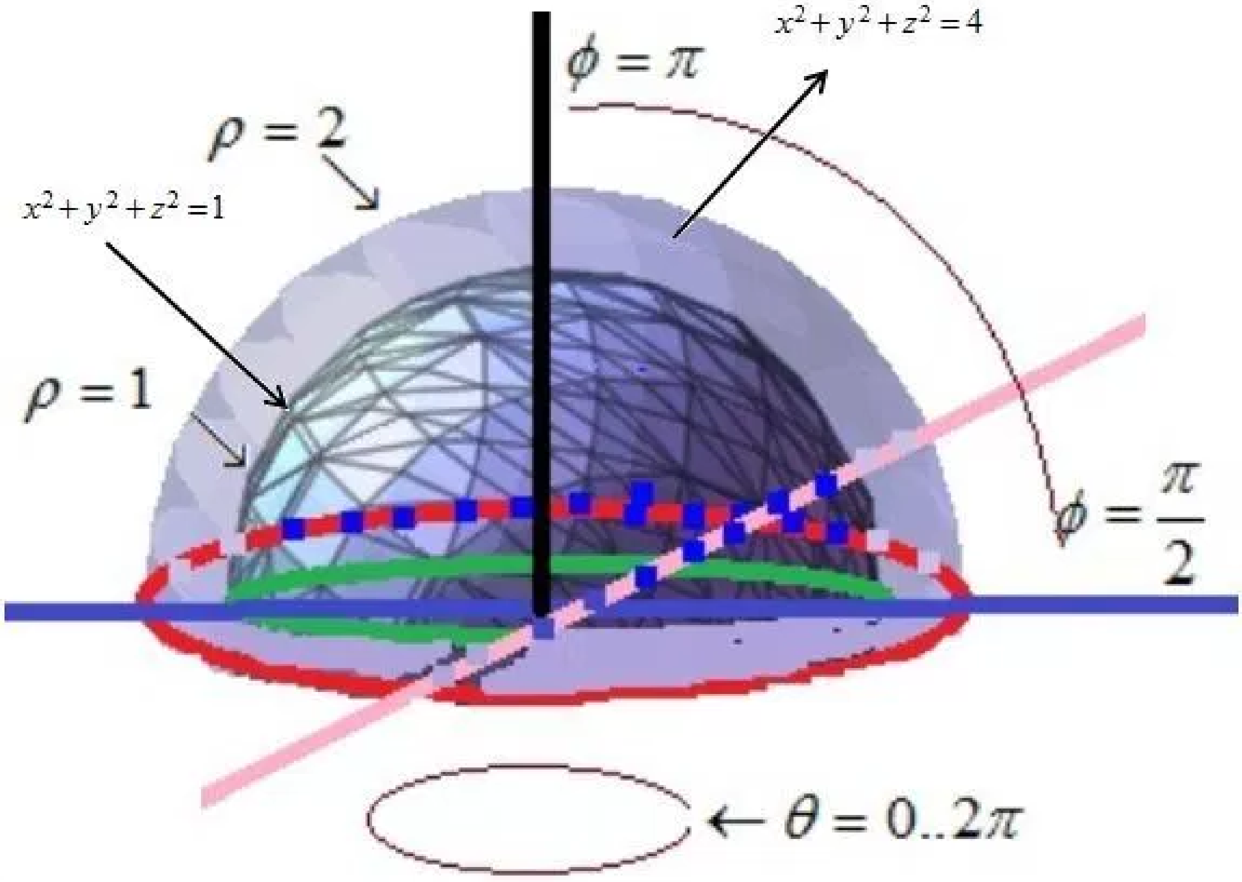
This implies that,

$$1\le\rho\le 2$$
$$1\le\sqrt{x^2+y^2+z^2}\le 2$$
$$1\le x^2+y^2+z^2\le 4$$

Observe that the region lies between two hemispheres of radius 1 and 2, and below the $xy-$ plane.

Since the volume is enclosed by a sphere of radius R is $\frac{4}{3}\pi R^3$.

The sketch of the solid is shown below:



Now the triple integral is evaluated as,

$$\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}\int_1^2 \rho^2 \sin(\phi) d\rho d\phi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}\left(\frac{\rho^3}{3}\right)_1^2 \sin(\phi) d\rho d\phi d\theta$$

$$= \frac{7}{3}\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \sin(\phi) d\phi d\theta$$

$$= \frac{7}{3}(-\cos(\phi))_{\frac{\pi}{2}}^{\pi}(\theta)_0^{2\pi}$$

$$= \frac{-7}{3}\left(\cos(\pi) - \cos\left(\frac{\pi}{2}\right)\right)(2\pi)$$

$$= \frac{-7}{3}(-1)2\pi$$

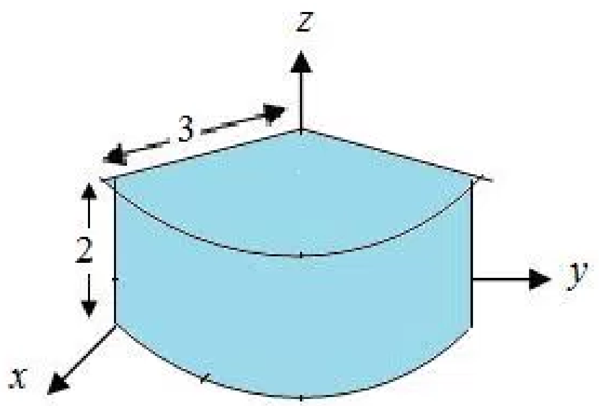
$$= \frac{14\pi}{3}$$

Hence, the value of the triple integral is

$$\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}\int_1^2 \rho^2 \sin(\phi) d\rho d\phi d\theta = \boxed{\frac{14\pi}{3}}.$$

Chapter 15 Multiple Integrals 15.9 19E

The shape shown is a section of cylinder, and is therefore best served with cylindrical coordinates.



Observe the figure, the shape goes through exactly one quarter of a full rotation in the xy-plane: resulting in our angle going from 0 to $\frac{\pi}{2}$.

The radius stops at 3, therefore, r will be in the interval [0, 3].

Finally, the z-coordinate goes from 0 to 2 (from the height of solid)

In cylindrical coordinates the volume element is calculated as follows

$$dv = dxdydz = rdrd\theta dz$$

The limits of cylindrical variables are,

r is from 0 to 3

θ is from 0 to $\frac{\pi}{2}$

z is from 0 to 2

Use the following parametric equations

$x = r\cos\theta;$

$y = r\sin\theta;$

$z = z;$

Use this information and convert the triple integral in x, y, z into cylindrical coordinates as follows,

$$\iiint f(x,y,z)\,dV = \int_0^{\frac{\pi}{2}}\int_0^{3}\int_0^{2} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Chapter 15 Multiple Integrals 15.9 20E

We know the spherical co-ordinates (ρ,θ,ϕ) of a point P in space are such that ρ is the distance of the point P from the origin say O, θ is the angle which the projection of OP in xy-plane makes with x-axis and ϕ is the angle which the line segment OP makes with the positive z-axis.

Now in the given solid, clearly ρ is varying from 1 to 2 (observe the solid is hollow for ρ less than 1), θ varies from $\frac{\pi}{2}$ to 2π (when we start from positive x-axis, there is no solid in the first octant thus starting from $\frac{\pi}{2}$ the solid rotates up to 2π) and the solid starts from xy-plane in positive z-direction, so ϕ is varying from 0 to $\frac{\pi}{2}$. Hence the integral is:

$$\int_0^{\pi/2}\int_{\pi/2}^{2\pi}\int_1^{2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

Chapter 15 Multiple Integrals 15.9 21E

$\iiint_B \left(x^2+y^2+z^2\right)^2 dV$ Ball with center at the origin and radius of 5.

$$\int_0^{\pi}\int_0^{2\pi}\int_0^5\left(\left(\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta+\rho^2\cos^2\phi\right)\frown 2\right)\rho^2\sin\phi \, d\rho \, d\theta \, d\phi$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^5\rho^6\sin\phi \, d\rho \, d\theta \, d\phi$$

$$\int_0^{\pi}\int_0^{2\pi}\left[\frac{\rho^7}{7}\right]_0^5\sin\phi \, d\theta \, d\phi=\frac{78125}{7}\int_0^{\pi}\int_0^{2\pi}\sin\phi \, d\theta \, d\phi$$

$$\frac{78125}{7}\int_0^{\pi}[\theta]_0^{2\pi}\sin\phi \, d\phi=\frac{156250\pi}{7}\int_0^{\pi}\sin\phi \, d\phi$$

$$\frac{156250\pi}{7}[-\cos\phi]_0^{\pi}=\frac{312500\pi}{7}$$

Chapter 15 Multiple Integrals 15.9 22E

Consider the following integral:

$$\iiint_H \left(9-x^2-y^2\right)dV$$

Here, *H* is the solid hemisphere $H=x^2+y^2+z^2\le 9,\ \ z\ge 0$ in the spherical coordinates.

Use spherical coordinates (ρ,θ,ϕ):

$$x=\rho\sin\phi\cos\theta, y=\rho\sin\phi\sin\theta, z=\rho\cos\theta$$

In this case, the hemisphere above the $xy-$plane and so, the angle with respect to $z-$axis is only half the total angle.

So, $0\le\phi\le\dfrac{\pi}{2}$

Since the base of the hemisphere is a circle upon $xy-$plane denoted by $x^2+y^2+z^2\le 9$, $z\ge 0$.

Then, $0\le\theta\le 2\pi$

Further, $0\le x^2+y^2+z^2=\rho^2\le 9$ says that $0\le\rho\le 3$.

Therefore, the spherical wedge *H* is given by,

$$H=\left\{(\rho,\theta,\phi)\,|\,0\le\rho\le 3, 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{2}\right\}$$

Use the spherical coordinates to evaluate the given triple integral:

$$f(x,y,z)dV=\int_g^h\int_c^d\int_a^b f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\phi$$

$$\begin{aligned}\iiint_H\left(9-x^2-y^2\right)dV &= \int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{3}\left(9-\rho^2\sin^2\phi\cos^2\theta-\rho^2\sin^2\phi\sin^2\theta\right)\rho^2\sin\phi d\rho d\theta d\phi\\ &= \int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{3}\left(9-\rho^2\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)\right)\rho^2\sin\phi d\rho d\theta d\phi\\ &= \int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{3}\left(9-\rho^2\sin^2\phi\right)\rho^2\sin\phi d\rho d\theta d\phi\\ &= \int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{3}\left(9\rho^2\sin\phi-\rho^4\sin^3\phi\right)d\rho d\theta d\phi\end{aligned}$$

$$= \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{2\pi} \left[\frac{9\rho^3 \sin\phi}{3} - \frac{\rho^5 \sin^3\phi}{5} \right]_0^3 d\theta d\phi$$

$$= \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{2\pi} \left(81\sin\phi - \frac{243\sin^3\phi}{5} \right) d\theta d\phi$$

$$= \int_{\theta=0}^{2\pi} \left[\int_{\phi=0}^{\frac{\pi}{2}} \left(81\sin\phi - \frac{243\sin^3\phi}{5} \right) d\phi \right] d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\int_{\phi=0}^{\frac{\pi}{2}} 81\sin\phi d\phi - \int_{\phi=0}^{\frac{\pi}{2}} \frac{243\sin^3\phi}{5} d\phi \right] d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[81[-\cos\phi]_0^{\frac{\pi}{2}} - \frac{243}{5} \left[-\frac{\sin^2\phi\cos\phi}{3} - \frac{2}{3}\cos\phi \right]_0^{\frac{\pi}{2}} \right] d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(81 - \frac{243}{5}\left(\frac{2}{3}\right) \right) d\theta$$

$$= (2\pi)\left(81 - \frac{162}{5} \right)$$

$$= (2\pi)\left(\frac{243}{5} \right)$$

$$\boxed{= \frac{486}{5}\pi}$$

Hence, the value of the given triple integral using spherical coordinates is $\boxed{\frac{486}{5}\pi}$.

Chapter 15 Multiple Integrals 15.9 23E

Consider the following triple integral:

$$\iiint_E \left(x^2 + y^2 \right) dV$$

Where *E* lies between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$.

The objective is to evaluate the given triple integral.

The volume of the region between the spheres is given by

$$V = \int_c^d \int_\alpha^\beta \int_a^b f\left(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\right)\rho^2 \sin\phi \; d\rho\, d\theta\, d\phi$$

Where E is a spherical wedge given by

$$E = \left\{(\rho,\theta,\phi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\right\}$$

Since the radius of the spheres is 2 and 3, so that the limits for ρ as (2, 3).

Use spherical coordinates.

$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$.

Then, $x^2 + y^2 = \rho^2\sin^2\phi$.

The region of integration can be expressed as a spherical rectangle as $2 \le \rho \le 3$, $0 \le \theta \le 2\pi$, and $0 \le \phi \le \pi$.

Therefore,

$$\begin{aligned}
\iiint_E \left(x^2 + y^2\right) dV &= \int_0^{2\pi}\int_0^{\pi}\int_2^3 \rho^2\sin^2\phi \cdot \rho^2\sin\phi \, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\int_2^3 \rho^4\sin^3\phi \, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\left(\frac{\rho^5}{5}\right)_2^3 \sin^3\phi \; d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\left(\frac{3^5 - 2^5}{5}\right)\sin^3\phi \; d\phi\, d\theta
\end{aligned}$$

$$= \frac{211}{5}\int_0^{2\pi}\int_0^{\pi}\sin^3\phi \; d\phi\, d\theta$$

$$= \frac{211}{5}\int_0^{2\pi}\int_0^{\pi}\left(\frac{3\sin\theta - \sin 3\theta}{4}\right) d\phi\, d\theta$$

$$= \frac{211}{20}\int_0^{2\pi}\left(-3\cos\theta + \frac{\cos 3\theta}{3}\right)_0^{\pi} d\phi$$

$$= \frac{211}{20}\int_0^{2\pi}\left[\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right] d\phi$$

Continuation to the above step,

$$\iiint_E \left(x^2+y^2\right)dV = \frac{211}{20}\times\frac{16}{3}\int_0^{2\pi} d\phi$$
$$=\frac{211}{20}\times\frac{16}{3}(\phi)_0^{2\pi}$$
$$=\frac{211}{20}\times\frac{16}{3}\times 2\pi$$
$$=\frac{1688}{15}\pi$$

Therefore, the volume of the region is $\frac{1688\pi}{15}$.

Chapter 15 Multiple Integrals 15.9 24E

Consider the triple integral,

$$\iiint_E y^2\,dV.$$

Where E is the solid hemisphere $x^2+y^2+z^2\le 9, y\ge 0.$

The objective is to find the above integral using spherical coordinates.

The volume of the region between the spheres is given by,

$$V=\int_c^d\int_\alpha^\beta\int_a^b f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\rho^2\sin\phi\ d\rho\,d\theta\,d\phi$$

Where E is a spherical wedge given by,

$$E=\{(\rho,\theta,\phi)\,|\,a\le\rho\le b,\alpha\le\theta\le\beta,c\le\phi\le d\}$$

In spherical coordinates,

$x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$ and $z=\rho\cos\phi$.

The region $E=\{x^2+y^2+z^2\le 9, y\ge 0\}$.

In spherical coordinates, the region is given by,

$$E=\left\{(\rho,\theta,\phi)\,|\,0\le\rho\le 3,-\frac{\pi}{2}\le\theta\le\frac{\pi}{2},0\le\phi\le\pi\right\}.$$

Therefore, the triple integral becomes,

$$\iiint_E y^2\,dV=\int_0^\pi\int_{-\pi/2}^{\pi/2}\int_0^3(\rho\sin\phi\sin\theta)^2\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$=\int_0^\pi\int_{-\pi/2}^{\pi/2}\int_0^3\rho^4\sin^3\phi\sin^2\theta\,d\rho\,d\theta\,d\phi$$
$$=\int_0^\pi\int_{-\pi/2}^{\pi/2}\left[\frac{\rho^5}{5}\right]_{\rho=0}^3\sin^3\phi\sin^2\theta\,d\theta\,d\phi$$
$$=\int_0^\pi\int_{-\pi/2}^{\pi/2}\frac{3^5}{5}\sin^3\phi\sin^2\theta\,d\theta\,d\phi$$

$$= \frac{3^5}{5} \int_0^{\pi} \int_{-\pi/2}^{\pi/2} \sin^3\phi \left(\frac{1-\cos 2\theta}{2} \right) d\theta \, d\phi$$

$$= \frac{243}{5\times 2} \int_0^{\pi} \sin^3\phi \left(\theta - \frac{\sin 2\theta}{2} \right)_{-\pi/2}^{\pi/2} d\phi$$

$$= \frac{243}{10} \int_0^{\pi} \sin^3\phi \left[\left(\frac{\pi}{2} - 0 \right) - \left(-\frac{\pi}{2} - 0 \right) \right] d\phi$$

$$= \frac{243}{10} \times \pi \int_0^{\pi} \sin^3\phi \, d\phi$$

$$= \frac{243\pi}{10} \int_0^{\pi} \frac{3\sin\phi - \sin 3\phi}{4} d\phi$$

$$= \frac{243\pi}{10} \left[-\cos\phi + \frac{\cos 3\phi}{3} \right]_0^{\pi}$$

$$= \frac{243\pi}{10} \left[\left(1 - \frac{1}{3} \right) - \left(-1 + \frac{1}{3} \right) \right]$$

$$= \frac{243\pi}{10} \left[\frac{4}{3} \right]$$

$$= \frac{162\pi}{5}$$

Thus,

$$\iiint_E y^2 \, dV = \boxed{\frac{162\pi}{5}}.$$

Chapter 15 Multiple Integrals 15.9 25E

Consider $\iiint_E x e^{x^2+y^2+z^2} dV$, where E is $x^2 + y^2 + z^2 \le 1$

Using spherical coordinates

$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta,$

$z = \rho \cos\phi$

And $\rho^2 = x^2 + y^2 + z^2$

Thus, $\rho^2 \le 1 \Rightarrow 0 \le \rho \le 1$

$$E = \left\{ 0 \le \rho \le 1, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le \frac{\pi}{2} \right\}$$

Evaluate the integral as follows:

$$\iiint_E xe^{x^2+y^2+z^2}\,dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho\sin\phi\cos\theta e^{\rho^2}\rho^2\sin\phi d\rho d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \sin^2\phi\cos\theta\rho^3 e^{\rho^2}\,d\rho d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \sin^2\phi\cos\theta\left[\rho^3 e^{\rho^2}\right]d\rho d\theta d\phi$$

Put

$$\rho^2 = t$$
$$2\rho d\rho = dt$$

When

$$\rho = 0,\ t = 0$$
$$\rho = 1,\ t = 1$$

Thus,

$$\iiint_E xe^{x^2+y^2+z^2}\,dV = \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \sin^2\phi\cos\theta\left[te^t\right]dt d\theta d\phi$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^2\phi\cos\theta\left[te^t - e^t\right]_0^1 d\theta d\phi$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^2\phi\cos\theta\left[e - e + 1\right]_0^1 d\theta d\phi$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^2\phi\cos\theta d\theta d\phi$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2\phi\left[\sin\theta\right]_0^{\frac{\pi}{2}} d\phi$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2\phi d\phi$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\phi}{2} d\phi$$

$$= \frac{1}{4}\left(\phi - \frac{\sin 2\phi}{2}\right)_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right) \qquad \text{since } \sin\pi = \sin 0 = 0$$

$$= \frac{\pi}{8}$$

Therefore, the value of the integral is $\boxed{\frac{\pi}{8}}$.

Chapter 15 Multiple Integrals 15.9 26E

Consider the integral $\iiint_E xyz\,dV$

And the region E is the region between the spheres $\rho = 2$ and $\rho = 4$ and above the cone $\phi = \frac{\pi}{3}$.

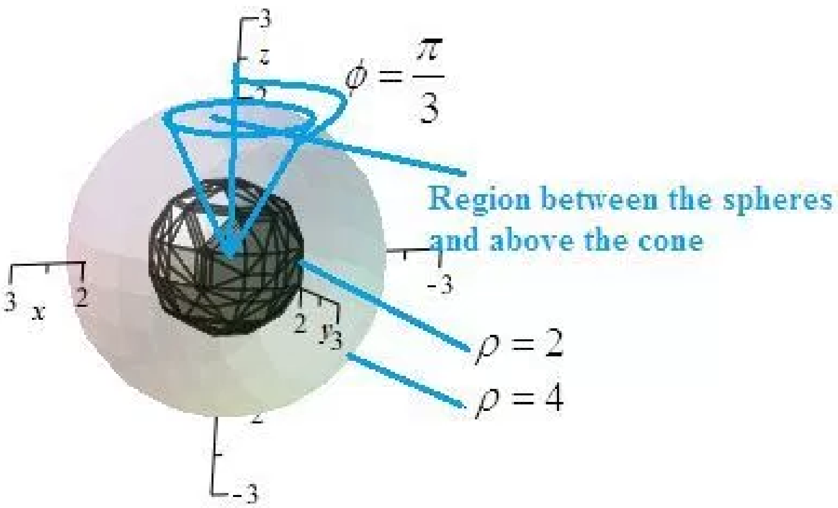
First convert the rectangular coordinates to spherical coordinates.

To convert the rectangular coordinates to spherical coordinates, use the transformations

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

And $\rho^2 = x^2 + y^2 + z^2$.

Sketch the region E bounded by $\rho = 2$ and ρ=4 and $\phi = \frac{\pi}{3}$.



In spherical coordinates, E is represented by $\left\{(\rho,\theta,\phi): 2 \le \rho \le 4, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{3}\right\}$

The volume element $dV = dxdydz$

Also convert volume element in terms of spherical coordinates.

$$dV = dxdydx = |J|\, d\rho d\theta d\phi$$

$$= \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\theta\sin\phi & \rho\cos\theta\sin\phi \\ \sin\phi\sin\theta & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} d\rho d\theta d\phi$$

$$= \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\theta\sin\phi & \rho\cos\theta\sin\phi \\ \sin\phi\sin\theta & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} d\rho d\theta d\phi$$

$$= \rho^2 \sin\phi\, d\rho d\theta d\phi$$

Convert the integral along with the limits to spherical coordinates and evaluate the integral.

Thus,

$$\iiint_E xyz\,dV = \int_0^{\frac{\pi}{3}}\int_0^{2\pi}\int_2^4 (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

As ρ,θ,ϕ are independent of each other evaluate the integral separately.

$$\iiint_E xyz\,dV = \left(\int_0^{\frac{\pi}{3}}\sin^3\phi\cos\phi\,d\phi\right)\left(\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\right)\left(\int_2^4\rho^5\,d\rho\right) \quad \ldots\ldots (1)$$

Use substitutions to evaluate the integrals $\int_0^{\frac{\pi}{3}}\sin^3\phi\cos\phi\,d\phi$, $\int_0^{2\pi}\sin\theta\cos\theta\,d\theta$ and $\int_2^4\rho^5\,d\rho$ separately.

Consider $\int_0^{\frac{\pi}{3}}\sin^3\phi\cos\phi\,d\phi$.

Use the substitution $\sin\phi = u$

Differentiate on each side.

$$\cos\phi\,d\phi = du$$

Change the limits of integration also.

When $\phi = 0 \Rightarrow \sin 0 = 0 \Rightarrow u = 0$

When $\phi = \frac{\pi}{3} \Rightarrow \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \Rightarrow u = \frac{\sqrt{3}}{2}$

$$\int_0^{\frac{\pi}{3}}\sin^3\phi\cos\phi\,d\phi = \int_0^{\frac{\sqrt{3}}{2}} u^3\,du$$

$$= \left[\frac{u^4}{4}\right]_0^{\frac{\sqrt{3}}{2}}$$ Use $\int u^n dx = \frac{u^{n+1}}{n+1}$

$$= \frac{\left(\frac{\sqrt{3}}{2}\right)^4}{4} - \frac{(0)^4}{4}$$

$$= \frac{9}{16\cdot 4}$$

$$= \frac{9}{64} \quad \ldots\ldots (2)$$

Consider $\int_0^{2\pi} \sin\theta\cos\theta\, d\theta$.

Use the substitution $\sin\theta = u$

Differentiate on each side.

$\cos\theta d\theta = du$

Change the limits of integration also.

When $\theta = 0 \Rightarrow \sin 0 = 0 \Rightarrow u = 0$

When $\theta = 2\pi \Rightarrow \sin 2\pi = 0 \Rightarrow u = 0$

$$\int_0^{2\pi} \sin\theta\cos\theta\, d\theta = \int_0^0 u du$$

$= 0$ ...... (3)

Consider the integral $\int_2^4 \rho^5\, d\rho$

$$\int_2^4 \rho^5\, d\rho = \left[\frac{\rho^6}{6}\right]_2^4$$

$$= \frac{1}{6}\left[4^6 - 2^6\right]$$

$$= \frac{1}{6}(4096 - 64)$$

$$= \frac{1}{6}(4032) \qquad \text{...... (4)}$$

$$= 672$$

Substitute (2), (3), (4) in (1)

$$\iiint_E xyz dV = \left(\frac{9}{64}\right)(0)(672)$$

$$= 0$$

Thus, the value of the integral over E is $\boxed{0}$.

Chapter 15 Multiple Integrals 15.9 27E

Consider the cone equations:

$$\phi = \frac{\pi}{6} \text{ and } \phi = \frac{\pi}{3}$$

The objective is to find the volume of the part of the ball $\rho \le a$ that lies between

$$\phi = \frac{\pi}{6} \text{ and } \phi = \frac{\pi}{3}$$

Write the Cartesian coordinates into spherical coordinates as:

$$x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi$$
$$x^2 + y^2 + z^2 = \rho^2; dV = \rho^2 \sin\phi d\rho d\phi d\theta$$

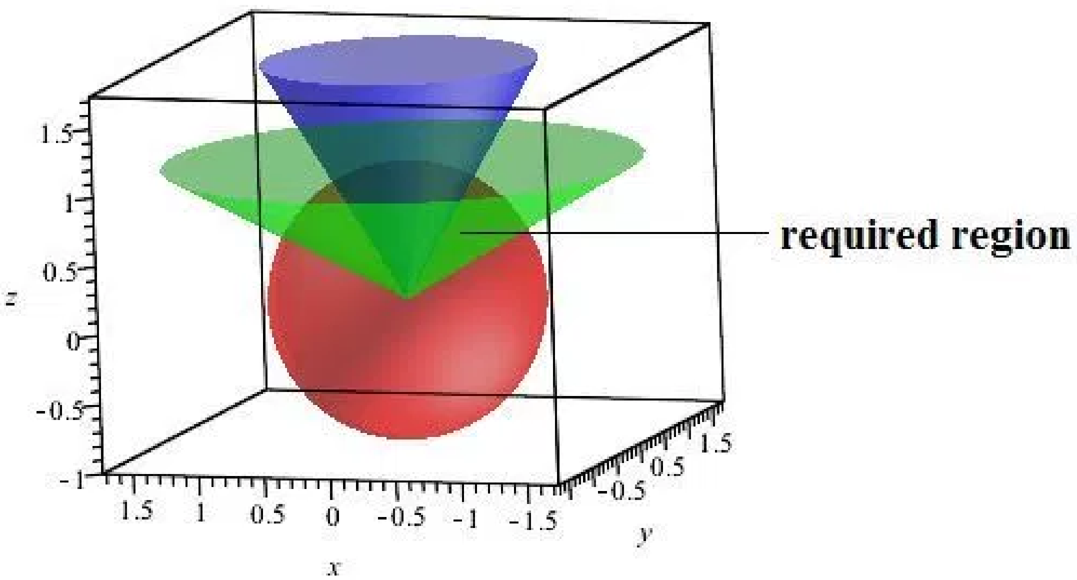
Given $\rho \le a$ and $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$

$$x^2 + y^2 + z^2 = a^2$$
$$\Rightarrow \rho^2 = a^2$$
$$\Rightarrow \rho = a$$

Hence, $0 \le \rho \le a$

Therefore, the region is $D = \left\{(\theta, \phi, \rho) : 0 \le \theta \le 2\pi, \frac{\pi}{6} \le \phi \le \frac{\pi}{3}, 0 \le \rho \le a\right\}$

The region is shown below.



Chapter 15 Multiple Integrals 15.9 28E

Let the average distance denoted by $\langle\rho\rangle$.

You are being for the average radius, which is another way of asking us to find the average ρ.

Because our function is going to be conducted over a sphere of radius a, we should normalize our function is $\dfrac{3\rho}{4\pi a^3}$.

Our region, by the description is defined as follows:

$$\begin{cases} 0 \le \rho \le a \\ 0 \le \theta \le \pi \\ 0 \le \varphi \le 2\pi \end{cases}$$

Therefore, the average distance is as follows:

$$\langle\rho\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho\frac{3}{4\pi a^3}\rho^2\sin(\theta)\,d\rho\,d\theta\,d\varphi \quad \cdots\cdots(1)$$

Compute (1), you get:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho\frac{3}{4\pi a^3}\rho^2\sin(\theta)\,d\rho\,d\theta\,d\varphi = \frac{3}{4\pi a^3}\int_0^{2\pi}\int_0^{\pi}\sin(\theta)\frac{\rho^4}{4}\,d\theta\,d\varphi\Big|_{\rho=0}^{\rho=a}$$

$$= -\frac{3}{4\pi a^3}\int_0^{2\pi}\cos(\theta)\frac{a^4}{4}\,d\varphi\Big|_{\theta=0}^{\theta=\pi}$$

$$= \frac{3}{4\pi a^3}\frac{2a^4\varphi}{4}\Big|_{\varphi=0}^{\varphi=2\pi}$$

$$= \frac{3}{4}a$$

Hence, the average distance is $\boxed{\langle\rho\rangle = \frac{3}{4}a}$.

Chapter 15 Multiple Integrals 15.9 29E

(a) The cone $\phi = \frac{\pi}{3}$ has z-axis as its axis and the sphere $\rho = 4\cos\phi$ is a sphere of diameter 4 with the z-axis as an axis and the diameter between the origin and the point $(x, y, z) = (0, 0, 4)$. The solid in our case is a cone-like object with vertex at the origin and opening upward, but with a convex base that is formed by the top of the sphere "cutting off" the opening cone. The limits of integration in ρ therefore go from 0 to the rounded base of the sphere, which is simply the sphere equation, $\rho = 4\cos\phi$. The solid is radially symmetric and goes all the way around the z-axis, so θ goes from 0 to 2π. Finally, the ϕ limits describe the cone's boundaries on the solid, so go from 0 to $\pi/3$.

To find a volume, the usual procedure is to integrate 1 as the integrand in the triple integral, since that will "count up" every small piece of the volume, multiply each by 1, and add them all up, resulting in the total volume. However, in converting to spherical coordinates a factor of $\rho^2 \sin\phi$ must be multiplied in. The integral is therefore:

$$\int_0^{\pi/3}\int_0^{2\pi}\int_0^{4\cos\phi}(1)(\rho^2\sin\phi)\,d\rho\,d\theta\,d\phi$$

Integrating in terms of ρ: $\displaystyle\int_0^{\pi/3}\int_0^{2\pi}\left(\frac{\rho^3\sin\phi}{3}\right)\Bigg|_0^{4\cos\phi} d\theta\,d\phi$

$$= \int_0^{\pi/3}\int_0^{2\pi}\left(\frac{(4\cos\phi)^3\sin\phi}{3} - 0\right)d\theta\,d\phi$$

Integrate in terms of θ: $\displaystyle\frac{64}{3}\int_0^{\pi/3}\theta\cos^3\phi\sin\phi\Big|_0^{2\pi} d\phi$

$$= \frac{128\pi}{3}\int_0^{\pi/3}\cos^3\phi\sin\phi\,d\phi \quad \text{...... (1)}$$

Suppose $\cos\phi = u, -\sin\phi\,d\phi = du$ and

When $\phi = 0$, we get $u = 1$ and when $\phi = \frac{\pi}{3}$, we get $u = \frac{1}{2}$

So, (1) becomes $\displaystyle\frac{128\pi}{3}\int_1^{1/2}u^3(-du)$

$$= \frac{128\pi}{3}\int_{1/2}^{1}u^3\,du$$

$$= \frac{128\pi}{3}\times\frac{u^4}{4}\Big\}_{1/2}^{1}$$

$$= \boxed{10\pi}$$

(b) Assume that the given region is of constant density. Since the solid has radial symmetry, the centroid must occur on the z-axis, and therefore its x- and y-coordinates are both 0. It remains to find the z-coordinate. The z-coordinate of the center of mass in three dimensions is given by

$\bar{z} = \frac{M_{xy}}{m}$ where M_{xy} is the moment about the xy - plane, given by $\iiint_E zk\,dV$ for constant density k, solid E, and m is the mass.

Since density equals mass over volume, we can write as $m = kv$ where v is the volume.
Since we found $v = 10\pi$ in part (a), we can plug all this into the z-coordinate of the centroid to get

$$\bar{z} = \frac{\iiint_E kz\,dV}{k(10\pi)}$$
$$= \frac{\iiint_E z\,dV}{10\pi}$$

The integral in the numerator is over the same solid as in part (a), so we use the same limits of integration. We convert to spherical coordinates, using the conversion $z = \rho\cos\phi$ and multiplying in the conversion factor $\rho^2 \sin\phi$:

$$\bar{z} = \frac{1}{10\pi}\int_0^{\pi/3}\int_0^{2\pi}\int_0^{4\cos\phi}(\rho\cos\phi)(\rho^2\sin\phi)\,d\rho\,d\theta\,d\phi$$
$$= \frac{1}{10\pi}\times\int_0^{\pi/3}\int_0^{2\pi}\left.\frac{\rho^4}{4}\right\}_0^{4\cos\phi}d\theta\sin\phi\cos\phi\,d\phi$$
$$= \frac{1}{40\pi}\int_0^{\pi/3}\left.\theta\right\}_0^{2\pi}256\cos^5\phi\sin\phi\,d\phi$$
$$= \frac{64}{5}\int_0^{\pi/3}\cos^5\phi\sin\phi\,d\phi \qquad \ldots\ldots (2)$$

Suppose $\cos\phi = u, -\sin\phi\,d\phi = du$ and

When $\phi = 0$, we get $u = 1$ and when $\phi = \frac{\pi}{3}$, we get $u = \frac{1}{2}$

Using these in (2), we get $\frac{64}{5}\int_1^{1/2} u^5(-du)$

$= \frac{64}{5} \times \frac{u^6}{6} \Big)_{1/2}^{1}$

$= \frac{32}{15}\left(1 - \frac{1}{64}\right)$

$= \frac{21}{10}$

Thus, the (x, y, z) coordinates of the centroid are $\boxed{(0, 0, \frac{21}{10})}$.

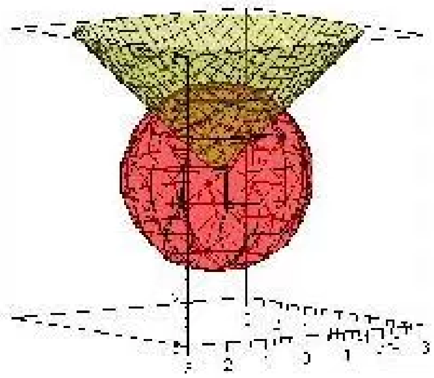
Chapter 15 Multiple Integrals 15.9 30E

The sphere equation $x^2 + y^2 + z^2 = 4$ is a sphere centered at the origin of radius 2. To visualize what the cone is doing; take the trace of the surface parallel to the xy - plane by plugging in $z = k$:

$k = \sqrt{x^2 + y^2}$

$k^2 = x^2 + y^2$

The cross-section of the cone parallel to the xy-plane is a circle of radius k, where k is the z-coordinate of the cross-section.



Convert to spherical coordinates. It is possible to find the converted equations using conversion factors, but because these are standard shapes, a shortcut is to think about what the spherical equations for these shapes would be. The equation for a sphere of radius 2 is $\rho = 2$. Since the cone has cross-sections that are circles of radius z, the radius of the cross section increases by 1 every time the z-coordinate increases by 1, and therefore the lateral side of the cone is at exactly a 45 degree angle between the xy - plane and the z-axis. The spherical equation for the cone is therefore $\phi = \pi / 4$.

Find the limits of integration in spherical coordinates. Throughout the solid, the radius reaches from the origin to the boundary of the sphere, so its limits are 0 and 2. The solid is radially symmetric around the z-axis and goes through every value of θ, the θ limits are 0 and 2π. The ϕ limits are from the boundary of the cone, which is $\phi = \pi/4$, to the xy - plane, where $\phi = \pi/2$.

To find a volume, standard procedure is to do a triple integral with integrand 1, which "adds up" all the small pieces of the volume to make the total volume. Since we are converting to spherical coordinates, we must multiply in the conversion factor $\rho^2 \sin\phi$.

Plug into the triple integral

$$\int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^2 (1)\rho^2 \sin\phi d\rho d\theta d\phi$$

$$= \int_{\pi/4}^{\pi/2}\int_0^{2\pi} \frac{\rho^3}{3}(\sin\phi)\Big|_0^2 d\theta d\phi$$

$$= \frac{8}{3}\int_{\pi/4}^{\pi/2} \sin\phi \times \theta)_0^{2\pi} d\phi$$

$$= \frac{16\pi}{3}(-\cos\phi)_{\pi/4}^{\pi/2}$$

$$= \frac{16\pi}{3}\left(-0+\cos\frac{\pi}{4}\right)$$

$$= \boxed{\frac{8\sqrt{2}\pi}{3}}$$

Chapter 15 Multiple Integrals 15.9 31E

The centroid of the mass is

$$\bar{x} = \frac{M_{yz}}{m}, \bar{y} = \frac{M_{xz}}{m}, \bar{z} = \frac{M_{xy}}{m}.$$

Where

$$M_{yz} = \iiint_E x\rho dV$$

$$\bar{y} = \frac{M_{xz}}{m}$$

$$M_{xz} = \iiint_E y\rho dV$$

$$\bar{z} = \frac{M_{xy}}{m}$$

$$M_{xy} = \iiint_E z\rho dV$$

Thus,

$$m=\iiint_E \rho dV$$

$$m=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\cos\phi} \rho dV$$

$$=k\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta$$

$$=\frac{k}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sin\phi\left[\rho^3\right]_0^{\cos\phi} d\phi d\theta$$

$$=\frac{k}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sin\phi\cos^3\phi d\phi d\theta$$

$$=\frac{k}{12}\int_0^{2\pi}\left[\cos^4\phi\right]_0^{\frac{\pi}{4}} d\theta$$

$$=\frac{2\pi k}{12}\left(1-\frac{1}{4}\right)$$

$$=\frac{\pi k}{8}$$

Therefore;

$$\bar{x}=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\cos\phi} x\rho dV$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\cos\phi} k\left(\rho\sin\phi\cos\theta\right)\rho^2\sin\phi d\rho d\phi d\theta$$

$$=\frac{k}{4}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \cos\theta\sin^2\phi\left[\rho^4\right]_0^{\cos\phi} d\phi d\theta$$

$$=\frac{k}{4}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \cos\theta\sin^2\phi\cos^4\phi d\phi d\theta$$

$$=\frac{k}{4}\int_0^{2\pi}\cos\theta\left[\frac{1}{32}\left(2\phi+\frac{1}{2}\sin 2\phi-\frac{1}{2}\sin 4\phi-\frac{1}{6}\sin 6\phi\right)\right]_0^{\frac{\pi}{4}} d\theta$$

$$=\frac{k}{4}\int_0^{2\pi}\cos\theta\left[\frac{1}{32}\left(2\left(\frac{\pi}{4}\right)+\frac{1}{2}\sin 2\left(\frac{\pi}{4}\right)-\frac{1}{2}\sin 4\left(\frac{\pi}{4}\right)-\frac{1}{6}\sin 6\left(\frac{\pi}{4}\right)\right)\right]d\theta$$

$$=\frac{k}{4}\int_0^{2\pi}\cos\theta\left[\frac{1}{32}\left(\left(\frac{\pi}{2}\right)+\frac{1}{2}\sin\sqrt{2}-0-\frac{1}{6}(-1)\right)\right]d\theta$$

$$=\left[\frac{1}{32}\left(\left(\frac{\pi}{64}\right)+\frac{1}{2}\sin\sqrt{2}+\frac{1}{6}\right)\right]\frac{k}{4}\int_0^{2\pi}\cos\theta d\theta$$

$$=\left[\frac{1}{32}\left(\left(\frac{\pi}{64}\right)+\frac{1}{2}\sin\sqrt{2}+\frac{1}{6}\right)\right]\frac{k}{4}\cdot 0$$

$$=0$$

$$\bar{y} = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\cos\phi} y\rho dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\cos\phi} k\left(\rho\sin\phi\sin\theta\right)\rho^2\sin\phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\cos\phi} k\left(\rho\sin\phi\sin\theta\right)\rho^2\sin\phi d\rho d\phi d\theta$$

$$= \frac{k}{4}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\sin\theta\sin^2\phi\left[\rho^4\right]_0^{\cos\phi} d\phi d\theta$$

$$= \frac{k}{4}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\sin\theta\sin^2\phi\cos^4\phi d\phi d\theta$$

$$= \frac{k}{4}\int_0^{2\pi}\sin\theta\left[\frac{1}{32}\left(2\phi+\frac{1}{2}\sin 2\phi-\frac{1}{2}\sin 4\phi-\frac{1}{6}\sin 6\phi\right)\right]_0^{\frac{\pi}{4}} d\theta$$

$$= \frac{k}{4}\int_0^{2\pi}\sin\theta\left[\frac{1}{32}\left(2\left(\frac{\pi}{4}\right)+\frac{1}{2}\sin 2\left(\frac{\pi}{4}\right)-\frac{1}{2}\sin 4\left(\frac{\pi}{4}\right)-\frac{1}{6}\sin 6\left(\frac{\pi}{4}\right)\right)\right]d\theta$$

$$= \frac{k}{4}\int_0^{2\pi}\sin\theta\left[\frac{1}{32}\left(\left(\frac{\pi}{2}\right)+\frac{1}{2}\sin\sqrt{2}-0-\frac{1}{6}(-1)\right)\right]d\theta$$

$$= \left[\frac{1}{32}\left(\left(\frac{\pi}{64}\right)+\frac{1}{2}\sin\sqrt{2}+\frac{1}{6}\right)\right]\frac{k}{4}\int_0^{2\pi}\sin\theta d\theta$$

$$= \left[\frac{1}{32}\left(\left(\frac{\pi}{64}\right)+\frac{1}{2}\sin\sqrt{2}+\frac{1}{6}\right)\right]\frac{k}{4}\cdot 0$$

$$= 0$$

$$\bar{z} = \frac{8}{\pi k}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\cos\phi} z\rho dV$$

$$= \frac{8}{\pi k}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\cos\phi} k\left(\rho\cos\phi\right)\rho^2 \sin\phi d\rho d\phi d\theta$$

$$= \frac{8}{\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\cos\phi} \rho^3 \sin\phi\cos\phi d\rho d\phi d\theta$$

$$= \frac{8}{\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\cos\phi} \rho^3 \sin\phi\cos\phi d\rho d\phi d\theta$$

$$= \frac{2}{\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sin\phi\cos\phi\left[\rho^4\right]_0^{\cos\phi} d\phi d\theta$$

$$= \frac{2}{\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sin\phi\cos^5\phi d\phi d\theta$$

$$= \frac{2}{\pi}\int_0^{2\pi}\left[\frac{\cos^6\phi}{6}\right]_0^{\frac{\pi}{4}} d\theta$$

$$= \frac{2}{6\pi}(2\pi)\left(\frac{7}{8}\right)$$

$$= \frac{7}{12}$$

Therefore the centroid is $\boxed{\left(0, 0, \frac{7}{12}\right)}$

Moments of inertia are

$$I_x = \iiint_E \left(y^2+z^2\right)\rho(x,y,z)\,dV$$
$$= \iiint_E \left(y^2+z^2\right)k\,dV$$

$$I_x = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\cos\phi} \left(y^2+z^2\right)k\,dV$$
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\cos\phi} \rho^2\left(\sin^2\phi\sin^2\theta+\cos^2\phi\right)k\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \frac{k}{5}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \left(\sin^2\phi\sin^2\theta+\cos^2\phi\right)\sin\phi\left[\rho^5\right]_0^{\cos\phi} d\phi\, d\theta$$

$$= \frac{k}{5}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \left(\sin^2\phi\sin^2\theta+\cos^2\phi\right)\sin\phi\cos^5\phi\, d\phi\, d\theta$$

$$= \frac{k}{5}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \left(\sin^3\phi\cos^5\phi\sin^2\theta+\sin\phi\cos^7\phi\right) d\phi\, d\theta$$

Put

$$\cos\phi = t$$
$$-\sin\phi\, d\phi = dt$$

Thus,

$$= \frac{k}{5}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \left(\left(1-\cos^2\phi\right)\sin\phi\cos^5\phi\sin^2\theta+\sin\phi\cos^7\phi\right) d\phi\, d\theta$$

$$= -\frac{k}{5}\int_0^{2\pi}\int_1^{\frac{1}{\sqrt{2}}} \left(t^5\left(1-t^2\right)\sin^2\theta+t^7\right) dt\, d\theta$$

$$= -\frac{k}{5}\int_0^{2\pi} \sin^2\theta \left[(1-t^2)\frac{t^6}{6} + 2t\frac{t^7}{42} - 2\frac{t^8}{42(8)} \right]_1^{\frac{1}{\sqrt{2}}} d\theta$$

$$= -\frac{k}{5}\int_0^{2\pi} \sin^2\theta \left[\left(\left(1-\frac{1}{2}\right)\frac{1}{48} + \frac{\sqrt{2}}{256\sqrt{2}} - \frac{1}{168(16)} \right) - \left((0)\frac{1}{6} + \frac{1}{21} - \frac{1}{42(4)} \right) \right] d\theta$$

$$= \left(-\frac{k}{5}\right)\left(-\frac{149}{5376}\right)\int_0^{2\pi} \sin^2\theta d\theta$$

$$= \frac{1}{2}\left(\frac{k}{5}\right)\left(\frac{149}{5376}\right)\int_0^{2\pi} (1-\cos 2\theta) d\theta$$

$$= \frac{149k}{53760}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{149k}{53760}(2\pi)$$

$$= \frac{298k\pi}{26880}$$

$$= \boxed{0.0111}$$

Similarly we can calculate

$$I_y = \iiint_E (x^2+z^2)\rho(x,y,z)\,dV$$

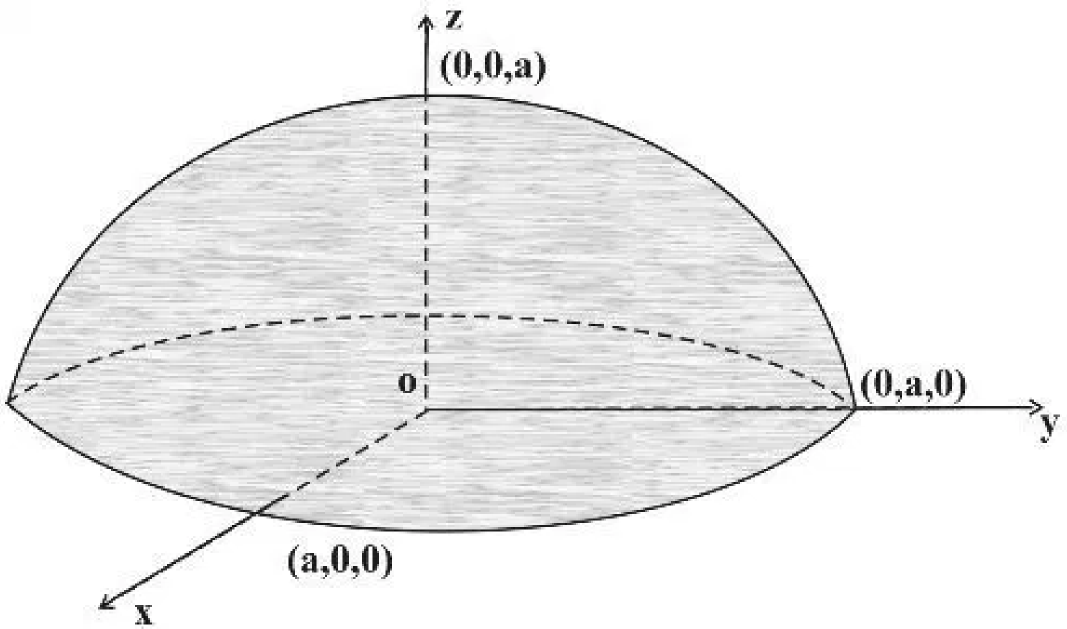
$$= \iiint_E (x^2+z^2)k\,dV$$

$$I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV$$

$$= \iiint_E (x^2+y^2)k\,dV$$

Chapter 15 Multiple Integrals 15.9 32E

Let the center of the base of hemisphere is at origin and axis along z – axis



Then $H = \left\{ (\rho, \theta, \phi):\ 0 \le \rho \le a,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{2} \right\}$

Now $\rho(x,y,z) \alpha \sqrt{x^2+y^2+z^2}$

Or $\rho(x,y,z) = k\sqrt{x^2+y^2+z^2}$

Where k is constant of proportionality

Then (A) mass of H is

$$m = \iiint_H \rho(x,y,z)\,dV$$

$$= k\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho.\,\rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

$$= k\int_0^{a} \rho^3 \, d\rho \int_0^{\pi/2} \sin\phi\, d\phi \int_0^{2\pi} d\theta$$

$$= k\left(\frac{\rho^4}{4}\right)_0^a (-\cos\phi)_0^{\pi/2} (\theta)_0^{2\pi}$$

$$= k\frac{a^4}{4} \times (-0+1)(2\pi)$$

$$= \boxed{\frac{1}{2}a^4 \pi k}$$

(B)

$$M_{yz} = \iiint_H x\,\rho(x,y,z)\,dV$$

$$= k\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho\sin\phi\cos\theta.\,\rho.\,\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= k\int_0^{a} \rho^4\, d\rho \int_0^{2\pi} \cos\theta\, d\theta \int_0^{\pi/2} \sin^2\phi\, d\phi$$

$$= k\left(\frac{\rho^5}{5}\right)_0^a (\sin\theta)_0^{2\pi} \left(\frac{\phi}{2} - \frac{1}{4}\sin 2\phi\right)_0^{\pi/2}$$

$$= k\left(\frac{a^5}{5}\right)(\sin 2\pi - \sin 0)\left(\frac{\pi}{4} - \frac{1}{4}\sin\pi\right)$$

$$= 0$$

$$M_{xz} = \iiint_H y\,\rho(x,y,z)\,dV$$

$$= k\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho\sin\phi\sin\theta.\rho.\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= k\int_0^{a}\rho^4\,d\rho\int_0^{\pi/2}\sin^2\phi\,d\phi\int_0^{2\pi}\sin\theta\,d\theta$$

$$= k\left(\frac{\rho^5}{5}\right)_0^{a}\left(\frac{\phi}{2}-\frac{1}{4}\sin 2\phi\right)_0^{\pi/2}\left(-\cos\theta\right)_0^{2\pi}$$

$$= k\left(\frac{a^5}{5}\right)\left(\frac{\pi}{4}-\frac{1}{4}\sin\pi-0\right)\left(-\cos 2\pi+\cos 0\right)$$

$$= k\left(\frac{a^5}{5}\right)\left(\frac{\pi}{4}\right)(-1+1)$$

$$= 0$$

$$M_{xy} = \iiint_H z\;\rho(x,y,z)\,dV$$

$$= k\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho\cos\phi.\rho.\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= k\int_0^{a}\rho^4\,d\rho\int_0^{2\pi}d\theta\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi$$

$$= k\left(\frac{\rho^5}{5}\right)(\theta)_0^{2\pi}\left(-\frac{1}{4}\cos 2\phi\right)_0^{\pi/2}$$

$$= k\left(\frac{a^5}{5}\right)(2\pi)\left(-\frac{1}{4}(-1)+\frac{1}{4}\right)$$

$$= k\left(\frac{a^5}{5}\right)(2\pi)\left(\frac{2}{4}\right)$$

$$= \frac{a^5\pi}{5}k$$

Then $\bar{x} = \dfrac{M_{yz}}{m} = 0$, $\bar{y} = \dfrac{M_{xz}}{m} = 0$

$$\bar{z} = \frac{M_{xy}}{m}$$

$$= \frac{a^5\pi k}{5}\times\frac{2}{a^5\pi k}$$

$$= \frac{2a}{5}$$

Thus the center of mass is $\boxed{\left(0,0,\dfrac{2a}{5}\right)}$

(C)

Then moment of inertia about $z-$axis is

$$I_z = \iiint_H (x^2+y^2)\rho(x,y,z)\,dV$$

$$= k\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} (\rho^2\sin^2\phi).\rho.\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= k\int_0^{a}\rho^5\,d\rho\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin^3\phi\, d\phi$$

$$= k\left(\frac{\rho^6}{6}\right)_0^a(\theta)_0^{2\pi}\left[\frac{-1}{3}(2+\sin^2\phi)\cos\phi\right]_0^{\pi/2}$$

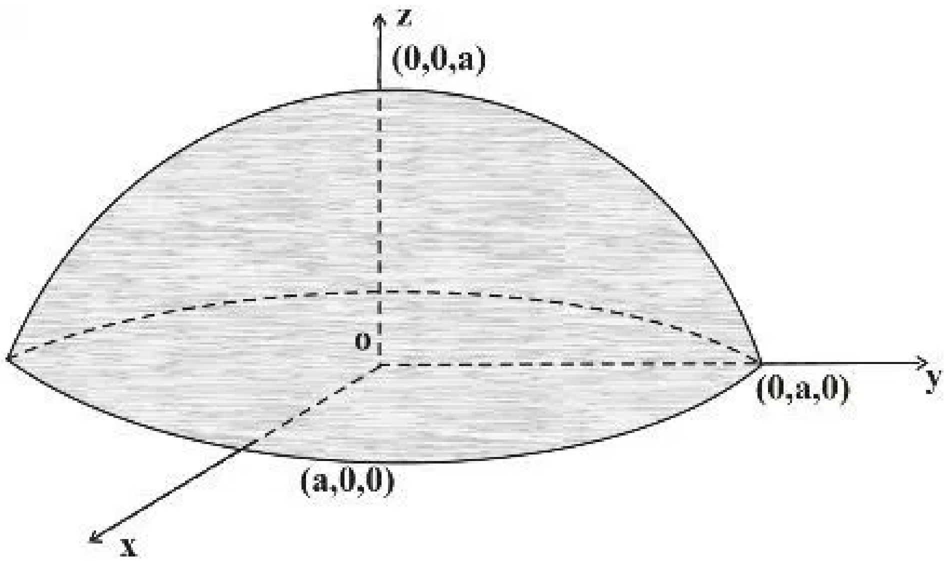
$$= k\left(\frac{a^6}{6}\right)(2\pi)\left(\frac{2}{3}\right)$$

$$= \frac{2}{9}a^6k\pi$$

Hence moment of inertia of solid H about is axis is $\boxed{\frac{2}{9}a^6k\pi}$, where k is constant of proportionality.

Chapter 15 Multiple Integrals 15.9 33E

Let the center of the base of hemisphere is at origin and the axis is along $z-$axis



(A)

As we know when the density of a solid is constant then the center of mass becomes centroid

Thus in this case $\rho(x,y,z) = k$ (constant)

The solid is given by

$$H = \left\{(\rho,\theta,\phi):\ 0\le\rho\le a,\ 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{2}\right\}$$

$$\text{Then } m = \iiint_H \rho(x,y,z)\,dV$$

$$= k\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= k\int_0^{a} \rho^2\, d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi\, d\phi$$

$$= k\left(\frac{\rho^3}{3}\right)_0^{a} (\theta)_0^{2\pi} (-\cos\phi)_0^{\pi/2}$$

$$= k\frac{a^3}{3}(2\pi)(1)$$

$$= \frac{2}{3}a^3\pi k$$

$$\text{Now } M_{yz} = \iiint_H x\,\rho(x,y,z)\,dV$$

$$= k\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho\sin\phi\cos\theta.\ \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= k\int_0^{a} \rho^2\, d\rho \int_0^{2\pi} \cos\theta\, d\theta \int_0^{\pi/2} \sin^2\phi\, d\phi$$

$$= k\left(\frac{\rho^4}{4}\right)_0^{a} (\sin\theta)_0^{2\pi} \left(\frac{\phi}{2}-\frac{1}{4}\sin 2\phi\right)_0^{\pi/2}$$

$$= k\frac{\rho^4}{4}(\sin 2\pi - \sin 0)\left(\frac{\pi}{4}-\frac{1}{4}\sin\pi\right)$$

$$= 0$$

$$M_{xz} = \iiint_H y\,\rho(x,y,z)\,dV$$

$$= k\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho\sin\phi\sin\theta.\ \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= k\int_0^{a} \rho^3\, d\rho \int_0^{2\pi} \sin\theta\, d\theta \int_0^{\pi/2} \sin^2\phi\, d\phi$$

$$= k\left(\frac{\rho^4}{4}\right)_0^{a} (-\cos\theta)_0^{2\pi} \left(\frac{\phi}{2}-\frac{1}{4}\sin 2\phi\right)_0^{\pi/2}$$

$$= k\left(\frac{a^4}{4}\right)(-\cos 2\pi + \cos 0)\left(\frac{\pi}{4}-\frac{1}{4}\sin\pi\right)$$

$$= 0$$

$$M_{xy} = \iiint_H z\,\rho(x,y,z)\,dV$$

$$= k\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho\cos\phi.\,\rho^2\sin\phi\; d\rho\, d\phi\, d\theta$$

$$= k\int_0^{2} \rho^3 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos\phi\sin\phi\; d\phi$$

$$= k\left(\frac{\rho^4}{4}\right)_0^a (\theta)_0^{2\pi}\left(-\frac{1}{4}\cos 2\phi\right)_0^{\pi/2}$$

$$= k\frac{a^4}{4}(2\pi)\left(\frac{2}{4}\right)$$

$$= \frac{a^4}{4}\pi k$$

Then $\bar{x} = \dfrac{M_{yz}}{m} = 0$

$\bar{y} = \dfrac{M_{xz}}{m} = 0$

$$\bar{z} = \frac{M_{xy}}{m}$$

$$= \frac{a^4\pi k}{4} \times \frac{3}{2a^3\pi k} = \frac{3}{8}a$$

Hence the centroid is $\boxed{\left(0,\ 0,\ \dfrac{3}{8}a\right)}$

(B)

Moment of inertia about diameter of the base:

The diameter is along y – axis, then we find moment of inertia about y – axis $I_y = \iiint_H (x^2+z^2)\,\rho(x,y,z)\,dV$

$$= k\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho^2\left(1-\sin^2\phi\sin^2\theta\right).\,\rho^2\sin\phi\; d\rho\, d\phi\, d\theta$$

$$= k\int_0^{a} \rho^4\, d\rho \int_0^{2\pi}\int_0^{\pi/2} \left(\sin\phi - \sin^3\phi\sin^2\theta\right) d\phi\, d\theta$$

$$= k\left(\frac{\rho^5}{5}\right)_0^a \int_0^{2\pi} (-\cos\phi)_0^{\pi/2} - \sin^2\theta\left(-\frac{1}{3}\left(2+\sin^2\phi\right)\cos\phi\right)_0^{\pi/2} d\theta$$

i.e. $I_y = k\frac{a^5}{5}\int_0^{2\pi}\left(1-\frac{2}{3}\sin^2\theta\right)d\theta$

$$= \frac{k\,a^5}{5}\left[\theta-\frac{2}{3}\left(\frac{\theta}{2}-\frac{1}{4}\sin 2\theta\right)\right]_0^{2\pi}$$

$$= \frac{ka^5}{5}\left[2\pi-\frac{2\pi}{3}\right]$$

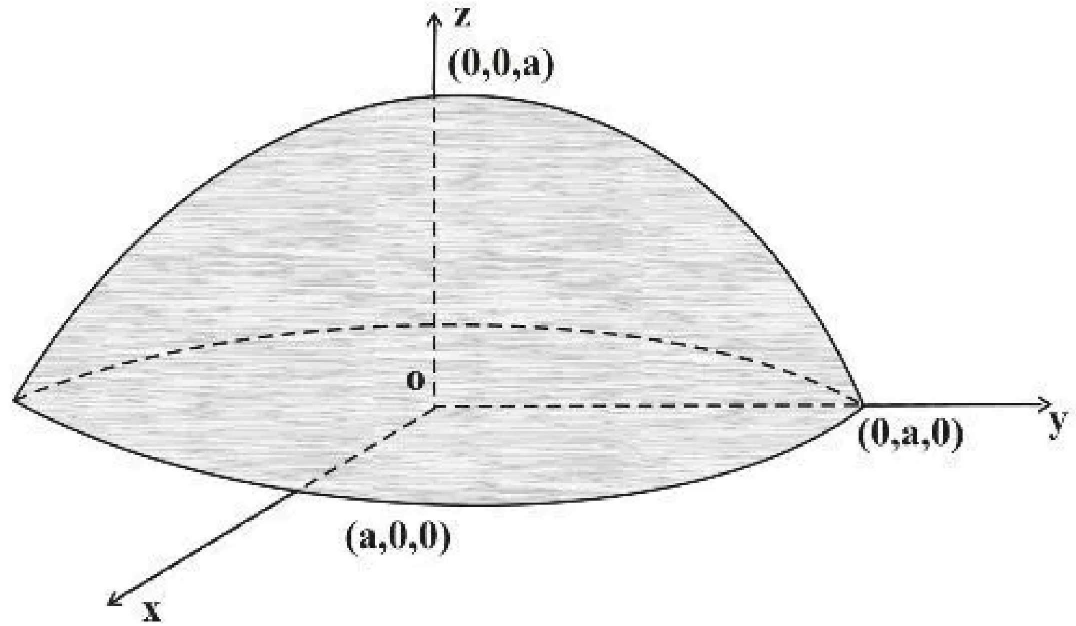
$$= \frac{ka^5}{5}\times\frac{4\pi}{3}$$

$$= \frac{k.4a^5\pi}{15}$$

Hence moment of inertia about diameter is $\boxed{\frac{4a^5 k\pi}{15}}$

Chapter 15 Multiple Integrals 15.9 34E

Let the center of the base be placed at the origin and then $\rho(x,y,z)=kz$ where k is a constant.



The solid hemisphere can be given as

$$H = \left\{ (\rho, \theta, \phi) : 0 \le \rho \le a,\ 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2} \right\}$$

Then mass of solid is

$$\begin{aligned} m &= \iiint_H \rho(x,y,z)\,dV \\ &= k\int_0^{2\pi}\int_0^{\pi/2}\int_0^a \rho\cos\phi.\rho^2\sin\phi\ d\rho\, d\phi\, d\theta \\ &= k\int_0^a \rho^3\ d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos\phi\sin\phi\, d\phi \\ &= k\left(\frac{\rho^4}{4}\right)_0^a (\theta)_0^{2\pi} \left(-\frac{1}{4}\cos 2\phi\right)_0^{\pi/2} \\ &= k\frac{a^4}{4}(2\pi)\left(\frac{1}{2}\right) \\ &= \frac{1}{4}a^4\pi k \end{aligned}$$

$$\begin{aligned} M_{xy} &= \iiint_H z\,\rho(x,y,z)\,dV \\ &= k\int_0^{2\pi}\int_0^{\pi/2}\int_0^a (\rho\cos\phi).(\rho\cos\phi).\rho^2\sin\phi\, d\rho\ d\phi\ d\theta \\ &= k\int_0^a \rho^4 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos^2\phi\sin\phi\, d\phi \\ &= k\left(\frac{\rho^5}{5}\right)_0^a (\theta)_0^{2\pi} \left(-\frac{1}{3}\cos^3\phi\right)_0^{\pi/2} \\ &= \frac{2}{15}\pi k a^5 \end{aligned}$$

$$\begin{aligned} M_{yz} &= \iiint_H x\,\rho(x,y,z)\,dV \\ &= k\int_0^{2\pi}\int_0^{\pi/2}\int_0^a (\rho\sin\phi\cos\theta)(\rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \\ &= k\int_0^a \rho^4\ d\rho \int_0^{2\pi} \cos\theta\, d\theta \int_0^{\pi/2} \sin^2\phi\cos\phi\, d\phi \\ &= k\left(\frac{\rho^5}{5}\right)_0^a (-\sin\theta)_0^{2\pi} \left(\frac{1}{3}\sin^3\phi\right)_0^{\pi/2} \\ &= k\frac{a^5}{5}(-\sin 2\pi + \sin 0)\left(\frac{1}{3}\right) \\ &= 0 \end{aligned}$$

$$M_{xz} = \iiint_H y\,\rho(x, y, z)\,dV$$

$$= k\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} (\rho \sin\phi \sin\theta).(\rho\cos\phi).\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= k\int_0^{a} \rho^4 d\rho \int_0^{2\pi} \sin\theta\, d\theta \int_0^{\pi/2} \sin^2\phi \cos\phi\, d\phi$$

$$= k\left(\frac{\rho^5}{5}\right)_0^{a} (-\cos\theta)_0^{2\pi} \left(\frac{1}{3}\sin^3\phi\right)_0^{\pi/2}$$

$$= k\left(\frac{a^5}{5}\right)(-\cos 2\pi + \cos 0)\left(\frac{1}{3}\right)$$

$$= 0$$

Then $\bar{x} = \frac{M_{yz}}{m} = 0$

$$\bar{y} = \frac{M_{xz}}{m} = 0$$

$$\bar{z} = \frac{M_{xy}}{m}$$

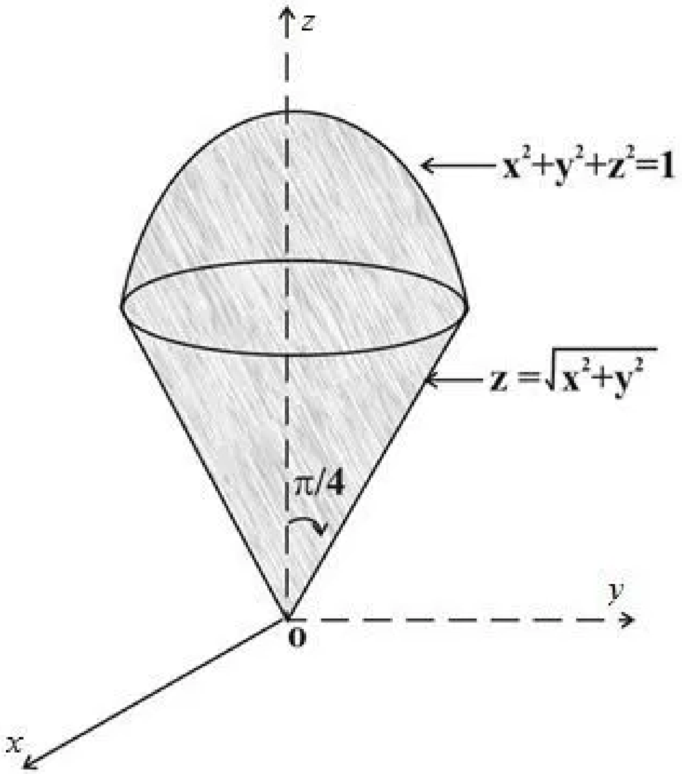
$$= \frac{2}{15}\pi a^5 k \, x \, \frac{4}{a^4 \pi k}$$

$$= \frac{8}{15}a$$

Hence the center of mass is $\boxed{\left(0, 0, \frac{8}{15}a\right)}$

Chapter 15 Multiple Integrals 15.9 35E

(a) The volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$



Notice that the sphere passes through the origin and has center $(0,0,0)$. We write the equation of the sphere in spherical coordinates as

$$\rho^2=\rho\cos\phi \quad \text{or} \quad \rho=\cos\phi$$

The equation of the cone can be written as

$$\rho\cos\phi=\rho\sin\phi$$

That is, $\cos\phi=\sin\phi$

This implies, $\phi=\dfrac{\pi}{4}$

Then the solid E is given by

$$E = \left\{ (\rho, \theta, \phi) : 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{4},\ 0 \le \rho \le 1 \right\}$$

Then volume of the solid is

$$V(E) = \iiint_E dv$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \, d\phi \int_0^1 \rho^2 \, d\rho$$

$$= (\theta)_0^{2\pi} (-\cos\phi)_0^{\pi/4} \left(\frac{\rho^3}{3} \right)_0^1$$

$$= (2\pi)\left(1 - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{3}\right)$$

$$= \frac{2\pi}{3}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{2}\pi}{3}\left(\sqrt{2}-1\right)$$

$$= \frac{\pi}{3}\left(2-\sqrt{2}\right)$$

Hence the volume is $\boxed{V = \frac{\pi}{3}\left(2-\sqrt{2}\right)}$

(b) Now we need to find the centroid. So we have density $\rho(x,y,z)=k$ (constant)

The mass is given by

$$m=\iiint_E \rho(x,y,z)\,dV$$

$$=k\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=k\frac{\pi}{3}\left(2-\sqrt{2}\right)$$

Now $M_{yz}=\iiint_E x\,\rho(x,y,z)\,dV$

$$=k\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho\sin\phi\cos\theta\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=k\int_0^1\rho^3\,d\rho\int_0^{2\pi}\cos\theta\,d\theta\int_0^{\pi/4}\sin^2\phi\,d\phi$$

$$=k\left(\frac{\rho^4}{4}\right)_0^1\left(\sin\theta\right)_0^{2\pi}\left(\frac{\phi}{2}-\frac{1}{4}\sin 2\theta\right)_0^{\pi/4}$$

$$=k\left(\frac{1}{4}\right)\left(\sin 2\pi-\sin 0\right)\left(\frac{\pi}{8}-\frac{1}{4}\sin\frac{\pi}{2}\right)$$

$$=0$$

$$M_{xz}=\iiint_E y\,\rho(x,y,z)\,dV$$

$$=k\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho\sin\phi\sin\theta.\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=k\int_0^1\rho^3\,d\rho\int_0^{2\pi}\sin\theta\,d\theta\int_0^{\pi/4}\sin^2\phi\,d\phi$$

$$= k\left(\frac{\rho^4}{4}\right)_0^1 (-\cos\theta)_0^{2\pi}\left(\frac{\phi}{2}-\frac{1}{4}\sin 2\phi\right)_0^{\pi/4}$$

$$= k\left(\frac{1}{4}\right)(-\cos 2\pi+\cos\theta)\left(\frac{\pi}{8}-\frac{1}{4}\sin\frac{\pi}{2}\right)$$

$$= 0$$

$$M_{xy} = \iiint_E z\,\rho(x,y,z)\,dv$$

$$= k\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho\cos\phi.\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= k\int_0^1 \rho^3\,d\rho\int_0^{2\pi} d\theta\int_0^{\pi/4}\sin\phi\cos\phi\,d\phi$$

$$= k\left(\frac{\rho^4}{4}\right)_0^1 (\theta)_0^{2\pi}\left(-\frac{1}{4}\cos 2\phi\right)_0^{\pi/4}$$

$$= k\left(\frac{1}{4}\right)(2\pi)\left(\frac{1}{4}\right)$$

$$= k\frac{\pi}{8}$$

Then $\overline{x} = \frac{M_{yz}}{m} = 0$

$$\overline{y} = \frac{M_{xz}}{m} = 0$$

$$\overline{z} = \frac{M_{xy}}{m}$$

$$= \frac{k\pi}{8}\times\frac{3}{k\pi\left(2-\sqrt{2}\right)}$$

$$= \frac{3}{8\left(2-\sqrt{2}\right)}$$

Hence the centroid is

$$\boxed{\left(0,0,\frac{3}{8\left(2-\sqrt{2}\right)}\right)}$$

Chapter 15 Multiple Integrals 15.9 36E

Let the center of the sphere is at (0, 0, 0) and let the diameter of intersection is along the z – axis. Then one of the intersecting planes will be xz – plane and the other be the plane whose angle with xz – plane is $\theta = \frac{\pi}{6}$

Then the required region can be given in spherical co – ordinates as

$$E = \left\{(\rho, \theta, \phi):\ 0 \le \rho \le a,\ 0 \le \theta \le \frac{\pi}{6},\ 0 \le \phi \le \pi\right\}$$

Then the required region can be given in spherical co – ordinates as

$$E = \left\{(\rho, \theta, \phi):\ 0 \le \rho \le a,\ 0 \le \theta \le \frac{\pi}{6},\ 0 \le \phi \le \pi\right\}$$

Then the volume is given by

$$V(E) = \iiint_E dv$$

$$= \int_0^{\pi/6}\int_0^{\pi}\int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\pi/6} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^{a} \rho^2 \, d\rho$$

$$= (\theta)_0^{\pi/6} (-\cos\phi)_0^{\pi} \left(\frac{\rho^3}{3}\right)_0^{a}$$

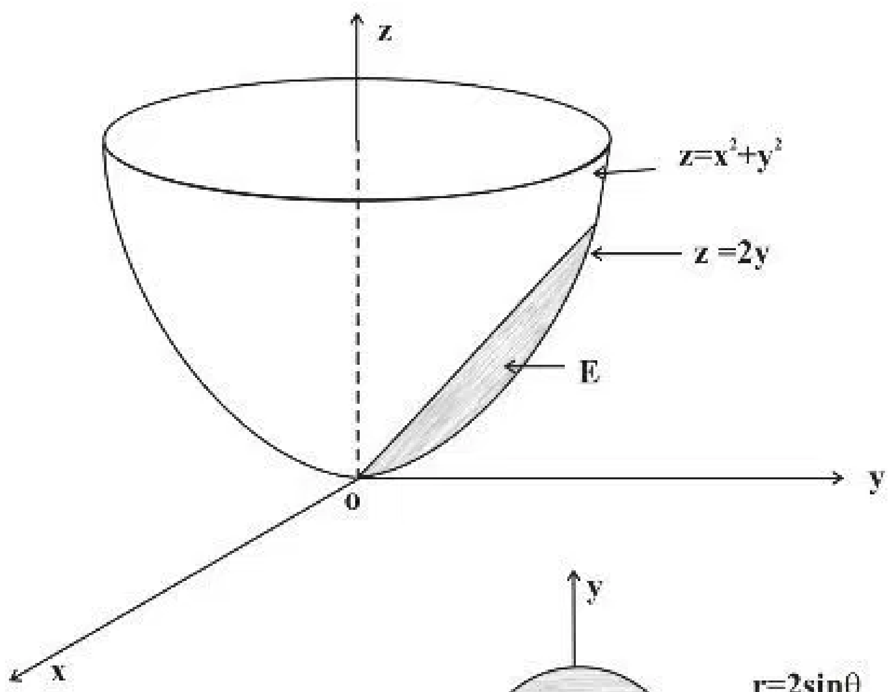
$$= \frac{\pi}{6}(2)\left(\frac{a^3}{3}\right)$$

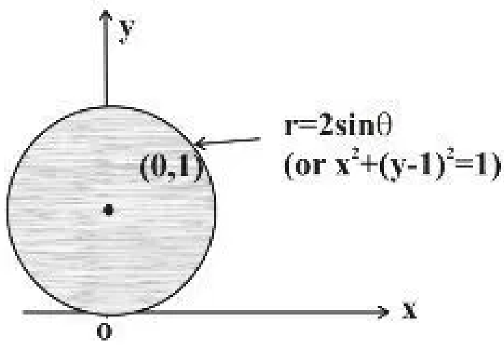
$$= \frac{1}{9}\pi a^3$$

Hence $\boxed{V(E) = \frac{1}{9}\pi a^3}$

Chapter 15 Multiple Integrals 15.9 37E

In cylindrical co – ordinates the parabolic is given by $z = r^2$ and the plane is $z = 2r\sin\theta$. These two meet in a circle $r = 2\sin\theta$





Therefore in cylindrical co − ordinates the region E can be given as

$$E = \{(r,\theta,z):\ 0 \le \theta \le \pi, 0 \le r \le 2\sin\theta,\ r^2 \le z \le 2r\sin\theta\}$$

Then $$\iiint_E z\,dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} zr\,dz\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\int_0^{2\sin\theta} r\left(z^2\right)_{r^2}^{2r\sin\theta} dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\int_0^{2\sin\theta} r\left(4r^2\sin\theta - r^4\right)dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\int_0^{2\sin\theta} \left(4r^3\sin\theta - r^5\right)dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\left[r^4\sin^2\theta - \frac{r^6}{6}\right]_0^{2\sin\theta} d\theta$$

i.e. $$\iiint_E z\,dV = \frac{1}{2}\int_0^{\pi}\left[16\sin^6\theta - \frac{64\sin^6\theta}{6}\right]d\theta$$

$$= \frac{1}{2}\times\frac{16}{3}\int_0^{\pi}\sin^6\theta\,d\theta$$

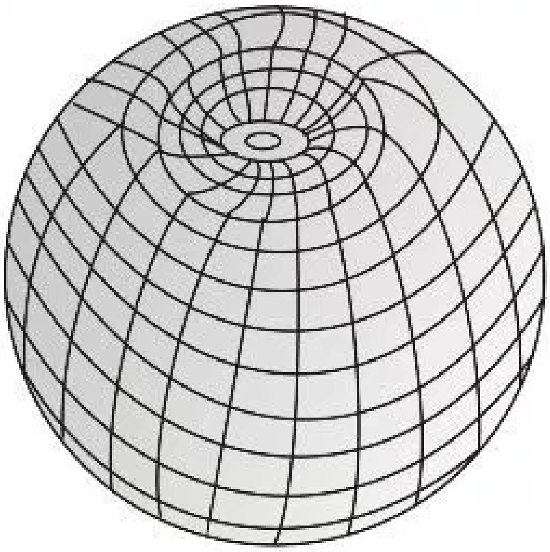
$$= \frac{8}{3}\left[\frac{-1}{6}\sin^5\theta\cos\theta - \frac{5}{24}\sin^3\theta\cos\theta + \frac{5\theta}{16} - \frac{5}{32}\sin 2\theta\right]_0^{\pi}$$

$$= \frac{8}{3}\times\frac{5\pi}{16}$$

$$= \frac{5\pi}{6}$$

Hence $$\boxed{\iiint_E z\,dv = \frac{5\pi}{6}}$$

Chapter 15 Multiple Integrals 15.9 38E



The torus is a single holed ring. The surface enclosed by the tours $\rho = \sin\phi$ is

$$E = \{(\rho, \theta, \phi) : 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi,\ 0 \le \rho \le \sin\phi\}$$

Therefore the volume is given by

$$V(E) = \iiint_E dv$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta . \int_0^{\pi} \left(\frac{\rho^3}{3}\right)_0^{\sin\phi} \sin\phi \, d\phi$$

$$= (\theta)_0^{2\pi} \frac{1}{3}\int_0^{\pi} \sin^4\phi \, d\phi$$

$$= 2\pi\left(\frac{1}{3}\right)\left[\frac{3}{8}\phi - \frac{1}{4}\sin 2\phi + \frac{1}{16}\sin 4\phi\right]_0^{\pi}$$

$$= \frac{2}{3}\pi \times \frac{3}{8}\pi$$

$$= \boxed{\frac{1}{4}\pi^2}$$

Chapter 15 Multiple Integrals 15.9 39E

Consider the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy \, dz \, dy \, dx.$

Need to evaluate the given integral by changing to spherical coordinates.

From the given integral notice that the boundary of the region is

$$E = \left\{(x, y, z) \middle| \sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2}, 0 \le y \le \sqrt{1-x^2}, 0 \le x \le 1\right\}.$$

Since $z = \sqrt{x^2+y^2}$ then $z^2 = x^2 + y^2$ this equation represents an equation of a cone.

So $z = \sqrt{x^2+y^2}$ represents an equation of the cone which is lies above the *xy*-plane.

And $z = \sqrt{2-x^2-y^2}$ then $x^2+y^2+z^2 = 2$ this equation represents an equation of a sphere.

So $z = \sqrt{2-x^2-y^2}$ represents the equation of a sphere which is lies above the *xy*-plane.

We have z varies from $\sqrt{x^2+y^2}$ to $\sqrt{2-x^2-y^2}$ that is the required region lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $z = \sqrt{2-x^2-y^2}$.

$$\sqrt{x^2+y^2} = \sqrt{2-x^2-y^2}$$

$$x^2+y^2 = 2-x^2-y^2$$

$$2(x^2+y^2) = 2$$

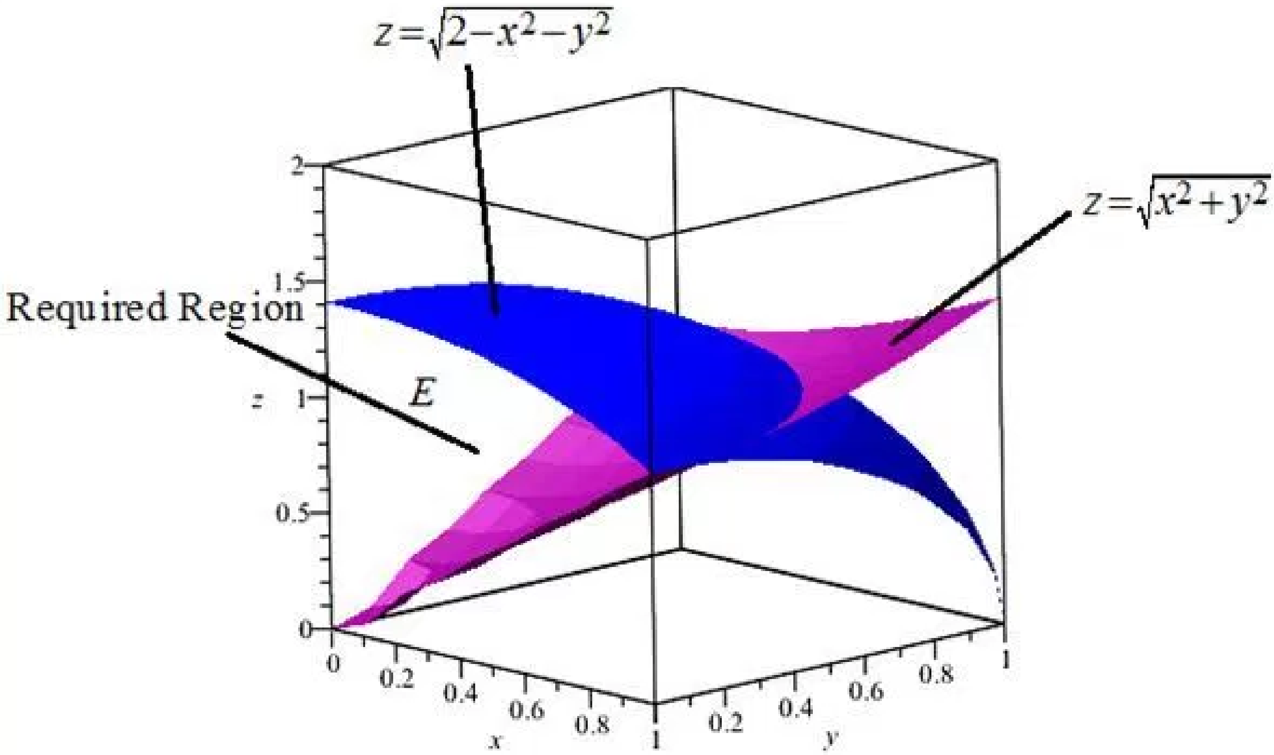
$$x^2+y^2 = 1$$

And the projection on the *xy*-plane is a circle of radius 1.

Since we have *y* varies from 0 to $\sqrt{1-x^2}$, and *x* varies from 0 to 1.

That is the required region is lies in the first octant.

The graph of the region E is shown below:



Since we know that

$$\rho^2 = x^2 + y^2 + z^2$$
$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

Use these formulas convert the given surfaces from rectangular to spherical coordinates.

Rewrite $z = \sqrt{x^2 + y^2}$ as

$$\begin{aligned}\rho\cos\phi &= \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2} \\ &= \rho\sqrt{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta} \\ &= \rho\sqrt{\sin^2\phi(\cos^2\theta + \sin^2\theta)} \\ &= \rho\sin\phi\end{aligned}$$

Then

$$\rho\cos\phi = \rho\sin\phi$$
$$\tan\phi = 1$$
$$\phi = \frac{\pi}{4}$$

Rewrite $\sqrt{2-x^2-y^2}=z$ as

$$\sqrt{2-(\rho\sin\phi\cos\theta)^2-(\rho\sin\phi\sin\theta)^2}=\rho\cos\phi$$
$$\sqrt{2-\rho^2\sin^2\phi\cos^2\theta-\rho^2\sin^2\phi\sin^2\theta}=\rho\cos\phi$$
$$\sqrt{2-\rho^2\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)}=\rho\cos\phi$$

$$\sqrt{2-\rho^2\sin^2\phi(1)}=\rho\cos\phi$$
$$\sqrt{2-\rho^2\sin^2\phi}=\rho\cos\phi$$
$$2-\rho^2\sin^2\phi=\rho^2\cos^2\phi$$

Continuation to the above

$$2=\rho^2\cos^2\phi+\rho^2\sin^2\phi$$
$$2=\rho^2\left(\cos^2\phi+\sin^2\phi\right)$$
$$2=\rho^2(1)$$
$$\rho=\sqrt{2}$$

So in the first octant ρ varies from 0 to $\sqrt{2}$, θ varies from 0 to $\frac{\pi}{2}$ and ϕ varies from 0 to $\frac{\pi}{4}$.

Therefore, the description of the solid E in spherical coordinates is given by:

$$E=\left\{(\rho,\theta,\phi)\middle|0\le\rho\le\sqrt{2},0\le\phi\le\frac{\pi}{4},0\le\theta\le\frac{\pi}{2}\right\}.$$

The given integral becomes

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}}(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)\left(\rho^2\sin\phi\right)\,d\rho\,d\phi\,d\theta$$
$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}}\left(\rho^4\sin^3\phi\cos\theta\sin\theta\right)\,d\rho\,d\phi\,d\theta$$
$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}}\left(\rho^4\sin^3\phi(2\cos\theta\sin\theta)\right)\,d\rho\,d\phi\,d\theta$$

$$=\frac{1}{2}\left[\frac{\rho^5}{5}\right]_0^{\sqrt{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\sin^2\phi\sin\phi\sin 2\theta\,d\phi\,d\theta$$

$$=\frac{1}{2}\left[\frac{\left(\sqrt{2}\right)^5-0}{5}\right]\int_0^{\frac{\pi}{2}}\sin 2\theta d\theta\int_0^{\frac{\pi}{4}}\left(1-\cos^2\phi\right)\sin\phi\,d\phi$$

$$=\frac{1}{2}\left[\frac{4\sqrt{2}}{5}\right]\left[\frac{-\cos 2(\theta)}{2}\right]_0^{\frac{\pi}{2}}\left[\frac{1}{3}\cos^3(x)-\cos x\right]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{2}\left[\frac{4\sqrt{2}}{5}\right]\left[\frac{-\cos\pi+\cos 0}{2}\right]\left[\frac{1}{3}\cos^3\left(\frac{\pi}{4}\right)-\cos\frac{\pi}{4}-\frac{1}{3}\cos^3(0)+\cos 0\right]$$

$$=\frac{1}{2}\left[\frac{4\sqrt{2}}{5}\right]\left[\frac{-(-1)+1}{2}\right]\left[\frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3-\frac{1}{\sqrt{2}}-\frac{1}{3}(1)+1\right]$$

$$=\frac{1}{2}\left[\frac{4\sqrt{2}}{5}\right][1]\left[\frac{1}{3}\left(\frac{1}{2\sqrt{2}}\right)-\frac{1}{\sqrt{2}}-\frac{1}{3}+1\right]$$

$$=\left[\frac{2\sqrt{2}}{5}\right]\left[\left(\frac{1}{6\sqrt{2}}\right)-\frac{1}{\sqrt{2}}+\frac{2}{3}\right]$$

$$=\frac{1}{15}-\frac{2}{5}+\frac{4\sqrt{2}}{15}$$

$$=\frac{4\sqrt{2}-5}{15}$$

Hence the required value of the given integral is $\boxed{\frac{4\sqrt{2}-5}{15}}$.

Chapter 15 Multiple Integrals 15.9 40E

Consider the integral, $\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}\left(x^2z+y^2z+z^3\right)\,dz\,dx\,dy.$

Evaluate the given integral by changing to spherical coordinates.

From the given integral, notice that the boundary of the region is as follows:

$$E=\left\{(x,y,z)\middle|-\sqrt{a^2-x^2-y^2}\le z\le\sqrt{a^2-x^2-y^2},-\sqrt{a^2-y^2}\le x\le\sqrt{a^2-y^2},-a\le y\le a\right\}.$$

Convert the rectangular coordinates into spherical coordinates is as shown below:

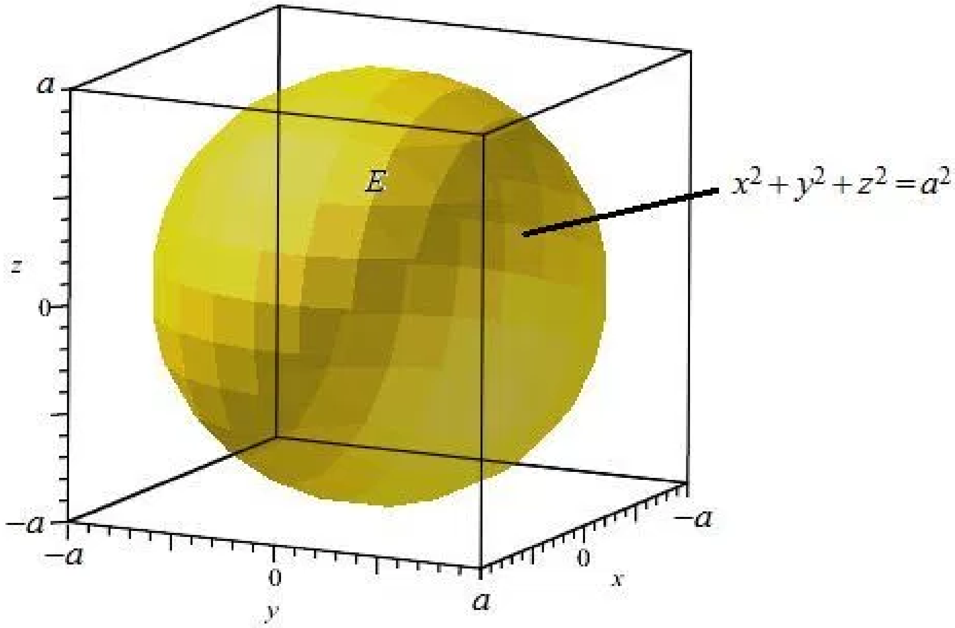
$$x=\rho\sin\phi\cos\theta$$
$$y=\rho\sin\phi\sin\theta$$
$$z=\rho\cos\phi$$
$$\rho^2=x^2+y^2+z^2 \qquad ......(1)$$

$$dxdydz=\rho^2\sin\phi d\rho d\phi d\theta$$

Notice that the region E is a sphere, the description of the solid E, in spherical coordinates, is as follows:

$$E=\left\{(\rho,\theta,\phi)\middle|0\le\rho\le a,0\le\phi\le\pi,0\le\theta\le 2\pi\right\}.$$

The graph of the region E is as shown below:



The given integral becomes the following:

$$\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}\left(x^2z+y^2z+z^3\right)\,dz\,dx\,dy$$

$$=\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}\left(x^2+y^2+z^2\right)z\,dz\,dx\,dy$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\left[\left(\rho^2\right)\rho\cos\phi\right]\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \qquad \text{from}(1)\ x^2+y^2+z^2=\rho^2$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\left(\rho^5\sin\phi\cos\phi\right)d\rho\,d\phi\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\left(\rho^5\cdot 2\cdot\sin\phi\cos\phi\right)d\rho\,d\phi\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\left(\rho^5\sin 2\phi\right)d\rho\,d\phi\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}d\theta\times\int_0^{\pi}\sin 2\phi\,d\phi\times\int_0^{a}\rho^5\,d\rho$$

$$=\frac{1}{2}\left[\theta\right]_0^{2\pi}\left[\frac{-\cos 2\phi}{2}\right]_0^{\pi}\left[\frac{\rho^6}{6}\right]_0^{a}$$

$$=\frac{1}{2}\left[\frac{a^6}{6}\right]\left[\frac{-\cos 2\pi+\cos 0}{2}\right]\left[2\pi\right]$$

$$=\frac{1}{2}\left[\frac{a^6}{6}\right]\left[\frac{-1+1}{2}\right]\left[2\pi\right]$$

$$=\frac{1}{2}\left[\frac{a^6}{6}\right]\left[\frac{0}{2}\right]\left[2\pi\right]$$

$$=0$$

Hence, the required value of the given integral is $\boxed{0}$.

Chapter 15 Multiple Integrals 15.9 42E

The mass is the integral of the density function over the volume.

We want the volume of the earth's atmosphere between the grounds, which is give as 6370 km from the center of the earth, to an altitude of 5 km, which is $6370+5=6375$ km from the center of the earth. In meters, this is a radius of 6,370,000 m to 6,375,000 m. We want the volume between these two concentric spheres. Therefore, the limit of integration of ρ will be 6,370,000 to 6,375,000. The problem specifies that the given density function is reasonable between these radii, so we will integrate δ between these two spheres.

For our other limits of integration, we want θ to go all the way around the z-axis, from 0 to 2π, and ϕ to go halfway around, from 0 to π—if ϕ goes halfway around, from the positive z-axis to the negative z-axis, the fact that θ goes all the way around will encompass the entire volume of the sphere without repetition.

In order for our integral to integrate over the volume, usually we would have an integrand of 1. In spherical coordinates, we must multiply in a factor of $\rho^2 \sin\phi$. Since we want the mass, we also multiply the density function into the integrand, giving the integral of density throughout the volume:

$$\int_0^{\pi}\int_0^{2\pi}\int_{6,370,000}^{6,375,000} \delta\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_{6,370,000}^{6,375,000} (619.09-0.000097\rho)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_{6,370,000}^{6,375,000} (619.09\rho^2 \sin\phi - 0.000097\rho^3 \sin\phi) \, d\rho \, d\theta \, d\phi$$

Integrate in terms of ρ:

$$\int_0^{\pi}\int_0^{2\pi}\int_{6,370,000}^{6,375,000} (619.09\rho^2 \sin\phi - 0.000097\rho^3 \sin\phi) \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\left(\frac{619.09\rho^3 \sin\phi}{3} - \frac{0.000097\rho^4 \sin\phi}{4}\right)\Bigg|_{6,370,000}^{6,375,000} d\theta \, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\left(\begin{array}{c}\dfrac{619.09(6,375,000)^3 \sin\phi}{3} - \dfrac{0.000097(6,375,000)^4 \sin\phi}{4} \\ -\left(\dfrac{619.09(6,370,000)^3 \sin\phi}{3} - \dfrac{0.000097(6,370,000)^4 \sin\phi}{4}\right)\end{array}\right) d\theta \, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\left(1.94402\times10^{17} \sin\phi\right) d\theta \, d\phi$$

Integrate in terms of θ:

$$\int_0^{\pi}\int_0^{2\pi}\left(1.94402\times10^{17} \sin\phi\right) d\theta \, d\phi$$

$$= \int_0^{\pi}\left(1.94402\times10^{17}\theta \sin\phi\right)\Big|_0^{2\pi} d\phi$$

$$= \int_0^{\pi}\left(1.94402\times10^{17}(2\pi)\sin\phi - 0\right) d\phi$$

$$= 2\pi\int_0^{\pi}\left(1.94402\times10^{17} \sin\phi\right) d\phi$$

Integrate in terms of ϕ:

$$2\pi\int_0^{\pi}\left(1.94402\times10^{17}\sin\phi\right)d\phi$$

$$=2\pi\left(-1.94402\times10^{17}\cos\phi\right)\Big|_0^{\pi}$$

$$=2\pi\left(-1.94402\times10^{17}\cos\pi-(-1.94402\times10^{17}\cos(0))\right)$$

$$=2\pi\left(-1.94402\times10^{17}(-1)+1.94402\times10^{17}(1)\right)$$

$$=4\pi(1.94402\times10^{17})$$

$$=2.44292\times10^{18}$$

The mass of this layer of atmosphere is approximately $\boxed{2.44292\times10^{18}\text{ kilograms}}$.

Chapter 15 Multiple Integrals 15.9 43E

Need to graph a silo consisting of a cylinder with radius 3 and height 10 surmounted by a hemisphere using graphing device.

The radius of the cylinder is 3 then the equation of the cylinder is $x^2+y^2=9.$

And the height of the cylinder is 10.

Also the radius of the hemisphere is 3.

Here we are using Maple.

Keystrokes:

with(plots);

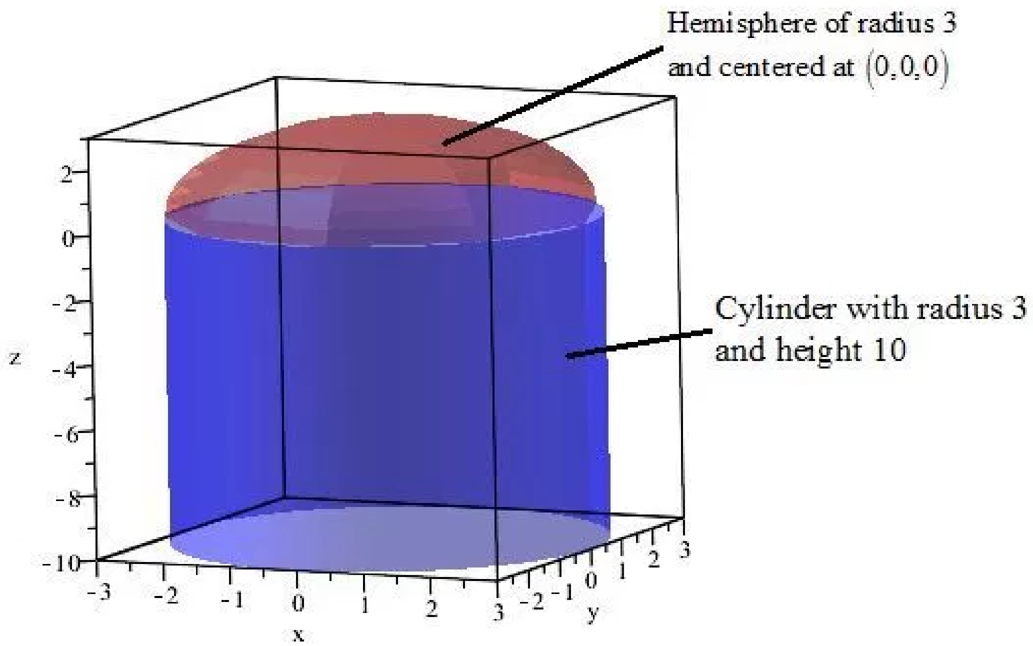
p1 := plot3d(3, theta = 0 .. 2*Pi, z = -10 .. 0, coords = cylindrical, axes = boxed, labels = ["x", "y", "z "], style = surface, color = blue);

p2 := implicitplot3d(rho = 3, rho = 0 .. 3.001, theta = 0 .. 2*Pi, phi = 0 .. (1/2)*Pi, coords = spherical, style = surface, color = orange);

display({p1, p2}, axes = boxed);

Maple result:

```
with(plots) :
p1 := plot3d(3, theta = 0 ..2·Pi, z = -10 ..0, coords = cylindrical, axes = boxed, labels = ["x",
    "y", "z "], style = surface, color = blue);
p2 := implicitplot3d(rho = 3, rho = 0 ..3.001, theta = 0 ..2·Pi, phi = 0 ..Pi/2, coords = spherical,
    style = surface, color = orange)
display({p1, p2}, axes = boxed);
```



Chapter 15 Multiple Integrals 15.9 44E

The radius of the earth is $\rho = 3960$ mi.

The latitude and longitude of a point P in the Northern Hemisphere are as follows:

$$\alpha = 90° - \phi°$$
$$\beta = 360° - \theta°$$

The latitude and longitude of a Los Angeles (L) are as follows:

$$\alpha = 90° - \phi° = 34.06°\text{N}$$
$$\phi° = 90° - 34.06°\text{N} = 55.94°$$
$$\beta = 360° - \theta° = 118.25°\text{W}$$
$$\theta° = 360° - 118.25°\text{W} = 241.75°$$

The spherical coordinates for Los Angeles is $(\rho, \theta, \phi) = (3960, 241.75°, 55.94°)$.

The rectangular coordinates for Los Angeles is

$$(x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$$
$$= (3960(\sin 55.94°)(\cos 241.75°), 3960(\sin 55.94°)(\sin 241.75°), 3960(\cos 55.94°))$$
$$\approx (-1552.804718706, -2889.91011737502, 2217.84062075)$$
$$\approx (-1552.80, -2889.91, 2217.84)$$

The latitude and longitude of a Montreal (M) are as follows:

$$\alpha = 90° - \phi° = 45.50°\text{N}$$
$$\phi° = 90° - 45.50°\text{N}$$
$$= 44.50°$$
$$\beta = 360° - \theta° = 73.60°\text{W}$$
$$\theta° = 360° - 73.60°\text{W}$$
$$= 286.40°$$

The spherical coordinates for Montreal is $(\rho, \theta, \phi) = (3960, 286.40°, 44.50°)$.

The rectangular coordinates for Los Angeles is

$$(x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$$
$$\approx (3960(\sin 44.50°)(\cos 286.40°), 3960(\sin 44.50°)(\sin 286.40°), 3960(\cos 44.50°))$$
$$\approx (783.6671414764, -2662.6725264246, 2824.47177865055903)$$
$$\approx (783.67, -2662.67, 2824.47)$$

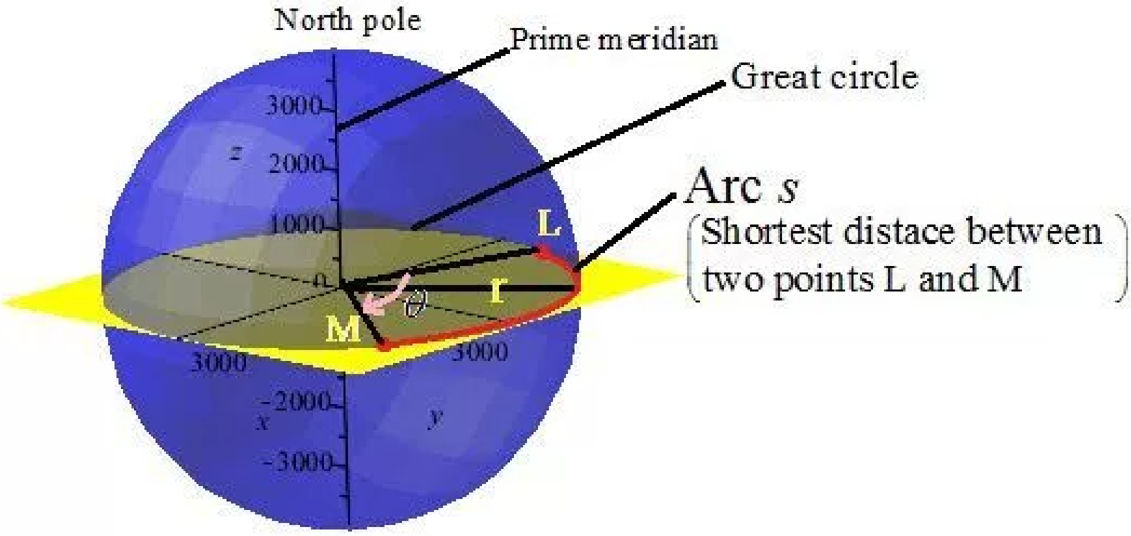
Let $\mathbf{u}$ be the vector from the origin to the Los Angeles (L), that is

$$\mathbf{u} = (-1552.80, -2889.91, 2217.84).$$

Let $\mathbf{v}$ be the vector from the origin to the Montreal (M), that is

$$\mathbf{v} = (783.67, -2662.67, 2824.47).$$

Let θ be an angle between these two vectors which lie on a great circle shown in the below graph.



By using dot product, we have

$$\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}||\mathbf{v}|\cos\theta$$

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\theta=\cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$$

$$\approx\cos^{-1}\left(\frac{(-1552.80,-2889.91,2217.84)\cdot(783.67,-2662.67,2824.47)}{\sqrt{(-1552.80)^2+(-2889.91)^2+(2217.84)^2}\sqrt{(783.67)^2+(-2662.67)^2+(2824.47)^2}}\right)$$

$$\approx\cos^{-1}\left(\frac{(-1552.80)(783.67)+(-2889.91)(-2662.67)+(2217.84)(2824.47)}{\sqrt{2411187.84+8351579.80+4918814.26}+\sqrt{614138.66+7089811.52+7977630.78}}\right)$$

$$\approx\cos^{-1}\left(\frac{-1216882.776+7694876.6597+6264222.5448}{\sqrt{15681581.9}\sqrt{15681580.96}}\right)$$

Continuation to the above

$$\approx\cos^{-1}\left(\frac{12742229.76}{(3959.9977)(3959.9975)}\right)$$

$$\approx\cos^{-1}\left(\frac{12742229.76}{15681580.992005}\right)$$

$$\approx\cos^{-1}(0.8125602749)$$

$$\approx 0.622265086$$

The greatest circle distance or (shortest distance) between Los Angeles and Montreal is

$$s=\text{Arc length of LM}$$

$$=r\cdot\theta \qquad (\text{Where } r \text{ is radius of the great circle})$$

$$=(3960)\cdot(0.622265086)$$

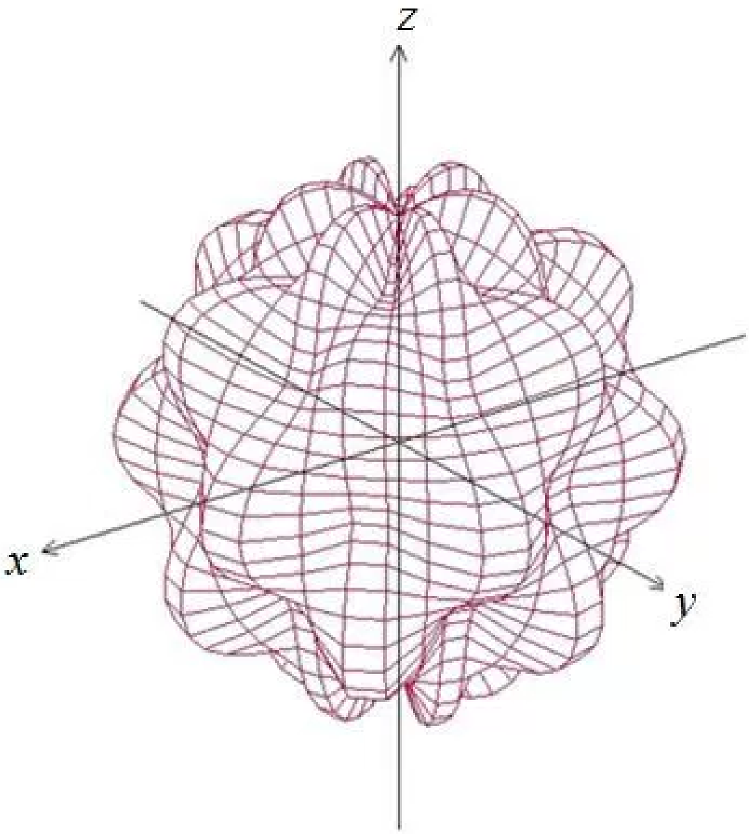
$$=2464.16974$$

$$\approx 2464.17$$

Therefore the required great-circle distance from Los Angeles to Montreal is $\boxed{2464.17\text{ mi}}$.

Chapter 15 Multiple Integrals 15.9 45E

To find the volume enclosed by the surface $\rho = 1 + \frac{1}{5}\sin(6\theta)\sin(5\phi)$, use the provided sketch and a Computer Algebra System (CAS).



First, set up a triple integral $\iiint_E \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$ in spherical coordinates that describes the volume of solid *E*.

Find the bounds of the integral.

Given that the equation of the surface as, $\rho = 1 + \frac{1}{5}\sin(6\theta)\sin(5\phi)$

So, the range of ρ is $0 \le \rho \le 1 + \frac{1}{5}\sin(6\theta)\sin(5\phi)$.

Since the solid goes all the way around the *z*-axis, the range is $0 \le \theta \le 2\pi$.

Since the solid is both above and below the *xy*-plane, obtain the range as $0 \le \phi \le \pi$.

Therefore, solve as follows:

$$\iiint_E \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1+\frac{1}{5}\sin(6\theta)\sin(5\phi)} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \boxed{\frac{136\pi}{99}}$$ By CAS

Chapter 15 Multiple Integrals 15.9 46E

Since the improper triple integral is defined as the limit of a triple integral over a solid sphere as the radius of the sphere increased indefinitely, then using spherical co – ordinates, the given integral can be written as:

Chapter 15 Multiple Integrals 15.9 47E

(a)

To show that the volume of a solid bound below by the cone $z = r\cot\phi_0$ and above by the sphere $r^2 + z^2 = a^2$ is $V = \frac{2\pi a^3}{3}(1-\cos\phi_0)$, begin by setting up the triple integral $\iiint_E r\,dz\,dr\,d\theta$ in cylindrical coordinates that describes the solid *E*.

To evaluate the integral and find the volume, first determine the bounds of the integral.

It is given that *z* ranges from $r\cot\phi_0 \le z \le \sqrt{a^2 - r^2}$.

Since the solid goes all the way around the *z*-axis, we have $0 \le \theta \le 2\pi$.

Find the remaining bounds for *r* by solving for *r* when the two surfaces intersect.

Set the *z* values of the surfaces $z = r\cot\phi_0$ and $z = \sqrt{a^2 - r^2}$ equal to each other

$$r\cot\phi_0 = \sqrt{a^2 - r^2}$$

$$r^2\cot^2\phi_0 = a^2 - r^2$$

$$r^2\left(1+\cot^2\phi_0\right) = a^2$$

$$r^2\left(\csc^2\phi_0\right) = a^2$$

$$r^2 = \frac{a^2}{\csc^2\phi_0}$$

$$r = \frac{a}{\csc\phi_0}$$

$$r = a\sin\phi_0$$

So the range for *r* is $0 \le r \le a\sin\phi_0$

Evaluate the integral $\iiint_E r\,dz\,dr\,d\theta$ using the bounds $r\cot\phi_0 \le z \le \sqrt{a^2-r^2}$, $0 \le \theta \le 2\pi$, and $0 \le r \le a\sin\phi_0$ to find the volume of the solid

$$\iiint_E r\,dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a\sin\phi_0}\int_{z=r\cot\phi_0}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a\sin\phi_0} r\left[z\right]_{z=r\cot\phi_0}^{z=\sqrt{a^2-r^2}} dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a\sin\phi_0} r\left[\sqrt{a^2-r^2} - r\cot\phi_0\right] dr\,d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{a\sin\phi_0} \left[r\sqrt{a^2-r^2} - r^2\cot\phi_0\right] dr\,d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{a\sin\phi_0} \left[\left(-\frac{1}{2}\right)\left(-2r\sqrt{a^2-r^2}\right) - r^2\cot\phi_0\right] dr\,d\theta$$

$$= \int_{0}^{2\pi}\left[-\frac{1}{2}\left(\frac{2}{3}\right)\left(a^2-r^2\right)^{3/2} - \frac{1}{3}r^3\cot\phi_0\right]_{r=0}^{r=a\sin\phi_0} d\theta$$

$$\left(\text{Use} \int f'(x)\sqrt{f(x)}\,dx = \frac{2}{3}\left(f(x)\right)^{\frac{3}{2}}\right)$$

By simplifying the above integral, we get

$$\iiint_E r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[-\frac{1}{3}\left(a^2-r^2\right)^{3/2}-\frac{1}{3}r^3\cot\phi_0\right]_{r=0}^{r=a\sin\phi_0} d\theta$$

$$= \int_0^{2\pi}\left[-\frac{1}{3}\left(a^2-\left(a\sin\phi_0\right)^2\right)^{3/2}-\frac{1}{3}\left(a\sin\phi_0\right)^3\cot\phi_0-\left(-\frac{1}{3}\left(a^2-0^2\right)^{3/2}-0\right)\right]d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\left[-\left(a^2-a^2\sin^2\phi_0\right)\left(\sqrt{a^2-a^2\sin^2\phi_0}\right)-a^3\sin^3\phi_0\cot\phi_0+a^3\right]d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\left[-a^2\left(1-\sin^2\phi_0\right)\left(a\sqrt{1-\sin^2\phi_0}\right)-a^3\sin^3\phi_0\frac{\cos\phi_0}{\sin\phi_0}+a^3\right]d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\left[-a^2\left(\cos^2\phi_0\right)\left(a\sqrt{\cos^2\phi_0}\right)-a^3\sin^2\phi_0\cos\phi_0+a^3\right]d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi}\left[-\left(\cos^2\phi_0\right)\left(\cos\phi_0\right)-\left(1-\cos^2\phi_0\right)\cos\phi_0+1\right]d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi}\left[-\cos^3\phi_0-\cos\phi_0+\cos^3\phi_0+1\right]d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi}\left[1-\cos\phi_0\right]d\theta$$

$$= \frac{a^3\left(1-\cos\phi_0\right)}{3}\left[\theta\right]_{\theta=0}^{\theta=2\pi}$$

$$= \frac{2\pi a^3}{3}\left(1-\cos\phi_0\right)$$

So the volume of the required solid is $V=\frac{2\pi a^3}{3}\left(1-\cos\phi_0\right)$

(b)

Consider a spherical wedge instead of the full volume $V = \frac{2\pi a^3}{3}(1-\cos\phi_0)$ found in part (a).

Breaking apart the result, the 2π part was found by integrating with respect to θ and evaluating at $\theta_2 - \theta_1 = 2\pi - 0$.

So to find the elemental volume of the wedge, replace 2π with $\theta_2 - \theta_1$.

Now consider the a^3. This represents the radial thickness of the wedge, and when evaluating at specific radii, $a^3 - 0^3$ turns into $\rho_2^3 - \rho_1^3$ where ρ_2 is the outer radius and ρ_1 is the inner radius.

Finally consider $(1-\cos\phi_0)$, which when evaluated at specific ϕ would turn into $\cos\phi_1 - \cos\phi_2$ where ϕ_2 is the outer angle and ϕ_1 is the inner angle.

Using these deductions, we have

$$V = \frac{2\pi a^3}{3}(1-\cos\phi_0)$$
$$\Delta V = \frac{(\theta_2-\theta_1)(\rho_2^3-\rho_1^3)}{3}(\cos\phi_1 - \cos\phi_2)$$

for $\rho_1 \le \rho \le \rho_2$, $\phi_1 \le \phi \le \phi_2$, and $\theta_1 \le \theta \le \theta_2$.

(c)

The Mean Value Theorem states that if a function is continuous on a closed interval $[a,b]$, differentiable on the open interval (a,b), and $f(a) = f(b)$, then there exists a number *c* in (a,b) such that $f'(c) = 0$.

The change in volume from part (b) is,

$$\Delta V = \frac{(\theta_2-\theta_1)(\rho_2^3-\rho_1^3)}{3}(\cos\phi_1 - \cos\phi_2)$$

Notice that $\theta_2 - \theta_1 = \Delta\theta$. The changes in the remaining variables are little bit difficult to simplify.

Use the Mean Value Theorem first on the change in the function $f(\rho) = \rho^3$.

Since the function is continuous and differentiable, there exists a value $\tilde{\rho}$ such that $f'(\tilde{\rho}) = 0$. Calculate $f'(\tilde{\rho})$

$$\begin{aligned} f'(\tilde{\rho}) &= \frac{d}{d\rho}(\rho^3)\Big]_{\rho=\tilde{\rho}} \\ &= 3\rho^2\Big]_{\rho=\tilde{\rho}} \\ &= 3\tilde{\rho}^2\Delta\rho \end{aligned}$$

So $f'(\tilde{\rho}) = 3\tilde{\rho}^2\Delta\rho$

Since the function $g(\phi)=\cos\phi$ is continuous and differentiable, there exists a value $\tilde{\phi}$ such that $g'(\tilde{\phi})=0$. Calculate $g'(\tilde{\phi})$

$$\begin{aligned} g'(\tilde{\phi}) &= \frac{d}{d\phi}(\cos\phi)\Big]_{\phi=\tilde{\phi}} \\ &= -\sin\phi\big]_{\phi=\tilde{\phi}} \\ &= -\sin\tilde{\phi}\Delta\phi \end{aligned}$$

So $g'(\tilde{\phi}) = -\sin\tilde{\phi}\Delta\phi$

Using the Mean Value Theorem then, the change in volume becomes

$$\begin{aligned} \Delta V &= \frac{(\theta_2-\theta_1)(\rho_2{}^3-\rho_1{}^3)}{3}(\cos\phi_1-\cos\phi_2) \\ &= \frac{\Delta\theta(3\tilde{\rho}^2\Delta\rho)}{3}\left(-\left(-\sin\tilde{\phi}\right)\Delta\phi\right) \\ &= \tilde{\rho}^2\sin\tilde{\phi}\Delta\rho\Delta\theta\Delta\phi \end{aligned}$$

Where $\rho_1 \le \tilde{\rho} \le \rho_2$, $\phi_1 \le \tilde{\phi} \le \phi_2$, $\Delta\theta = \theta_2 - \theta_1$, $\Delta\phi = \phi_2 - \phi_1$, and $\Delta\rho = \rho_2 - \rho_1$