DAY FOUR

Quadratic Equation and Inequalities

Learning & Revision for the Day

- Quadratic Equation
- Relation between Roots and Coefficients
- Formation of an Equation
- Transformation of Equations
- Maximum and Minimum Value of $ax^2 + bx + c$
- Sign of Quadratic Expression
- Position of Roots
- Inequalities
- Arithmetic-Geometric-Harmonic Mean Inequality
- Logarithm Inequality

Quadratic Equation

- An equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, a, b and c, $x \in R$, is called a **real quadratic equation**. Here a, b and c are called the **coefficients** of the equation.
- The quantity $D = b^2 - 4ac$ is known as the **discriminant** of the equation $ax^2 + bx + c = 0$ and its roots are given by $x = \dfrac{-b \pm \sqrt{D}}{2a}$
- An equation of the form $az^2 + bz + c = 0$, where $a \neq 0$, a, b and c, $z \in C$ (complex) is called a **complex quadratic equation** and its roots are given by $z = \dfrac{-b \pm \sqrt{D}}{2a}$.

Nature of Roots of Quadratic Equation

Let $a, b, c \in R$ and $a \neq 0$, then the equation $ax^2 + bx + c = 0$

(i) has real and distinct roots if and only if $D > 0$.

(ii) has real and equal roots if and only if $D = 0$.

(iii) has complex roots with non-zero imaginary parts if and only if $D < 0$.

Some Important Results

(i) If $p + iq$ (where, $p, q \in R$, $q \neq 0$) is one root of $ax^2 + bx + c = 0$, then second root will be $p - iq$

(ii) If $a, b, c \in Q$ and $p + \sqrt{q}$ is an **irrational root** of $ax^2 + bx + c = 0$, then other root will be $p - \sqrt{q}$.

(iii) If $a, b, c \in Q$ and D is a perfect square, then $ax^2 + bx + c = 0$ has **rational roots.**

(iv) If $a = 1$, b, $c \in I$ and roots of $ax^2 + bx + c = 0$ are rational numbers, then these roots must be integers.

(v) If the roots of $ax^2 + bx + c = 0$ are both positive, then the signs of a and c should be like and opposite to the sign of b.

(vi) If the roots of $ax^2 + bx + c = 0$ are both negative, then signs of a, b and c should be like.

(vii) If the roots of $ax^2 + bx + c = 0$ are reciprocal to each other, then $c = a$.

(viii) In the equation $ax^2 + bx + c = 0$ $(a, b, c \in R)$, if $a + b + c = 0$, then the roots are $1, \frac{c}{a}$ and if $a - b + c = 0$, then the roots are -1 and $-\frac{c}{a}$.

Relation between Roots and Coefficients

Quadratic Roots

If α and β are the roots of quadratic equation $ax^2 + bx + c = 0$; $a \neq 0$, then sum of roots $= \alpha + \beta = -\frac{b}{a}$

and product of roots $= \alpha\beta = \frac{c}{a}$.

And, also $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

Cubic Roots

If α, β and γ are the roots of cubic equation $ax^3 + bx^2 + cx + d = 0$; $a \neq 0$, then $\alpha + \beta + \gamma = -\frac{b}{a}$

$$\beta\gamma + \gamma\alpha + \alpha\beta = \frac{c}{a}$$

and $$\alpha\beta\gamma = -\frac{d}{a}$$

Common Roots (Conditions)

Suppose that the quadratic equations are $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$.

(i) When **one root** is common, then the condition is $(a'c - ac')^2 = (bc' - b'c)(ab' - a'b)$.

(ii) When **both roots** are common, then the condition is $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

Formation of an Equation

Quadratic Equation

If the roots of a quadratic equation are α and β, then the equation will be of the form $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

Cubic Equation

If α, β and γ are the roots of the cubic equation, then the equation will be form of

$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)\,x - \alpha\beta\gamma = 0.$$

Transformation of Equations

Let the given equation be

$$a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0 \qquad \dots(A)$$

Then, the equation

(i) whose roots are k $(\neq 0)$ times roots of the Eq. (A), is obtained by replacing x by $\frac{x}{k}$ in Eq. (A).

(ii) whose roots are the negatives of the roots of Eq. (A), is obtained by replacing x by $-x$ in Eq. (A).

(iii) whose roots are k more than the roots of Eq. (A), is obtained by replacing x by $(x - k)$ in Eq. (A).

(iv) whose roots are reciprocals of the roots of Eq. (A), is obtained by replacing x by $1/x$ in Eq. (A) and then multiply both the sides by x^n.

Maximum and Minimum Value of $ax^2 + bx + c$

(i) When $a > 0$, then minimum value of $ax^2 + bx + c$ is $\frac{-D}{4a}$ or $\frac{4ac - b^2}{4a}$ at $x = \frac{-b}{2a}$

(ii) When $a < 0$, then maximum value of $ax^2 + bx + c$ is $\frac{-D}{4a}$ or $\frac{4ac - b^2}{4a}$ at $x = \frac{-b}{2a}$

Sign of Quadratic Expression

Let $f(x) = ax^2 + bx + c$, where a, b and $c \in R$ and $a \neq 0$.

(i) If $a > 0$ and $D < 0$, then $f(x) > 0, \forall x \in R$.

(ii) If $a < 0$ and $D < 0$, then $f(x) < 0, \forall x \in R$.

(iii) If $a > 0$ and $D = 0$, then $f(x) \geq 0, \forall x \in R$.

(iv) If $a < 0$ and $D = 0$, then $f(x) \leq 0, \forall x \in R$.

(v) If $a > 0$, $D > 0$ and $f(x) = 0$ have two real roots α and β, where $(\alpha < \beta)$, then $f(x) > 0$, $\forall\ x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) < 0$, $\forall\ x \in (\alpha, \beta)$.

(vi) If $a < 0$, $D > 0$ and $f(x) = 0$ have two real roots α and β, where $(\alpha < \beta)$, then $f(x) < 0$, $\forall\ x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0$, $\forall\ x \in (\alpha, \beta)$.

Position of Roots

Let $ax^2 + bx + c = 0$ has roots α and β. Then, we have the following conditions:

(i) with respect to one real number (k).

Situation	Required conditions
$\alpha < \beta < k$	$D \ge 0$, $af(k) > 0$, $k > -b/2a$
$k < \alpha < \beta$	$D \ge 0$, $af(k) > 0$, $k < -b/2a$
$\alpha < k < \beta$	$D > 0$, $af(k) < 0$

(ii) with respect to two real numbers k_1 and k_2.

Situation	Required conditions
$k_1 < \alpha < \beta < k_2$	$D \ge 0$, $af(k_1) > 0$, $af(k_2) > 0$, $k_1 < -b/2a < k_2$
$\alpha < k_1 < k_2 < \beta$	$D > 0$, $af(k_1) < 0$, $af(k_2) < 0$
$k_1 < \alpha < k_2 < \beta$	$D > 0$, $f(k_1) f(k_2) < 0$

Inequalities

Let a and b be two real numbers. If $a - b$ is negative, we say that a is less than b ($a < b$) and if $a - b$ is positive, then a is greater than b ($a > b$). This shows the inequalities concept.

Important Results on Inequalities

(i) If $a > b$, then $a \pm c > b \pm c, \forall c \in R$.

(ii) If $a > b$, then

(a) for $m > 0$, $am > bm, \frac{a}{m} > \frac{b}{m}$

(b) for $m < 0$, $am < bm, \frac{a}{m} < \frac{b}{m}$

(iii) (a) If $a > b > 0$, then

- $a^2 > b^2$
- $|a| > |b|$
- $\frac{1}{a} < \frac{1}{b}$

(b) If $a < b < 0$, then

- $a^2 > b^2$
- $|a| > |b|$
- $\frac{1}{a} > \frac{1}{b}$

(iv) If $a < 0 < b$, then

(a) $a^2 > b^2$, if $|a| > |b|$ (b) $a^2 < b^2$, if $|a| < |b|$

(v) If $a < x < b$ and a, b are positive real numbers, then $a^2 < x^2 < b^2$.

(vi) If $a < x < b$ and a is negative number and b is positive number, then

(a) $0 \le x^2 < b^2$, if $|b| > |a|$ (b) $0 \le x^2 < a^2$, if $|a| > |b|$

(vii) If $a_i > b_i > 0$, where $i = 1, 2, 3, \ldots, n$, then

$$a_1 a_2 a_3 \ldots a_n > b_1 b_2 b_3 \ldots b_n.$$

(viii) If $a_i > b_i$, where $i = 1, 2, 3, \ldots, n$, then

$$a_1 + a_2 + a_3 + \ldots + a_n > b_1 + b_2 + \ldots + b_n.$$

(ix) If $|x| < a$, then

(a) for $a > 0, -a < x < a$.

(b) for $a < 0, x \in \phi$.

Arithmetic-Geometric-Harmonic Mean Inequality

The Arithmetic-Geometric-Harmonic Mean of positive real numbers is defined as follows

Arithmetic Mean $\ge$ Geometric Mean $\ge$ Harmonic Mean

(i) If $a, b > 0$ then $\frac{a+b}{2} \ge \sqrt{ab} \ge \frac{2}{\frac{1}{a} + \frac{1}{b}}$

(ii) If $a_i > 0$, where $i = 1, 2, 3, \ldots, n$, then

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \ge (a_1 \cdot a_2 \ldots a_n)^{1/n} \ge \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}$$

Logarithm Inequality

If a is a positive real number other than 1 and $a^x = m$, then x is called the logarithm of m to the base a, written as $\log_a m$. In $\log_a m$, m should always be positive.

(i) If $m \le 0$, then $\log_a m$ will be meaningless.

(ii) $\log_a m$ exists, if $m, a > 0$ and $a \ne 1$.

Important Results on Logarithm

(i) $a^{\log_a x} = x$; $a > 0, \ne 1, x > 0$

(ii) $a^{\log_b x} = x^{\log_b a}$; $a, b > 0, \ne 1, x > 0$

(iii) $\log_a a = 1, a > 0, \ne 1$

(iv) $\log_a x = \frac{1}{\log_x a}$; $x, a > 0, \ne 1$

(v) $\log_a x = \log_a b \ \log_b x = \frac{\log_b x}{\log_b a}$; $a, b > 0, \ne 1, x > 0$

(vi) For $x > 0$; $a > 0, \ne 1$

(a) $\log_{a^n}(x) = \frac{1}{n} \log_a x$

(b) $\log_{a^n} x^m = \left(\frac{m}{n}\right) \log_a x$

(vii) For $x > y > 0$

(a) $\log_a x > \log_a y$, if $a > 1$

(b) $\log_a x < \log_a y$, if $0 < a < 1$

(viii) If $a > 1$ and $x > 0$, then

(a) $\log_a x > p \Rightarrow x > a^p$

(b) $0 < \log_a x < p \Rightarrow 0 < x < a^p$

(ix) If $0 < a < 1$, then

(a) $\log_a x > p \Rightarrow 0 < x < a^p$

(b) $0 < \log_a x < p \Rightarrow a^p < x < 1$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 If $\sqrt{3x^2-7x-30}+\sqrt{2x^2-7x-5}=x+5$, then x is equal to

(a) 2 (b) 3 (c) 6 (d) 5

2 The number of solutions for equation $x^2-5|x|+6=0$ is

(a) 4 (b) 3 (c) 2 (d) 1

3 The roots of the equation $|2x-1|^2-3|2x-1|+2=0$ are

(a) $\left\{-\frac{1}{2},0,\frac{1}{2}\right\}$ (b) $\left\{-\frac{1}{2},0,\frac{3}{2}\right\}$ (c) $\left\{-\frac{3}{2},\frac{1}{2},0,1\right\}$ (d) $\left\{-\frac{1}{2},0,1,\frac{3}{2}\right\}$

4 The product of all the values of x satisfying the equation $(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10$ is

(a) 4 (b) 6 (c) 8 (d) 19

5 The root of the equation $2(1+i)x^2-4(2-i)x-5-3i=0$, where $i=\sqrt{-1}$, which has greater modulus, is

(a) $\frac{3-5i}{2}$ (b) $\frac{5-3i}{2}$ (c) $\frac{3+i}{2}$ (d) $\frac{3i+1}{2}$

6 $x^2+x+1+2k(x^2-x-1)=0$ is perfect square for how many value of k

(a) 2 (b) 0 (c) 1 (d) 3

7 If the roots of $(a^2+b^2)x^2-2(bc+ad)x+c^2+d^2=0$ are equal, then

(a) $\frac{a}{b}=\frac{c}{d}$ (b) $\frac{a}{c}+\frac{b}{d}=0$ (c) $\frac{a}{d}=\frac{b}{c}$ (d) $a+b=c+d$

8 The least value of $|\alpha|$ for which $\tan\theta$ and $\cot\theta$ are roots of the equation $x^2+ax+1=0$, is

(a) 2 (b) 1 (c) 1/2 (d) 0

9 If one root of the equation $x^2-\lambda x+12=0$ is even prime while $x^2+\lambda x+\mu=0$ has equal roots, then μ is equal to

(a) 8 (b) 16 (c) 24 (d) 32

10 If $a+b+c=0$, then the roots of the equation $4ax^2+3bx+2c=0$, where $a, b, c\in R$ are

(a) real and distinct (b) imaginary
(c) real and equal (d) infinite

11 The equation $(\cos\beta-1)x^2+(\cos\beta)x+\sin\beta=0$ in the variable x has real roots, then β lies in the interval

(a) $(0,2\pi)$ (b) $(-\pi,0)$ (c) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (d) $(0,\pi)$

12 If $ax^2+2bx-3c=0$ has no real root and $\frac{3c}{4}<a+b$, then the range of c is

(a) $(-1,1)$ (b) $(0,1)$
(c) $(0,\infty)$ (d) $(-\infty,0)$

13 If a, b and c are real numbers in AP, then the roots of $ax^2+bx+c=0$ are real for

(a) all a and c (b) no a and c
(c) $\left|\frac{c}{a}-7\right|\ge 4\sqrt{3}$ (d) $\left|\frac{a}{c}+7\right|\ge 2\sqrt{3}$

14 If $P(x)=ax^2+bx+c$ and $Q(x)=-ax^2+dx+c=0$; $ac\ne 0$, then the equation $P(x)\cdot Q(x)=0$ has

(a) four real roots (b) exactly two real roots
(c) either two or four real roots (d) atmost two real roots

15 The rational values of a in $ax^2+bx+1=0$ if $\frac{1}{4+\sqrt{3}}$ is a root, are

(a) $a=13, b=-8$ (b) $a=-13, b=8$
(c) $a=13, b=8$ (d) $a=-13, b=-8$

16 If $1-i$, is a root of the equation $x^2+ax+b=0$, where $a,b\in R$, then the values of a and b are

(a) 1,–1 (b) 2,– 2
(c) 3,–3 (d) None of these

17 The values of p for which one root of the equation $x^2-30x+p=0$ is the square of the other, is/are

(a) Only 125 (b) 125 and – 216
(c) 125 and 215 (d) Only 216

18 If the roots of the quadratic equation $\frac{x-m}{mx+1}=\frac{x+n}{nx+1}$ are reciprocal to each other, then

(a) $n=0$ (b) $m=n$ (c) $m+n=1$ (d) $m^2+n^2=1$

19 Let α and α^2 be the roots of $x^2+x+1=0$, then the equation whose roots are α^{31} and α^{62}, is

(a) $x^2-x+1=0$ (b) $x^2+x-1=0$
(c) $x^2+x+1=0$ (d) $x^{60}+x^{30}+1=0$

20 If α and β are the roots of $x^2-a(x-1)+b=0$, then the value of $\frac{1}{\alpha^2-a\alpha}+\frac{1}{\beta^2-a\beta}+\frac{2}{a+b}$ is

(a) $\frac{4}{a+b}$ (b) $\frac{1}{a+b}$ (c) 0 (d) –1

21 The value of a for which the sum of the squares of the roots of the equation $x^2-(a-2)x-a-1=0$ assume the least value is ➜ **AIEEE 2005**

(a) 2 (b) 3 (c) 0 (d) 1

22 If α and β be the roots of the equation $2x^2+2(a+b)x+a^2+b^2=0$, then the equation whose roots are $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$, is

(a) $x^2-2abx-(a^2-b^2)^2=0$ (b) $x^2-4abx-(a^2-b^2)^2=0$
(c) $x^2-4abx+(a^2-b^2)^2=0$ (d) None of these

23 Let α,β be the roots of $x^2-2x\cos\phi+1=0$, then the equation whose roots are α^n and β^n, is

(a) $x^2-2x\cos n\phi-1=0$ (b) $x^2-2x\cos n\phi+1=0$
(c) $x^2-2x\sin n\phi+1=0$ (d) $x^2+2x\sin n\phi-1=0$

24 The harmonic mean of the roots of the equation $(5+\sqrt{2})x^2-(4+\sqrt{5})x+8+2\sqrt{5}=0$ is

(a) 2 (b) 4 (c) 6 (d) 8

25 If the ratio of the roots of $\lambda x^2 + \mu x + \nu = 0$ is equal to the ratio of the roots of $x^2 + x + 1 = 0$, then λ, μ and ν are in

(a) AP (b) GP
(c) HP (d) None of these

26 If the roots of the equation $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$ are equal in magnitude but opposite in sign, then the product of the roots is

(a) $-2(p^2+q^2)$ (b) $-(p^2+q^2)$
(c) $\frac{-(p^2+q^2)}{2}$ (d) $-pq$

27 If the roots of the equation $ax^2+bx+c=0$ of the form $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$, then $(a+b+c)^2$ is equal to

(a) $2b^2-ac$ (b) Σa^2 (c) b^2-4ac (d) b^2-2ac

28 If α and β are the roots of the equation $ax^2+bx+c=0$ such that $\beta<\alpha<0$, then the quadratic equation whose roots are $|\alpha|, |\beta|$, is given by

(a) $|a|x^2+|b|x+|c|=0$ (b) $ax^2-|b|x+c=0$
(c) $|a|x^2-|b|x+|c|=0$ (d) $a|x|^2+b|x+c=0$

29 If α and β be the roots of $x^2+px+q=0$, then $\frac{(\omega\alpha+\omega^2\beta)(\omega^2\alpha+\omega\beta)}{\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}}$ is equal to

(a) $-\frac{q}{p}$ (b) $\alpha\beta$ (c) $-\frac{p}{q}$ (d) ω

30 If α and β are roots of the equation $x^2+px+3.\frac{p}{4}=0$, such that $|\alpha-\beta|=\sqrt{10}$, then p belongs → **JEE Mains 2013**

(a) $\{2,-5\}$ (b) $\{-3,2\}$ (c) $\{-2,5\}$ (d) $\{3,-5\}$

31 Sachin and Rahul attemped to solve a quadratic equation. Sachin made a mistake in writing down the constant term and ended up in roots $(4,3)$. Rahul made a mistake in writing down coefficient of x to get roots $(3,2)$. The correct roots of equation are → **AIEEE 2011**

(a) $-4,-3$ (b) 6, 1 (c) 4, 3 (d) $-6,-1$

32 If α and β are the roots of $x^2+x+2=0$ and γ,δ are the roots of $x^2+3x+4=0$, then $(\alpha+\gamma)(\alpha+\delta)(\beta+\gamma)(\beta+\delta)$ is equal to

(a) -18 (b) 18 (c) 24 (d) 44

33 In $\Delta PQR, R=\frac{\pi}{2}$. If $\tan\frac{P}{2}$ and $\tan\frac{Q}{2}$ are the roots of the equation $ax^2+bx+c=0$, then

(a) $a=b+c$ (b) $b=c+a$
(c) $c=a+b$ (d) $b=c$

34 If the equation $k(6x^2+3)+rx+2x^2-1=0$ and $6k(2x^2+1)+px+4x^2-2=0$ have both roots common, then $2r-p$ is equal to

(a) 2 (b) 1 (c) 0 (d) k

35 If the equations $x^2+2x+3=0$ and $ax^2+bx+c=0$, $a,b,c\in R$, have a common root, then $a:b:c$ is → **AIEEE 2012**

(a) 1 : 2 : 3 (b) 3 : 2 : 1 (c) 1 : 3 : 2 (d) 3 : 1 : 2

36 The equation formed by decreasing each root of $ax^2+bx+c=0$ by 1 is $2x^2+8x+2=0$, then

(a) $a=-b$ (b) $b=-c$ (c) $c=-a$ (d) $b=a+c$

37 If $f(x)=x^2+2bx+2c^2$ and $g(x)=-x^2-2cx+b^2$ such that min $f(x)>$ max $g(x)$, then the relation between b and c is

(a) $|c|<|b|\sqrt{2}$ (b) $0<c<b\sqrt{2}$
(c) $|c|<|b|\,2$ (d) $|c|>|b|\sqrt{2}$

38 If $a\in R$ and $a_1,a_2,a_3\ldots,a_n\in R$, then $(x-a_1)^2+(x-a_2)^2+\ldots+(x-a_n)^2$ assumes its least value at $x=$

(a) $a_1+a_2+\ldots+a_n$ (b) $2(a_1+a_2+a_3+\ldots+a_n)$
(c) $n(a_1+a_2+\ldots+a_n)$ (d) None of these

39 If the roots of the equation $bx^2+cx+a=0$ is imaginary, then for all real values of x, the expression $3b^2x^2+6bcx+2c^2$ is → **AIEEE 2009**

(a) greater than $4ab$ (b) less than $4ab$
(c) greater than $-4ab$ (d) less than $-4ab$

40 If $x^2+2ax+10-3a>0$ for all $x\in R$, then

(a) $-5<a<2$ (b) $a<-5$
(c) $a>5$ (d) $2<a<5$

41 If the expression $\left(ax-1+\frac{1}{x}\right)$ is non-negative for all positive real x, then the minimum value of a must be

(a) 0 $\frac{1}{2}$
(c) $\frac{1}{4}$ (d) None of these

42 The number of real solutions of the equation $\left(\frac{9}{10}\right)^x=-3+x-x^2$ is

(a) 0 (b) 1
(c)2 (d) None of these

43 If α and β be the roots of the quadratic equation $ax^2+bx+c=0$ and k be a real number, then the condition, so that $\alpha<k<\beta$ is given by

(a) $ac>0$ (b) $ak^2+bk+c=0$
(c) $ac<0$ (d) $a^2k^2+abk+ac<0$

44 $|2x-3|<|x+5|$, then x belongs to

(a) $(-3,5)$ (b) $(5,9)$ (c) $\left(-\frac{2}{3},8\right)$ (d) $\left(-8,\frac{2}{3}\right)$

45 The least integral value α of x such that $\frac{x-5}{x^2+5x-14}>0$, satisfies → **JEE Mains 2013**

(a) $\alpha^2+3\alpha-4=0$ (b) $\alpha^2-5\alpha+4=0$
(c) $\alpha^2-7\alpha+6=0$ (d) $\alpha^2+5\alpha-6=0$

46 The solution set of $\frac{|x-2|-1}{|x-2|-2} \le 0$ is

(a) $[0, 1] \cup (3, 4)$ (b) $[0, 1] \cup [3, 4]$
(c) $[-1, 1) \cup (3, 4]$ (d) None of these

47 Number of integral solutions of $\frac{x+2}{x^2+1} > \frac{1}{2}$ is

(a) 0 (b) 1 (c) 2 (d) 3

48 If the product of n positive numbers is 1, then their sum is

(a) a positive integer (b) divisible by n
(c) equation to $n + \frac{1}{n}$ (d) greater than or equal to n

49 The minimum value of $P = bcx + cay + abz$, when $xyz = abc$, is

(a) $3abc$ (b) $6abc$
(c) abc (d) $4abc$

50 If a, b and c are distinct three positive real numbers, then $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ is

(a) > 1 (b) > 9
(c) < 9 (d) None of these

51 If a, b, c are positive real numbers such that $a + b + c = 1$, then the greatest value of $(1-a)(1-b)(1-c)$, is

(a) $\frac{1}{27}$ (b) $\frac{8}{27}$ (c) $\frac{4}{27}$ (d) 9

52 If a, b, c, d are positive real numbers such that $a + b + c + d = 2$, then $M = (a+b)(c+d)$ satisfies the relation

(a) $0 \le M \le 1$ (b) $1 \le M \le 2$ (c) $2 \le M \le 3$ (d) $3 \le M \le 4$

53 $\log_2(x^2 - 3x + 18) < 4$, then x belongs to

(a) (1, 2) (b) (2, 16)
(c) (1, 16) (d) None of these

54 If $\log_{0.3}(x-1) > \log_{0.09}(x-1)$, then x lies in

(a) (1, 2) (b) $(-\infty, 1)$
(c) $(2, \infty)$ (d) None of these

55 What is the solution set of the following inequality?

$$\log_x\left(\frac{x+5}{1-3x}\right) > 0$$

(a) $0 < x < \frac{1}{3}$ (b) $x \ge 3$
(c) $\frac{1}{3} < x < 1$ (d) None of these

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 Let $S = \{x \in R : x \ge 0 \text{ and } 2\left|\sqrt{x}-3\right| + \sqrt{x}(\sqrt{x}-6) + 6 = 0\}$. Then, S → **JEE Mains 2018**

(a) is an empty set
(b) contains exactly one element
(c) contains exactly two elements
(d) contains exactly four elements

2 The roots of the equation $2^{x+2} \cdot 3^{3x/(x-1)} = 9$ are given by

(a) $1 - \log_2 3, 2$ (b) $\log_2\left(\frac{2}{3}\right), 1$
(c) $2, -2$ (d) $-2, 1 - \frac{(\log 3)}{(\log 2)}$

3 Let α and β be the roots equation $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$ for $n \ge 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is equal to → **JEE Mains 2015**

(a) 6 (b) -6 (c) 3 (d) -3

4 If $x^2 - 5x + 1 = 0$, then $\frac{x^{10}+1}{x^5}$ is equal to

(a) 2424 (b) 3232
(c) 2525 (d) None of these

5 If $a < b < c < d$, then the roots of the equation $(x-a)(x-c) + 2(x-b)(x-d) = 0$ are

(a) real and distinct (b) real and equal
(c) imaginary (d) None of these

6 Let α and β be the roots of equation $px^2 + qx + r = 0$, $p \ne 0$. If p, q and r are in AP and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, then the value of $|\alpha - \beta|$ is → **JEE Mains 2014**

(a) $\frac{\sqrt{61}}{9}$ (b) $\frac{2\sqrt{17}}{9}$ (c) $\frac{\sqrt{34}}{9}$ (d) $\frac{2\sqrt{13}}{9}$

7 If α and β are the roots of the equation $ax^2 + bx + c = 0$ ($a \ne 0$, a, b, c being different), then $(1+\alpha+\alpha^2)(1+\beta+\beta^2)$ is

(a) zero (b) positive
(c) negative (d) None of these

8 The minimum value of the sum of real numbers $a^{-5}, a^{-4}, 3a^{-3}, 1, a^8$ and a^{10} $a > 0$ is

(a) 9 (b) 8 (c) 2 (d) 1

9 For $a > 0$, the roots of the equation $\log_{ax} a + \log_x a^2 + \log_{a^2 x} a^3 = 0$ is given by

(a) $a^{-4/3}$ (b) $a^{-3/4}$ (c) $a^{1/2}$ (d) a^{-1}

10 If a, b and c are in AP and if the equations $(b-c)x^2+(c-a)x+(a-b)=0$ and $2(c+a)x^2+(b+c)x=0$ have a common root, then

(a) a^2, b^2 and c^2 are in AP (b) a^2, c^2 and b^2 are in AP
(c) c^2, a^2 and b^2 are in AP (d) None of these

11 If the equations $x^2+ax+12=0$, $x^2+bx+15=0$ and $x^2+(a+b)x+36=0$ have a common positive root, then the ordered pair (a, b) is

(a) $(-6,-7)$ (b) $(-7,-8)$
(c) $(-6,-8)$ (d) $(-8,-7)$

12 If x is real, then the maximum and minimum value of the expression $\frac{x^2-3x+4}{x^2+3x+4}$ will be

(a) 2, 1 (b) $5, \frac{1}{5}$
(c) $7, \frac{1}{7}$ (d) None of these

13 If $a \in R$ and the equation $-3(x-[x])^2+2(x-[x])+a^2=0$ (where, $[x]$ denotes the greatest integer $\le x$) has no integral solution, then all possible values of a lie in the interval ➜ **JEE Mains 2014**

(a) $(-1,0)\cup(0,1)$ (b) $(1,2)$
(c) $(-2,-1)$ (d) $(-\infty,-2)\cup(2,\infty)$

14 If $a=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$, then the quadratic equation whose roots are $\alpha=a+a^2+a^4$ and $\beta=a^3+a^5+a^6$, is

(a) $x^2-x+2=0$ (b) $x^2+2x+2=0$
(c) $x^2+x+2=0$ (d) $x^2+x-2=0$

15 If α and β are roots of $375x^2-25x-2=0$ and $S_n=\alpha^n+\beta^n$, then $\lim_{n\to\infty}\sum_{r=1}^{n}S_r$ is equal to

(a) $\frac{7}{116}$ (b) $\frac{1}{12}$
(c) $\frac{29}{358}$ (d) None of these

16 If $S=\{a\in N, 1\le a\le 100\}$ and $[\tan^2 x]-\tan x-a=0$ has real roots, where $[.]$ denotes the greatest integer function, then number of elements in set S equals

(a) 2 (b) 5 (c) 6 (d) 9

17 The sum of all real values of x satisfying the equation $(x^2-5x+5)^{x^2+4x-60}=1$ is ➜ **JEE Mains 2016**

(a) 3 (b) -4 (c) 6 (d) 5

18 If λ is an integer and α, β are the roots of $4x^2-16x+\frac{\lambda}{4}=0$ such that $1<\alpha<2$ and $2<\beta<3$, then the possible values of λ are

(a) {60, 64, 68} (b) {61, 62, 63}
(c) {49, 50,…, 62, 63} (d) {62, 65, 68, 71, 75}

19 If α and β are the roots of the equation $ax^2+bx+c=0$, then the quadratic equation whose roots are $\frac{\alpha}{1+\alpha}$ and $\frac{\beta}{1+\beta}$ is

(a) $ax^2-b(1-x)+c(1-x)^2=0$
(b) $ax^2-b(x-1)+c(x-1)^2=0$
(c) $ax^2+b(1-x)+c(1-x)^2=0$
(d) $ax^2+b(x+1)+c(1+x)^2=0$

20 If both the roots of the quadratic equation $x^2-2kx+k^2+k-5=0$ are less than 5, then k lies in the interval ➜ **AIEEE 2005**

(a) $[4,5]$ (b) $(-\infty,4)$ (c) $(6,\infty)$ (d) $(5,6)$

21 All the values of m for which both roots of the equation $x^2-2mx+m^2-1=0$ are greater than -2 but less than 4 lie in the interval

(a) $m>3$ (b) $-1<m<3$
(c) $1<m<4$ (d) $-2<m<0$

22 Let α and β be real and z be a complex number. If $z^2+\alpha z+\beta=0$ has two distinct roots on the line $\text{Re}(z)=1$, then it is necessary that ➜ **AIEEE 2011**

(a) $\beta\in(-1,0)$ (b) $|\beta|=1$ (c) $\beta\in[1,\infty)$ (d) $\beta\in(0,1)$

23 The equation $e^{\sin x}-e^{-\sin x}-4=0$ has ➜ **AIEEE 2012**

(a) infinite number of real roots
(b) no real root
(c) exactly one real root
(d) exactly four real roots

24 If a, b, c, d are positive real numbers such that $a+\frac{1}{b}=4$, $b+\frac{1}{c}=1$, $c+\frac{1}{d}=4$ and $d+\frac{1}{a}=1$, then

(a) $a=c$ and $b=d$ (b) $b=d$ but $a\ne c$
(c) $ab=1$ and $cd\ne 1$ (d) $cd=1$ and $ab\ne 1$

(ω and ω^2 are complex cube roots of unity)

25 Let $f: R\to R$ be a continuous function defined by $f(x)=\frac{1}{e^x+2e^{-x}}$

Statement I $f(c)=\frac{1}{3}$, for some $c\in R$.

Statement II $0<f(x)\le\frac{1}{2\sqrt{2}}, \forall x\in R$. ➜ **AIEEE 2010**

(a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true. Statement II is a correct explanation of Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II is false

ANSWERS

SESSION 1

1. (c)	**2.** (a)	**3.** (d)	**4.** (c)	**5.** (a)	**6.** (a)	**7.** (c)	**8.** (a)	**9.** (b)	**10.** (a)
11. (d)	**12.** (d)	**13.** (c)	**14.** (c)	**15.** (a)	**16.** (d)	**17.** (b)	**18.** (a)	**19.** (c)	**20.** (c)
21. (d)	**22.** (b)	**23.** (b)	**24.** (b)	**25.** (b)	**26.** (c)	**27.** (c)	**28.** (c)	**29.** (a)	**30.** (c)
31. (b)	**32.** (d)	**33.** (c)	**34.** (c)	**35.** (a)	**36.** (b)	**37.** (d)	**38.** (d)	**39.** (c)	**40.** (a)
41. (c)	**42.** (a)	**43.** (d)	**44.** (c)	**45.** (d)	**46.** (b)	**47.** (d)	**48.** (d)	**49.** (a)	**50.** (b)
51. (b)	**52.** (a)	**53.** (a)	**54.** (a)	**55.** (d)					

SESSION 2

1. (c)	**2.** (d)	**3.** (c)	**4.** (c)	**5.** (a)	**6.** (d)	**7.** (b)	**8.** (b)	**9.** (a)	**10.** (b)
11. (b)	**12.** (c)	**13.** (a)	**14.** (c)	**15.** (b)	**16.** (d)	**17.** (a)	**18.** (c)	**19.** (c)	**20.** (b)
21. (b)	**22.** (c)	**23.** (b)	**24.** (a)	**25.** (b)					

Hints and Explanations

SESSION 1

1 We have,

$\sqrt{3x^2-7x-30}+\sqrt{2x^2-7x-5}=x+5$

$\Rightarrow \sqrt{3x^2-7x-30}=(x+5)-\sqrt{2x^2-7x-5}$

On squaring both sides, we get

$3x^2-7x-30$
$= x^2+25+10x+(2x^2-7x-5)$
$-2(x+5)\sqrt{2x^2-7x-5}$

$\Rightarrow -10x-50=-2(x+5)\sqrt{2x^2-7x-5}$

$x+5\neq 0,\ \sqrt{2x^2-7x-5}=5$

[$\because x=-5$ does not satisfy the given equation]

$\Rightarrow\ 2x^2-7x-30=0$

$\therefore\ x=6$

2 Given equation is $x^2-5|x|+6=0$

When $x\geq 0,\ x^2-5x+6=0$

and when $x<0,\ x^2+5x+6=0$

$\Rightarrow\ x^2-3x-2x+6=0;\ x\geq 0$

and $x^2+3x+2x+6=0;\ x<0$

$\Rightarrow\ (x-3)(x-2)=0,\ x\geq 0$

and $(x+3)\cdot(x+2)=0,\ x<0$

$\therefore\ x=3,\ x=2$ and $x=-3,\ x=-2$

There are four solutions of this equation.

3 Given equation is

$|2x-1|^2-3|2x-1|+2=0$

Let $|2x-1|=t$, then

$t^2-3t+2=0$

$\Rightarrow\ (t-1)(t-2)=0\ \Rightarrow\ t=1,2$

$\Rightarrow\ |2x-1|=1$ and $|2x-1|=2$

$\Rightarrow\ 2x-1=\pm1$ and $2x-1=\pm2$

$\Rightarrow\ x=1,0$ and $x=\dfrac{3}{2},-\dfrac{1}{2}$

4 $\because\ 5-2\sqrt{6}=\dfrac{1}{5+2\sqrt{6}}$

$\therefore\ t+\dfrac{1}{t}=10,$

where $t=(5+2\sqrt{6})^{x^2-3}$ ...(i)

$\Rightarrow\ t^2-10t+1=0$

$\Rightarrow\ t=5\pm2\sqrt{6}$

or $t=(5+2\sqrt{6})^{\pm1}$ ...(ii)

From Eqs. (i) and (ii),

$x^2-3=\pm1$

$\Rightarrow\ x^2=2,4$

$\Rightarrow\ x=-\sqrt{2},\sqrt{2},-2,2$

$\therefore$ Required product $=8$

5 The given equation is

$2(1+i)x^2-4(2-i)x-5-3i=0$

$\Rightarrow x=\dfrac{4(2-i)\pm\sqrt{16(2-i)^2+8(1+i)(5+3i)}}{4(1+i)}$

$=-\dfrac{i}{1+i}$ or $\dfrac{4-i}{1+i}=\dfrac{-1-i}{2}$ or $\dfrac{3-5i}{2}$

Now, $\left|\dfrac{-1-i}{2}\right|=\sqrt{\dfrac{1}{4}+\dfrac{1}{4}}=\sqrt{\dfrac{1}{2}}$

and $\left|\dfrac{3-5i}{2}\right|=\sqrt{\dfrac{9}{4}+\dfrac{25}{4}}=\sqrt{\dfrac{17}{2}}$

Also, $\sqrt{\dfrac{17}{2}}>\sqrt{\dfrac{1}{2}}$

Hence, required root is $\dfrac{3-5i}{2}$.

6 Given equation

$(1+2k)x^2+(1-2k)x+(1-2k)=0$

If equation is a perfect square then root are equal

i.e. $(1-2k)^2-4(1+2k)(1-2k)=0$

i.e. $k=\dfrac{1}{2},\dfrac{-3}{10}.$

Hence, total number of values = 2.

7 Since, roots are real.

$\therefore\ \{2(bc+ad)\}^2=4(a^2+b^2)(c^2+d^2)$

$\Rightarrow 4b^2c^2+4a^2d^2+8abcd=4a^2c^2+4a^2d^2+4b^2c^2+4b^2d^2$

$\Rightarrow\ 4a^2c^2+4b^2d^2-8abcd=0$

$\Rightarrow\ 4(ac-bd)^2=0$

$\Rightarrow\ ac=bd$

$\Rightarrow\ \dfrac{a}{d}=\dfrac{b}{c}$

8 Given equation is $x^2+ax+1=0$

Since, roots are real

$\therefore\ a^2-4\geq 0\ \Rightarrow\ |a|\geq 2$

Thus, the least value of $|a|$ is 2.

9 We know that only even prime is 2,

$\therefore\ (2)^2-\lambda(2)+12=0.$

$\Rightarrow\ \lambda=8$ ...(i)

$\because x^2+\lambda x+\mu=0$ has equal roots.

$\therefore\ \lambda^2-4\mu=0$ [$\because D=0$]

$\Rightarrow\ (8)^2-4\mu=0\ \Rightarrow\ \mu=16$

10 Here, $D=(3b)^2-4(4a)(2c)$

$=9b^2-32ac=9(-a-c)^2-32ac$

$=9a^2-14ac+9c^2$

$=9c^2\left[\left(\dfrac{a}{c}\right)^2-\dfrac{14}{9}\cdot\dfrac{a}{c}+1\right]$

$=9c^2\left[\left(\dfrac{a}{c}-\dfrac{7}{9}\right)^2-\dfrac{49}{81}+1\right]>0$

Hence, the roots are real and distinct.

11 For real roots, discriminant,

$$D = b^2 - 4ac \ge 0$$
$$= \cos^2\beta - 4(\cos\beta - 1)\sin\beta \ge 0$$
$$= \cos^2\beta + 4(1 - \cos\beta)\sin\beta \ge 0$$

So, $\sin\beta$ should be > 0.

$[\because \cos^2\beta \ge 0, 1 - \cos\beta \ge 0]$

$\Rightarrow \quad \beta \in (0, \pi)$

12 Here, $D = 4b^2 + 12ca < 0$

$\Rightarrow \quad b^2 + 3ca < 0 \quad \ldots(i)$

$\Rightarrow \quad ca < 0$

If $c > 0$, then $a < 0$

Also, $\dfrac{3c}{4} < a + b$

$\Rightarrow \quad 3ca > 4a^2 + 4ab$

$\Rightarrow b^2 + 3ca > 4a^2 + 4ab + b^2$

$= (2a + b)^2 \ge 0 \quad \ldots(ii)$

From (i) and (ii), $c > 0$, is not true.

$\therefore \quad c < 0$

13 Since, $D \ge 0$

$\therefore \quad b^2 - 4ac \ge 0$

$\Rightarrow \left(\dfrac{c + a}{2}\right)^2 - 4ac \ge 0 \quad [\because 2b = a + c]$

$\Rightarrow \quad c^2 - 14ca + a^2 \ge 0$

$\Rightarrow \left(\dfrac{c}{a}\right)^2 - 14\left(\dfrac{c}{a}\right) + 1 \ge 0$

$\Rightarrow \quad \left(\dfrac{c}{a} - 7\right)^2 \ge 48$

$\Rightarrow \quad \left|\dfrac{c}{a} - 7\right| \ge 4\sqrt{3}$

14 Let D_1 and D_2 be the discriminants of given equation, respectively. Then

$$D_1 + D_2 = b^2 - 4ac + d^2 + 4ac$$
$$= b^2 + d^2 > 0$$

So, either D_1 and D_2 are positive or atleast one D's is positive.

Therefore, $P(x) \cdot Q(x) = 0$ has either two or four real roots.

15 One root $= \dfrac{1}{4 + \sqrt{3}} \times \dfrac{4 - \sqrt{3}}{4 - \sqrt{3}} = \dfrac{4 - \sqrt{3}}{13}$

$\therefore$ Other root $= \dfrac{4 + \sqrt{3}}{13}$

$\therefore$ The quadratic equation is

$$x^2 - \left(\frac{4 + \sqrt{3}}{13} + \frac{4 - \sqrt{3}}{13}\right)x + \frac{4 + \sqrt{3}}{13} \cdot \frac{4 - \sqrt{3}}{13} = 0$$

or $\quad 13x^2 - 8x + 1 = 0$

This equation must be identical with $ax^2 + bx + 1 = 0$;

$\therefore \quad a = 13$ and $b = -8$.

16 Sum of roots

$$\frac{-a}{1} = (1 - i) + (1 + i) \Rightarrow a = -2.$$

[since, non-real complex roots occur in conjugate pairs]

Product of roots,

$$\frac{b}{1} = (1 - i)(1 + i) \Rightarrow b = 2$$

17 Let roots be α and α^2.

Then, $\alpha + \alpha^2 = 30$ and $\alpha^3 = p$

$\Rightarrow \quad \alpha^2 + \alpha - 30 = 0$

$\Rightarrow \quad (\alpha + 6)(\alpha - 5) = 0$

$\therefore \quad \alpha = -6, 5$

$\Rightarrow \quad p = \alpha^3 = (-6)^3 = -216$

and $\quad p = (5)^3 = 125$

$\therefore \quad p = 125$ and -216

18 Given, $\dfrac{x - m}{mx + 1} = \dfrac{x + n}{nx + 1}$

$\Rightarrow x^2(m - n) + 2mnx + (m + n) = 0$

Roots are α, $\dfrac{1}{\alpha}$ respectively, then

$$\alpha \cdot \frac{1}{\alpha} = \frac{m + n}{m - n}$$

$\Rightarrow m - n = m + n \Rightarrow n = 0.$

19 Since, α, α^2 be the roots of the equation

$$x^2 + x + 1 = 0$$

$\therefore \quad \alpha + \alpha^2 = -1 \quad \ldots(i)$

and $\alpha^3 = 1 \quad \ldots(ii)$

Now, $\quad \alpha^{31} + \alpha^{62} = \alpha^{31}(1 + \alpha^{31})$

$\Rightarrow \quad \alpha^{31} + \alpha^{62} = \alpha^{30} \cdot \alpha(1 + \alpha^{30} \cdot \alpha)$

$\Rightarrow \quad \alpha^{31} + \alpha^{62} = (\alpha^3)^{10} \cdot \alpha\{1 + (\alpha^3)^{10} \cdot \alpha\}$

$\Rightarrow \quad \alpha^{31} + \alpha^{62} = \alpha(1 + \alpha) \quad$ [from Eq. (ii)]

$\Rightarrow \quad \alpha^{31} + \alpha^{62} = -1 \quad$ [from Eq. (i)]

Again, $\quad \alpha^{31} \cdot \alpha^{62} = \alpha^{93}$

$\Rightarrow \quad \alpha^{31} \cdot \alpha^{62} = [\alpha^3]^{31} = 1$

$\therefore$ Required equation is

$$x^2 - (\alpha^{31} + \alpha^{62})x + \alpha^{31} \cdot \alpha^{62} = 0$$

$\Rightarrow \quad x^2 + x + 1 = 0$

20 Since, α and β are the roots of $x^2 - ax + a + b = 0$, then

$$\alpha + \beta = a$$

and $\quad \alpha\beta = a + b$

$\Rightarrow \quad \alpha^2 + \alpha\beta = a\alpha$

$\Rightarrow \quad \alpha^2 - a\alpha = -(a + b)$

and $\quad \alpha\beta + \beta^2 = a\beta$

$\Rightarrow \quad \beta^2 - a\beta = -(a + b)$

$\therefore \quad \dfrac{1}{\alpha^2 - a\alpha} + \dfrac{1}{\beta^2 - a\beta} + \dfrac{2}{a + b}$

$$= \frac{1}{-(a + b)} + \frac{1}{-(a + b)} + \frac{2}{(a + b)} = 0$$

21 Let α and β be the roots of equation

$$x^2 - (a - 2)x - a - 1 = 0$$

Then, $\quad \alpha + \beta = a - 2$ and $\alpha\beta = -a - 1$

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\Rightarrow \quad \alpha^2 + \beta^2 = (a - 2)^2 + 2(a + 1)$

$\Rightarrow \quad \alpha^2 + \beta^2 = a^2 - 2a + 6$

$\Rightarrow \quad \alpha^2 + \beta^2 = (a - 1)^2 + 5$

The value of $\alpha^2 + \beta^2$ will be least, if

$$a - 1 = 0$$

$\Rightarrow \quad a = 1$

22 Since, α and β are the roots of the equation

$$2x^2 + 2(a + b)x + a^2 + b^2 = 0$$

$\therefore \quad (\alpha + \beta)^2 = (a + b)^2$ and $\alpha\beta = \dfrac{a^2 + b^2}{2}$

Now, $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$= (a + b)^2 - 4\left(\frac{a^2 + b^2}{2}\right)$$
$$= -(a - b)^2$$

Now, the required equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$ is

$$x^2 - \{(\alpha + \beta)^2 + (\alpha - \beta)^2\}x + (\alpha + \beta)^2(\alpha - \beta)^2 = 0$$

$\Rightarrow \quad x^2 - \{(a + b)^2 - (a - b)^2\}x - (a + b)^2(a - b)^2 = 0$

$\Rightarrow \quad x^2 - 4abx - (a^2 - b^2)^2 = 0$

23 The given equation is

$x^2 - 2x\cos\phi + 1 = 0$

$\therefore \quad x = \dfrac{2\cos\phi \pm \sqrt{4\cos^2\phi - 4}}{2}$

$= \cos\phi \pm i\sin\phi$

Let $\alpha = \cos\phi + i\sin\phi$, then

$\beta = \cos\phi - i\sin\phi$

$\therefore \alpha^n + \beta^n = (\cos\phi + i\sin\phi)^n + (\cos\phi - i\sin\phi)^n$

$= 2\cos n\phi$

and $\alpha^n\beta^n = (\cos n\phi + i\sin n\phi) \cdot (\cos n\phi - i\sin n\phi)$

$= \cos^2 n\phi + \sin^2 n\phi = 1$

$\therefore$ Required equation is

$$x^2 - 2x\cos n\phi + 1 = 0$$

24 Given equation is

$(5 + \sqrt{2})x^2 - (4 + \sqrt{5})x + 8 + 2\sqrt{5} = 0$

Let x_1 and x_2 are the roots of the equation, then

$$x_1 + x_2 = \frac{4 + \sqrt{5}}{5 + \sqrt{2}} \quad \ldots(i)$$

and $\quad x_1x_2 = \dfrac{8 + 2\sqrt{5}}{5 + \sqrt{2}} = \dfrac{2(4 + \sqrt{5})}{5 + \sqrt{2}}$

$= 2(x_1 + x_2) \quad \ldots(ii)$

Clearly, harmonic mean

$$= \frac{2x_1x_2}{x_1 + x_2} = \frac{4(x_1 + x_2)}{(x_1 + x_2)} = 4$$

[from Eq. (ii)]

25 Let α, β and α', β' be the roots of the given equations, respectively.

$\therefore \alpha + \beta = -\dfrac{\mu}{\lambda}, \alpha\beta = \dfrac{\nu}{\lambda} \quad \ldots(i)$

and $\alpha' = \omega$ and $\beta' = \omega^2$

$\because \quad \dfrac{\alpha}{\beta} = \dfrac{\alpha'}{\beta'} \quad$ [given]

$\Rightarrow \quad \dfrac{\alpha}{\beta} = \dfrac{\omega}{\omega^2} \Rightarrow \beta = \alpha\omega$

From Eq. (i),

$$\alpha + \alpha\omega = -\frac{\mu}{\lambda}, \alpha^2\omega = \frac{\nu}{\lambda}$$

$$\Rightarrow \quad -\alpha\omega^2 = -\frac{\mu}{\lambda}, \alpha^2\omega = \frac{\nu}{\lambda}$$

$$[\because -\omega^2 = 1 + \omega]$$

$$\Rightarrow \quad \frac{\mu^2}{\lambda^2} = \frac{\nu}{\lambda} \Rightarrow \mu^2 = \lambda\nu$$

26 Simplified form of given equation is

$$(2x + p + q)r = (x + p)(x + q)$$

$$\Rightarrow x^2 + (p + q - 2r)x - (p + q)r + pq = 0$$

Since, sum of roots = 0

$$\Rightarrow -(p + q - 2r) = 0$$

$$\Rightarrow \quad r = \frac{p + q}{2}$$

and product of roots

$$= -(p + q)r + pq$$

$$= -\frac{(p + q)^2}{2} + pq$$

$$= -\frac{1}{2}(p^2 + q^2)$$

27 Clearly, sum of roots,

$$\frac{k + 1}{k} + \frac{k + 2}{k + 1} = -\frac{b}{a} \quad \text{... (i)}$$

and product of roots,

$$\frac{k + 1}{k} \times \frac{k + 2}{k + 1} = \frac{c}{a}$$

$$\Rightarrow \quad \frac{k + 2}{k} = \frac{c}{a}$$

$$\Rightarrow \quad \frac{2}{k} = \frac{c}{a} - 1 = \frac{c - a}{a} \Rightarrow k = \frac{2a}{c - a}$$

On putting the value of k in Eq. (i), we get

$$\frac{c + a}{2a} + \frac{2c}{c + a} = -\frac{b}{a}$$

$$\Rightarrow \quad (c + a)^2 + 4ac = -2b(a + c)$$

$$\Rightarrow \quad (a + c)^2 + 2b(a + c) = -4ac$$

$$\Rightarrow (a + c)^2 + 2b(a + c) + b^2 = b^2 - 4ac$$

$$\Rightarrow \quad (a + b + c)^2 = b^2 - 4ac$$

28 Since, α and β are the roots of the equation $ax^2 + bx + c = 0$

$$\therefore \quad \alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

Now, sum of roots $= |\alpha| + |\beta|$

$$= -\alpha - \beta \quad (\because \beta < \alpha < 0)$$

$$= -\left(-\frac{b}{a}\right) = \left|\frac{b}{a}\right| \quad (\because |\alpha| + |\beta| > 0)$$

and product of roots $= |\alpha||\beta| = \left|\frac{c}{a}\right|$

Hence, required eqution is

$$x^2 - \left|\frac{b}{a}\right|x + \left|\frac{c}{a}\right| = 0$$

$$\Rightarrow \quad |a|x^2 - |b|x + |c| = 0$$

29 Since, α and β are the roots of the equation $x^2 + px + q = 0$ therefore $\alpha + \beta = -p$ and $\alpha\beta = q$

Now, $(\omega\alpha + \omega^2\beta)(\omega^2\alpha + \omega\beta)$

$$= \alpha^2 + \beta^2 + (\omega^4 + \omega^2)\alpha\beta \quad (\because \omega^3 = 1)$$

$$= \alpha^2 + \beta^2 - \alpha\beta \quad (\because \omega + \omega^2 = -1)$$

$$= (\alpha + \beta)^2 - 3\alpha\beta = p^2 - 3q$$

Also, $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta}$

$$= \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{\alpha\beta} = \frac{p(3q - p^2)}{q}$$

$\therefore$ The given expression

$$= \frac{(p^2 - 3q)}{\frac{p(3q - p^2)}{q}} = -\frac{q}{p}$$

30 Clearly, $\alpha + \beta = -p$ and $\alpha\beta = \frac{3p}{4}$

Also, $\quad (\alpha - \beta)^2 = 10$

$$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = 10$$

$$\Rightarrow \quad p^2 - 3p = 10$$

$$\Rightarrow \quad (p + 2)(p - 5) = 0$$

$$\therefore \quad p = -2, 5$$

31 Let the quadratic equation be $ax^2 + bx + c = 0$ and its roots are α and β.

Sachin made a mistake in writing down constant term.

$\therefore$ Sum of roots is correct i.e. $\alpha + \beta = 7$

Rahul made mistake in writing down coefficient of x.

$\therefore$ Product of roots is correct.

i.e. $\alpha \cdot \beta = 6$

$\Rightarrow$ Correct quadratic equation is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0.$$

$\Rightarrow$ $x^2 - 7x + 6 = 0$ having roots 1 and 6.

32 Since, $\alpha + \beta = -1, \alpha\beta = 2$,

$\gamma + \delta = -3$, and $\gamma\delta = 4$

$$\therefore (\alpha + \gamma)(\alpha + \delta)(\beta + \gamma)(\beta + \delta)$$

$$= (\alpha^2 - 3\alpha + 4)(\beta^2 - 3\beta + 4)$$

$$= 4 - 3\alpha\beta^2 + 4\beta^2 - 3\alpha^2\beta + 9\alpha\beta - 12\beta + 4\alpha^2 - 12\alpha + 16$$

$$= 4 - 3(2)\beta + 4\beta^2 + 4\alpha^2 - 3(2)\alpha + 9(2) - 12(\beta + \alpha) + 16$$

$$= 4 - 6\beta + 4(\alpha^2 + \beta^2) - 6\alpha + 18 + 12 + 16$$

$$= 50 + 6 + 4[(\alpha + \beta)^2 - 2\alpha\beta]$$

$$= 56 - 12 = 44$$

33 Given, $R = \frac{\pi}{2} \Rightarrow P + Q = \frac{\pi}{2}$

$$\Rightarrow \frac{P}{2} + \frac{Q}{2} = \frac{\pi}{4}$$

$$\Rightarrow \quad 1 = \tan\frac{\pi}{4} = \frac{\tan\frac{P}{2} + \tan\frac{Q}{2}}{1 - \tan\frac{P}{2}\tan\frac{Q}{2}}$$

$$\Rightarrow \quad 1 = \frac{-\frac{b}{a}}{1 - \frac{c}{a}} = \frac{b}{c - a} \Rightarrow a + b = c$$

34 Given, $(6k + 2)x^2 + rx + 3k - 1 = 0$

and $(12k + 4)x^2 + px + 6k - 2 = 0$

For both common roots,

$$\frac{6k + 2}{12k + 4} = \frac{r}{p} = \frac{3k - 1}{6k - 2}$$

$$\Rightarrow \quad \frac{r}{p} = \frac{1}{2} \Rightarrow 2r - p = 0$$

35 Given equations are

$$x^2 + 2x + 3 = 0 \quad \text{... (i)}$$

and $\quad ax^2 + bx + c = 0 \quad$ ... (ii)

Since, Eq. (i) has imaginary roots.

So, Eq. (ii) will also have both roots same as Eq. (i)

Thus, $\frac{a}{1} = \frac{b}{2} = \frac{c}{3}$

Hence, $a:b:c$ is 1:2:3.

36 α, β be the roots of $ax^2 + bx + c = 0$

$\therefore \quad \alpha + \beta = -b/a, \alpha\beta = c/a$

Now roots are $\alpha - 1, \beta - 1$

Their sum, $\alpha + \beta - 2 = (-b/a) - 2$

$$= -8/2 = -4$$

Their product,

$$(\alpha - 1)(\beta - 1) = \alpha\beta - (\alpha + \beta) + 1$$

$$= c/a + b/a + 1 = 1$$

$\therefore b/a = 2$ i.e. $b = 2a$

also $c + b = 0 \Rightarrow b = -c$.

37 $\min f(x) = -\frac{D}{4a} = -\frac{4b^2 - 8c^2}{4}$

$$= -(b^2 - 2c^2)$$

(upward parabola)

$\max g(x) = -\frac{D}{4a} = \frac{4c^2 + 4b^2}{4}$

$$= b^2 + c^2$$

(downward parabola)

Now, $\quad 2c^2 - b^2 > b^2 + c^2$

$$\Rightarrow \quad c^2 > 2b^2 \Rightarrow |c| > \sqrt{2}\,|b|$$

38 We have,

$$(x - a_1)^2 + (x - a_2)^2 + \dots + (x - a_n)^2$$

$$= n\,x^2 - 2x(a_1 + a_2 + \dots + a_n) + (a_1^2 + a_2^2 + \dots + a_n^2)$$

So, it attains its minimum value at

$$x = \frac{2(a_1 + a_2 + \dots + a_n)}{2n}$$

$$= \frac{a_1 + a_2 + \dots + a_n}{n} \quad \left[\text{using} : x = \frac{-b}{2a}\right]$$

39 Given $bx^2 + cx + a = 0$ has imaginary roots.

$$\Rightarrow \quad c^2 - 4ab < 0 \Rightarrow c^2 < 4ab \quad \text{...(i)}$$

Let $f(x) = 3b^2x^2 + 6bcx + 2c^2$

Here, $3b^2 > 0$

So, the given expression has a minimum value.

$\therefore$ Minimum value $= \frac{-D}{4a}$

$$= \frac{4ac - b^2}{4a} = \frac{4(3b^2)(2c^2) - 36b^2c^2}{4(3b^2)}$$

$$= -\frac{12b^2c^2}{12b^2} = -c^2 > -4ab$$

[from Eq. (i)]

40 According to given condition,

$$4a^2 - 4(10 - 3a) < 0$$
$$\Rightarrow a^2 + 3a - 10 < 0$$
$$\Rightarrow (a+5)(a-2) < 0$$
$$\Rightarrow -5 < a < 2.$$

41 We have, $ax - 1 + \frac{1}{x} \geq 0$

$$\Rightarrow \frac{ax^2 - x + 1}{x} \geq 0$$

$\Rightarrow ax^2 - x + 1 \leq 0$ as $x > 0$

It will hold if $a > 0$ and $D \leq 0$

$a > 0$ and $1 - 4a \leq 0 \Rightarrow a \geq \frac{1}{4}$

Therefore, the minimum value of a is $\frac{1}{4}$.

42 Let $f(x) = -3 + x - x^2$.

Then, $f(x) < 0$ for all x, because coeff. of $x^2 < 0$ and disc. < 0.

Thus, LHS of the given equation is always positive whereas the RHS is always less than zero. Hence, the given equation has no solution.

43 Let $f(x) = ax^2 + bx + c$.

Then, k lies between α and β, if a $f(k) < 0$

$$\Rightarrow a(ak^2 + bk + c) < 0$$
$$\Rightarrow a^2k^2 + abk + ac < 0.$$

44 We have, $|2x - 3| < |x + 5|$

$$\Rightarrow |2x-3| - |x+5| < 0$$

$$\Rightarrow \begin{cases} 3 - 2x + x + 5 < 0, & x \leq -5 \\ 3 - 2x - x - 5 < 0, & -5 < x \leq \frac{3}{2} \\ 2x - 3 - x - 5 < 0, & x > \frac{3}{2} \end{cases}$$

$$\Rightarrow \begin{cases} x > 8, & x \leq -5 \\ x > -\frac{2}{3}, & -5 < x \leq \frac{3}{2} \\ x < 8, & x > \frac{3}{2} \end{cases}$$

$$\Rightarrow x \in \left(-\frac{2}{3}, \frac{3}{2}\right] \cup \left(\frac{3}{2}, 8\right)$$

$$\Rightarrow x \in \left(-\frac{2}{3}, 8\right)$$

45

$$\frac{x-5}{x^2 + 5x - 14} > 0$$

$$\Rightarrow \frac{(x+7)(x-2)(x-5)}{(x-2)^2(x+7)^2} > 0$$

$\Rightarrow x \in (-7, 2) \cup (5, \infty)$

So, the least integral value α of x is -6, which satisfy the equation

$$\alpha^2 + 5\alpha - 6 = 0$$

46 Given, $\frac{|x-2|-1}{|x-2|-2} \leq 0$

Let $|x - 2| = k$

Then, given equation,

$$\frac{k-1}{k-2} \leq 0 \Rightarrow \frac{(k-1)(k-2)}{(k-2)^2} \leq 0$$

$$\Rightarrow (k-1)(k-2) \leq 0 \Rightarrow 1 \leq k \leq 2$$

$$\Rightarrow 1 \leq |x-2| \leq 2$$

***Case* I** When $1 \leq |x - 2|$

$$\Rightarrow |x-2| \geq 1$$

$\Rightarrow$ $x - 2 \geq 1$ or $x - 2 \leq -1$

$\Rightarrow$ $x \geq 3$ and $x \leq 1$ ...(i)

***Case* II** When $|x - 2| \leq 2$

$$\Rightarrow -2 \leq x - 2 \leq 2$$
$$\Rightarrow -2 + 2 \leq x \leq 2 + 2$$

$\Rightarrow$ $0 \leq x \leq 4$ ...(ii)

From (i) and (ii), $x \in [0, 1] \cup [3, 4]$

47

$$\frac{x+2}{x^2+1} - \frac{1}{2} > 0$$

$$\Rightarrow \frac{-x^2 - 1 + 2x + 4}{2(x^2+1)} > 0$$

$$\Rightarrow \frac{3 + 2x - x^2}{2(x^2+1)} > 0$$

Since, denominator is positive

$\therefore 3 + 2x - x^2 > 0$

$$\Rightarrow -1 < x < 3$$
$$\Rightarrow x = 0, 1, 2$$

48 Let $a_1, a_2, \dots, a_n$ be n positive integers such that $a_1 a_2 \dots a_n = 1$.

Using AM $\geq$ GM, we have

$$\Rightarrow \frac{a_1 + a_2 + \dots + a_n}{n} \geq (a_1 a_2 \dots a_n)^{1/n}$$

$\Rightarrow a_1 + a_2 + \dots a_n \geq n.$

49 Since, $AM \geq GM$

$$\therefore \frac{bcx + cay + abz}{3} \geq (a^2b^2c^2 \cdot xyz)^{1/3}$$

$\Rightarrow bcx + cay + abz \geq 3abc$

[$\because xyz = abc$]

50 We know that, $AM > GM$

$\therefore \frac{a+b+c}{3} > (abc)^{1/3}$ ... (i)

and $\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3} > \left(\frac{1}{abc}\right)^{1/3}$

$\Rightarrow \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) > \frac{3}{(abc)^{1/3}}$ ... (ii)

From (i) and (ii), we get

$$\left(\frac{a+b+c}{3}\right)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) > \frac{3}{(abc)^{1/3}} \cdot (abc)^{1/3}$$

$$\Rightarrow (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) > 9$$

51 Using GM $\leq$ AM, we have

$$\{(1-a)(1-b)(1-c)^{1/3}\} \leq \frac{1-a+1-b+1-c}{3}$$

$$\Rightarrow (1-a)(1-b)(1-c) \leq \frac{8}{27}$$

Hence, the greatest value is $\frac{8}{27}$.

52 Using AM $\geq$ GM, we have

$$\frac{(a+b)+(c+d)}{2} \geq \{(a+b)(c+d)\}^{1/2}$$

$$\Rightarrow \frac{2}{2} \geq M^{1/2} \Rightarrow M \leq 1$$

As $a, b, c, d > 0$.

Therefore, $M = (a+b)(c+d) > 0$.

Hence, $0 \leq M \leq 1$.

53 $\log_2(x^2 - 3x + 18) < 4$

$$\Rightarrow x^2 - 3x + 18 < 16$$

($\because \log_a b < c \Leftrightarrow b < a^c$, if $a > 1$)

$$\Rightarrow x^2 - 3x + 2 < 0$$
$$\Rightarrow (x-1)(x-2) < 0 \Rightarrow x \in (1, 2)$$

54. Clearly, $x - 1 > 0 \Rightarrow x > 1$

and $\log_{0.3}(x-1) > \log_{(0.3)^2}(x-1)$

$$\Rightarrow \log_{0.3}(x-1) > \frac{1}{2}\log_{0.3}(x-1)$$

$$\Rightarrow \log_{0.3}(x-1) > 0 \Rightarrow x < 2$$

Hence, $x \in (1, 2)$

55 By definition of log x, $x > 0$ and

$$\left(\frac{x+5}{1-3x}\right) > 0$$

$$\Rightarrow \frac{(x+5)(1-3x)}{(1-3x)^2} > 0$$

$$\Rightarrow (x+5)(1-3x) > 0$$
$$\Rightarrow (x+5)(3x-1) < 0$$
$$\Rightarrow -5 < x < 1/3$$

As $x > 0$, $0 < x < 1/3$

$$\therefore \frac{x+5}{1-3x} < 1 \Rightarrow x < -1$$

This does not satisfy $0 < x < 1/3$.

Hence, there is no solution.

SESSION 2

1 We have

$$2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0$$

Let $\sqrt{x} - 3 = y$

$$\Rightarrow \sqrt{x} = y + 3$$

$$\therefore 2|y| + (y+3)(y-3) + 6 = 0$$
$$\Rightarrow 2|y| + y^2 - 3 = 0$$
$$\Rightarrow |y|^2 + 2|y| - 3 = 0$$
$$\Rightarrow (|y| + 3)(|y| - 1) = 0$$
$$\Rightarrow |y| \neq -3 \Rightarrow |y| = 1$$

$\Rightarrow \quad y = \pm 1$

$\Rightarrow \quad \sqrt{x} - 3 = \pm 1$

$\Rightarrow \quad \sqrt{x} = 4, 2$

$\Rightarrow \quad x = 16, 4$

2 We have, $2^{x+2} \cdot 3^{3x/(x-1)} = 9$

Taking log on both sides, we get

$$(x+2)\log 2 + \frac{3x}{(x-1)}\log 3 = 2\log 3$$

$$\Rightarrow (x^2 + x - 2)\log 2 + 3x\log 3 = 2(x-1)\log 3$$

$$\Rightarrow x^2 \log 2 + (\log 2 + \log 3)x - 2\log 2 + 2\log 3 = 0$$

$$\therefore x = \frac{-(\log 2 + \log 3) \pm \sqrt{\{(\log 2 + \log 3)^2 - 4\log 2 (-2\log 2 + 2\log 3)\}}}{2\log 2}$$

$$= \frac{-(\log 2 + \log 3) \pm \sqrt{\{(3\log 2)^2 - 6\log 2 \log 3 + (\log 3)^2\}}}{2\log 2}$$

$$= \frac{-(\log 2 + \log 3) \pm (3\log 2 - \log 3)}{2\log 2}$$

$$\therefore x = -2, 1 - \frac{\log 3}{\log 2}$$

3 Given, α and β are the roots of the equation $x^2 - 6x - 2 = 0$

$\because a_n = \alpha^n - \beta^n, n \geq 1 \Rightarrow a_{10} = \alpha^{10} - \beta^{10}$

$a_8 = \alpha^8 - \beta^8$ and $a_9 = \alpha^9 - \beta^9$

Now, consider

$$\frac{a_{10} - 2a_8}{2a_9}$$

$$= \frac{\alpha^{10} - \beta^{10} - 2(\alpha^8 - \beta^8)}{2(\alpha^9 - \beta^9)}$$

$$= \frac{\alpha^8(\alpha^2 - 2) - \beta^8(\beta^2 - 2)}{2(\alpha^9 - \beta^9)}$$

$$= \frac{\alpha^8 \cdot 6\alpha - \beta^8 \cdot 6\beta}{2(\alpha^9 - \beta^9)}$$

$$= \frac{6\alpha^9 - 6\beta^9}{2(\alpha^9 - \beta^9)} = \frac{6}{2} = 3$$

[$\because \alpha$ and β are the roots of $x^2 - 6x - 2 = 0$ or $x^2 = 6x + 2$ $\Rightarrow \alpha^2 = 6\alpha + 2$ $\Rightarrow \alpha^2 - 2 = 6\alpha$ and $\beta^2 = 6\beta + 2$ $\Rightarrow \beta^2 - 2 = 6\beta$]

4 We have, $x^2 - 5x + 1 = 0$

$$x + \frac{1}{x} = 5 \Rightarrow x^2 + \frac{1}{x^2} = 5^2 - 2 = 23$$

and $$x^3 + \frac{1}{x^3} = 5^3 - 3 \times 5 = 110$$

Now, $$\left(x^2 + \frac{1}{x^2}\right)\left(x^3 + \frac{1}{x^3}\right) = 23 \times 110 = 2530$$

$$\Rightarrow x^5 + \frac{1}{x^5} = 2530 - \left(x + \frac{1}{x}\right) = 2525$$

5 Given equation can be rewritten as

$3x^2 - (a + c + 2b + 2d)x + ac + 2bd = 0$

Now, discriminant, D

$= (a + c + 2b + 2d)^2 - 4 \cdot 3(ac + 2bd)$

$= \{(a + 2d) + (c + 2b)\}^2 - 12(ac + 2bd)$

$= \{(a + 2d) - (c + 2b)\}^2 + 4(a + 2d)(c + 2b) - 12(ac + 2bd)$

$= \{(a + 2d) - (c + 2b)\}^2 - 8ac + 8ab + 8dc - 8bd$

$= \{(a + 2d) - (c + 2b)\}^2 + 8(c - b)(d - a)$

Which is +ve, since $a < b < c < d$.

Hence, roots are real and distinct.

6 Since, α and β are roots of

$px^2 + qx + r = 0, p \neq 0$

$$\therefore \quad \alpha + \beta = \frac{-q}{p}, \alpha\beta = \frac{r}{p}$$

Since, p, q and r are in AP.

$\therefore \quad 2q = p + r$

Also, $$\frac{1}{\alpha} + \frac{1}{\beta} = 4 \Rightarrow \frac{\alpha + \beta}{\alpha\beta} = 4$$

$$\Rightarrow \alpha + \beta = 4\alpha\beta \Rightarrow \frac{-q}{p} = \frac{4r}{p}$$

$\because \quad 2q = p + r$

$\Rightarrow \quad 2(-4r) = p + r \Rightarrow p = -9r$

$$\because \quad \alpha + \beta = \frac{-q}{p} = \frac{4r}{p} = \frac{4r}{-9r} = -\frac{4}{9}$$

and $$\alpha\beta = \frac{r}{p} = \frac{r}{-9r} = \frac{1}{-9}$$

Now, consider

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \frac{16}{81} + \frac{4}{9} = \frac{16 + 36}{81}$$

$$\Rightarrow (\alpha - \beta)^2 = \frac{52}{81}$$

$$\Rightarrow |\alpha - \beta| = \frac{2}{9}\sqrt{13}$$

7 $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$

Now, $(1 + \alpha + \alpha^2)(1 + \beta + \beta^2)$

$= 1 + (\alpha + \beta) + (\alpha^2 + \beta^2) + \alpha\beta + \alpha\beta(\alpha + \beta) + (\alpha\beta)^2$

$= 1 + (\alpha + \beta) + (\alpha + \beta)^2 - \alpha\beta + \alpha\beta(\alpha + \beta) + (\alpha\beta)^2$

$$= 1 - \frac{b}{a} + \frac{b^2}{a^2} - \frac{c}{a} + \left(\frac{c}{a}\right)\left(-\frac{b}{a}\right) + \frac{c^2}{a^2}$$

$$= \frac{(a^2 + b^2 + c^2 - ab - bc - ca)}{a^2}$$

$= [(a - b)^2 + (b - c)^2 + (c - a)^2] / 2a^2$

which is positive.

8 Using AM $\geq$ GM

$$\therefore \frac{a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + 1 + a^8 + a^{10}}{8}$$

$\geq (a^{-5} \cdot a^{-4} \cdot a^{-3} \cdot a^{-3} \cdot a^{-3} \cdot 1 \cdot a^8 \cdot a^{10})^{1/8}$

$\Rightarrow a^{-5} + a^{-4} + 3a^{-3} + 1 + a^8 + a^{10} \geq 8 \cdot 1$

Hence, minimum value is 8.

9 We have,

$$\frac{\log_a a}{\log_a a + \log_a x} + \frac{2\log_a a}{\log_a x} + \frac{3\log_a a}{2\log_a a + \log_a x} = 0$$

$$\Rightarrow \frac{1}{1 + t} + \frac{2}{t} + \frac{3}{2 + t} = 0$$

(let $\log_a x = t$)

$$\Rightarrow \frac{2t + t^2 + 2t^2 + 6t + 4 + 3t^2 + 3t}{t(1 + t)(2 + t)} = 0$$

$\Rightarrow \quad 6t^2 + 11t + 4 = 0$

$\Rightarrow \quad 6t^2 + 8t + 3t + 4 = 0$

$\Rightarrow \quad (2t + 1)(3t + 4) = 0$

$$\Rightarrow \quad t = -\frac{1}{2}, -\frac{4}{3}$$

$$\Rightarrow \quad \log_a x = -\frac{1}{2}, -\frac{4}{3}$$

$\therefore \quad x = a^{-1/2}, a^{-4/3}$

10 Since, $x = 1$ is a root of first equation. If α is another root of first equation, then

$$\alpha \times 1 = \alpha = \frac{a - b}{b - c}$$

(Product of roots)

$$= \frac{2a - 2b}{2b - 2c} = \frac{2a - (a + c)}{a + c - 2c} = 1$$

So, the roots of first equation are 1 and 1.

Since, the equations have a common root, 1 is also a root of second equation.

$\Rightarrow \quad 2(c + a) + b + c = 0$

$\Rightarrow \quad 2(2b) + b + c = 0$

[since, a, b and c are in AP]

$\Rightarrow \quad c = -5b$

Also, $\quad a + c = 2b \Rightarrow a = 2b - c = 2b + 5b = 7b$

$\therefore \quad a^2 = 49b^2, c^2 = 25b^2$

Hence, a^2, c^2 and b^2 are in AP.

11 On adding first and second equations, we get

$2x^2 + (a + b)x + 27 = 0$

On subtracting above equation from given third equation, we get

$x^2 - 9 = 0 \Rightarrow x = 3, -3$

Thus, common positive root is 3.

Now, $(3)^2 + 3a + 12 = 0 \Rightarrow a = -7$

and $9 + 3b + 15 = 0 \Rightarrow b = -8$

Hence, the order pair (a, b) is $(-7, -8)$.

12 Let $$y = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$$

$\Rightarrow \quad (y - 1)x^2 + 3(y + 1)x + 4(y - 1) = 0$

$\because x$ is real.

$\therefore \quad D \geq 0$

$\Rightarrow \quad 9(y + 1)^2 - 16(y - 1)^2 \geq 0$

$\Rightarrow \quad -7y^2 + 50y - 7 \geq 0$

$\Rightarrow \quad 7y^2 - 50y + 7 \leq 0$

$\Rightarrow \quad (y - 7)(7y - 1) \leq 0$

$\Rightarrow \quad y \le 7$ and $y \ge \frac{1}{7}$

$\Rightarrow \quad \frac{1}{7} \le y \le 7$

Hence, maximum value is 7 and minimum value is $\frac{1}{7}$.

13 Here, $a \in R$ and equation is

$-3\{x-[x]\}^2 + 2\{x-[x]\} + a^2 = 0$

Let $t = x - [x]$, then

$-3t^2 + 2t + a^2 = 0$

$\Rightarrow \quad t = \frac{1 \pm \sqrt{1+3a^2}}{3}$

$\because \quad t = x - [x] = \{x\}$ [fractional part]

$\therefore \quad 0 \le t \le 1$

$\Rightarrow \quad 0 \le \frac{1 \pm \sqrt{1+3a^2}}{3} < 1$ $[\because 0 \le \{x\} < 1]$

But $1 - \sqrt{1+3a^2} < 0$ therefore

$0 \le \frac{1 + \sqrt{1+3a^2}}{3} < 1$

$\Rightarrow \quad \sqrt{1+3a^2} < 2$

$\Rightarrow \quad 1 + 3a^2 < 4 \Rightarrow a^2 - 1 < 0$

$\Rightarrow \quad (a+1)(a-1) < 0$

$\Rightarrow \quad a \in (-1, 1)$

For no integral solution we consider the interval $(-1, 0) \cup (0, 1)$.

14 Given, $a = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$

$\therefore \quad a^7 = \cos 2\pi + i \sin 2\pi = 1$ $[\because e^{i\theta} = \cos\theta + i \sin\theta]$

Also, $\alpha = a + a^2 + a^4$,

$\beta = a^3 + a^5 + a^6$

Then, the sum of roots,

$S = \alpha + \beta = a + a^2 + a^3 + a^4 + a^5 + a^6$

$\Rightarrow \quad S = \frac{a(1-a^6)}{1-a} = \frac{a - a^7}{1-a}$

$= \frac{a-1}{1-a} = -1$ $[\because a^7 = 1]$

and product of the roots,

$P = \alpha\beta = (a + a^2 + a^4)(a^3 + a^5 + a^6)$

$= a^4 + a^5 + 1 + a^6 + 1 + a^2 + 1 + a + a^3$ $[\because a^7 = 1]$

$= 3 + (a + a^2 + a^3 + a^4 + a^5 + a^6)$

$= 3 - 1 = 2$

Hence, the required quadratic equation is $x^2 + x + 2 = 0$

15 Since, α and β are the roots of $375x^2 - 25x - 2 = 0$.

$\therefore \quad \alpha + \beta = \frac{25}{375} = \frac{1}{15}$

and $\quad \alpha\beta = -\frac{2}{375}$

Now, consider

$\lim_{n\to\infty} \sum_{r=1}^{n} S_r = \lim_{n\to\infty} \sum_{r=1}^{n} (\alpha^r + \beta^r)$

$= (\alpha + \alpha^2 + \alpha^3 + \dots \infty) + (\beta + \beta^2 + \beta^3 + \dots \infty)$

$= \frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta} = \frac{\alpha - \alpha\beta + \beta - \alpha\beta}{(1-\alpha)(1-\beta)}$

$= \frac{\alpha + \beta - 2\alpha\beta}{1 - (\alpha+\beta) + \alpha\beta}$

$= \frac{\frac{1}{15} + \frac{4}{375}}{1 - \frac{1}{15} - \frac{2}{375}}$

$= \frac{25 + 4}{375 - 25 - 2}$

$= \frac{29}{348} = \frac{1}{12}$

16 Here, $[\tan^2 x]$ = integer

and a = integer

So, $\tan x$ is also an integer.

Then, $\tan^2 x - \tan x - a = 0$

$\Rightarrow \quad a = \tan x (\tan x - 1) = I(I-1)$

= Product of two consecutive integers

$\therefore \quad a = 2, 6, 12, 20, 30, 42, 56, 72, 90$

Hence, set S has 9 elements.

17 Given, $(x^2 - 5x + 5)^{x^2+4x-60} = 1$

Clearly, this is possible when

I. $x^2 + 4x - 60 = 0$ and $x^2 - 5x + 5 \ne 0$

or

II. $x^2 - 5x + 5 = 1$

or

III. $x^2 - 5x + 5 = -1$ and $x^2 + 4x - 60 =$ Even integer

Case I When $x^2 + 4x - 60 = 0$

$\Rightarrow \quad x^2 + 10x - 6x - 60 = 0$

$\Rightarrow \quad x(x+10) - 6(x+10) = 0$

$\Rightarrow \quad (x+10)(x-6) = 0$

$\Rightarrow \quad x = -10$ or $x = 6$

Note that, for these two values of x, $x^2 - 5x + 5 \ne 0$

Case II When $x^2 - 5x + 5 = 1$

$\Rightarrow \quad x^2 - 5x + 4 = 0$

$\Rightarrow \quad x^2 - 4x - x + 4 = 0$

$\Rightarrow \quad (x-4)(x-1) = 0$

$\Rightarrow \quad x = 4$ or $x = 1$

Case III When $x^2 - 5x + 5 = -1$

$\Rightarrow \quad x^2 - 5x + 6 = 0$

$\Rightarrow \quad x^2 - 2x - 3x + 6 = 0$

$\Rightarrow \quad x(x-2) - 3(x-2) = 0$

$\Rightarrow \quad (x-2)(x-3) = 0$

$\Rightarrow \quad x = 2$ or $x = 3$

Now, when $x = 2$,

$x^2 + 4x - 60 = 4 + 8 - 60 = -48$, which is an even integer.

When $x = 3$,

$x^2 + 4x - 60 = 9 + 12 - 60 = -39$, which is not an even integer.

Thus, in this case, we get $x = 2$

Hence, the sum of all real values of

$x = -10 + 6 + 4 + 1 + 2 = 3$

18 Given, $4x^2 - 16x + \frac{\lambda}{4} = 0$

$\therefore \quad x = \frac{16 \pm \sqrt{(256 - 4\lambda)}}{8}$

$= \frac{8 \pm \sqrt{(64 - \lambda)}}{4}$

$\Rightarrow \quad \alpha, \beta = 2 \pm \frac{\sqrt{(64-\lambda)}}{4}$

Here, $64 - \lambda > 0$

$\therefore \quad \lambda < 64$

Also, $\quad 1 < \alpha < 2$ and $2 < \beta < 3$

$\therefore \quad 1 < 2 - \frac{\sqrt{64-\lambda}}{4} < 2$

and $\quad 2 < 2 + \frac{\sqrt{64-\lambda}}{4} < 3$

$\Rightarrow \quad -1 < -\frac{\sqrt{64-\lambda}}{4} < 0$

and $\quad 0 < \frac{\sqrt{(64-\lambda)}}{4} < 1$

$\Rightarrow \quad 1 > \frac{\sqrt{(64-\lambda)}}{4} > 0$

and $\quad 0 < \frac{\sqrt{(64-\lambda)}}{4} < 1$

i.e. $\quad 0 < \frac{\sqrt{(64-\lambda)}}{4} < 1$

$\Rightarrow \quad 0 < \sqrt{(64-\lambda)} < 4$

$\Rightarrow \quad 0 < 64 - \lambda < 16 \Rightarrow \lambda > 48$

or $\quad 48 < \lambda < 64$

$\therefore \quad \lambda = \{49, 50, 51, 52, \dots, 63\}$

19 Since, roots of $ax^2 + bx + c = 0$ are α and β. Hence, roots of $cx^2 + bx + a = 0$, will be $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. Now, if we replace x by $x - 1$, then roots of $c(x-1)^2 + b(x-1) + a = 0$ will be $1 + \frac{1}{\alpha}$ and $1 + \frac{1}{\beta}$. Now, again replace x by $\frac{1}{x}$, we will get $c(1-x)^2 + b(1-x) + ax^2 = 0$, whose roots are $\frac{\alpha}{1+\alpha}$ and $\frac{\beta}{1+\beta}$.

20 Let $f(x) = x^2 - 2kx + k^2 + k - 5$

Since, both roots are less than 5.

$\therefore \quad D \ge 0, -\frac{b}{2a} < 5$ and $f(5) > 0$

Here, $\quad D = 4k^2 - 4(k^2 + k - 5)$

$= -4k + 20 \ge 0$

$\Rightarrow \quad k \le 5$ ... (i)

$-\frac{b}{2a} < 5 \Rightarrow k < 5$ ... (ii)

and $f(5) > 0$

$\Rightarrow 25 - 10k + k^2 + k - 5 > 0$

$\Rightarrow k^2 - 9k + 20 > 0$

$\Rightarrow (k-5)(k-4) > 0$

$\Rightarrow k < 4$ and $k > 5$ ...(iii)

From (i), (ii) and (iii), we get

$k < 4$

21 Since, both roots of equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4.

$\therefore D \geq 0, -2 < -\frac{b}{2a} < 4,$

$f(4) > 0$ and $f(-2) > 0$

Now, $D \geq 0$

$\Rightarrow 4m^2 - 4m^2 + 4 \geq 0$

$\Rightarrow 4 > 0 \Rightarrow m \in R$ ... (i)

$-2 < -\frac{b}{2a} < 4$

$\Rightarrow -2 < \left(\frac{2m}{2 \cdot 1}\right) < 4$

$\Rightarrow -2 < m < 4$ ... (ii)

$f(4) > 0$

$\Rightarrow 16 - 8m + m^2 - 1 > 0$

$\Rightarrow m^2 - 8m + 15 > 0$

$\Rightarrow (m-3)(m-5) > 0$

$\Rightarrow -\infty < m < 3$ and $5 < m < \infty$ ... (iii)

and $f(-2) > 0$

$\Rightarrow 4 + 4m + m^2 - 1 > 0$

$\Rightarrow m^2 + 4m + 3 > 0$

$(m+3)(m+1) > 0$

$\Rightarrow -\infty < m < -3$ and

$-1 < m < \infty$ ... (iv)

From (i), (ii), (iii) and (iv), we get m lie between -1 and 3.

22 Let $z = x + iy$, given $\text{Re}(z) = 1$

$\therefore x = 1 \Rightarrow z = 1 + iy$

Since, the complex roots are conjugate to each other.

So, $z = 1 + iy$ and $1 - iy$ are two roots of $z^2 + \alpha z + \beta = 0$.

$\because$ Product of roots $= \beta$

$\Rightarrow (1 + iy)(1 - iy) = \beta$

$\therefore \beta = 1 + y^2 \geq 1 \Rightarrow \beta \in [1, \infty)$

23 Given equation is

$e^{\sin x} - e^{-\sin x} = 4 \Rightarrow e^{\sin x} - \frac{1}{e^{\sin x}} = 4$

Let $e^{\sin x} = t$, then $t - \frac{1}{t} = 4$

$\Rightarrow t^2 - 1 - 4t = 0 \Rightarrow t^2 - 4t - 1 = 0$

$\Rightarrow t = \frac{4 \pm \sqrt{16+4}}{2}$

$t = 2 \pm \sqrt{5} \Rightarrow e^{\sin x} = 2 \pm \sqrt{5}$

But $-1 \leq \sin x \leq 1 \Rightarrow e^{-1} \leq e^{\sin x} \leq e^1$

$\Rightarrow e^{\sin x} \in \left[\frac{1}{e}, e\right]$

Also, $0 < e < 2 + \sqrt{5}$

Hence, given equation has no solution.

24 Using AM > GM, we have

$a + \frac{1}{b} > 2\sqrt{\frac{a}{b}},\ b + \frac{1}{c} > 2\sqrt{\frac{b}{c}}$

$c + \frac{1}{d} > 2\sqrt{\frac{c}{d}}$ and $d + \frac{1}{a} > 2\sqrt{\frac{d}{a}}$

$\left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right)\left(c + \frac{1}{d}\right)\left(d + \frac{1}{a}\right) > 16$

But, $\left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right)\left(c + \frac{1}{d}\right)$

$\left(d + \frac{1}{a}\right) = 4 \times 1 \times 4 \times 1 = 16$

$\therefore a = \frac{1}{b}, b = \frac{1}{c}, c\ \ \frac{1}{d}$ and $d = \frac{1}{a}$

$\Rightarrow a = \frac{1}{b} = 2,\ b = \frac{1}{c} = \frac{1}{2},\ c = \frac{1}{d} = 2$

and $d = \frac{1}{a} = \frac{1}{2}$

$\Rightarrow a = 2, b = \frac{1}{2}, c = 2$ and $d = \frac{1}{2}$

$\Rightarrow a = c$ and $b = d$

25 We have, $f(x) = \frac{1}{e^x + \frac{2}{e^x}}$

Using $AM \geq GM$, we get

$\frac{e^x + \frac{2}{e^x}}{2} \geq \left(e^x \cdot \frac{2}{e^x}\right)^{1/2}$, as $e^x > 0$

$\Rightarrow e^x + \frac{2}{e^x} \geq 2\sqrt{2}$

$\Rightarrow 0 < \frac{1}{e^x + \frac{2}{e^x}} \leq \frac{1}{2\sqrt{2}}$

$\therefore 0 < f(x) \leq \frac{1}{2\sqrt{2}}, \forall x \in R$

Statement II is true and Statement I is also true as for some 'c'.

$\Rightarrow f(c) = \frac{1}{3}$ [for $c = 0$]

which lies betwen 0 and $\frac{1}{2\sqrt{2}}$.

So, Statement II is correct explanation of Statement I.