# CHAPTER 11 Limits, Continuity and Differentiability

## Section-A JEE Advanced/ IIT-JEE

### A Fill in the Blanks

1. Let $f(x)=\begin{cases}(x-1)^2\sin\frac{1}{(x-1)}-|x| & \text{if } x\neq 1\\ -1, & \text{if } x=1\end{cases}$

be a real-valued function. Then the set of points where $f(x)$ is not differentiable is ...................... *(1981 - 2 Marks)*

2. Let $f(x)=\begin{cases}\frac{(x^3+x^2-16x+20)}{(x-2)^2}, & \text{if } x\neq 2\\ k, & \text{if } x=2\end{cases}$

If $f(x)$ is continuous for all $x$, then $k=$ .......... *(1981 - 2 Marks)*

3. A discontinuous function $y=f(x)$ satisfying $x^2+y^2=4$ is given by $f(x)=$ ...................... *(1982 - 2 Marks)*

4. $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}=$ ........ *(1984 - 2 Marks)*

5. If $f(x)=\sin x,\ x\neq n\pi,\ n=0,\pm1,\pm2,\pm3,$ .............
$=2$, otherwise

and $g(x)=x^2+1,\ x\neq 0, 2$
$=4, x=0$
$=5, x=2,$

then $\lim_{x\to 0} g[f(x)]$ is ...................... *(1986 - 2 Marks)*

6. $\lim_{x\to-\infty}\left[\frac{x^4\sin\left(\frac{1}{x}\right)+x^2}{(1+|x|^3)}\right]=$ ............... *(1987 - 2 Marks)*

7. If $f(9)=9,\ f'(9)=4$, then $\lim_{x\to 9}\frac{\sqrt{f(x)}-3}{\sqrt{x}-3}$ equals............ *(1988 - 2 Marks)*

8. $ABC$ is an isosceles triangle inscribed in a circle of radius $r$. If $AB=AC$ and $h$ is the altitude from $A$ to $BC$ then the triangle $ABC$ has perimeter $P=2(\sqrt{2hr-h^2})+\sqrt{2hr})$ and area $A=$ ............. also $\lim_{h\to 0}\frac{A}{P^3}=$ ...................... *(1989 - 2 Marks)*

9. $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}=$ ...................... *(1990 - 2 Marks)*

10. Let $f(x)=x\,|x|$. The set of points where $f(x)$ is twice differentiable is .................... *(1992 - 2 Marks)*

11. Let $f(x)=[x]\sin\left(\frac{\pi}{[x+1]}\right)$, where [•] denotes the greatest integer function. The domain of $f$ is... and the points of discontinuity of f in the domain are..... *(1996 - 2 Marks)*

12. $\lim_{x\to 0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2}=$ ... *(1996 - 1 Mark)*

13. Let $f(x)$ be a continuous function defined for $1\le x\le 3$. If $f(x)$ takes rational values for all $x$ and $f(2)=10$, then $f(1.5)=$............ *(1997 - 2 Marks)*

### B True / False

1. If $\lim_{x\to a}[f(x)g(x)]$ exists then both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. *(1981 - 2 Marks)*

### C MCQs with One Correct Answer

1. If $f(x)=\sqrt{\frac{x-\sin x}{x+\cos^2 x}}$, then $\lim_{x\to\infty} f(x)$ is *(1979)*
   (a) 0 (b) $\infty$
   (c) 1 (d) none of these

2. For a real number $y$, let $[y]$ denotes the greatest integer less than or equal to $y$: Then the function $f(x)=\frac{\tan(\pi[x-\pi])}{1+[x]^2}$ is *(1981 - 2 Marks)*
   (a) discontinuous at some $x$
   (b) continuous at all $x$, but the derivative $f'(x)$ does not exist for some $x$
   (c) $f'(x)$ exists for all $x$, but the second derivative $f'(x)$ does not exist for some $x$
   (d) $f'(x)$ exists for all $x$

3. There exist a function $f(x)$, satisfying $f(0) = 1, f'(0) = -1$, $f(x) > 0$ for all $x$, and *(1982 - 2 Marks)*
(a) $f''(x) > 0$ for all $x$ (b) $-1 < f''(x) < 0$ for all $x$
(c) $-2 \le f''(x) \le -1$ for all (d) $f''(x) < -2$ for all $x$

4. If $G(x) = -\sqrt{25-x^2}$ then $\lim_{x\to 1} \frac{G(x)-G(1)}{x-1}$ has the value *(1983 - 1 Mark)*
(a) $\frac{1}{24}$ (b) $\frac{1}{5}$
(c) $-\sqrt{24}$ (d) none of these

5. If $f(a) = 2$, $f'(a) = 1$, $g(a) = -1$, $g'(a) = 2$, then the value of $\lim_{x\to a} \frac{g(x)f(a) - g(a)f(x)}{x-a}$ is *(1983 - 1 Mark)*
(a) $-5$ (b) $\frac{1}{5}$
(c) 5 (d) none of these

6. The function $f(x) = \frac{\ln(1+ax) - \ln(1-bx)}{x}$ is not defined at $x = 0$. The value which should be assigned to $f$ at $x = 0$ so that it is continuous at $x = 0$, is *(1983 - 1 Mark)*
(a) $a - b$ (b) $a + b$
(c) $\ln a - \ln b$ (d) none of these

7. $\lim_{n\to\infty} \left\{ \frac{1}{1-n^2} + \frac{2}{1-n^2} + .... + \frac{n}{1-n^2} \right\}$ is equal to *(1984 - 2 Marks)*
(a) 0 (b) $-\frac{1}{2}$
(c) $\frac{1}{2}$ (d) none of these

8. If $f(x) = \frac{\sin[x]}{[x]}$, $[x] \ne 0$ *(1985 - 2 Marks)*
$= 0$, $[x] = 0$
Where $[x]$ denotes the greatest integer less than or equal to $x$. then $\lim_{x\to 0} f(x)$ equals –
(a) 1 (b) 0
(c) $-1$ (d) none of these

9. Let $f : R \to R$ be a differentiable function and $f(1) = 4$. Then the value of $\lim_{x\to 1} \int_4^{f(x)} \frac{2t}{x-1} dt$ is *(1990 - 2 Marks)*
(a) $8f'(1)$ (b) $4f'(1)$ (c) $2f'(1)$ (d) $f'(1)$

10. Let [.] denote the greatest integer function and $f(x) = [\tan^2 x]$, then: *(1993 - 1 Mark)*
(a) $\lim_{x\to 0} f(x)$ does not exist
(b) $f(x)$ is continuous at $x = 0$
(c) $f(x)$ is not differentiable at $x = 0$
(d) $f'(0) = 1$

11. The function $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$, [.] denotes the greatest integer function, is discontinuous at *(1995S)*
(a) All $x$ (b) All integer points
(c) No $x$ (d) $x$ which is not an integer

12. $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2+r^2}}$ equals *(1997 - 2 Marks)*
(a) $1+\sqrt{5}$ (b) $-1+\sqrt{5}$ (c) $-1+\sqrt{2}$ (d) $1+\sqrt{2}$

13. The function $f(x) = [x]^2 - [x^2]$ (where $[y]$ is the greatest integer less than or equal to $y$), is discontinuous at *(1999 - 2 Marks)*
(a) all integers
(b) all integers except 0 and 1
(c) all integers except 0
(d) all integers except 1

14. The function $f(x) = (x^2-1)|x^2-3x+2| + \cos(|x|)$ is NOT differentiable at *(1999 - 2 Marks)*
(a) $-1$ (b) 0 (c) 1 (d) 2

15. $\lim_{x\to 0} \frac{x \tan 2x - 2x \tan x}{(1-\cos 2x)^2}$ is *(1999 - 2 Marks)*
(a) 2 (b) $-2$ (c) 1/2 (d) $-1/2$

16. For $x \in R$, $\lim_{x\to\infty} \left(\frac{x-3}{x+2}\right)^x =$ *(2000S)*
(a) $e$ (b) $e^{-1}$ (c) $e^{-5}$ (d) $e^5$

17. $\lim_{x\to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ equals *(2001S)*
(a) $-\pi$ (b) $\pi$ (c) $\pi/2$ (d) 1

18. The left-hand derivative of $f(x) = [x] \sin(\pi x)$ at $x = k$, $k$ an integer, is *(2001S)*
(a) $(-1)^k(k-1)\pi$ (b) $(-1)^{k-1}(k-1)\pi$
(c) $(-1)^k k\pi$ (d) $(-1)^{k-1}k\pi$

19. Let $f : R \to R$ be a function defined by $f(x) = \max\{x, x^3\}$. The set of all points where $f(x)$ is NOT differentiable is *(2001S)*
(a) $\{-1, 1\}$ (b) $\{-1, 0\}$ (c) $\{0, 1\}$ (d) $\{-1, 0, 1\}$

20. Which of the following functions is differentiable at $x = 0$?
(a) $\cos(|x|) + |x|$ (b) $\cos(|x|) - |x|$ *(2001S)*
(c) $\sin(|x|) + |x|$ (d) $\sin(|x|) - |x|$

21. The domain of the derivative of the function
$$f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases}$$ is *(2002S)*
(a) $R - \{0\}$ (b) $R - \{1\}$
(c) $R - \{-1\}$ (d) $R - \{-1, 1\}$

**22.** The integer n for which $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite non-zero number is *(2002S)*

(a) 1 (b) 2 (c) 3 (d) 4

**23.** Let $f: R\to R$ be such that $f(1)=3$ and $f'(1)=6$. Then $\lim_{x\to 0}\left(\frac{f(1+x)}{f(1)}\right)^{1/x}$ equals *(2002S)*

(a) 1 (b) $e^{1/2}$ (c) $e^2$ (d) $e^3$

**24.** If $\lim_{x\to 0}\frac{((a-n)nx-\tan x)\sin nx}{x^2}=0$, where n is nonzero real number, then $a$ is equal to *(2003S)*

(a) 0 (b) $\frac{n+1}{n}$ (c) $n$ (d) $n+\frac{1}{n}$

**25.** $\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$, given that $f'(2)=6$ and $f'(1)=4$

(a) does not exist (b) is equal to – 3/2
(c) is equal to 3/2 (d) is equal to 3 *(2003S)*

**26.** If $f(x)$ is differentiable and strictly increasing function, then the value of $\lim_{x\to 0}\frac{f(x^2)-f(x)}{f(x)-f(0)}$ is *(2004S)*

(a) 1 (b) 0 (c) –1 (d) 2

**27.** The function given by $y=||x|-1|$ is differentiable for all real numbers except the points *(2005S)*

(a) $\{0,1,-1\}$ (b) $\pm 1$ (c) 1 (d) –1

**28.** If $f(x)$ is continuous and differentiable function and $f(1/n)=0\ \forall\ n\ge 1$ and $n\in I$, then *(2005S)*

(a) $f(x)=0, x\in(0,1]$
(b) $f(0)=0, f'(0)=0$
(c) $f(0)=0=f'(0),\ x\in(0,1]$
(d) $f(0)=0$ and $f'(0)$ need not to be zero

**29.** The value of $\lim_{x\to 0}\left((\sin x)^{1/x}+(1+x)^{\sin x}\right)$, where $x>0$ is *(2006 - 3M, –1)*

(a) 0 (b) –1 (c) 1 (d) 2

**30.** Let $f(x)$ be differentiable on the interval $(0,\infty)$ such that $f(1)=1$, and $\lim_{t\to x}\frac{t^2f(x)-x^2f(t)}{t-x}=1$ for each $x>0$. Then $f(x)$ is *(2007 - 3 marks)*

(a) $\frac{1}{3x}+\frac{2x^2}{3}$ (b) $\frac{-1}{3x}+\frac{4x^2}{3}$ (c) $\frac{-1}{x}+\frac{2}{x^2}$ (d) $\frac{1}{x}$

**31.** $\lim_{x\to\frac{\pi}{4}}\frac{\int_2^{\sec^2 x}f(t)dt}{x^2-\frac{\pi^2}{16}}$ equals *(2007 - 3 marks)*

(a) $\frac{8}{\pi}f(2)$ (b) $\frac{2}{\pi}f(2)$ (c) $\frac{2}{\pi}f\left(\frac{1}{2}\right)$ (d) $4f(2)$

**32.** Let $g(x)=\frac{(x-1)^n}{\log\cos^m(x-1)}$; $0<x<2$, m and n are integers, $m\ne 0$, $n>0$, and let $p$ be the left hand derivative of $|x-1|$ at $x=1$. If $\lim_{x\to 1^+}g(x)=p$, then *(2008)*

(a) $n=1, m=1$ (b) $n=1, m=-1$
(c) $n=2, m=2$ (d) $n>2, m=n$

**33.** If $\lim_{x\to 0}\left[1+x\,\ell n(1+b^2)\right]^{1/x}=2b\sin^2\theta, b>0$ and $\theta\in(-\pi,\pi]$, then the value of $\theta$ is *(2011)*

(a) $\pm\frac{\pi}{4}$ (b) $\pm\frac{\pi}{3}$ (c) $\pm\frac{\pi}{6}$ (d) $\pm\frac{\pi}{2}$

**34.** If $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$, then *(2012)*

(a) $a=1, b=4$ (b) $a=1, b=-4$
(c) $a=2, b=-3$ (d) $a=2, b=3$

**35.** Let $f(x)=\begin{cases}x^2\left|\cos\frac{\pi}{x}\right|, & x\ne 0\\ 0, & x=0\end{cases}$, $x\in R$ then $f$ is *(2012)*

(a) differentiable both at $x=0$ and at $x=2$
(b) differentiable at $x=0$ but not differentiable at $x=2$
(c) not differentiable at $x=0$ but differentiable at $x=2$
(d) differentiable neither at $x=0$ nor at $x=2$

**36.** Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $\left(\sqrt[3]{1+a}-1\right)x^2+\left(\sqrt{1+a}-1\right)x+\left(\sqrt[6]{1+a}-1\right)=0$ where $a>-1$. Then $\lim_{a\to 0^+}\alpha(a)$ and $\lim_{a\to 0^+}\beta(a)$ are *(2012)*

(a) $-\frac{5}{2}$ and 1 (b) $-\frac{1}{2}$ and – 1
(c) $-\frac{7}{2}$ and 2 (d) $-\frac{9}{2}$ and 3

## D MCQs with One or More than One Correct

**1.** If $x+|y|=2y$, then $y$ as a function of $x$ is *(1984 - 3 Marks)*

(a) defined for all real $x$
(b) continuous at $x=0$
(c) differentiable for all $x$
(d) such that $\frac{dy}{dx}=\frac{1}{3}$ for $x<0$

**2.** If $f(x)=x(\sqrt{x}-\sqrt{x+1})$, then– *(1985 - 2 Marks)*

(a) $f(x)$ is continuous but not differentiable at $x=0$
(b) $f(x)$ is differentiable at $x=0$
(c) $f(x)$ is not differentiable at $x=0$
(d) none of these

3. The function $f(x) = 1 + |\sin x|$ is *(1986 - 2 Marks)*
(a) continuous nowhere
(b) continuous everywhere
(c) differentiable nowhere
(d) not differentiable at $x = 0$
(e) not differentiable at infinite number of points.

4. Let $[x]$ denote the greatest integer less than or equal to $x$. If $f(x) = [x \sin \pi x]$, then $f(x)$ is *(1986 - 2 Marks)*
(a) continuous at $x = 0$ (b) continuous in $(-1, 0)$
(c) differentiable at $x = 1$ (d) differentiable in $(-1, 1)$
(e) none of these

5. The set of all points where the function $f(x) = \dfrac{x}{(1+|x|)}$ is differentiable, is *(1987 - 2 Marks)*
(a) $(-\infty, \infty)$ (b) $[0, \infty)$
(c) $(-\infty, 0) \cup (0, \infty)$ (d) $(0, \infty)$
(e) None

6. The function $f(x) = \begin{cases} |x-3|, & x \ge 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$ is *(1988 - 2 Marks)*
(a) continous at $x = 1$ (b) differentiable at $x = 1$
(c) continous at $x = 3$ (d) differentiable at $x = 3$.

7. If $f(x) = \frac{1}{2}x - 1$, then on the interval $[0, \pi]$ *(1989 - 2 Marks)*
(a) $\tan[f(x)]$ and $1/f(x)$ are both continuous
(b) $\tan[f(x)]$ and $1/f(x)$ are both discontinuous
(c) $\tan[f(x)]$ and $f^{-1}(x)$ are both continuous
(d) $\tan[f(x)]$ is continuous but $1/f(x)$ is not.

8. The value of $\lim\limits_{x \to 0} \dfrac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x}$ *(1991 - 2 Marks)*
(a) 1 (b) $-1$
(c) 0 (d) none of these

9. The following functions are continuous on $(0, \pi)$. *(1991 - 2 Marks)*
(a) $\tan x$
(b) $\displaystyle\int_0^x t \sin \frac{1}{t}\, dt$
(c) $\begin{cases} 1, & 0 < x \le \dfrac{3\pi}{4} \\ 2\sin\dfrac{2}{9}x, & \dfrac{3\pi}{4} < x < \pi \end{cases}$
(d) $\begin{cases} x \sin x, & 0 < x \le \dfrac{\pi}{2} \\ \dfrac{\pi}{2}\sin(\pi + x), & \dfrac{\pi}{2} < x < \pi \end{cases}$

10. Let $f(x) = \begin{cases} 0, & x < 0 \\ x^2, & x \ge 0 \end{cases}$ then for all $x$ *(1994)*
(a) $f'$ is differentiable (b) $f$ is differentiable
(c) $f'$ is continuous (d) $f$ is continuous

11. Let $g(x) = x f(x)$, where $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$. At $x = 0$
(a) $g$ is differentiable but $g'$ is not continuous *(1994)*
(b) $g$ is differentiable while $f$ is not
(c) both $f$ and $g$ are differentiable
(d) $g$ is differentiable and $g'$ is continuous

12. The function $f(x) = \max\{(1-x), (1+x), 2\}$, $x \in (-\infty, \infty)$ is
(a) continuous at all points *(1995)*
(b) differentiable at all points
(c) differentiable at all points except at $x = 1$ and $x = -1$
(d) continuous at all points except at $x = 1$ and $x = -1$, where it is discontinuous

13. Let $h(x) = \min\{x, x^2\}$, for every real number of $x$, Then *(1998 - 2 Marks)*
(a) $h$ is continuous for all $x$
(b) $h$ is differentiable for all $x$
(c) $h'(x) = 1$, for all $x > 1$
(d) $h$ is not differentiable at two values of $x$.

14. $\lim\limits_{x \to 1} \dfrac{\sqrt{1-\cos 2(x-1)}}{x-1}$ *(1998 - 2 Marks)*
(a) exists and it equals $\sqrt{2}$
(b) exists and it equals $-\sqrt{2}$
(c) does not exist because $x - 1 \to 0$
(d) does not exist because the left hand limit is not equal to the right hand limit.

15. If $f(x) = \min\{1, x^2, x^3\}$, then *(2006 - 5M, –1)*
(a) $f(x)$ is continuous $\forall x \in R$
(b) $f(x)$ is continuous and differentiable everywhere.
(c) $f(x)$ is not differentiable at two points
(d) $f(x)$ is not differentiable at one point

16. Let $L = \lim\limits_{x \to 0} \dfrac{a - \sqrt{a^2 - x^2} - \dfrac{x^2}{4}}{x^4}$, $a > 0$.
If $L$ is finite, then *(2009)*
(a) $a = 2$ (b) $a = 1$ (c) $L = \dfrac{1}{64}$ (d) $L = \dfrac{1}{32}$

17. Let $f: R \to R$ be a function such that $f(x+y) = f(x) + f(y)$, $\forall\, x, y \in R$. If $f(x)$ is differentiable at $x = 0$, then *(2011)*
(a) $f(x)$ is differentiable only in a finite interval containing zero
(b) $f(x)$ is continuous $\forall\, x \in R$
(c) $f'(x)$ is constant $\forall\, x \in R$
(d) $f(x)$ is differentiable except at finitely many points.

**18.** If $f(x)=\begin{cases}-x-\frac{\pi}{2}, & x\le -\frac{\pi}{2}\\ -\cos x, & -\frac{\pi}{2}<x\le 0\\ x-1, & 0<x\le 1\\ \ell nx, & x>1\end{cases}$, then *(2011)*

(a) $f(x)$ is continuous at $x=-\frac{\pi}{2}$

(b) $f(x)$ is not differentiable at $x=0$

(c) $f(x)$ is differentiable at $x=1$

(d) $f(x)$ is differentiable at $x=-\frac{3}{2}$

**19.** For every integer $n$, let $a_n$ and $b_n$ be real numbers. Let function $f: IR\to IR$ be given by *(2012)*

$$f(x)=\begin{cases}a_n+\sin\pi x, & \text{for } x\in[2n,2n+1]\\ b_n+\cos\pi x, & \text{for } x\in(2n-1,2n)\end{cases}$$

for all integers $n$. If $f$ is continuous, then which of the following hold(*s*) for all $n$?

(a) $a_{n-1}-b_{n-1}=0$ (b) $a_n-b_n=1$

(c) $a_n-b_{n+1}=1$ (d) $a_{n-1}-b_n=-1$

**20.** For $a\in\mathbb{R}$ (the set of all real numbers), $a\ne -1$, *(JEE Adv. 2013)*

$$\lim_{n\to\infty}\frac{(1^a+2^a+...+n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+...+(na+n)]}=\frac{1}{60}.$$

Then $a=$

(a) 5 (b) 7 (c) $\frac{-15}{2}$ (d) $\frac{-17}{2}$

**21.** Let $f:[a,b]\to[1,\infty)$ be a continuous function and let $g: R\to R$ be defined as *(JEE Adv. 2014)*

$$g(x)=\begin{cases}0, & \text{if } x<a,\\ \int_a^x f(t)dt, & \text{if } a\le x\le b;\\ \int_a^b f(t)dt, & \text{if } x>b.\end{cases}$$ then

(a) $g(x)$ is continuous but not differentiable at $a$

(b) $g(x)$ is differentiable on $R$

(c) $g(x)$ is continuous but not differentiable at $b$

(d) $g(x)$ is continuous and differentiable at either (*a*) or (*b*) but not both

**22.** For every pair of continuous functions $f, g:[0,1]\to R$ such that max $\{f(x): x\in[0,1]\}$ = max $\{g(x): x\in[0,1]\}$, the correct statement(s) is (are): *(JEE Adv. 2014)*

(a) $(f(c))^2+3f(c)=(g(c))^2+3g(c)$ for some $c\in[0,1]$

(b) $(f(c))^2+f(c)=(g(c))^2+3g(c)$ for some $c\in[0,1]$

(c) $(f(c))^2+3f(c)=(g(c))^2+g(c)$ for some $c\in[0,1]$

(d) $(f(c))^2=(g(c))^2$ for some $c\in[0,1]$

**23.** Let $g: R\to R$ be a differentiable function with $g(0)=0$, $g'(0)=0$ and $g'(1)\ne 0$. Let $f(x)=\begin{cases}\frac{x}{|x|}g(x), & x\ne 0\\ 0, & x=0\end{cases}$ and $h(x)=e^{|x|}$ for all $x\in R$. Let $(foh)(x)$ denote $f(h(x))$ and $(hof)(x)$ denote $h(f(x))$. Then which of the following is (are) true? *(JEE Adv. 2015)*

(a) $f$ is differentiable at $x=0$

(b) $h$ is differentiable at $x=0$

(c) $foh$ is differentiable at $x=0$

(d) $hof$ is differentiable at $x=0$

**24.** Let $a, b\in\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=a\cos(|x^3-x|)+b|x|\sin(|x^3+x|)$. Then f is *(JEE Adv. 2016)*

(a) differentiable at x=0 if a=0 and b=1

(b) differentiable at x=1 if a=1 and b=0

(c) NOT differentiable at x=0 if a=1 b=0

(d) NOT differentiable at x=1 if a=1 and b=1

**25.** Let $f:\left[-\frac{1}{2},2\right]\to\mathbb{R}$ and $g:\left[-\frac{1}{2},2\right]\to\mathbb{R}$ be functions defined by $f(x)=[x^2-3]$ and $g(x)=|x|f(x)+|4x-7|f(x)$, where $[y]$ denotes the greatest integer less than or equal to y for $y\in R$. Then *(JEE Adv. 2016)*

(a) $f$ is discontinuous exactly at three points in $\left[-\frac{1}{2},2\right]$

(b) $f$ is discontinuous exactly at four points in $\left[-\frac{1}{2},2\right]$

(c) $g$ is NOT differentiable exactly at four points in $\left(-\frac{1}{2},2\right)$

(d) $g$ is NOT differentiable exactly at five points in $\left(-\frac{1}{2},2\right)$

## E Subjective Problems

**1.** Evaluate $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}, (a\ne 0)$ *(1978)*

**2.** $f(x)$ is the integral of $\frac{2\sin x-\sin 2x}{x^3}, x\ne 0$, find $\lim_{x\to 0}f'(x)$ *(1979)*

3. Evaluate : $\lim_{h \to 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$ *(1980)*

4. Let $f(x+y) = f(x) + f(y)$ for all $x$ and $y$. If the function $f(x)$ is continuous at $x = 0$, then show that $f(x)$ is continuous at all $x$. *(1981 - 2 Marks)*

5. Use the formula $\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$ to find

$$\lim_{x \to 0} \frac{2^x - 1}{(1+x)^{1/2} - 1}$$ *(1982 - 2 Marks)*

6. Let $f(x) = \begin{cases} 1+x, 0 \le x \le 2 \\ 3-x, 2 \le x \le 3 \end{cases}$ *(1983 - 2 Marks)*

Determine the form of $g(x) = f[f(x)$ and hence find the points of discontinuity of $g$, if any

7. Let $f(x) = \begin{cases} \frac{x^2}{2} & ,0 \le x < 1 \\ 2x^2 - 3x + \frac{3}{2}, 1 \le x \le 2 \end{cases}$ *(1983 - 2 Marks)*

Discuss the continuity of $f, f'$ and $f''$ on [ 0, 2].

8. Let $f(x) = x^3 - x^2 + x + 1$ and

$g(x) = \max\{f(t); 0 \le t \le x\}, \ 0 \le x \le 1$ *(1985 - 5 Marks)*

$\quad = 3 - x \qquad 0 \le x \le 2$

Discuss the continuity and differentiability of the function g(x) in the interval (0, 2).

9. Let $f(x)$ be defined in the interval [–2, 2] such that

$$f(x) = \begin{cases} -1, -2 \le x \le 0 \\ x-1, 0 < x \le 2 \end{cases}$$

and $g(x) = f(|x|) + |f(x)|$

Test the differentiability of $g(x)$ in $(-2, 2)$. *(1986 - 5 Marks)*

10. Let $f(x)$ be a continuous and $g(x)$ be a discontinuous function. prove that $f(x) + g(x)$ is a discontinuous function. *(1987 - 2 Marks)*

11. Let $f(x)$ be a function satisfying the condition $f(-x) = f(x)$ for all real $x$. If $f'(0)$ exists, find its value. *(1987 - 2 Marks)*

12. Find the values of a and b so that the function

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x, & 0 \le x < \pi/4 \\ 2x\cot x + b & \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x & \pi/2 < x \le \pi \end{cases}$$

is continuous for $0 \le x \le \pi$. *(1989 - 2 Marks)*

13. Draw a graph of the function $y = [x] + |1 - x|, \ -1 \le x \le 3$. Determine the points, if any, where this function is not differentiable. *(1989 - 4 Marks)*

14. Let $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2}, & x < 0 \\ a, & x = 0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4}, & x > 0 \end{cases}$ *(1990 - 4 Marks)*

Determine the value of $a$, if possible, so that the function is continuous at $x = 0$

15. A function $f: R \to R$ satisfies the equation $f(x+y) = f(x)f(y)$ for all $x, y$ in $R$ and $f(x) \neq 0$ for any $x$ in $R$. Let the function be differentiable at $x = 0$ and $f'(0) = 2$. Show that $f'(x) = 2f(x)$ for all $x$ in $R$. Hence, determine $f(x)$. *(1990 - 4 Marks)*

16. Find $\lim_{x \to 0} \{\tan(\pi/4 + x)\}^{1/x}$ *(1993 - 2 Marks)*

17. Let $f(x) = \begin{cases} \{1 + |\sin x|\}^{a/|\sin x|} & ; \ \frac{\pi}{6} < x < 0 \\ b & ; \ x = 0 \\ e^{\tan 2x/\tan 3x} & ; \ 0 < x < \frac{\pi}{6} \end{cases}$

*(1994 - 4 Marks)*

Determine $a$ and $b$ such that $f(x)$ is continuous at $x = 0$

18. Let $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for all real $x$ and $y$. If $f'(0)$ exists and equals $-1$ and $f(0) = 1$, find $f(2)$. *(1995 - 5 Marks)*

19. Determine the values of $x$ for which the following function fails to be continuous or differentiable: *(1997 - 5 Marks)*

$f(x) = \begin{cases} 1-x, & x < 1 \\ (1-x)(2-x), & 1 \le x \le 2 \\ 3-x, & x > 2 \end{cases}$ Justify your answer.

20. Let $f(x), x \ge 0$, be a non–negative continuous function, and let $F(x) = \int_0^x f(t)\,dt, x \ge 0$. If for some $c > 0, f(x) \le cF(x)$ for all $x \ge 0$, then show that $f(x) = 0$ for all $x \ge 0$. *(2001 - 5 Marks)*

21. Let $\alpha \in R$. Prove that a function $f: R \to R$ is differentiable at $\alpha$ if and only if there is a function $g: R \to R$ which is continuous at $\alpha$ and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$ for all $x \in R$. *(2001 - 5 Marks)*

22. Let $f(x) = \begin{cases} x + a & \text{if } x < 0 \\ |x - 1| & \text{if } x \ge 0, \end{cases}$ and *(2002 - 5 Marks)*

$g(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ (x-1)^2 + b & \text{if } x \ge 0, \end{cases}$ where $a$ and $b$ are non-negative real numbers. Determine the composite function $g$ o $f$. If $(g \text{ o } f)(x)$ is continuous for all real $x$, determine the values of $a$ and $b$. Further, for these values of $a$ and $b$, is g o f differentiable at $x = 0$? Justify your answer.

**23.** If a function $f:[-2a, 2a]\to R$ is an odd function such that $f(x)=f(2a-x)$ for $x\in[a, 2a]$ and the left hand derivative at $x=a$ is 0 then find the left hand derivative at $x=-a$.
***(2003 - 2 Marks)***

**24.** $f'(0)=\lim\limits_{n\to\infty} nf\left(\frac{1}{n}\right)$ and $f(0)=0$. Using this find

$$\lim_{n\to\infty}\left((n+1)\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right)-n\right),\left|\cos^{-1}\frac{1}{n}\right|<\frac{\pi}{2}$$

***(2004 - 2 Marks)***

**25.** If $|c|\le\frac{1}{2}$ and $f(x)$ is a differentiable function at $x=0$ given

by $f(x)=\begin{cases} b\sin^{-1}\left(\frac{c+x}{2}\right) & , \quad -\frac{1}{2}<x<0 \\ \frac{1}{2} & , \quad x=0 \\ \frac{e^{ax/2}-1}{x} & , \quad 0<x<\frac{1}{2} \end{cases}$.

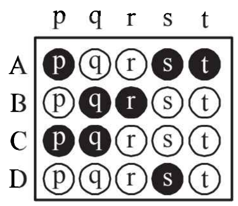
Find the value of '$a$' and prove that $64\,b^2=4-c^2$
***(2004 - 4 Marks)***

**26.** If $f(x-y)=f(x)\cdot g(y)-f(y)\cdot g(x)$ and

$g(x-y)=g(x)\cdot g(y)-f(x)\cdot f(y)$ for all $x, y\in R$.
If right hand derivative at $x=0$ exists for $f(x)$. Find derivative of $g(x)$ at $x=0$ ***(2005 - 4 Marks)***

## F Integer Value Correct Type

**DIRECTIONS (Q. 1 and 2) :** *Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :*
*If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.*



**1.** In this questions there are entries in columns I and II. Each entry in column I is related to exactly one entry in column II. Write the correct letter from column II against the entry number in column I in your answer book. ***(1992 - 2 Marks)***

| *Column I* | *Column II* |
|---|---|
| (A) $\sin(\pi[x])$ | (p) differentiable everywhere |
| (B) $\sin(\pi(x-[x])$ | (q) nowhere differentiable |
| | (r) not differentiable at 1 and – 1 |

**2.** In the following $[x]$ denotes the greatest integer less than or equal to $x$.

Match the functions in **Column I** with the properties in **Column II** and indicate your answer by darkening the appropriate bubbles in the $4\times4$ matrix given in the ORS. ***(2007 - 6 marks)***

| *Column I* | *Column II* |
|---|---|
| (A) $x\,\vert x\vert$ | (p) continuous in (–1, 1) |
| (B) $\sqrt{\vert x\vert}$ | (q) differentiable in (–1, 1) |
| (C) $x+[x]$ | (r) strictly increasing in (–1, 1) |
| (D) $\vert x-1\vert+\vert x+1\vert$ | (s) not differentiable at least at one point in (–1, 1) |

**DIRECTIONS (Q. 3) :** Following question has matching lists. The codes for the list have choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

**3.** Let $f_1:R\to R$, $f_2:[0,\infty)\to R$, $f_3:R\to R$ and $f_4:R\to[0,\infty)$ be defined by $f_1(x)=\begin{cases}|x| & \text{if } x<0, \\ e^x & \text{if } x\ge 0;\end{cases}$

$f_2(x)=x^2$; $f_3(x)=\begin{cases}\sin x & \text{if } x<0, \\ x & \text{if } x\ge 0;\end{cases}$ and $f_4(x)=\begin{cases}f_2(f_1(x)) & \text{if } x<0, \\ f_2(f_1(x))-1 & \text{if } x\ge 0.\end{cases}$ ***(JEE Adv. 2014)***

| List-I | | List-II |
|---|---|---|
| **P.** $f_4$ is | **1.** | Onto but not one-one |
| **Q.** $f_3$ is | **2.** | Neither continuous nor one-one |
| **R.** $f_2 o f_1$ is | **3.** | Differentiable but not one-one |
| **S.** $f_2$ is | **4.** | Continuous and one-one |

| | P | Q | R | S |
|---|---|---|---|---|
| (a) | 3 | 1 | 4 | 2 |
| (b) | 1 | 3 | 4 | 2 |
| (c) | 3 | 1 | 2 | 4 |
| (d) | 1 | 3 | 2 | 4 |

## I Integer Value Correct Type

**1.** Let $f: [1, \infty) \to [2, \infty)$ be a differentiable function such that $f(1) = 2$. If $6\int_1^x f(t)dt = 3xf(x) - x^3$ for all $x \geq 1$, then the value of $f(2)$ is *(2011)*

**2.** The largest value of non-negative integer a for which

$$\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$$ is *(JEE Adv. 2014)*

**3.** Let $f: R \to R$ and $g: R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h: R \to R$ by

$$h(x) = \begin{cases} \max\ \{f(x), g(x)\} & \text{if } x \leq 0, \\ \min\ \{f(x), g(x)\} & \text{if } x > 0. \end{cases}$$

The number of points at which $h(x)$ is not differentiable is *(JEE Adv. 2014)*

**4.** Let $m$ and $n$ be two positive integers greater than 1. If

$$\lim_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right) = -\left(\frac{e}{2}\right)$$ then the value of $\frac{m}{n}$ is

*(JEE Adv. 2015)*

**5.** Let $\alpha, \beta \in \mathbb{R}$ be such that $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x - \sin x} = 1$. Then $6(\alpha + \beta)$ equals. *(JEE Adv. 2016)*

## Section-B JEE Main / AIEEE

**1.** $\lim_{x\to 0}\frac{\sqrt{1-\cos 2x}}{\sqrt{2}x}$ is **[2002]**

(a) 1 (b) –1
(c) zero (d) does not exist

**2.** $\text{Lim}_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+3}\right)^x$ **[2002]**

(a) $e^4$ (b) $e^2$ (c) $e^3$ (d) 1

**3.** Let $f(x) = 4$ and $f'(x) = 4$. Then $\lim_{x\to 2}\frac{xf(2) - 2f(x)}{x-2}$ is given by **[2002]**

(a) 2 (b) –2 (c) – 4 (d) 3

**4.** $\lim_{n\to\infty}\frac{1^p + 2^p + 3^p + ..... + n^p}{n^{p+1}}$ is **[2002]**

(a) $\frac{1}{p+1}$ (b) $\frac{1}{1-p}$ (c) $\frac{1}{p} - \frac{1}{p-1}$ (d) $\frac{1}{p+2}$

**5.** $\lim_{x\to 0}\frac{\log x^n - [x]}{[x]}, n \in N$, ([x] denotes greatest integer less than or equal to $x$) **[2002]**

(a) has value –1 (b) has value 0
(c) has value 1 (d) does not exist

**6.** If $f(1) = 1, f^1(1) = 2$, then $\lim_{x\to 1}\frac{\sqrt{f(x)} - 1}{\sqrt{x} - 1}$ is **[2002]**

(a) 2 (b) 4 (c) 1 (d) 1/2

**7.** $f$ is defined in [-5, 5] as **[2002]**

$f(x) = x$ if $x$ is rational
$= -x$ if $x$ is irrational. Then

(a) $f(x)$ is continuous at every $x$, except $x = 0$
(b) $f(x)$ is discontinuous at every $x$, except $x = 0$
(c) $f(x)$ is continuous everywhere
(d) $f(x)$ is discontinuous everywhere

**8.** $f(x)$ and $g(x)$ are two differentiable functions on [0, 2] such that $f''(x) - g''(x) = 0$, $f'(1) = 2g'(1) = 4f(2) = 3g(2) = 9$ then $f(x)-g(x)$ at $x = 3/2$ is **[2002]**

(a) 0 (b) 2 (c) 10 (d) 5

**9.** If $f(x+y)=f(x).f(y)\forall x.y$ and $f(5)=2, f'(0)=3$, then $f'(5)$ is **[2002]**

(a) 0 (b) 1 (c) 6 (d) 2

**10.** $\lim_{n\to\infty}\frac{1+2^4+3^4+...n^4}{n^5}-\lim_{n\to\infty}\frac{1+2^3+3^3+...n^3}{n^5}$ **[2003]**

(a) $\frac{1}{5}$ (b) $\frac{1}{30}$ (c) Zero (d) $\frac{1}{4}$

**11.** If $\lim_{x\to 0}\frac{\log(3+x)-\log(3-x)}{x}=k$, the value of $k$ is **[2003]**

(a) $-\frac{2}{3}$ (b) 0 (c) $-\frac{1}{3}$ (d) $\frac{2}{3}$

**12.** The value of $\lim_{x\to 0}\frac{\int_0^{x^2}\sec^2 t\,dt}{x\sin x}$ is **[2003]**

(a) 0 (b) 3 (c) 2 (d) 1

**13.** Let $f(a)=g(a)=k$ and their nth derivatives $f^n(a), g^n(a)$ exist and are not equal for some $n$. Further if

$$\lim_{x\to a}\frac{f(a)g(x)-f(a)-g(a)f(x)+f(a)}{g(x)-f(x)}=4$$

then the value of $k$ is **[2003]**

(a) 0 (b) 4 (c) 2 (d) 1

**14.** $\lim_{x\to\frac{\pi}{2}}\frac{\left[1-\tan\left(\frac{x}{2}\right)\right][1-\sin x]}{\left[1+\tan\left(\frac{x}{2}\right)\right][\pi-2x]^3}$ is **[2003]**

(a) $\infty$ (b) $\frac{1}{8}$ (c) 0 (d) $\frac{1}{32}$

**15.** If $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & ,x\neq 0 \\ 0 & ,x=0\end{cases}$ then f(x) is **[2003]**

(a) discontinuous every where
(b) continuous as well as differentialble for all $x$
(c) continuous for all $x$ but not differentiable at $x=0$
(d) neither differentiable nor continuous at $x=0$

**16.** If $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x}=e^2$, then the values of $a$ and $b$, are **[2004]**

(a) $a=1$ and $b=2$ (b) $a=1, b\in \boldsymbol{R}$
(c) $a\in\boldsymbol{R}, b=2$ (d) $a\in\boldsymbol{R}, b\in\boldsymbol{R}$

**17.** Let $f(x)=\frac{1-\tan x}{4x-\pi}, x\neq\frac{\pi}{4}, x\in\left[0,\frac{\pi}{2}\right]$. If $f(x)$ is continuous in $\left[0,\frac{\pi}{2}\right]$, then $f\left(\frac{\pi}{4}\right)$ is **[2004]**

(a) $-1$ (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 1

**18.** $\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}\text{...........}+\frac{1}{n}\sec^2 1\right]$ equals **[2005]**

(a) $\frac{1}{2}\sec 1$ (b) $\frac{1}{2}\operatorname{cosec} 1$
(c) $\tan 1$ (d) $\frac{1}{2}\tan 1$

**19.** Let $\alpha$ and $\beta$ be the distinct roots of $ax^2+bx+c=0$, then $\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is equal to **[2005]**

(a) $\frac{a^2}{2}(\alpha-\beta)^2$ (b) 0
(c) $\frac{-a^2}{2}(\alpha-\beta)^2$ (d) $\frac{1}{2}(\alpha-\beta)^2$

**20.** Suppose $f(x)$ is differentiable at $x=1$ and $\lim_{h\to 0}\frac{1}{h}f(1+h)=5$, then $f'(1)$ equals **[2005]**

(a) 3 (b) 4 (c) 5 (d) 6

**21.** Let $f$ be differentiable for all $x$. If $f(1)=-2$ and $f'(x)\geq 2$ for $x\in[1,6]$, then **[2005]**

(a) $f(6)\geq 8$ (b) $f(6)<8$ (c) $f(6)<5$ (d) $f(6)=5$

**22.** If $f$ is a real valued differentiable function satisfying $|f(x)-f(y)|\leq(x-y)^2, x,y\in R$ and $f(0)=0$, then $f(1)$ equals **[2005]**

(a) $-1$ (b) 0 (c) 2 (d) 1

**23.** Let $f:R\to R$ be a function defined by $f(x)=\min\{x+1,|x|+1\}$, Then which of the following is true? **[2007]**

(a) $f(x)$ is differentiable everywhere
(b) $f(x)$ is not differentiable at $x=0$
(c) $f(x)\geq 1$ for all $x\in R$
(d) $f(x)$ is not differentiable at $x=1$

**24.** The function $f:R/\{0\}\to R$ given by **[2007]**

$$f(x)=\frac{1}{x}-\frac{2}{e^{2x}-1}$$

can be made continuous at $x=0$ by defining $f(0)$ as

(a) 0 (b) 1 (c) 2 (d) $-1$

25. Let $f(x)=\begin{cases}(x-1)\sin\dfrac{1}{x-1} & \text{if } x\neq 1\\ 0 & \text{if } x=1\end{cases}$ **[2008]**

Then which one of the following is true?
(a) $f$ is neither differentiable at $x=0$ nor at $x=1$
(b) $f$ is differentiable at $x=0$ and at $x=1$
(c) $f$ is differentiable at $x=0$ but not at $x=1$
(d) $f$ is differentiable at $x=1$ but not at $x=0$

26. Let f : **R** → **R** be a positive increasing function with $\lim\limits_{x\to\infty}\dfrac{f(3x)}{f(x)}=1$. Then $\lim\limits_{x\to\infty}\dfrac{f(2x)}{f(x)}=$ **[2010]**

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) 3 (d) 1

27. $\lim\limits_{x\to 2}\left(\dfrac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}\right)$ **[2011]**

(a) equals $\sqrt{2}$ (b) equals $-\sqrt{2}$
(c) equals $\frac{1}{\sqrt{2}}$ (d) does not exist

28. The values of $p$ and $q$ for which the function **[2011]**

$f(x)=\begin{cases}\dfrac{\sin(p+1)x+\sin x}{x}, & x<0\\ q, & x=0\\ \dfrac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}}, & x>0\end{cases}$ is continuous for all $x$ in $R$, are

(a) $p=\frac{5}{2}, q=\frac{1}{2}$ (b) $p=-\frac{3}{2}, q=\frac{1}{2}$
(c) $p=\frac{1}{2}, q=\frac{3}{2}$ (d) $p=\frac{1}{2}, q=-\frac{3}{2}$

29. Let $f : R \to [0,\infty)$ be such that $\lim\limits_{x\to 5} f(x)$ exists and $\lim\limits_{x\to 5}\dfrac{(f(x))^2-9}{\sqrt{|x-5|}}=0$. Then $\lim\limits_{x\to 5} f(x)$ equals :

(a) 0 (b) 1 (c) 2 (d) 3

30. If $f : R \to R$ is a function defined by $f(x)=[x]\cos\left(\dfrac{2x-1}{2}\right)\pi$, where $[x]$ denotes the greatest integer function, then $f$ is . **[2012[**
(a) continuous for every real $x$.
(b) discontinuous only at $x=0$
(c) discontinuous only at non-zero integral values of $x$.
(d) continuous only at $x=0$.

31. Consider the function, $f(x)=|x-2|+|x-5|, x\in R$.
**Statement-1 :** $f'(4)=0$
**Statement-2 :** $f$ is continuous in [2,5], differentiable in (2,5) and $f(2)=f(5)$. **[2012]**
(a) Statement-1 is false, Statement-2 is true.
(b) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.
(c) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for Statement-1.
(d) Statement-1 is true, statement-2 is false.

32. $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to **[JEE M 2013]**

(a) $-\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) 2

33. $\lim\limits_{x\to 0}\dfrac{\sin\left(\pi\cos^2 x\right)}{x^2}$ is equal to: **[JEE M 2014]**

(a) $-\pi$ (b) $\pi$ (c) $\frac{\pi}{2}$ (d) 1

34. $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to : **[JEE M 2015]**

(a) 2 (b) $\frac{1}{2}$ (c) 4 (d) 3

35. If the function.

$g(x)=\begin{cases}k\sqrt{x+1}, & 0\le x\le 3\\ mx+2, & 3<x\le 5\end{cases}$ is differentiable, then the value of $k+m$ is : **[JEE M 2015]**

(a) $\frac{10}{3}$ (b) 4 (c) 2 (d) $\frac{16}{5}$

36. For $x\in\mathbb{R}$, $f(x)=|\log 2-\sin x|$ and $g(x)=f(f(x))$, then : **[JEE M 2016]**
(a) $g'(0)=-\cos(\log 2)$
(b) g is differentiable at $x=0$ and $g'(0)=-\sin(\log 2)$
(c) g is not differentiable at $x=0$
(d) $g'(0)=\cos(\log 2)$

37. $\lim\limits_{n\to\infty}\left(\dfrac{(n+1)(n+2)\ldots 3n}{n^{2n}}\right)^{\frac{1}{n}}$ is equal to: **[JEE M 2016]**

(a) $\frac{9}{e^2}$ (b) $3\log 3-2$
(c) $\frac{18}{e^4}$ (d) $\frac{27}{e^2}$

38. Let $p=\lim\limits_{x\to 0^+}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ then $\log p$ is equal to : **[JEE M 2016]**

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$
(c) 2 (d) 1

# 11 Limits, Continuity and Differentiability

## Section-A : JEE Advanced/ IIT-JEE

**A** **1.** $\{0\}$ **2.** $k=7$ **3.** $f(x)=\sqrt{4-x^2}, -2\le x\le 0 = -\sqrt{4-x^2},\ 0\le x\le 2$

**4.** $\frac{2}{\pi}$ **5.** 1 **6.** $-1$ **7.** 4 **8.** $\sqrt{2rh-h^2}, \frac{I}{128r}$ **9.** $e^5$

**10.** $R-\{0\}$ **11.** $(-\infty,-1)\cup[0,\infty)$, $I-\{0\}$ where $I$ is the set of ingeger except $n=-1$

**12.** $e^2$ **13.** 10

**B** **1.** $F$

**C**

| | | | | | | |
|---|---|---|---|---|---|---|
| **1.** (c) | **2.** (d) | **3.** (a) | **4.** (d) | **5.** (c) | **6.** (b) | **7.** (b) |
| **8.** (d) | **9.** (a) | **10.** (b) | **11.** (c) | **12.** (b) | **13.** (d) | **14.** (d) |
| **15.** (c) | **16.** (c) | **17.** (b) | **18.** (a) | **19.** (d) | **20.** (d) | **21.** (d) |
| **22.** (c) | **23.** (c) | **24.** (d) | **25.** (d) | **26.** (c) | **27.** (a) | **28.** (b) |
| **29.** (c) | **30.** (a) | **31.** (a) | **32.** (c) | **33.** (d) | **34.** (b) | **35.** (b) |
| **36.** (b) | | | | | | |

**D**

| | | | | | |
|---|---|---|---|---|---|
| **1.** (a, b, d) | **2.** (b) | **3.** (b, d, e) | **4.** (a, b, d) | **5.** (a) | **6.** (a, b, c) |
| **7.** (b) | **8.** (d) | **9.** (b, c) | **10.** (b, c, d) | **11.** (a, b) | **12.** (a, c) |
| **13.** (a, c, d) | **14.** (d) | **15.** (a, d) | **16.** (a, c) | **17.** (b, c) | **18.** (a, b, c, d) |
| **19.** (b, d) | **20.** (b, d) | **21.** (a, c) | **22.** (a, d) | **23.** (a, d) | **24.** (a, b) |
| **25.** (b, c) | | | | | |

**E** **1.** $\frac{2}{3\sqrt{3}}$ **2.** 1 **3.** $a^2\cos a + 2a\sin a$ **5.** $2\ln 2$

**6.** $g(x)=\begin{cases}2+x, & 0\le x\le 1\\ 2-x, & 1<x\le 2\\ 4-x, & 2<x\le 3\end{cases}$ discontinuity at $x=1, 2$ **7.** $f$ and $f'$ are continuous and $f''$ is discontinuous on $[0, 2]$

**8.** cont. on $(0, 2)$ and differentiable on $(0, 2)-\{1\}$ **9.** not differentiable at $x=1$ **11.** $f'(0)=0$

**12.** $a=\frac{\pi}{6}, b=-\frac{\pi}{12}$ **13.** $x=0, 1, 2, 3$ **14.** $a=8$ **15.** $f(x)=e^{2x}$

**16.** $e^2$ **17.** $a=\frac{2}{3}, b=e^{2/3}$ **18.** $f(2)=-1$

**19.** $f$ is continuous and differentiable at all points except at $x=2$.

**22.** $g(f(x))=\begin{cases}x+a+1. & \text{if } x<-a\\ (x+a-1)^2+b & \text{if } a\le x<0\\ x^2+b & \text{if } 0\le x\le 1\\ (x-2)^2+b & \text{if } x>1\end{cases}$, $a=1, b=0$, gof is differentiable at $x=0$ **23.** 0

**24.** $\frac{\pi-2}{\pi}$ **25.** 1 **26.** 0

**F** **1.** (A) - p, (B) - r **2.** (A) - p, q, r ; (B) - p, s ; (C) - r, s ; (D) - p, q **3.** (d)

**I** **1.** 6 **2.** 2 **3.** 3 **4.** 2 **5.** 7

**Section-B : JEE Main/ AIEEE**

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1.** | (d) | **2.** | (a) | **3.** | (c) | **4.** | (a) | **5.** | (d) | **6.** | (a) | **7.** | (b) |
| **8.** | (d) | **9.** | (c) | **10.** | (a) | **11.** | (d) | **12.** | (d) | **13.** | (b) | **14.** | (d) |
| **15.** | (c) | **16.** | (b) | **17.** | (c) | **18.** | (d) | **19.** | (a) | **20.** | (c) | **21.** | (a) |
| **22.** | (b) | **23.** | (a) | **24.** | (b) | **25.** | (c) | **26.** | (d) | **27.** | (d) | **28.** | (b) |
| **29.** | **(d)** | **30.** | (a) | **31.** | (c) | **32.** | (d) | **33.** | (b) | **34.** | (a) | **35.** | (c) |
| **36.** | (d) | **37.** | (d) | **38.** | (a) | | | | | | | | |

# Section-A JEE Advanced/ IIT-JEE

## A. Fill in the Blanks

**1.** Given $f(x)=\begin{cases}(x-1)^2\sin\dfrac{1}{x-1}-|x|, x\neq 1\\ -1, \quad x=1\end{cases}$

We know that $|x|$ is not differentiable at $x=0$

$\therefore\ (x-1)^2\sin\dfrac{1}{x-1}-|x|$ is not differentiable at $x=0$.

At all other values of $x, f(x)$ is differentiable.

$\therefore$ The req. set of points is $\{0\}$.

**2.** It will be continuous at $x=2$ if

$$\lim_{x\to 2} f(x)=f(2) \Rightarrow \lim_{x\to 2}\frac{x^3+x^2-16x+20}{(x-2)^2}=k$$

$$\Rightarrow k=\lim_{x\to 2}\frac{(x-2)^2(x+5)}{(x-2)^2}=\lim_{x\to 2}(x+5)=7$$

$\therefore\ k=7$

**3.** $f(x)=\sqrt{4-x^2}, -2\le x\le 0 \ =-\sqrt{4-x^2}, 0\le x\le 2$

By choosing any arcs of circle $x^2-y^2=4$, we can define a discontinuous function, one of which is

$$f(x)=\begin{cases}\sqrt{4-x^2}, -2\le x\le 0\\ -\sqrt{4-x^2}, 0\le x\le 2\end{cases}$$

**4.** **KEY CONCEPT**

**(L' Hospital rule)**

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if $\dfrac{f(a)}{g(a)}$ is of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ or $0\times\infty$.

$$\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2} \quad [\text{form } 0\times\infty]$$

$$=\lim_{x\to 1}\frac{1-x}{\cot(\pi x/2)} \quad \left[\text{Form}\frac{0}{0}\right]$$

$$=\lim_{x\to 1}\frac{-1}{\frac{-\pi}{2}\text{cosec}^2\left(\frac{\pi x}{2}\right)} \quad [\text{Applying } L'\text{Hospital's rule}]$$

$$=\frac{2}{\pi}.$$

**5.** Given that,

$f(x)=\sin x, x\neq\pi, n=0,\pm1,\pm2,...=2$, otherwise

And $g(x)=x^2+1, x\neq 0,2$

$=4, x=0=5, \quad x=2$

Then $\lim_{x\to 0} g[f(x)]=\lim_{x\to 0} g(\sin x)\Rightarrow \lim_{x\to 0}(\sin^2 x+1)=1$

**6.** $$\lim_{x\to-\infty}\left[\frac{x^4\sin\left(\frac{1}{x}\right)+x^2}{(1+|x|^3)}\right]$$

Let $L=\lim_{x\to-\infty}\dfrac{x^3}{1+|x|^3}\left[x\sin\left(\dfrac{1}{x}\right)+\dfrac{1}{x}\right]$

$$=\lim_{x\to-\infty}\frac{x^3}{|x|^3}\left[\frac{1}{1+\frac{1}{|x|x^2}}\right]\left[x\sin\left(\frac{1}{x}\right)+\frac{1}{x}\right] \quad ...(1)$$

$$=\lim_{x\to-\infty}\frac{x^3}{|x|^3}.1=\lim_{x\to-\infty}\frac{x^3}{-x^3}=-1$$

**7.** Given that $f(9)=9, f'(9)=4$

Then,

$$\lim_{x\to 9}\frac{\sqrt{f(x)}-3}{\sqrt{x}-3}=\lim_{x\to 9}\frac{(\sqrt{f(x)}-3)(\sqrt{f(x)}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)}$$

$$\lim_{x\to 9}\frac{\sqrt{x}+3}{\sqrt{f(x)}+3}=\lim_{x\to 9}\frac{f(x)-9}{x-9}.\left[\frac{3+3}{3+3}\right]$$

$$=\lim_{x\to 9}\frac{f(x)-f(9)}{x-9}.1=f'(9)=4$$

**8.** In $\Delta ABC, AB=AC$

$AD\perp BC$ ($D$ is mid pt of $BC$)

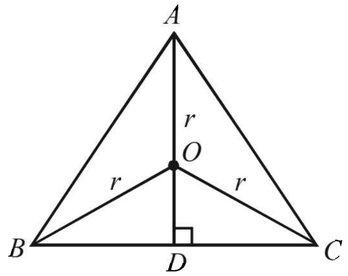
Let $r=$ radius of circumcircle

$\therefore\ OA=OB=OC=r$

Now $BD=\sqrt{BO^2-OD^2}=\sqrt{r^2-(h-r)^2}$

$=\sqrt{2rh-h^2}$

$\therefore\ BC=2\sqrt{2rh-h^2}$




$\therefore$ Area of $\Delta ABC = \frac{1}{2} \times BC \times AD = h\sqrt{2rh - h^2}$

Also $\lim_{h\to 0} \frac{A}{P^3} = \lim_{h \to 0} \frac{h\sqrt{2rh-h^2}}{8(\sqrt{2rh-h^2}+\sqrt{2rh})^3}$

$= \lim_{h\to 0} \frac{h^{3/2}\sqrt{2r-h}}{8h^{3/2}(\sqrt{2r-h})+\sqrt{2r})^3}$

$= \lim_{h\to 0} \frac{\sqrt{2r-h}}{8(2r-h+\sqrt{2r})^3}$

$= \frac{\sqrt{2r}}{8(\sqrt{2r}+\sqrt{2r})^3} = \frac{\sqrt{2r}}{8.8.2r.\sqrt{2r}} = \frac{1}{128r}$

**9.** $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4} = \lim_{x\to\infty}\left\{\left[1+\frac{5}{x+1}\right]^{\frac{x+1}{5}}\right\}^{5\left(\frac{x+4}{x+1}\right)}$

[Using $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$]

$\lim_{e^{x\to\infty}} 5\left(\frac{x+4}{x+1}\right) = e^{5\lim_{x\to\infty}\left(\frac{1+4/x}{1+1/x}\right)} = e^5$

**10.** We have,

$f(x) = x|x| = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

$f'(x) = \begin{cases} -2x, & x < 0 \\ 2x, & x \geq 0 \end{cases}$

$f''(x) = \begin{cases} -2, & x < 0 \\ 2, & x \geq 0 \end{cases}$

Clearly $f''(x)$ exists at every pt. except at $x = 0$

Thus $f(x)$ is twice differentiable on $R - \{0\}$.

**11.** Thus function is not defined for those values of $x$ for which $[x+1] = 0$. In other words it means that

$0 \leq x+1 < 1$ or $-1 \leq x < 0$ .......(1)

Hence the function is defined outside the region given by (1).

Required domain is $]-\infty, -1[\cup[0,\infty]$

Now, consider integral values of $x$ say $x = n$

R.H.L. $= \lim_{h\to 0}[n+h]\sin\frac{\pi}{[n+1+h]} = n\sin\frac{\pi}{(n+1)}$

L.H.L. $= \lim_{h\to 0}[n-h]\sin\frac{\pi}{[n+1-h]} = (n-1)\frac{\pi}{n}$

Clearly RHL $\neq$ LHL. Hence the given function is not continuous for integral values of $n$ $(n \neq 0, -1)$.

At $x = 0, f(0) = 0$,

$\lim f(0+h) = \lim_{h\to 0}[h]\sin\frac{\pi}{[h+1]} = 0$

The function is not defined for $x < 0$. Hence we cannot find $\lim f(0-h)$. Thus $f(x)$ is continuous at $x = 0$. Hence the points of discontinuity are given by $I - \{0\}$ where $I$ is set of integers $n$ except $n = -1$

**12.** **KEY CONCEPT**

$\lim_{x\to 0}[f(x)]^{g(x)} = e^{\lim_{x\to 0} g(x)\log f(x)}$

$\lim_{x\to 0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2} = e^{\lim_{x\to 0}\frac{1}{x^2}\log\left[\frac{1+5x^2}{1+3x^2}\right]}$

$= e^{\lim_{x\to 0}\left[5.\frac{\log(1+5x^2)}{5x^2} - 3.\frac{\log(1+3x^2)}{3x^2}\right]}$

$= e^{5-3} = e^2$

**13.** Since $f(x)$ is given continuous on the closed bounded interval $[1, 3]$, $f(x)$ is bounded and assumes all the values lying in the interval $[m, M]$ where

$m = \min f(x)$ and $M = \max f(x)$

$1 \leq x \leq 3 \Rightarrow f(1) \leq f(x) \leq (3)$

If $m \gtrless M$, then $f(x)$ must assume all the irrational values lying in the $[m, M]$. But since $f(x)$ takes only rational values, we must have $m = M$ i.e., $f(x)$ must be a constant function. As $f(2) = 10$, we get

$f(x) = 10 \; \forall \; x \in [1,3] \Rightarrow f(1.5) = 10$

## B. True/False

**1.** Consider $f(x) = \frac{|x-a|}{x-a}, g(x) = \frac{x-a}{|x-a|}$

then $\lim_{x\to a}(f(x)g(x))$ exists but neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

## C. MCQs with ONE Correct Answer

**1.** **(c)** $f(x) = \sqrt{\frac{x-\sin x}{x+\cos^2 x}}$;

$\Rightarrow \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\sqrt{\frac{1-\frac{\sin x}{x}}{1+\frac{\cos^2 x}{x}}} = \sqrt{\frac{1-0}{1+0}} = 1$

**2.** **(d)** $f(x) = \frac{\tan(\pi[x-\pi])}{1+[x]^2}$

By def. $[x-\pi]$ is an integer whatever be the value of x.

And so $\pi[x-\pi]$ is an integral multiple of $\pi$.

Consequently $\tan(\pi[x-\pi]) = 0, \forall$ x.

And since $1+[x]^2 \neq 0$ for any x, we conclude that $f(x) = 0$.

Thus $f(x)$ is constant function and so, it is continuous and differentiable any no. of times, that is $f'(x), f''(x), f'''(x),...$ all exist for every x, their value being 0 at every pt. x. Hence, out of all the alternatives only (d) is correct.

**3.** **(a)** $f(x) = e^{-x}$ is one such function.

Here $f(0) = 1, f'(0) = -1, f(x) > 0, \forall x$.

$\therefore \quad f''(x) > 0 \ \forall x$

**4.** **(d)** $\lim_{x \to 1} \frac{-\sqrt{25 - x^2} - (-\sqrt{24})}{x - 1}$

$$= \lim_{x \to 1} \frac{\sqrt{24} - \sqrt{25 - x^2}}{x - 1} \times \frac{\sqrt{24} + \sqrt{25 - x^2}}{\sqrt{24} + \sqrt{25 - x^2}}$$

$$= \lim_{x \to 1} \frac{x^2 - 1}{(x-1)[\sqrt{24} + \sqrt{25 - x^2}]}$$

$$= \lim_{x \to 1} \frac{x+1}{[\sqrt{24} + \sqrt{25 - x^2}]} = \frac{2}{2\sqrt{24}} = \frac{1}{2\sqrt{6}}$$

**5.** **(c)** $\lim_{x \to a} \frac{g(x) f(a) - g(a) f(x)}{x - a}$

$$= \lim_{h \to 0} \frac{g(a+h) f(a) - g(a) f(a+h)}{h} \quad [\text{For } x = a + h]$$

$$= \lim_{h \to 0} \frac{g(a+h) f(a) - g(a) f(a) + g(a) f(a) - g(a) f(a+h)}{h}$$

$$= \lim_{h \to 0} f(a) \left[\frac{g(a+h) - g(a)}{h}\right] - \lim_{h \to 0} g(a) \left[\frac{f(a+h) - f(a)}{h}\right]$$

$= f(a)\, g'(a) - g(a)\, f'(a) = 2 \times 2 - (-1) \times 1 = 5$

**6.** **(b)** For $f(x)$ to be continuous at $x = 0$

$$f(0) = \lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{\ln(1 + ax) - \ln(1 - bx)}{x}$$

$$\left[\text{Using } \lim_{x \to 0} \frac{\ln(1+x)}{x} = 1\right]$$

$$= a + b$$

**7.** **(b)** $\lim_{n \to \infty} \left(\frac{1}{1 - n^2} + \frac{2}{1 - n^2} + ... + \frac{n}{1 - n^2}\right)$

$$= \lim_{n \to \infty} \frac{1 + 2 + 3 + ... + n}{1 - n^2} = \lim_{n \to \infty} \frac{\Sigma n}{1 - n^2} = \lim_{n \to \infty} \frac{n(n+1)}{2(1 - n^2)}$$

$$= \lim_{n \to \infty} \frac{1 + 1/n}{2\left[\frac{1}{n^2} - 1\right]} = -1/2$$

**8.** **(d)** The given function can be restated as

$$f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & \text{if } x \in (-\infty, 0) \cup [1, \infty] \\ 0, & \text{if } x \in [0, 1) \end{cases}$$

$$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{h \to 0} \frac{\sin[-h]}{[-h]}$$

$$= \lim_{x \to 0^-} \frac{\sin(-1)}{(-1)} = \sin 1$$

And $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} 0 = 0$

$\because \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$ as $\sin 1 \neq 0$

$\therefore \quad \lim_{x \to 0} f(x)$ does not exist.

**9.** **(a)** We have $f: R \to R$, a differentiable function and $f(1) = 4$

**NOTE THIS STEP**

$$\lim_{x \to 1} \int_4^{f(x)} \frac{2t}{x - 1} dt = \lim_{x \to 1} \left[\frac{t^2}{x - 1}\right]_4^{f(x)}$$

$$= \lim_{x \to 1} \frac{(f(x))^2 - 16}{x - 1} = \lim_{x \to 1} \frac{f(x) - 4}{x - 1} . \lim_{x \to 1} (f(x) + 4)$$

$= f'(1) . (f(1) + 4) = 8 f'(1) \quad$ [Using $f(1) = 4$]

**10.** **(b)** We have $f(x) = [\tan^2 x]$

$\tan x$ is an increasing function for $-\frac{\pi}{4} < x < \frac{\pi}{4}$

$\therefore \quad \tan\left(-\frac{\pi}{4}\right) < \tan x < \tan\left(\frac{\pi}{4}\right)$ **NOTE THIS STEP**

$\Rightarrow \quad -1 < \tan x < 1 \Rightarrow 0 < \tan^2 x < 1$

$\Rightarrow \quad [\tan^2 x] = 0$

Hence, $\lim_{x \to 0} f(x) = \lim_{x \to 0} [\tan^2 x] = 0$

Also $f(0) = 0$

$\therefore \quad f(x)$ is continuous at $x = 0$

**11.** **(c)** When $x$ is not an integer, both the functions $[x]$ and $\cos\left(\frac{2x - 1}{2}\right)\pi$ are continuous.

$\therefore \quad f(x)$ is continuous on all non integral points.

For $x = n \in I$

$$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [x] \cos\left(\frac{2x - 1}{2}\right)\pi$$

$$= (n - 1) \cos\left(\frac{2n - 1}{2}\right)\pi = 0$$

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [x] \cos\left(\frac{2x - 1}{2}\right)\pi$$

$$= n \cos\left(\frac{2n - 1}{2}\right)\pi = 0$$

Also $f(n) = n \cos \frac{(2n - 1)\pi}{2} = 0$

$\therefore f$ is continuous at all integral pts as well.

Thus, $f$ is continuous everywhere.

**12.** **(b)** We have $= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}}$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{n\sqrt{1 + (r/n)^2}}$$

$$= \int_0^2 \frac{x}{\sqrt{1 + x^2}} dx \quad \left[\because \lim_{n \to \infty} \frac{1}{r} \sum_{r=0}^{a_n} f\left(\frac{r}{n}\right) = \int_0^a f(x) dx\right]$$

$$= \left[\sqrt{1 + x^2}\right]_0^2 = \sqrt{5} - 1$$

**13.** **(d)** We have $f(x) = [x]^2 - [x^2]$

At $x = 0$,

L.H.L. $= \lim_{h\to 0} f(-h) = \lim_{h\to 0}[-h]^2 - [(-h)^2]$

$= \lim_{h\to 0} f(-1)^2 - [h^2] = \lim_{h\to 0} 1 - 0 = 1$

R.H.L. $= \lim_{h\to 0} f(h) = \lim_{h\to 0}[h]^2 - [h^2]$

$= \lim_{h\to 0} 0 - 0 = 0$

$\because$ L.H.L. $\neq$ R.H.L.

$\therefore$ $f(x)$ is not continuous at $x = 0$.

**At $x = 1$**

L.H.L. $= \lim_{h\to 0} f(1-h) = \lim_{h\to 0}[1-h]^2 - [(1-h)^2]$

$= \lim_{h\to 0} 0 - 0 = 0$

R.H.L. $= \lim_{h\to 0} f(1+h) = \lim_{h\to 0}[1+h]^2 - [(1+h)^2]$

$= \lim_{h\to 0} 1 - 1 = 0$

$f(1) = [1]^2 - [1^2] = 1 - 1 = 0$

$\therefore$ L.H.L. = R.H.L. = f(1)

$\therefore$ f(x) is continuous at x = 1.

Clearly at other integral pts f(x) is not continuous.

**14.** **(d)** We have $|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

Also, $|x^2 - 3x + 2| = |(x-1)(x-2)|$

$$= \begin{cases} (1-x)(2-x) & \text{if } x < 1 \\ (x-1)(2-x) & \text{if } 1 \leq x < 2 \\ (x-1)(x-2) & \text{if } x \geq 2 \end{cases}$$

As $\cos(-\theta) = \cos\theta \Rightarrow \cos|x| = \cos x$

$\therefore$ Given function can be written as

$$f(x) = \begin{cases} (x^2-1)(x-1)(x-2) + \cos x & \text{if } x \leq 1 \\ -(x^2-1)(x-1)(x-2) + \cos x & \text{if } 1 \leq x < 2 \\ (x^2-1)(x-1)(x-2) + \cos x & \text{if } x \geq 2 \end{cases}$$

This function is differentiable at all points except possibly at x = 1 and x = 2.

$Lf'(1) = \left\{\frac{d}{dx}[(x^2-1)(x-1)(x-2) + \cos x]\right\}_{x=1}$

$= -\sin 1$

$Rf'(1) = \left\{\frac{d}{dx}[-(x^2-1)(x-1)(x-2) + \cos x]\right\}_{x=1}$

$= -\sin 1$

$\because$ $Lf'(1) = Rf'(1)$

$\therefore$ f is differentiable at x = 1.

$Lf'(2) = \left\{\frac{d}{dx}[-(x^2-1)(x-1)(x-2) + \cos x]\right\}_{x=2}$

$= -3 - \sin 2$

$Rf'(2) = \left\{\frac{d}{dx}((x^2-1)(x-1)(x-2) + \cos x)\right\}_{x=2}$

$= 3 - \sin 2$

$Lf'(2) \neq Rf'(2)$

$\therefore$ f is not differentiable at x = 2.

**15.** **(c)** $\lim_{x\to 0} \frac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$

$$= \lim_{x\to 0} \frac{x\left\{2x + \frac{8x^3}{3} + \frac{64x^5}{15} + \ldots\right\} - 2x\left\{x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\right\}}{4\sin^4 x}$$

$$= \lim_{x\to 0} \frac{x^4\left\{\frac{8}{3} - \frac{2}{3} + \text{terms containing higher positive powers of } x\right\}}{4\sin^4 x}$$

$= \frac{1}{4}.2 = \frac{1}{2}$

**16.** **(c)** For $x \in R$,

$$\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x = \lim_{x\to\infty}\left\{\left[1 - \frac{5}{x+2}\right]^{\frac{-(x+2)}{5}}\right\}^{\frac{-5x}{x+2}}$$

$$= e^{\lim_{x\to\infty} -\frac{5}{1+\frac{2}{x}}} = e^{-5}$$

**17.** **(b)** $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2} = \lim_{x\to 0}\frac{\sin(\pi - \pi\sin^2 x)}{x^2}$

$[\sin(\pi - \theta) = \sin\theta]$

$= \lim_{x\to 0}\frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x} \times \frac{(\pi\sin^2 x)}{x^2} = \pi$

**18.** **(a)** At $\underset{\text{at } x=k}{LHD} = \lim_{h\to 0}\frac{f(k) - f(k-h)}{h}$ (k = integer)

$= \lim_{h\to 0}\frac{[k]\sin k\pi - [k-h]\sin(k-h)\pi}{h}$

$= \lim_{h\to 0}\frac{-(k-1)\sin(k-h)\pi}{h}$ $[\because \sin k\pi = 0]$

$= \lim_{h\to 0}\frac{-(k-1)\sin(k\pi - h\pi)}{h}$

$[\sin(k\pi - \theta) = (-1)^{k-1}\sin\theta]$

$= \lim_{h\to 0}\frac{-(k-1)(-1)^{k-1}\sin h\pi}{h\pi} \times \pi = \pi(k-1)(-1)^k$

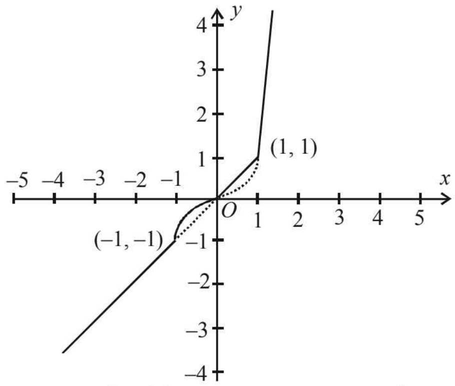
**19.** **(d)** $f(x) = \max.\{x, x^3\}$

$$= \begin{cases} x\ ; & x < -1 \\ x^3; & -1 \leq x \leq 0 \\ x\ ; & 0 \leq x \leq 1 \\ x^3; & x \geq 1 \end{cases}$$

**KEY CONCEPT**

A continuous function $f(x)$ is not differentiable at $x = a$ if graphically it takes a sharp turn at $x = a$.

Graph of $f(x) = \max\{x, x^3\}$ is as shown with solid lines.




From graph of $f(x)$ at $x = -1, 0, 1$, we have sharp turns.

$\therefore$ $f(x)$ is not differentiable at $x = -1, 0, 1$.

**20. (d)** Let us test each of four options.

(a) $f(x) = \cos|x| + |x| = \begin{cases} \cos x - x, & x < 0 \\ \cos x + x, & x \geq 0 \end{cases}$

$f'(x) = \begin{cases} -\sin x - 1, & x < 0 \\ -\sin x + 1, & x \geq 0 \end{cases}$

At $x = 0, LD = -1, RD = 1$

$\therefore$ Not differentiable

(b) $f(x) = \cos|x| - |x| = \begin{cases} \cos x + x, & x < 0 \\ \cos x - x, & x \geq 0 \end{cases}$

$\therefore$ Not differentiable at $x = 0$

(c) $f(x) = \sin|x| + |x| = \begin{cases} -\sin x - x, & x < 0 \\ \sin x - x, & x \geq 0 \end{cases}$

$\therefore$ Not differentiable at $x = 0$

(d) $f(x) = \sin|x| - |x| = \begin{cases} -\sin x + x, & x < 0 \\ \sin x - x, & x \geq 0 \end{cases}$

$f'(x) = \begin{cases} -\cos x + 1, & x < 0 \\ \cos x - 1, & x \geq 0 \end{cases}$

At $x = 0, LD = 0, RD = 0$

$\therefore$ $f$ is differentiable at $x = 0$.

**21. (d)** The given function is

$$f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \leq 1 \\ \frac{1}{2}(|x| - 1) & \text{if } |x| > 1 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} \frac{1}{2}(-x-1) & \text{if } x < -1 \\ \tan^{-1} x & \text{if } -1 \leq x \leq 1 \\ \frac{1}{2}(x-1) & \text{if } x > 1 \end{cases}$$

Clearly L.H.L. at $(x = -1) = \lim_{h \to 0} f(-1-h) = 0$

R.H.L. at $(x = -1) = \lim_{h \to 0} f(-1+h)$

$= \lim_{h \to 0} \tan^{-1}(-1+h) = 3\pi/4$

$\therefore$ L.H.L. $\neq$ R.H.L. at $x = -1$

$\therefore$ $f(x)$ is discontinuous at $x = -1$

Also we can prove in the same way, that $f(x)$ is discontinuous at $x = 1$

$\therefore$ $f'(x)$ can not be found for $x = \pm 1$ or domain of $f'(x) = R - \{-1, 1\}$

**22. (c)** Given that,

$$\lim_{x \to 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n} = \text{finite non zero number}$$

$$= \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)(e^x - \cos x)}{x^n (1 + \cos x)}$$

$$= \lim_{x \to 0} \left(\frac{\sin^2 x}{x^2}\right) \cdot \left(\frac{e^x - \cos x}{x^{n-2}}\right) \cdot \left(\frac{1}{1 + \cos x}\right)$$

$$= \lim_{x \to 0} 1^2 . \frac{e^x - \cos x}{x^{n-2}} . \frac{1}{2}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{e^x - \sin x}{(x-2)x^{n-3}} \quad \text{[using L' Jospital's rule]}$$

For this limit to be finite, $n - 3 = 0 \Rightarrow n = 3$

**23. (c)** Given that $f; R \to R$ such that $f(1) = 3$ and $f'(1) = 6$

Then $\lim_{x \to 0} \left[\frac{f(1+x)}{f(1)}\right]^{1/x}$

$$= e^{\lim_{x \to 0} \frac{1}{x}[\log f(1+x) - \log f(1)]}$$

$$= e^{\lim_{x \to 0} \frac{\frac{1}{f(1+x)} f'(1+x)}{1}} \quad \text{[Using L' Hospital rule]}$$

$$= e^{\frac{f'(1)}{f(1)}} = e^{6/3} = e^2$$

**24. (d)** We are given that

$$\lim_{x \to 0} \frac{[(a-n)nx - \tan x]\sin nx}{x^2} = 0$$

where $n$ is non zero real number

$$\Rightarrow \lim_{x \to 0} n . \frac{\sin nx}{nx}\left[\left\{(a-n)n - \frac{\tan x}{x}\right\}\right] = 0$$

$$\Rightarrow 1.n[(a-n)n - 1] = 0 \Rightarrow a = \frac{1}{n} + n$$

**25. (d)** Let $L = \lim_{h \to 0} \frac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)}$ $\left[\frac{0}{0} \text{form}\right]$

$\therefore$ Applying L Hospital's rule, we get

$$L = \lim_{h \to 0} \frac{f'(2h + 2 + h^2).(2 + 2h)}{f'(h - h^2 + 1).(1 - 2h)}$$

$$= \frac{f'(2).2}{f'(1).1} = \frac{6 \times 2}{4 \times 1} = 3$$

**26. (c)** Let $L = \lim_{x \to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$ $\left[\because f'(a) > 0, f \text{ being strictly increasing}\right]$

Using L.H. Rule, we get

$$L = \lim_{x \to 0} \frac{f'(x^2).2x - f'(x)}{f'(x)} = \lim_{x \to 0} \frac{f'(x^2).2x}{f'(x)} - 1 = 0 - 1$$

$= -1$

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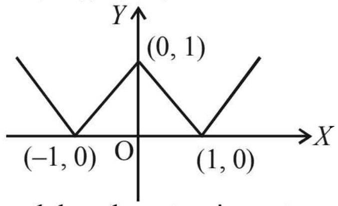
**27. (a)** Graph of $y = ||x| - 1|$ is as follows :




The graph has sharp turnings at $x = -1, 0$ and $1$; and hence not differentiable at $x = -1, 0, 1$.

**28. (b)** Given that $f(x)$ is a continuous and differentiable function and $f\left(\frac{1}{x}\right) = 0, x = n, n \in I$

$\therefore \quad f(0^+) = f\left(\frac{1}{\infty}\right) = 0$

Since R.H.L. = 0,

$\therefore \quad f(0) = 0$ for $f(x)$ to be continuous.

Also $f'(0) = \lim\limits_{h\to 0} \frac{f(h) - f(0)}{h - 0} = \lim\limits_{h\to 0} \frac{f(h)}{h} = 0$

$= 0$ [Using $f(0) = 0$ and $f(0^+) = 0$]

Hence $f(0) = 0, f'(0) = 0$

**29. (c)** $\lim\limits_{x\to 0} [(\sin x)^{1/x} + (1/x)^{\sin x}]$

$= \lim\limits_{x\to 0} (\sin x)^{1/x} + \lim\limits_{x\to 0} \left(\frac{1}{x}\right)^{\sin x}$

$= 0 + e^{\lim\limits_{x\to 0} \sin x \log\left(\frac{1}{x}\right)}$

[$\because |\sin x| < 1$ when $x \to 0$]

$= e^{\lim\limits_{x\to 0} \frac{-\log x}{\operatorname{cosec} x}} = e^{\lim\limits_{x\to 0} \frac{-1/x}{-\operatorname{cosec} x \cot x}}$

[Using L' Hospital rule]

$= e^{\lim\limits_{x\to 0} \frac{\sin x}{x} . \tan x} = e^0 = 1$

**30. (a)** Given that $f(x)$ is differentiable on $(0, \infty)$ with

$f(1) = 1$ and $\lim\limits_{t\to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$

$\Rightarrow \quad \lim\limits_{t\to x} \frac{2t f(x) - x^2 f'(t)}{1} = 1$ [Using L' Hospital rule]

$\Rightarrow \quad 2xf(x) - x^2 f'(x) = 1 \Rightarrow f'(x) - \frac{2}{x} f(x) = -\frac{1}{x^2}$

[Linear differential equation]

Integrating factor

**NOTE THIS STEP**

$e^{\int -\frac{2}{x} dx} = e^{-2\log x} = e^{\log 1/x^2} = \frac{1}{x^2}$

$\therefore \quad$ Solution is $f(x) \times \frac{1}{x^2} = \int \left(-\frac{1}{x^2}\right) \times \frac{1}{x^2} dx$

$\Rightarrow \quad \frac{f(x)}{x^2} = \frac{1}{3x^3} + C \quad \Rightarrow f(x) = Cx^2 + \frac{1}{3x}$

Also $f(1) = 1$

$\Rightarrow \quad 1 = C + \frac{1}{3} \quad \Rightarrow \quad C = 2/3 \quad \therefore f(x) = \frac{2}{3}x^2 + \frac{1}{3x}$

**31. (a) KEY CONCEPT**

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\, dt\right] = f(h(x))\, h'(x) - f(g(x)).g'(x)$$

$$= \frac{f(2) \times 2 \times 2 \times 1}{2 \times \frac{\pi}{4}} = \frac{8}{\pi} f(2)$$

Let $L = \lim\limits_{x\to \frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t)dt}{x^2 - \frac{\pi^2}{16}}$ $\left[\frac{0}{0}\text{form}\right]$

On applying L' Hospital's rule, we get

$$L = \lim_{x\to \frac{\pi}{4}} \frac{\frac{d}{dx}\left[\int_2^{\sec^2 x} f(t)dt\right]}{\frac{d}{dx}\left(x^2 - \frac{\pi^2}{16}\right)}$$

$$L = \lim_{x\to \frac{\pi}{4}} \frac{f(\sec^2 x).2\sec^2 x \tan x}{2x}$$

**32. (c)** As per question,

$p$ = left hand derivative of $|x - 1|$ at $x = 1 \Rightarrow p = -1$

Also $\lim\limits_{x\to 1+} g(x) = p$

Where $g(x) = \frac{(x-1)^n}{\log \cos^m (x-1)}$, $0 < x < 2$,

$m, n$ are integers, $m \neq 0, n > 0$

$\therefore$ we get,

$\lim\limits_{x\to 1^+} \frac{(x-1)^n}{\log \cos^m (x-1)} = -1 \Rightarrow \lim\limits_{h\to 0} \frac{h^n}{\log \cos^m h} = -1$

$\Rightarrow \lim\limits_{h\to 0} \frac{h^n}{m(\log \cos h)} = -1$ [Using L' Hospital's rule]

$\Rightarrow \lim\limits_{h\to 0} \frac{n h^{n-1} \cos h}{m(-\sin h)} = -1$ [Using L' Hospital's rule]

$\Rightarrow \lim\limits_{h\to 0} \frac{n h^{n-2} \cos h}{m\left(\frac{\sin h}{h}\right)} = 1 \Rightarrow n = 2$ and $m = 2$

**33. (d)** $\lim\limits_{x\to 0} [1 + x\ell n(1 + b^2)]^{\frac{1}{x}} = 2b \sin^2 \theta$

$\Rightarrow e^{\lim\limits_{x\to 0} \frac{1}{x} \ell n[1 + x\ell n(1+b^2)]} = 2b \sin^2 \theta$

$\Rightarrow e^{\lim\limits_{x\to 0} \frac{\ell n[1 + x\ell n(1+b^2)]}{x\ell n(1+b^2)} \times \ell n(1+b^2)} = 2b \sin^2 \theta$

$\Rightarrow e^{\ell n(1+b^2)} = 2b \sin^2 \theta$

$\Rightarrow 1 + b^2 = 2b \sin^2 \theta \Rightarrow 2 \sin^2 \theta = b + \frac{1}{b}$

We know that $2 \sin^2 \theta \le 2$ and $b + \frac{1}{b} \ge 2$ for $b > 0$

$\therefore 2\sin^2\theta = b + \frac{1}{b} = 2 \Rightarrow \sin^2\theta = 1$

As $\theta \in (-\pi, \pi]$, $\therefore \theta = \pm\frac{\pi}{2}$

**34.** **(b)** Given: $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1} - ax - b\right) = 4$

$\Rightarrow \lim_{x\to\infty}\frac{x^2+x+1-ax^2-ax-bx-b}{x+1} = 4$

$\Rightarrow \lim_{x\to\infty}\frac{(1-a)x^2+(1-a-b)x+(1-b)}{x+1} = 4$

For this limit to be finite $1-a=0 \Rightarrow a=1$
then given limit reduces to

$\lim_{x\to\infty}\frac{-bx+(1-b)}{x+1} = 4 \Rightarrow \lim_{x\to\infty}\frac{-b+\frac{(1-b)}{x}}{1+\frac{1}{x}} = 4$

$\Rightarrow -b = 4$ or $b = -4$
Hence $a = 1$, $b = -4$.

**35.** **(b)** We have $f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0}\frac{h^2\left|\cos\frac{\pi}{h}\right|}{h} = \lim_{h\to 0} h\left|\cos\frac{\pi}{h}\right|$
$= 0 \times$ some finite value $= 0$

Also, $f'(0^-) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{h^2\left|\cos\frac{\pi}{-h}\right|}{-h}$

$= \lim_{h\to 0} -h\left|\cos\frac{\pi}{h}\right| = 0 \times$ some finite value $= 0$

$\because$ $f'(0^+) = f'(0^-) \Rightarrow f$ is differentiable at $x=0$

Now $f'(2^+) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$

$= \lim_{h\to 0}\frac{(2+h)^2\left|\cos\frac{\pi}{2+h}\right| - 4\left|\cos\frac{\pi}{2}\right|}{h}$

$= \lim_{h\to 0}\frac{(2+h)^2\left(\cos\frac{\pi}{2+h}\right)}{h}$

$= \lim_{h\to 0}\frac{(2+h)^2}{h}\sin\left(\frac{\pi}{2}-\frac{\pi}{2+h}\right)$

$= \lim_{h\to 0}\frac{(2+h)^2}{h}\sin\left(\frac{\pi h}{2(2+h)}\right)$

$= \lim_{h\to 0}\frac{(2+h)^2}{h}\times\frac{\sin\left(\frac{\pi h}{2(2+h)}\right)}{\left(\frac{\pi h}{2(2+h)}\right)}\times\frac{\pi h}{2(2+h)} = \pi$

Also $f'(2^-) = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$

$= \lim_{h\to 0}\frac{(2-h)^2\left|\cos\left(\frac{\pi}{2-h}\right)\right| - 0}{-h}$

$= \lim_{h\to 0}\frac{-(2-h)^2\cos\left(\frac{\pi}{2-h}\right)}{-h}$

$= \lim_{h\to 0}\frac{(2-h)^2\sin\left(\frac{\pi}{2}-\frac{\pi}{2-h}\right)}{h}$

$= \lim_{h\to 0}\frac{(2-h)^2}{h}\times\frac{\sin\left(\frac{-\pi h}{2(2-h)}\right)}{\left(\frac{-\pi h}{2(2-h)}\right)}\times\left(\frac{-\pi h}{2(2-h)}\right) = -\pi$

As $f'(2^+) \neq f'(2^-) \Rightarrow f$ is not differentiable at $x = 2$.

**36.** **(b)** The given equation is

$(\sqrt[3]{1+a}-1)x^2 + (\sqrt{1+a}-1)x + (\sqrt[6]{1+a}-1) = 0$

Let $a + 1 = y$, then equation reduces to
$(y^{1/3}-1)x^2 + (y^{1/2}-1)x + (y^{1/6}-1) = 0$
Dividing both sides by $y - 1$, we get

$\left(\frac{y^{1/3}-1}{y-1}\right)x^2 + \left(\frac{y^{1/2}-1}{y-1}\right)x + \left(\frac{y^{1/6}-1}{y-1}\right) = 0$

Taking limit as $y \to 1$ i.e. $a \to 0$ on both sides we get

$\frac{1}{3}x^2 + \frac{1}{2}x + \frac{1}{6} = 0 \Rightarrow 2x^2 + 3x + 1 = 0$

$\Rightarrow x = -1, -\frac{1}{2}$ (roots of the equation)

Thus $\lim_{a\to 0^+}\alpha(a) = -1$, $\lim_{a\to 0^+}\beta(a) = -\frac{1}{2}$

## D. MCQs with ONE or MORE THAN ONE Correct

**1.** **(a, b, d)** Given that $x + |y| = 2y$
If $y < 0$ then $x - y = 2y$
$\Rightarrow y = x/3 \Rightarrow x < 0$
If $y = 0$ then $x = 0$. If $y > 0$ then $x + y = 2y$
$\Rightarrow y = x \Rightarrow x > 0$

Thus we can define $f(x) = y = \begin{cases} x/3, & x < 0 \\ x, & x \geq 0 \end{cases}$

**Continuity at $x = 0$**

$LL = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}(-h/3) = 0$

$RL = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} h = 0$

$f(0) = 0$
As $LL = RL = f(0)$
$\therefore$ $f(x)$ is continuous at $x = 0$
**Differentiability at $x = 0$**
$Lf' = 1/3$ ; $Rf' = 1$

As $Lf' \neq Rf' \Rightarrow f(x)$ is not differentiable at $x = 0$

But for $x < 0$, $\frac{dy}{dx} = \frac{1}{3}$.

**2.** **(b)** We have $f(x) = x(\sqrt{x} - \sqrt{x+1})$

Let us check differentiability of $f(x)$ at $x = 0$

$$Lf'(0) = \lim_{h\to 0} \frac{(0-h)[\sqrt{0-h} - \sqrt{0-h+1}] - 0}{-h}$$

$$= \lim_{h\to 0}[\sqrt{-h} - \sqrt{-h+1}] = 0 - \sqrt{1} = -1$$

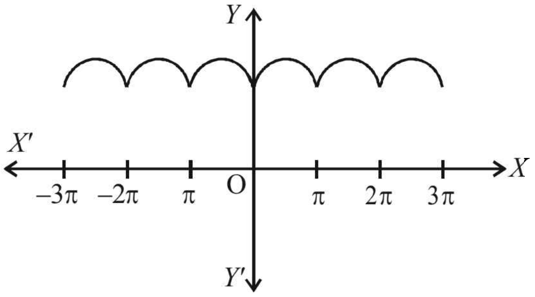
$$Rf'(0) = \lim_{h\to 0} \frac{(0+h)[\sqrt{0+h} - \sqrt{0+h+1}] - 0}{h}$$

$$= \lim_{h\to 0} \sqrt{h} - \sqrt{h+1} = -1$$

Since $Lf'(0) = Rf'(0)$

$\therefore$ $f$ is differentiable at $x = 0$.

**3.** **(b,d,e)** The graph of $f(x) = 1 + |\sin x|$ is as shown in the fig.




From graph it is clear that function is continuous every where but not differentiable at integral multiples of $\pi$ ($\because$ at these points curve has sharp turnings)

**4.** **(a,b,d)** We have, for $-1 \le x \le 1 \Rightarrow 0 \le x\sin \pi x \le 1/2$

$\therefore$ $f(x) = [x \sin \pi x] = 0$

Also $x \sin \pi x$ becomes negative and numerically less than 1 when $x$ is slightly greater than 1 and so by definition of $[x]$,

$f(x) = [x \sin \pi x] = -1$ when $1 < x < 1 + h$

Thus $f(x)$ is constant and equal to 0 in the closed interval $[-1, 1]$ and so $f(x)$ is continuous and differentiable in the open interval $(-1, 1)$.

At $x = 1$, $f(x)$ is clearly discontinuous, since $f(1-0) = 0$ and $f(1+0) = -1$ and $f(x)$ is non-differentiable at $x = 1$.

**5.** **(a)** The given function is,

$$f(x) = \frac{x}{1+|x|} = \begin{cases} \frac{x}{1-x}, & x < 0 \\ \frac{x}{1+x}, & x \ge 0 \end{cases}$$

For $x < 0$, $f'(x) = \frac{1(1-x)-(-1)x}{(1-x)^2} = \frac{1}{(1-x)^2}$,

For $x > 0$, $f'(x) = \frac{1(1+x)-1(x)}{(1-x)^2} = \frac{1}{(1+x)^2}$,

For $x = 0$,

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{\frac{-h}{1+h} - 0}{-h} = 1$$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\frac{h}{1+h} - 0}{h} = 1$$

$Lf'(0) = Rf'(0)$

$\Rightarrow$ $f$ is differentiable at $x = 0$

Hence $f$ is differentiable in $(-\infty, \infty)$.

**6.** **(a,b,c)** $f(x) = \begin{cases} |x-3| & , x \ge 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4} & , x < 1 \end{cases}$

$$= \begin{cases} \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4} & , x < 1 \\ 3 - x & , 1 \le x < 3 \\ x - 3 & , x \ge 3 \end{cases}$$

$Lf'(1) = \frac{2x}{4} - \frac{3}{2} = -1$

$Rf'(1) = -1$. Thus $Lf'(1) = Rf'(1)$

$\therefore$ $f$ is differentiable at $x = 1$ and hence continuous at $x = 1$.

Again, $Lf'(3) = -1$ and $Rf'(3) = 1$

$\Rightarrow$ $Lf'(3) \neq Rf'(3)$

$\Rightarrow$ $f$ is not differentiable at $x = 3$

Let us now check the continuity at $x = 3$

L.H.L. $= \lim_{h\to 0} f(3-h) = \lim_{h\to 0}[3-(3-h)] = 0$

R.H.L. $= \lim_{h\to 0} f(3+h) = \lim_{h\to 0}[3+h-3] = 0$

$f(3) = 0$

$\therefore$ $f$ is continuous at $x = 3$

**7.** **(b)** We have, $f(x) = \frac{x}{2} - 1$

$\therefore$ $[f(x)] = \left[\frac{x}{2} - 1\right] = -1$, $0 \le x < 2$

$\tan[f(x)] = \tan(-1), 0 \le x < 2 = 0, 2 \le x \le \pi$

$\therefore$ The function $\tan[f(x)]$ is discontinuous at $x = 2$.

Also the function $\frac{1}{f(x)} = \frac{1}{\frac{x}{2} - 1} = \frac{2}{x-2}$ is discontinuous at $x = 2$.

Thus both the given functions $\tan[f(x)]$ as well as $\frac{1}{f(x)}$ are discontinuous on the interval $[0, \pi]$.

Also $f^{-1}(x) = y$

$\Rightarrow x = f(y) = \frac{y}{2} - 1 \Leftrightarrow y = 2(x+1)$

$\therefore$ $f^{-1}(x) = 2(x+1)$ is continuous on $[0, \pi]$

**8.** **(d)** $\lim_{x\to 0} \frac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x}$

$$= \lim_{x\to 0} \frac{\sqrt{\frac{1}{2}.2\sin^2 x)}}{x} = \lim_{x\to 0} \frac{|\sin x|}{x}$$

$\therefore$ L.H.L. $= \lim\limits_{h\to 0} \dfrac{|\sin(0-h)|}{0-h} = \lim\limits_{h\to 0} \dfrac{|-\sin h|}{-h}$

$= \lim\limits_{h\to 0} \dfrac{\sin h}{-h} = -1$

R.H.L. $= \lim\limits_{h\to 0} \dfrac{|\sin(0+h)|}{0+h} = \lim\limits_{h\to 0} \dfrac{\sin h}{h} = 1$

As L.H.L. $\neq$ R.H.L.

$\therefore$ The given limit does not exist.

**9.** **(b,c)** On $(0, \pi)$

(a) $\tan x = f(x)$

We know that $\tan x$ is discontinuous at $x = \pi/2$

(b) $f(x) = \int_0^x t \sin\left(\dfrac{1}{t}\right) dt$

**NOTE THIS STEP**

$\Rightarrow f'(x) = x \sin\left(\dfrac{1}{x}\right)$ which exists on $(0, \pi)$

$\therefore f(x)$, being differentiable, is continuous on $(0, \pi)$.

(c) $f(x) = \begin{cases} 1 & , 0 < x \le 3\pi/4 \\ 2\sin\dfrac{2x}{9} & , 3\pi/4 < x < \pi \end{cases}$

Clearly $f(x)$ is continuous on $(0, \pi)$ except possibly at $x = \dfrac{3\pi}{4}$, where,

L.H.L. $= \lim\limits_{h\to 0} f\left(\dfrac{3\pi}{4} - h\right) = \lim\limits_{x\to 0} 1 = 1$

R.H.L. $= \lim\limits_{h\to 0} f\left(\dfrac{3\pi}{4} + h\right) = \lim\limits_{x\to 0} 2\sin\dfrac{2}{9}\left(\dfrac{3\pi}{4} + h\right)$

$= \lim\limits_{h\to 0} 2\sin\left(\dfrac{\pi}{6} + \dfrac{2h}{9}\right) = 2\sin\dfrac{\pi}{6} = 2.\dfrac{1}{2} = 1$

Also $f\left(\dfrac{3\pi}{4}\right) = 1$

As L.H.L. = R.H.L. $= f\left(\dfrac{3\pi}{4}\right)$

$\therefore f(x)$ is continuous on $(0, \pi)$

(d) $f(x) = \begin{cases} x\sin x & , 0 < x \le \pi/2 \\ \dfrac{\pi}{2}\sin(\pi + x) & , \dfrac{\pi}{2} < x < \pi \end{cases}$

Here $f(x)$ will be continuous on $(0, \pi)$ if it is continuous at $x = \pi/2$.

At $x = \pi/2$,

L.H.L. $= \lim\limits_{h\to 0} f\left(\dfrac{\pi}{2} - h\right)$

$= \lim\limits_{h\to 0} \left(\dfrac{\pi}{2} - h\right)\sin\left(\dfrac{\pi}{2} - h\right) = \dfrac{\pi}{2}\sin\dfrac{\pi}{2} = \dfrac{\pi}{2}$

R.H.L. $= \lim\limits_{h\to 0} f\left(\dfrac{\pi}{2} + h\right) = \lim\limits_{h\to 0} \dfrac{\pi}{2}\sin\left(\pi + \dfrac{\pi}{2} + h\right)$

$= \dfrac{\pi}{2}\sin\left(\pi + \dfrac{\pi}{2}\right) = \dfrac{-\pi}{2}\sin\dfrac{\pi}{2} = -\dfrac{\pi}{2}$

As L.H.L. $\neq$ R.H.L.

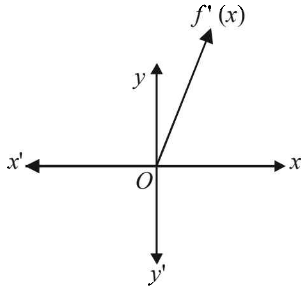
$\therefore$ $f(x)$ is not continuous on $(0, \pi)$.

**10.** **(b,c,d)** $f(x) = \begin{cases} 0, & x < 0 \\ x^2, & x \ge 0 \end{cases}$

$\therefore$ $f'(x) = \begin{cases} 0, & x < 0 \\ 2x, & x \ge 0 \end{cases}$

which exists $\forall\, x$ except possibly at $x = 0$.

At $x = 0$, $Lf' = 0 = Rf' \Rightarrow f$ is differentiable.




From graph of $f'(x)$, it is clear that $f'(x)$ is continuous but not differentiable at $x = 0$.

**11.** **(a, b)** We have $g(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0 & , x = 0 \end{cases}$

If $x \neq 0$, $g'(x) = x^2 \cos(1/x)\left(-\dfrac{1}{x^2}\right) + 2x\sin\dfrac{1}{x}$

$= -\cos\left(\dfrac{1}{x}\right) + 2x\sin\left(\dfrac{1}{x}\right)$

which exists for $\forall x \neq 0$.

If $x = 0$,

$g'(0) = \lim\limits_{x\to 0} \dfrac{g(x) - g(0)}{x - 0}$

$= \lim\limits_{x\to 0} \dfrac{x^2\sin(1/x) - 0}{x - 0} = \lim\limits_{x\to 0} x\sin\left(\dfrac{1}{x}\right) = 0$

$\Rightarrow$ $g'(x) = \begin{cases} -\cos\left(\dfrac{1}{x}\right) + 2x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

At $x = 0$, $\cos\left(\dfrac{1}{x}\right)$ is not continuous, therefore $g'(x)$ is not continuous at $x = 0$.

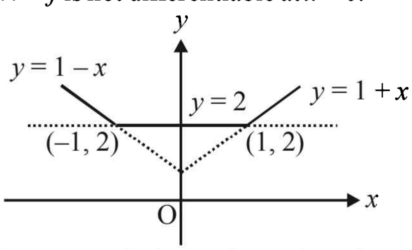
At $x = 0$

Lf' $= \lim\limits_{x\to 0} \dfrac{0 - (-x)\sin\left(-\dfrac{1}{x}\right)}{x} = -\sin\left(\dfrac{1}{x}\right)$

which does not exist.

$\therefore$ $f$ is not differentiable at $x = 0$.

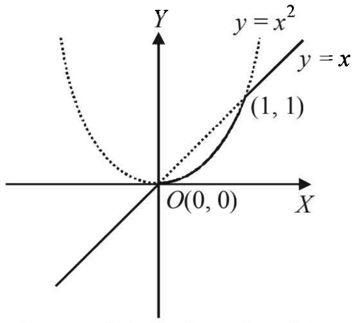
**12.** **(a, c)**




From graph it is clear that $f(x)$ is continuous every where and also differentiable everywhere except at $x = 1$ and $-1$.

**13.** **(a, c, d)** From the figure it is clear that

$$h(x)=\begin{cases} x, & \text{if } x\le 0 \\ x^2, & \text{if } 0<x<1 \\ x, & \text{if } x\ge 1 \end{cases}$$




From the graph it is clear that $h$ is continuous for all $x\in R$, $h'(x)=1$ for all $x>1$ and $h$ is not differentiable at $x=0$ and 1.

**14.** **(d)** L.H.L. $=\lim\limits_{x\to 1^-}\dfrac{\sqrt{1-\cos[2(x-1)]}}{x-1}$

$=\lim\limits_{x\to 1^-}\dfrac{\sqrt{2\sin^2(x-1)}}{x-1}=\sqrt{2}.\lim\limits_{x\to 1^-}\dfrac{\sqrt{\sin^2(x-1)}}{x-1}$

$=\sqrt{2}\lim\limits_{x\to 1^-}\dfrac{|\sin(x-1)|}{x-1}=\sqrt{2}.\lim\limits_{h\to 0}\dfrac{|\sin(-h)|}{-h}$

$=\sqrt{2}\lim\limits_{h\to 0}\dfrac{\sin h}{-h}=-\sqrt{2}$

Again,

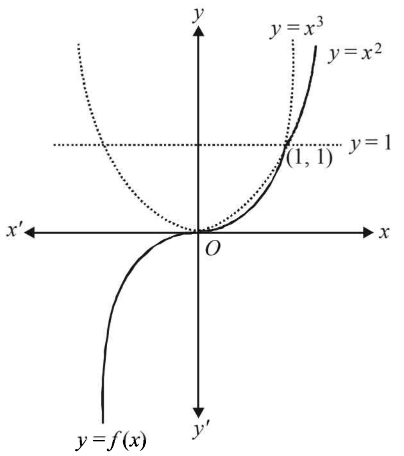
R.H.L. $=\lim\limits_{x\to 1^+}\dfrac{\sqrt{1-\cos(x-1)}}{x-1}=\lim\limits_{x\to 1^+}\sqrt{2}\dfrac{|\sin(x-1)|}{x-1}$

Put $x=1+h$, $h>0$ for $x\to 1^+$, $h\to 0$.

$=\lim\limits_{h\to 0}\sqrt{2}\dfrac{|\sin h|}{h}=\lim\limits_{h\to 0}\sqrt{2}\dfrac{\sin h}{h}=\sqrt{2}$

L.H.L. $\ne$ R.H.L. Therefore $\lim\limits_{x\to 1}f(x)$ does not exist.

**15.** **(a, d)** From graph, $f(x)$ is continuous everywhere but not differentiable at $x=1$.




**16.** **(a,c)** Given that $L=\lim\limits_{x\to 0}\dfrac{a-\sqrt{a^2-x^2}-\dfrac{x^2}{4}}{x^4}, a>0$

and $L$ is finite.

Now $L=\lim\limits_{x\to 0}\dfrac{\dfrac{x}{\sqrt{a^2-x^2}}-\dfrac{x}{2}}{4x^3}$ (Using L'Hospital's rule)

$=\lim\limits_{x\to 0}\dfrac{\dfrac{1}{\sqrt{a^2-x^2}}-\dfrac{1}{2}}{4x^2}$

$\because$ $L$ is finite, limiting value of numerator should be zero which is so when $\dfrac{1}{\sqrt{a^2}}-\dfrac{1}{2}=0$

i.e. $a=2$ $(\because a>0)$

Applying $L$ 'Hospital's rule again, we get

$L=\lim\limits_{x\to 0}\dfrac{\dfrac{x}{(a^2-x^2)^{3/2}}}{8x}=\lim\limits_{x\to 0}\dfrac{1}{8(a^2-x^2)^{3/2}}$

$=\dfrac{1}{8\times a^3}=\dfrac{1}{8\times 8}$ (using $a=2$)

$=\dfrac{1}{64}$

**17.** **(b, c)** $\because f(x+y)=f(x)+f(y)\ \forall\ x,y\in R$

$\therefore$ Putting $x=y=0$, we get

$f(0)=0$

Also $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$=\lim\limits_{h\to 0}\dfrac{f(h)}{h}=f'(0)=k$ (say)

$\Rightarrow f(x)=kx+c$

But $f(0)=0\Rightarrow c=0$

$\therefore f(x)=kx$

Which is continuous and differentiable $\forall\ x\in R$.

$\therefore$ b and c are the correct options.

**18.** **(a, b, c, d)**

At $x=-\dfrac{\pi}{2}$

LHL $=\lim\limits_{x\to -\frac{\pi^{-1}}{2}}-x-\dfrac{\pi}{2}=0$

RHL $=\lim\limits_{x\to -\frac{\pi^+}{2}}-\cos x=0$ and $f\left(\dfrac{-\pi}{2}\right)=0$

$\therefore$ LHL = RHL = $f\left(\dfrac{-\pi}{2}\right)$

$\Rightarrow f(x)$ is continuous at $x=-\dfrac{\pi}{2}$

Also at $x=0$

$Lf'(0)=\sin 0=0$; $Rf'(0)=1-0=1$

$\because Lf'(0)\ne Rf'(0)$

$\Rightarrow f$ is not differentiable at $x=0$

At $x=1$

$Lf'(1)=Rf'(1)\Rightarrow f$ is differentiable at $x=1$.

At $x=\dfrac{-3}{2}$, $f(x)=-\cos x$ which is differentiable.

$\therefore$ All four options are correct.

**19. (b,d)**

We have $f(x) = \begin{cases} a_n + \sin \pi x, & x \in [2n, 2n+1] \\ b_n + \cos \pi x, & x \in (2n-1, 2n) \end{cases}$

As $f$ is continuous for all $n$

$\therefore$ At $x = 2n$, LHL = RHL = $f(2n)$

$\Rightarrow b_n + \cos 2\pi n = a_n + \sin 2\pi n = a_n + \sin 2\pi n$

$\Rightarrow b_n + 1 = a_n \Rightarrow a_n - b_n = 1$

$\therefore$ $b$ is correct.

Also at $x = 2n + 1$, LHL = RHL = $f(2n + 1)$

$\Rightarrow \lim_{h \to 0} a_n + \sin \pi (2n + 1 - h)$

$= \lim_{h \to 0} b_{n+1} + \cos \pi(2n + 1 - h) = a_n + \sin (2n + 1)\pi$

$\Rightarrow a_n = b_{n+1} - 1 = a_n \Rightarrow a_n - b_{n+1} = -1$

$\therefore$ $c$ is incorrect

$\Rightarrow a_{n-1} - b_n = -1$

$\therefore$ $d$ is correct.

**20. (b,d)**

$$\lim_{n \to \infty} \frac{1^a + 2^a + --- + n^a}{(n+1)^{a-1}[(na+1) + (na+2) + --- + (na+n)]} = \frac{1}{60}$$

$$\Rightarrow \lim_{n \to \infty} \frac{n^a}{(n+1)^{a-1}} \frac{\left(\frac{1}{n}\right)^a + \left(\frac{2}{n}\right)^a + ---- + \left(\frac{n}{n}\right)^a}{n^2 a + \frac{n(n+1)}{2}} = \frac{1}{60}$$

$$\Rightarrow \lim_{n \to \infty} \frac{n^{a-1}}{(n+1)^{a-1}} \frac{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^a}{a + \frac{1}{2}\left(1 + \frac{1}{n}\right)} = \frac{1}{60}$$

$$\Rightarrow \lim_{n \to \infty} \left(\frac{1}{1 + \frac{1}{n}}\right)^{a-1} \frac{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^a}{a + \frac{1}{2}\left(1 + \frac{1}{n}\right)} = \frac{1}{60}$$

$$\Rightarrow \frac{\int_0^1 x^a dx}{a + \frac{1}{2}} = \frac{1}{60} \Rightarrow \frac{\left[x^{a+1}\right]_0^1}{(a+1)\left(a + \frac{1}{2}\right)} = \frac{1}{60}$$

$$\Rightarrow \frac{1}{(a+1)\left(a + \frac{1}{2}\right)} = \frac{1}{60}$$

$$\Rightarrow 2a^2 + 3a - 119 = 0 \Rightarrow a = 7 \text{ or } -\frac{17}{2}$$

**21. (a,c)** $g(x)$ may be discontinuous at $x = a$ or $x = b$.

Let us check the continuity of $g(x)$ at $x = a$ and $x = b$

$$\lim_{x \to a^-} g(x) = 0$$

$$\lim_{x \to a^+} g(x) = \lim_{x \to a^+} \int_a^x f(t)\,dt = \int_a^a f(t)\,dt = 0$$

$$g(a) = \int_a^a f(t)\,dt = 0$$

$\therefore$ $g(x)$ is continuous at $x = a$

Also $\lim_{x \to b^-} g(x) = \lim_{x \to b^-} \int_a^x f(t)\,dt = \int_a^b f(t)\,dt$

$$\lim_{x \to b^+} g(x) = \int_a^b f(t)\,dt \Rightarrow g(b) = \int_a^b f(t)\,dt$$

$\therefore$ $g(x)$ is continuous at $x = b$

Hence $g(x)$ is continuous $\forall x \in R$

Now $g'(x) = \begin{cases} 0, & x < a \\ f(x), & a \le x \le b \\ 0, & x > b \end{cases}$

$g'(a^-) = 0$ and $g'(a^+) = f(a)$

$g'(b^-) = f(b)$ and $g'(b^+) = 0$

As $f(a), f(b) \in [1, \infty)$ $\therefore f(a), f(b) \neq 0$

Hence $g'(a^-) \neq g'(a^+)$ and $g'(b^-) \neq g'(b^+)$

$\therefore$ $g$ is not differentiable at $a$ and $b$.

**22. (a, d)** Let $f$ and $g$ be maximum at $c_1$ and $c_2$ respectively, $c_1, c_2 \in (0,1)$

Then, $f(c_1) = g(c_2)$

Let $h(x) = f(x) - g(x)$

Then, $h(c_1) = f(c_1) - g(c_1) > 0$

and $h(c_2) = f(c_2) - g(c_2) < 0$

$\therefore$ $h(x) = 0$ has atleast one root in $(c_1, c_2)$

$C \in (c_1, c_2)$ i.e. for $f(c) = g(c)$

which shows that (a) and (d) are correct.

**23. (a, d)** $f(x) = \begin{cases} \frac{x}{|x|} g(x), & x \neq 0 \\ 0 & , x = 0 \end{cases}$

$$= \begin{cases} -g(x) & , x < 0 \\ 0 & , x = 0 \\ g(x) & , x > 0 \end{cases}$$

$$f'(x) = \begin{cases} -g'(x) & , x < 0 \\ 0 & , x = 0 \\ g'(x) & , x > 0 \end{cases}$$

$\therefore$ $Lf'(0) = -g'(0) = 0$

$Rf'(0) = g'(0) = 0$

$\therefore$ $f$ is differentiable at $x = 0$

$h(x) = e^{|x|} = \begin{cases} e^{-x} & , x < 0 \\ e^x & , x \ge 0 \end{cases}$

$h'(x) = \begin{cases} -e^{-x} & , x < 0 \\ e^x & , x \ge 0 \end{cases} \Rightarrow Lh'(0) = -1, Rh'(0) = 1$

$\therefore$ $h$ is not differentiable at $x = 0$

$f\text{o}h(x) = f(h(x)) = g(e^{|x|})$ as $e^{|x|} > 0$

$$= \begin{cases} g(e^{-x}) & if \quad x<0 \\ g(1) & if \quad x=0 \\ g(e^{x}) & if \quad x>0 \end{cases}$$

$$f'[h(x)] = \begin{cases} -g'(e^{-x}).e^{-x} & , \quad x<0 \\ 0 & , \quad x=0 \\ g'(e^{x})e^{x} & , \quad x>0 \end{cases}$$

$Lf'(h(0)) = -g'(1), Rf'(h(0)) = g'(1)$

$\because \quad g'(1) \neq 0, \therefore Lf'(h(0)) \neq Rf'(h(0))$

$\therefore \quad foh$ is not differentiable at $x=0$.

$$hof(x) = \begin{cases} e^{|f(x)|} & , \quad x \neq 0 \\ 1 & , \quad x=0 \end{cases}$$

$$Lh'(f(0)) = \lim_{k \to 0} \frac{h(f(0)) - h(f(0-k))}{k}$$

$$= \lim_{k \to 0} \frac{1-e^{|g(-k)|}}{k} = \lim_{k \to 0} \frac{1-e^{|g(-k)|}}{|g(-k)|} \times \frac{|g(-k)|}{k}$$

$$= 1 \times 0 = 0 \left( \because g'(0) = 0 \Rightarrow \lim_{k \to 0} \frac{g(-k)}{k} = \lim_{k \to 0} \frac{g(k)}{k} = 0 \right)$$

$$Rh'(f(0)) = \lim_{k \to 0} \frac{h(f(0+k)) - h(f(0))}{k}$$

$$= \lim_{k \to 0} \frac{e^{|g(k)|}-1}{k} = \lim_{k \to 0} \frac{e^{|g(k)|}-1}{|g(k)|} \times \frac{|g(k)|}{k} = 0$$

$\therefore \quad hof$ is differentiable at $x=0$.

**24. (a, b)** $f(x) = a\cos^3(|x^3 - x|) + b|x|\sin(|x^3+x|)$

(a) If $a=0, b=1$

$\Rightarrow \quad f(x) = |x|\sin|x^3+x|$

$= x\sin(x^3+x), x \in R$

$\therefore$ f is differentiable every where.

(b), (c) If $a=1, b=0 \Rightarrow f(x) = \cos^3(|x^3-x|)$

$= \cos^3(x^3-x)$

which is differentiable every where.

(d) when $a=1, b=1, f(x) = \cos(x^3-x) + x\sin(x^3+x)$

which is differentiable at $x=1$

$\therefore$ Only a and b are the correct options.

**25. (b, c)** $f(x) = [x^2-3]$ is discontinuous at all integral points in $\left[-\frac{1}{2}, 2\right]$

Which happens when $x = 1, \sqrt{2}, \sqrt{3}, 2$

$\therefore$ f is discontinuous exactly at four points in $\left[-\frac{1}{2}, 2\right]$

Also $g(x) = (|x| + |4x-7|)f(x)$

Here f is not differentiable at $x = 1, \sqrt{2}, \sqrt{3} \in \left(-\frac{1}{2}, 2\right)$

and $|x| + |4x-7|$ is not differentiable at 0 and $\frac{7}{4}$

But $f(x) = 0, \forall x \in \left[\sqrt{3}, 2\right]$

$\therefore \quad$ g(x) becomes differentiable at $x = \frac{7}{4}$

Hence g(x) is non-differentiable at four points i.e., $0, 1, \sqrt{2}, \sqrt{3}$

## E. SUBJECTIVE PROBLEMS

**1.** $$\lim_{x \to a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$$

$$= \lim_{x \to a} \frac{(\sqrt{a+2x}-\sqrt{3x})(\sqrt{a+2x}+\sqrt{3x})(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{3a+x}-2\sqrt{x})(\sqrt{3a+x}+2\sqrt{x})(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \lim_{x \to a} \frac{(a+2x-3x)(\sqrt{3a+x}+2\sqrt{x})}{(3a+x-4x)(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \lim_{x \to a} \frac{(a-x)(\sqrt{3a+x}+2\sqrt{x})}{3(a-x)(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \lim_{x \to a} \frac{(\sqrt{3a+x}+2\sqrt{x})}{3(\sqrt{a+2x}+\sqrt{3x})} = \frac{\sqrt{3a+a}+2\sqrt{a}}{3(\sqrt{a+2a}+\sqrt{3a})}$$

$$= \frac{4\sqrt{a}}{3 \times 2\sqrt{3a}} = \frac{2}{3\sqrt{3}}$$

**NOTE :** The given limit is of the form $\frac{0}{0}$. Hence limit of the function can also be find out by using L' Hospital's Rule.

**2.** $$f(x) = \int \frac{2\sin x - \sin 2x}{x^3} dx, \; x \neq 0$$

$$\therefore \quad f'(x) = \frac{2\sin x - \sin 2x}{x^3}, \; x \neq 0$$

$$\therefore \quad \lim_{x \to 0} f'(x) = \lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3}$$

$$= \lim_{x \to 0} \frac{2\sin x(1-\cos x)(1+\cos x)}{x^3(1+\cos x)}$$

$$= \lim_{x \to 0} 2.\frac{\sin^3 x}{x^3}.\frac{1}{1+\cos x}$$

$$= 2 \times (1)^3 \times \frac{1}{2} = 1$$

**3.** $$\lim_{h \to 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$$

$$= \lim_{h \to 0} \frac{a^2[\sin(a+h) - \sin a] + 2ah\sin(a+h) + h^2\sin(a+h)}{h}$$

$$= \lim_{h \to 0} \frac{a^2\left[2\cos\left(a+\frac{h}{2}\right)\sin\frac{h}{2}\right]}{2 \times \frac{h}{2}} + 2a\sin(a+h) + h\sin(a+h)$$

$= a^2\cos a + 2a\sin a$

**4.** As f(x) is continuous at x = 0, we have

LHL = RHL = f(0)

$$\Rightarrow \quad \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(0+h) = f(0)$$

$$\Rightarrow \quad f(0) + \lim_{h \to 0} f(-h) = f(0) + \lim_{h \to 0} f(h) = f(0)$$

[Using the given property $f(x+y) = f(x) + f(y)$]

$$\Rightarrow \quad \lim_{h \to 0} f(-h) = \lim_{h \to 0} f(h) = 0 \qquad \text{...(1)}$$

Now let x = a be any arbitrary point then at x = a,

LHL $= \lim_{h\to 0} f(a-h) = \lim_{h\to 0}[f(a)+f(-h)]$

[Using, $f(x+y)=f(x)+f(y)$]

$= f(a) + \lim_{h\to 0} f(-h) = f(a)$ [using $eq^n$ (1)]

Similarly, R.H.L. $= \lim_{h\to a} f(a+h) = f(a)$

Thus, we get

$\lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(a+h) = f(a)$

$\Rightarrow$ f is continuous at x = a. But a is any arbitrary point

$\therefore$ f is continuous $\forall x \in R$.

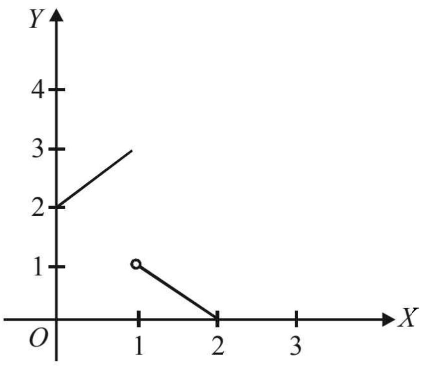
5. $$\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1} = \lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}\times\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$$

$$= \lim_{x\to 0}\frac{(2^x-1)(\sqrt{1+x}+1)}{1+x-1}$$

$$= \lim_{x\to 0}\frac{2^x-1}{x}.\lim_{x\to 0}(\sqrt{1+x}+1)$$

$= \ln 2 . (1 + 1) = 2 \ln 2.$

6. Graph of $f(f(x))$ is




Clearly form graph $f(f(x))$ is discontinuous at $x = 1$ and 2.

7. We have $f(x) = \frac{x^2}{2}, 0 \le x < 1 = 2x^2 - 3x + \frac{3}{2}, 1 \le x \le 2$

Here $f(x)$ is continuous everywhere except possibly at $x = 1$

$\Rightarrow$ At $x = 1$, $Lf' = \frac{2}{2}\times 1 = 1$; $Rf' = 4\times 1 - 3 = 1$

$\Rightarrow$ $f$ is differentiable and hence continuous at $x = 1$

$\therefore$ $f(x)$ is continuous on [0, 2]

$f'(x) = x,\ 0 \le x < 1$

$= 4x - 3,\ 1 \le x \le 2$

**At $x = 1$,**

$\lim_{x\to 1^-} f'(x) = \lim_{h\to 0} f'(1-h) = \lim_{h\to 0}(1-h) = 1$

$\lim_{x\to 1^+} f'(x) = \lim_{h\to 0} f'(1+h) = \lim_{h\to 0} 4(1+h) - 3 = 1$

$f'(1) = 4 - 3 = 1$

$\therefore$ $f'$ is continuous at $x = 1$

$\therefore$ $f'$ is continuous on [0, 2]

$$f''(x) = \begin{cases} 2 & , 0 \le x < 1 \\ 4 & , 1 \le x \le 2 \end{cases}$$

Clearly $f''(x)$ is discontinuous at $x = 1$,

$\therefore$ $f''(x)$ is discontinuous on [0, 2].

8. Given $f(x) = x^3 - x^2 + x + 1$

$\therefore$ $f'(x) = 3x^2 - 2x + 1 = 3\left(x^2 - \frac{2}{3}x + \frac{1}{3}\right)$

$$= 3\left[\left(x-\frac{1}{3}\right)^2 - \frac{1}{9} + \frac{1}{3}\right]$$

$$= 3\left[\left(x-\frac{1}{3}\right)^2 + \frac{2}{9}\right] > 0 \forall x \in R.$$

Hence $f(x)$ is an increasing function of $x$ for all real values of $x$.

Now max $[f(t): 0 \le t \le x]$ means the greatest value of $f(t)$ in $0 \le t \le x$ which is obtained at $t = x$, since $f(t)$ is increasing for all t.

$\therefore$ max. $[f(t): 0 \le t \le x] = x^3 - x^2 + x + 1$

Hence the function g is defined as follows :

$g(x) = x^3 - x^2 + x + 1$ when $0 \le x \le 1$

$= 3 - x$ when $1 < x \le 2$

Now it is sufficient to discuss the continuity and differentiability of $g(x)$ at $x = 1$. Since for all other values of $x$, $g(x)$ is clearly continuous and differentiable, being a polynomial function of $x$.

We have, $g(1) = 2$

$g(1-0) = \lim_{h\to 0}[(1-h)^3 - (1-h)^2 + (1-h) + 1] = 2$

$g(1+0) = \lim_{h\to 0}[3 - (1+h)] = 2$

Hence $g(x)$ is continuous at $x = 1$

Now,

$$Lg'(1) = \lim_{h\to 0}\frac{[(1-h)^3 - (1-h)^2 + (1-h) + 1] - 2}{-h}$$

$$= \lim_{h\to 0}\frac{1 - 3h + 3h^2 - h^3 - 1 + 2h - h^2 + 1 - h + 1 - 2}{-h}$$

$$= \lim_{h\to 0}\frac{-2h + 2h^2 - h^3}{-h} = \lim_{h\to 0}[2 - 2h + h^2] = 2$$

$$Rg'(1) = \lim_{h\to 0}\frac{[3 - (1+h) - 2]}{h} = \lim_{h\to 0}\frac{-h}{h} = -1$$

Since $Lg'(1) \ne Rg'(1)$, the function $g(x)$ is not differentiable at $x = 1$

Hence $g(x)$ is continuous on (0, 2). It is also differentiable on (0, 2) except at $x = 1$.

9. We have $f(x) = -1,\ -2 \le x \le 0$

$= x - 1,\ 0 < x \le 2$

and $g(x) = f(|x|) + |f(x)|$

Hence $g(x)$ involves $|x|$ and $|x-1|$ or $|-1| = 1$

Therefore we should divide the given interval (–2, 2) into the following intervals.

| $I_1$ | $I_2$ | $I_3$ |
|---|---|---|
| $[-2, 2] = [-2, 0)$ | $[0, 1)$ | $[1, 2]$ |
| $x = -\text{ve}$ | $+\text{ve}$ | $+\text{ve}$ |
| $\|x\| = -x$ | $x$ | $x$ |
| $f(x) = -1$ | $x-1$ | $x-1$ |
| $f(\|x\|) = -1$ | $= x-1$ | $= x-1$ |
| $\|f(x)\| = \|-1\|$ | $\|x-1\|$ | $\|x-1\|$ |
| $= 1$ | $= -(x-1)$ | $= x-1$ |

$\therefore$ Using above we get

$g(x) = f|x| + |f(x)|$

$= -1 + 1 = 0$ in $I_1$

$= x-1 - (x-1) = 0$ in $I_2$

$= x-1 + x-1 = 2(x-1)$ in $I_3$

Hence g (x) is defined as follows :

$$g(x) = \begin{cases} 0, & -2 \le x < 1 \\ 2(x-1), & 1 \le x \le 2 \end{cases}$$

$Lg'(1) = 0; Rg'(1) = 2$ (not equal )

Hence $g(x)$ is not differentiable at $x = 1$.

**10.** Let $h(x) = f(x) + g(x)$ be continuous.

Then, $g(x) = h(x) - f(x)$

Now, $h(x)$ and $f(x)$ both are continuous functions.

$\therefore$ $h(x) - f(x)$ must also be continuous. But it is a contradiction as given that $g(x)$ is discontinuous. Therefore our assumption that $f(x) + g(x)$ a continous function is wrong and hence $f(x) + g(x)$ is discontinuous.

**11.** Given that $f(x)$ is a function satisfying

$$f(-x) = f(x), \forall x \in R \quad ...(1)$$

Also $f'(0)$ exists

$\Rightarrow f'(0) = Rf'(0) = Lf'(0)$

Now, $Rf'(0) = f'(0)$

$$\Rightarrow \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = f'(0)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(h) - f(0)}{h} = f'(0)$$

Again $Lf'(0) = f'(0)$ ...(2)

$$\Rightarrow \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = f'(0)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = f'(0)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(h) - f(0)}{h} = f'(0) \quad ...(3)$$

[Using eq. (1)]

From equations (2) and (3), we get

$\Rightarrow f'(0) = 0$

**12.** Given that,

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & , 0 \le x < \frac{\pi}{4} \\ 2x\cot x + b & , \frac{\pi}{4} \le x \le \frac{\pi}{2} \\ a\cos 2x - b\sin x & , \frac{\pi}{2} < x \le \pi \end{cases}$$

is continuous for $0 \le x \le \pi$.

$\therefore$ $f(x)$ must be continuous at $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$

$$\lim_{x \to \left(\frac{\pi}{4}\right)^-} f(x) = f\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \lim_{h \to 0} f\left(\frac{\pi}{4} - h\right) = \frac{2\pi}{4}\cot\frac{\pi}{4} + b$$

$$\Rightarrow \lim_{h \to 0}\left(\frac{\pi}{4} - h\right) + a\sqrt{2}\sin\left(\frac{\pi}{4} - h\right) = \frac{\pi}{2} + b$$

$$\Rightarrow \frac{\pi}{4} + a = \frac{\pi}{2} + b$$

$$\Rightarrow a - b = \frac{\pi}{4} \quad ...(1)$$

Also, $\lim_{x \to \left(\frac{\pi}{2}\right)^+} f(x) = f\left(\frac{\pi}{2}\right)$

$$\Rightarrow \lim_{h \to 0} f\left(\frac{\pi}{2} + h\right) = 2.\frac{\pi}{2}\cot\frac{\pi}{2} + b$$

$$\Rightarrow \lim_{h \to 0} a\cos 2\left(\frac{\pi}{2} + h\right) - b\sin\left(\frac{\pi}{2} + h\right) = b$$

$$\Rightarrow a\cos\pi - b\sin\frac{\pi}{2} = b \Rightarrow -a - b = b$$

$$\Rightarrow a + 2b = 0 \quad ...(2)$$

Solving (1) and (2), we get $a = \frac{\pi}{6}$ and $b = \frac{-\pi}{12}$.

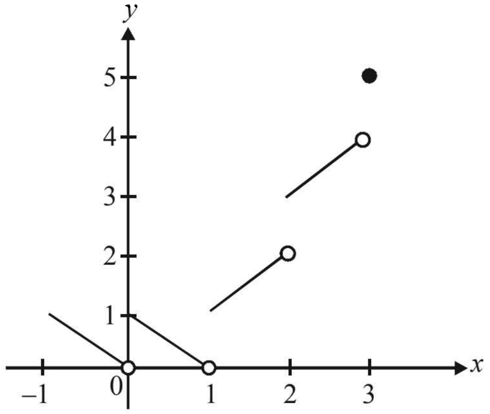
**13.** We have, $[x] + |1-x|, -1 \le x \le 3$

**NOTE THIS STEP**

$$\text{or } y = \begin{cases} -1+1-x & , -1 \le x < 0 \\ 0+1-x & , 0 \le x < 1 \\ 1-1+x & , 1 \le x < 2 \\ 2-1+x & , 2 \le x < 3 \\ 3-1+x & , x = 3 \end{cases}$$

$$\text{or } y = \begin{cases} -x , & -1 \le x < 0 \\ 1-x , & 0 \le x < 1 \\ x , & 1 \le x < 2 \\ 1+x , & 2 \le x < 3 \\ 2+x , & x = 3 \end{cases}$$

From graph we can say that given functions is not differentiable at $x = 0, 1, 2, 3$.



**14.** We are given that,

$$f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2} & ,x<0\\ a & ,x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}-4}} & ,x>0\end{cases}$$

Here L.H.L at $(x=0)$

$$=\lim_{h\to 0}\frac{1-\cos 4(0-h)}{(0-h)^2}=\lim_{h\to 0}\frac{1-\cos 4h}{h^2}$$

$$=\lim_{h\to 0}\frac{2\sin^2 2h}{4h^2}.4=8$$

R.H.L at $(x=0)$

$$=\lim_{h\to 0}\frac{\sqrt{0+h}}{\sqrt{16+\sqrt{0+h}}-4}=\lim_{h\to 0}\frac{\sqrt{h}(\sqrt{16+h}+4)}{16+\sqrt{h}-16}$$

$$=\lim_{h\to 0}\sqrt{16+\sqrt{h}}+4=\sqrt{16}+4=8$$

For continuity of function $f(x)$, we must have

L.H.L. = R.H.L = f(0)

$\Rightarrow$ f(0) = 8 $\Rightarrow$ $a=8$

**15.** We are given

$f(x+y)=f(x)\,f(y),\ \forall\ x,y\in R$

$f(x)\neq 0$, for any $x$

$f$ is differentiable at $x=0, f'(0)=2$

To prove that $f'(x)=2f(x),\ \forall\ x\in R$ and to find $f(x)$.

We have for $x=y=0$

$f(0+0)=f(0)f(0)$

$\Rightarrow f(0)=[f(0)]^2 \Rightarrow f(0)=1$

Again $f'(0)\neq 2$

$$\Rightarrow \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=2 \Rightarrow \lim_{h\to 0}\frac{f(0)f(h)-f(0)}{h}=2$$

$$\Rightarrow \lim_{h\to 0}\frac{f(0)[f(h)-1]}{h}=2$$

$$\Rightarrow \lim_{h\to 0}\frac{f(h)-1}{h}=2 \quad ...(1) \quad [\text{Using } f(0)=1]$$

$$\text{Now, } f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}=\lim_{h\to 0}f(x)\left(\frac{f(h)-1}{h}\right)$$

$$=f(x)\lim_{h\to 0}\left[\frac{f(h)-1}{h}\right]$$

$=f(x).2$ [Using eq. (1)]

$=2f(x)$

$$\text{Also, }\frac{f'(x)}{f(x)}=2$$

Integrating on both sides with respect to $x$, we get

$\log|f(x)|=2x+C$

At $x=0$, $\log f(0)=C \Rightarrow C=\log 1=0$

$\therefore \log|f(x)|=2x \Rightarrow f(x)=e^{2x}$

**16.** $$\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}\right)+x\right\}^{\frac{1}{x}}=e^{\lim\limits_{x\to 0}\log\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{\frac{1}{x}}}$$

$$\left[\text{Using } \lim_{x\to a}f(x)=e^{\lim\limits_{x\to 0}\log f(x)}\right]$$

$$=e^{\lim\limits_{x\to 0}\frac{\log\tan\left(\frac{\pi}{4}+x\right)}{x}} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$=e^{\lim\limits_{x\to 0}\left[\frac{\sec^2\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}\right)+x}\right]} \qquad [\text{Using } L'\text{ Hospital's rule}]$$

$$=e^{\frac{2}{1}}=e^2$$

**17.** Given that, $$f(x)=\begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}} & \frac{-\pi}{6}<x<0\\ b & .\ x=0\\ e^{\frac{\tan 2x}{\tan 3x}} & 0<x<\frac{\pi}{6}\end{cases}$$

is continuous at $x=0$

$$\therefore \lim_{h\to 0}f(0-h)=f(0)=\lim_{h\to 0}f(0+h)$$

We have,

$$\lim_{h\to 0}f(0-h)=\lim_{h\to 0}[1+|\sin(-h)|]^{\frac{a}{|\sin(-h)|}}$$

$$=\lim_{h\to 0}[1+\sin h]^{\frac{a}{\sin h}}$$

$$e^{\lim\limits_{h\to 0}\frac{a}{\sin h}\log(1+\sin h)}=e^a$$

and $f(0)=b$

$\therefore\ e^a=b$ ...(1)

$$\text{Also }\lim_{h\to 0}f(0+h)=\lim_{h\to 0}e^{\frac{\tan 2h}{\tan 3h}}$$

$$=e^{\lim\limits_{h\to 0}\frac{\tan 2h}{2h}\times\frac{3h}{\tan 3h}\times\frac{2}{3}}=e^{\frac{2}{3}}$$

$\therefore\ e^{\frac{2}{3}}=b$ ...(2)

From (1) and (2)

$$e^a=b=e^{\frac{2}{3}} \Rightarrow a=\frac{2}{3}\text{ and } b=e^{\frac{2}{3}}$$

**18.** $$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2} \quad ...(1)$$

Putting $y=0$ and $f(0)=1$ in (1), we get

$$f\left(\frac{x}{2}\right)=\frac{1}{2}[f(x)+1]$$

$$\therefore\ f(x)=2f\left(\frac{x}{2}\right)-1 \quad ...(2)$$

$$\text{Now, } f'(x)=\mathop{Lt}_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{f(2x)+f(2h)}{2}-f(x)\right], \text{ by (1)}$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{(2f(x)-1)+(2f(h)-1)}{2}-f(x)\right], \text{ by (2)}$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}[f(h)-1]$$

$$= \underset{h\to 0}{Lt}\frac{f(h)-f(0)}{h} = f'(0) = -1$$

Hence $f'(x) = -1$, integrating, we get
$f(x) = -x + c$. Putting $x = 0$, we get
$f(0) = c = 1$ by (1) $\quad \therefore f(x) = 1 - x$
$f(2) = 1 - 2 = -1$

**19.** By the given definition it is clear that the function $f$ is continuous and differentiable at all points except possible at $x = 1$ and $x = 2$.

Continuity at $x = 1$

$$\text{L.H.L.} = \lim_{h\to 0}[1-(1-h)] = \lim_{h\to 0} h = 0$$

$$\text{R.H.L.} = \lim_{h\to 0}[1-(1+h)][2-(1+h)]$$

$$= \lim_{h\to 0}\{-h(1-h)\} = 0$$

Also, $f(1) = 0$

$\therefore$ L.H.L. = R.H.L. = $f(1) = 0$

Therefore, $f$ is continuous at $x = 1$

Now, differentiability at $x = 1$

$$Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}, h>0$$

$$= \lim_{h\to 0}\frac{(1-(1-h)-0}{-h} = \lim_{h\to 0}\left(\frac{h}{-h}\right) = -1$$

$$\text{and } Rf'(1) = \lim_{h\to 0}\frac{f((1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{\{1-(1-h)\}\{2-(1-h)\}-0}{h}$$

$$= \lim_{h\to 0}\frac{-h(1-h)}{h} = \lim_{h\to 0}(h-1) = -1$$

Since $Lf'(1) = Rf'(1)$

Hence, f is differentiable at $x = 1$

Continuous at $x = 2$

$$\text{L.H.L.} = \lim_{h\to 0}[1-(2-h)][2-(2-h)]$$

$$\lim_{h\to 0}\{(-1+h)\}\{h\} = 0$$

$$\text{and R.H.L.} = \lim_{h\to 0}[3-(2+h)] = \lim_{h\to 0}(1-h) = 1$$

Since L.H.L. ≠ R.H.L., therefore $f$ is not continuous at $x = 2$. As such $f$ cannot be differentiable at $x = 2$. Hence $f$ is continuous and differentiable at all points except at $x = 2$.

**20.** Given that, $F(x) = \int_0^x f(t)dt$

**NOTE THIS STEP**

$\therefore \quad F'(x) = f(x).1 - f(0).0$

[Using Leibnitz theorem]

$\Rightarrow F'(x) = f(x)$ ... (1), $\forall\, x \ge 0$

Also $F(0) = \int_0^0 f(t)dt = 0$

But given that $f(x) \le cF(x), \forall\, x \ge 0$

$\therefore$ We get $f(0) \le cF(0) = 0$

$\therefore \quad f(0) \le 0 \quad$ ....(2)

But ATQ $f(x)$ is non- negative continuous function on $[0, \infty)$

$\therefore \quad f(x) \ge 0$

$\therefore \quad f(0) \ge 0 \quad$ ....(3)

$\therefore$ From (2) and (3) $f(0) = 0$

Again $f(x) \le cF(x) \forall\, x \ge 0$, we get

$$f(x) - cF(x) \le 0$$

$\Rightarrow F'(x) - cF(x) \le 0, \forall\, x \ge 0$ [Using equation (1)]

$$e^{cx}F'(x) - ce^{-cx}F(x) \le 0$$

[Multiplying both sides by $e^{-cx}$ (I.F.) and keeping in mind that $e^{-cx} > 0, \forall\, x$]

$$\Rightarrow \frac{d}{dx}[e^{-cx}F(x)] \le 0$$

$\Rightarrow g(x) = e^{-cx}F(x)$ is a decreasing function on $[0, \infty)$.

That is $g(x) \le g(0)$ for all $x \ge 0$

But $g(0) = F(0) = 0$

$\therefore \quad g(x) \le 0, \forall\, x \ge 0$

$\Rightarrow \quad e^{-cx}F(x) \le 0, \forall\, x \ge 0$

$\Rightarrow \quad F(x) \le 0, \forall\, x \ge 0$

$\therefore \quad f(x) \le cF(x) \le 0, \forall\, x \ge 0$

$[\therefore c > 0$ and using $f(x) \le cF(x)]$

$\Rightarrow \quad f(x) \le 0, \forall\, x \ge 0$

But given $f(x) \ge 0$

$\Rightarrow \quad f(x) = 0, \forall\, x \ge 0.$

**21.** (I) $g$ is continuous at $\alpha$ and

$$f(x) - f(\alpha) = g(x)(x-\alpha), \forall\, x \in R$$

$\Rightarrow$ Since $g$ is continuous at $x = \alpha$

$$\text{and } g(x) = \frac{f(x)-f(\alpha)}{x-\alpha}$$

We should have, $\lim_{x\to\alpha} g(x) = g(\alpha)$

$$\Rightarrow \lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{x-\alpha} = g(\alpha) \Rightarrow f'(\alpha) = g(\alpha)$$

$\Rightarrow f'(\alpha)$ exists and is equal to $g(\alpha)$.

(II) $\because \; f(x)$ is differentiable at $x = \alpha$

$$\therefore \lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{x-\alpha} = f'(\alpha)$$

exists and is finite.

Let us define,

$$g(x)=\begin{cases}\dfrac{f(x)-f(\alpha)}{x-\alpha}, & x\neq\alpha\\ f'(\alpha), & x=\alpha\end{cases}$$

Then, $f(x)-f(\alpha)=(x-\alpha)\,g(x)$, $\forall\ x\neq\alpha$.

Now for continuity of $g(x)$ at $x=\alpha$

$$\lim_{x\to\alpha} g(x)=\lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{x-\alpha}=f'(\alpha)=g(\alpha)$$

$\therefore$ $g$ is continuous at $x=\alpha$.

**22.** Given that

$$f(x)=\begin{cases}x+a, & \text{if } x<0\\ |x-1|, & \text{if } x\geq 0\end{cases}=\begin{cases}x+a, & \text{if } x<0\\ 1-x, & \text{if } 0\leq x<1\\ x-1, & \text{if } x\geq 1\end{cases}$$

and $g(x)=\begin{cases}(x+1), & \text{if } x<0\\ (x-1)^2+b, & \text{if } x\geq 0\end{cases}$

where $a,b\geq 0$

Then $(gof)(x)=g[f(x)]$

**NOTE THIS STEP**

$$=\begin{cases}f(x)+1, & \text{if } f(x)<0\\ [f(x)-1]^2+b, & \text{if } f(x)\geq 0\end{cases}$$

(Using definition of $g(x)$

Now, $f(x)<0$ when $x+a<0$ i.e. $x<-a$

$f(x)=0$ when $x=-a$ or $x=1$

$f(x)>0$ when $-a<x<1$ or $x>1$

$$g(f(x))=\begin{cases}f(x)+1, & \text{if } x<-a\\ [f(x)-1]^2+b, & \text{if } x=-a \text{ or } x=1\\ [f(x)-1]^2+b, & \text{if } -a<x<0\\ [f(x)-1]^2+b, & \text{if } 0\leq x<1\\ [f(x)-1]^2+b, & \text{if } x>1\end{cases}$$

[Keeping in mind that $x=0$ and 1 are also the breaking points because of definition of $f(x)$]

$$\therefore g[f(x)]=\begin{cases}x+a+1, & \text{if } x<-a\\ (x+a-1)^2+b, & \text{if } -a\leq x<0\\ (1+x)-1)^2+b, & \text{if } 0\leq x\leq 1\\ (x-1-1)^2+b, & \text{if } x>1\end{cases}$$

Substituting the value of $f(x)$ under different conditions).

$$\therefore\quad g[f(x)]=\begin{cases}x+a+1, & \text{if } x<-a\\ (x+a-1)^2+b, & \text{if } -a\leq x<0=F(x)\text{(say)}\\ x^2+b, & \text{if } 0\leq x\leq 1\\ (x-2)^2+b, & \text{if } x>1\end{cases}$$

Now given that $gof(x)\equiv F(x)$ is continuous for all real numbers, therefore it will be continuous at $-a$

$\Rightarrow$ L.H.S = R.H.L = $f(-a)$

$$\lim_{h\to 0}F(-a-h)=\lim_{h\to 0}F(-a+h)=f(-a)$$

Now, $\lim_{h\to 0}F(-a-h)=\lim_{h\to 0}(-a-h+a+1)=1$

$$\lim_{h\to 0}F(-a+h)=\lim_{h\to 0}(-a+h+a-1)^2+b=1+b$$

$F(-a)=1+b$

Thus we should have $1=1+b\Rightarrow b=0$.

Again for continuity at $x=0$

L.H.L $=f(0)$

$\Rightarrow \lim_{h\to 0}f(0-h)=f(0)$

$\Rightarrow \lim_{h\to 0}f(-h+a-1)^2+b=b\Rightarrow (a-1)^2=0\Rightarrow a=1$

For $a=1$ and $b=0$, $gof$ becomes

$$gof(x)=\begin{cases}x+2, & x<-1\\ x^2, & -1\leq x\leq 1\\ (x-2)^2 & x>1\end{cases}$$

Now to check differentiability of $gof(x)$ at $x=0$

We see, $gof(x)=x^2=F(x)$

$\Rightarrow$ $F'(x)=2x$ which exists clearly at $x=0$.

$gof$ is differentiable at $x=0$

**23.** Given that $f:[-2a, 2a]\to R$

$f$ is an odd function.

$Lf'$ at $x=a$ is 0.

$$\Rightarrow \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}=0$$

$$\Rightarrow \lim_{h\to 0}\frac{f(a-h)-f(a)}{h}=0 \qquad ...(1)$$

To find $Lf'$ at $x=-a$ which is given by

$$\lim_{h\to 0}\frac{f(-a-h)-f(-a)}{-h}=\lim_{h\to 0}\frac{-f(a+h)+f(a)}{-h}$$

$[\because f(-x)=-f(x)]$

$$=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Again for $x\in[a, 2a]$

$f(x)=f(2a-x)$

$\therefore$ $f(a+h)=f(2a-a-h)=f(a-h)$

Substiting this values in last expression we get

$$Lf'(-a)=\lim_{h\to 0}\frac{f(a-h)-f(a)}{h}$$

$=0$ [Using equation (1)]

Hence $Lf'(-a)=0$

**24.** To find,

$$\lim_{n\to\infty}\left[(n+1)\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right)-n\right]$$

$$=\lim_{n\to\infty}n\left[\left(1+\frac{1}{n}\right)\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right)-1\right]=\lim_{n\to\infty}nf\left(\frac{1}{n}\right)$$

where $f(x)=\left[(1+x)\frac{2}{\pi}\cos^{-1}x-1\right]$ such that

$$f(0)=\left[(1+0)\frac{2}{\pi}\cos^{-1}0-1\right]=\frac{2}{\pi}.\frac{\pi}{2}-1=0$$

$\therefore$ Using given relation as $\lim_{n\to\infty} nf\left(\frac{1}{n}\right)=f'(0)$

then given limit becomes

$$=f'(0)=\frac{d}{dx}\left[(1+x)\frac{2}{\pi}\cos^{-1}x-1\right]\Bigg|_{x=0}$$

$$=\frac{2}{\pi}\left[\cos^{-1}x-\frac{1-x}{\sqrt{1-x^2}}\right]\Bigg|_{x=0}$$

$$=\frac{2}{\pi}\left[\frac{\pi}{2}-1\right]=1-\frac{2}{\pi}=\frac{\pi-2}{\pi}.$$

**25.** Given that, $f(x)$ is differentiable at $x=0$.
Hence, $f(x)$ will also be continuous at $x=0$

$$\Rightarrow \lim_{h\to 0} f(0+h)=f(0) \Rightarrow \lim_{h\to 0}\frac{e^{\frac{ah}{2}}-1}{h}=\frac{1}{2}$$

$$\Rightarrow \lim_{h\to 0}\frac{e^{\frac{ah}{2}}-1}{\frac{ah}{2}}\times\frac{a}{2}=\frac{1}{2} \Rightarrow a=1$$

Also differentiability of $f(x)$ at $x=0$, gives
$Lf'(0)=Rf'(0)$

$$\Rightarrow \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{b\sin^{-1}\left(\frac{c-h}{2}\right)-\frac{1}{2}}{-h}=\lim_{h\to 0}\frac{e^{\frac{ah}{2}}-1-\frac{1}{2}}{h}$$

$$=\lim_{h\to 0}\frac{2e^{\frac{ah}{2}}-2-h}{2h^2} \quad [\text{form } \tfrac{0}{0}]$$

$$\lim_{h\to 0}\frac{\frac{b}{\sqrt{1-\left(\frac{c-h}{2}\right)^2}}.\left(-\frac{1}{2}\right)}{-1} \quad \text{[Using L Hospital's rule]}$$

$$=\lim_{h\to 0}\frac{2.e^{\frac{ah}{2}}.\frac{a}{2}-1}{4h}=\lim_{h\to 0}\frac{e^{\frac{h}{2}}-1}{8\left(\frac{h}{2}\right)} \quad [\text{Putting } a=1]$$

$$\Rightarrow \frac{b}{\sqrt{1-\frac{c^2}{4}}}=\frac{1}{8} \Rightarrow 4b=\sqrt{1-\frac{c^2}{4}} \Rightarrow 16b^2=\frac{4-c^2}{4}$$

$\Rightarrow 64b^2=4-c^2$ Hence proved.

**26.** Given that,

$f(x-y)=f(x).g(y)-f(y).g(x)$ ...(i)

$g(x-y)=g(x).g(y)+f(x)f(y)$ ...(ii)

In eqn. (i), putting $x=y$, we get

$f(0)=f(x)g(x)-f(x)g(x) \Rightarrow f(0)=0$

Putting $y=0$, in eqn. (i), we get

$f(x)=f(x)g(0)-f(0)g(x)$

$\Rightarrow f(x)=f(x)g(0)$ [using $f(0)=0$]

$\Rightarrow g(0)=1$

Putting $x=y$ in eqn. (ii), we get

$g(0)=g(x)g(x)+f(x)f(x)$

$\Rightarrow 1=[g(x)]^2+[f(x)]^2$ [using $g(0)=1$]

$\Rightarrow [g(x)]^2=1-[f(x)]^2$ ...(iii)

Clearly $g(x)$ will be differentiable only if $f(x)$ is differentiable.

$\therefore$ First we will check the differentiability of $f(x)$

Given that $Rf'(0)$ exists

i.e., $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ exists

i.e., $\lim_{h\to 0}\frac{f(0)g(-h)-f(-h)g(0)}{h}$ exists

i.e., $\lim_{h\to 0}\frac{-f(-h)}{h}$ exists (using $f(0)=0$ and $g(0)=1$)

Which can be written as,

$$\lim_{h\to 0}\frac{f(0)-f(-h)}{-h}=Lf'(0)$$

$\Rightarrow Lf'(0)=Rf'(0)$

$\therefore$ $f$ is differentiable, at $x=0$

Differentiating equation (iii), we get

$2g(x).g'(x)=-2f(x)f'(x)$

For $x=0$

$\Rightarrow g(0).g'(0)=-f(0)f'(0)$

$\Rightarrow g'(0)=0$ [Using $f(0)=0$ and $g(0)=1$]

## F. Match the Following

**1.** (A) $\sin(\pi[x])=0,\ \forall\ x\in R$

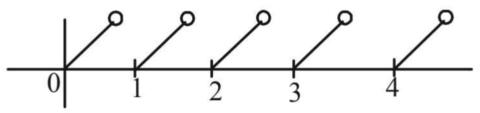
$\therefore$ Differentiable everywhere.

$\therefore$ (A) $\to$ (p)

(B) $\sin(\pi(x-[x]))=f(x)$

We know that

$$x-[x]\begin{cases}=x, \text{if } 0\le x<1\\ =x-1, \text{if } 1\le x<2\\ =x-2, \text{if } 2\le x<3\end{cases}$$

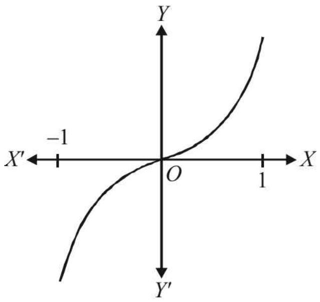



It's graph is, as shown in figure which is discontinuous at $\forall\ x\in z$. Clearly $x-[x]$ and hence $\sin(\pi(x-[x]))$ is not differentiable $\forall\ x\in z$

(B) $\to$ r

**2.** (A) $y=x|x|=\begin{cases}-x^2 & \text{if } x<0\\ x^2 & \text{if } x\ge 0\end{cases}$

Graph is as follows :



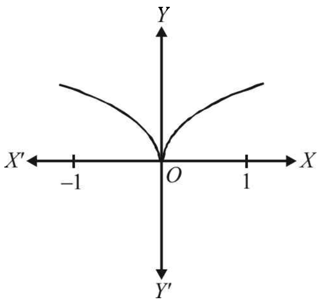

From graph $y = x|x|$ is continuous in $(-1, 1)$ **(p)**
differentiable in $(-1, 1)$ **(q)**
Strictly increasing in $(-1, 1)$. **(r)**

(B) $y = \sqrt{|x|} = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$

$\Rightarrow \ y^2 = -x, x < 0$
{where $y$ can take only + ve values}

and $y^2 = x, \ x \geq 0$

$\therefore$ Graph is as follows :

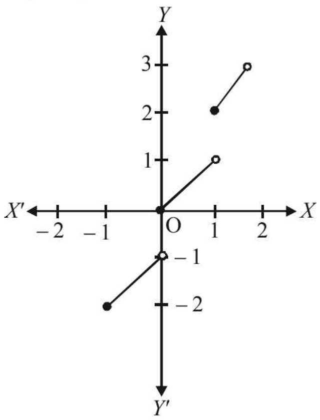



From graph $y = \sqrt{|x|}$ is continuous in $(-1, 1)$ ***(p)***
not differentiable at $x = 0$ ***(s)***

(C) **NOTE THIS STEP**

$$y = x + [x] = \begin{cases} - \quad - \quad - \\ x-1, & -1 \leq x < 0 \\ x, & 0 \leq x < 1 \\ x+1, & 1 \leq x < 2 \\ - \quad - \quad - \end{cases}$$

$\therefore$ Graph of $y = x + [x]$ is as follows :

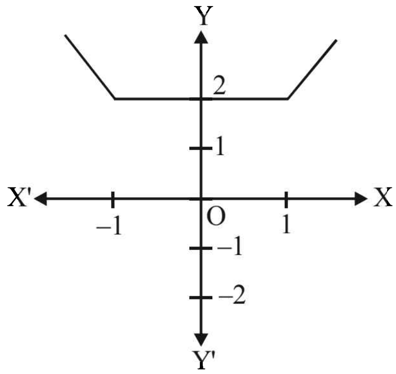



From graph, $y = x + [x]$ is neither continuous, nor differentiable at $x = 0$ and hence in $(-1, 1)$. ***(s)***

Also it is strictly increasing in $(-1, 1)$ ***(r)***

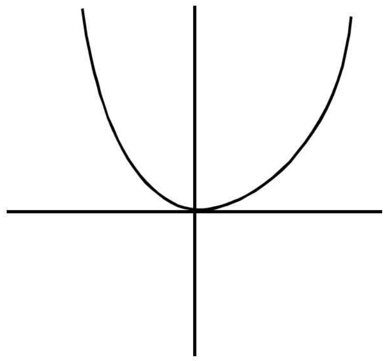
(D) $y = |x-1| + |x+1| = \begin{cases} -2x, & x < -1 \\ 2, & -1 \leq x < 1 \\ 2x, & x \geq 1 \end{cases}$

Graph of function is as follows :




From graph, $y = f(x)$ is continuous **(p)** and differentiable **(q)** in $(-1, 1)$ but not strictly increasing in $(-1, 1)$.

**3.** **(d)** $P(1)$: $f_4(x) = \begin{cases} x^2, & x < 0 \\ e^{2x} - 1, & x \geq 0 \end{cases}$

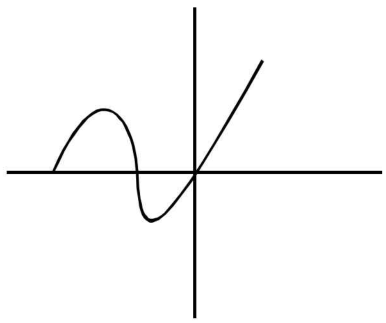


Range of $f_4 = [0, \infty)$

$\therefore f_4$ is onto

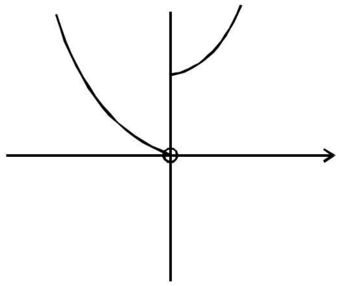
From graph $f_4$ is not one one.

$Q(3)$: $f_3(x) = \begin{cases} \sin x, & x < 0 \\ x, & x \geq 0 \end{cases}$



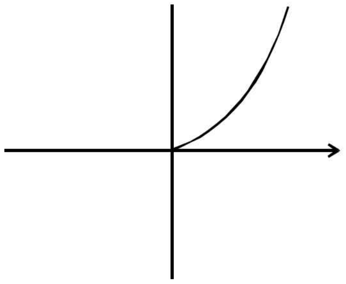
From graph $f$ is differentiable but not one one.

$R(2)$: $f_2 o f_1(x) = \begin{cases} x^2, & x < 0 \\ e^{2x}, & x \geq 0 \end{cases}$



From graph $f_2of_1$ is neither continuous nor one one.

$S(4)$: $f_2(x) = x^2, x \in [0, \infty)$



It is continuous and one one.

### I. Integer Value Correct Type

**1. (6)**

$6\int_1^x f(t)dt = 3xf(x) - x^3$

Differentiating, we get $6f(x) = 3f(x) + 3xf'(x) - 3x^2$

$\Rightarrow f'(x) - \frac{1}{x}f(x) = x$

I.F. $= \frac{1}{x}$

$\therefore$ Solution is $f(x).\frac{1}{x} = \int 1.dx = x + c$

$\therefore f(x) = x^2 + cx$

But $f(1) = 2 \Rightarrow c = 1$ $\therefore f(x) = x^2 + x$

Hence $f(2) = 4 + 2 = 6$

**Note :** Putting $x = 1$ in given integral equation, we get

$f(1) = \frac{1}{3}$ while given $f(1) = 2$.

$\therefore$ Data given in the question is inconsistent.

**2. (2)** $\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$

$\Rightarrow \lim_{x\to 1}\left\{\frac{a(1-x)+\sin(x-1)}{(x-1)+\sin(x-1)}\right\}^{1+\sqrt{x}}$

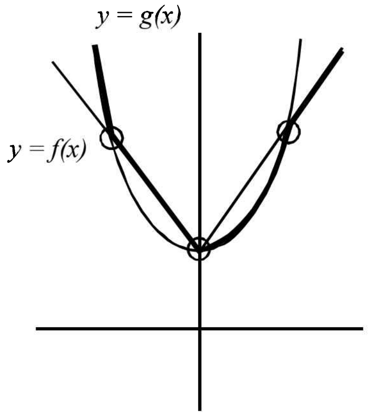
$\Rightarrow \lim_{x\to 1}\left\{\frac{-a+\frac{\sin(x-1)}{x-1}}{1+\frac{\sin(x-1)}{x-1}}\right\}^{1+\sqrt{x}} \Rightarrow \left(\frac{-a+1}{2}\right)^2 = \frac{1}{4}$

$\Rightarrow a = 0$ or $2$

$\therefore$ Largest value of a is 2.

**3. (3)** $f(x) = |x| + 1 = \begin{cases} x+1, & x \ge 0 \\ -x+1, & x < 0 \end{cases}$

$g(x) = x^2 + 1$




From graph there are 3 points at which $h(x)$ is not differentiable.

**4. (2)** $\lim_{\alpha\to 0}\frac{e^{\cos\alpha^n} - e}{\alpha^m} = \frac{-e}{2}$

$\Rightarrow \lim_{\alpha\to 0}\frac{e\left[e^{\cos\alpha^n - 1} - 1\right]}{\cos\alpha^n - 1} \times \frac{\cos\alpha^n - 1}{\alpha^m} = \frac{-e}{2}$

$\Rightarrow e\lim_{\alpha\to 0}\frac{-2\sin^2\frac{\alpha^n}{2}}{\left(\frac{\alpha^n}{2}\right)^2} \times \frac{\left(\frac{\alpha^n}{2}\right)^2}{\alpha^m} = \frac{-e}{2}$

$\Rightarrow \frac{-e}{2}\alpha^{2n-m} = \frac{-e}{2}$ or $\alpha^{2n-m} = 1$

$\Rightarrow 2n - m = 0 \Rightarrow \frac{m}{n} = 2$

**5. (7)** $\lim_{x\to 0}\frac{x^2\sin\beta x}{\alpha x - \sin x} = 1$

$\Rightarrow \lim_{x\to 0}\frac{x^3\beta}{\alpha x - \sin x} = 1$

$\Rightarrow \lim_{x\to 0}\frac{x^3\beta}{\alpha x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ....\infty\right)} = 1$

$\Rightarrow \lim_{x\to 0}\frac{x^3\beta}{(\alpha-1)x + \frac{x^3}{3!} - \frac{x^5}{5!} + ....\infty} = 1$

For above to be possible, we should have

$\alpha - 1 = 0$ and $\beta = \frac{1}{3!}$

$\Rightarrow \alpha = 1$ and $\beta = \frac{1}{6}$

$\therefore 6(\alpha+\beta) = 6\left(1+\frac{1}{6}\right) = 7$

## Section-B JEE Main/ AIEEE

1. (d) $\lim \frac{\sqrt{1-\cos 2x}}{\sqrt{2}x} \Rightarrow \lim \frac{\sqrt{1-(1-2\sin^2 x)}}{\sqrt{2}x}$;

$$\lim_{x\to 0} \frac{\sqrt{2\sin^2 x}}{\sqrt{2}x} \Rightarrow \lim_{x\to 0} \frac{|\sin x|}{x}$$

The limit of above does not exist as
LHS = –1 ≠ RHL = 1

2. (a) $\lim_{x\to\infty} \left(\frac{x^2+5x+3}{x^2+x+2}\right)^x = \lim_{x\to\infty}\left(1+\frac{4x+1}{x^2+x+2}\right)^x$

$$= \lim_{x\to\infty}\left[\left(1+\frac{4x+1}{x^2+x+2}\right)^{\frac{x^2+x+2}{4x+1}}\right]^{\frac{(4x+1)x}{x^2+x+2}}$$

$$= e^{\lim_{x\to\infty}\frac{4x^2+x}{x^2+x+2}} \left[\because \lim_{x\to\infty}(1+\lambda x)^{\frac{1}{x}} = e^{\lambda}\right]$$

$$= e^{\lim_{x\to\infty}\frac{4+\frac{1}{x}}{1+\frac{1}{x}+\frac{2}{x^2}}} = e^4$$

3. (c) Apply L H Rule

We have, $\lim_{x\to 2} \frac{xf(2)-2f(x)}{x-2} \quad \left(\frac{0}{0}\right)$

$= \lim_{x\to 2} f(2)-2f'(x) = f(2)-2f'(2)$
$= 4 - 2 \times 4 = -4.$

4. (a) We have $\lim_{n\to\infty} \frac{1^p+2^p+....+n^p}{n^{p+1}}$;

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^p}{n^p\cdot n} = \int_0^1 x^p dx = \left[\frac{x^{p+1}}{p+1}\right]_0^1 = \frac{1}{p+1}$$

5. (d) Since $\lim_{x\to 0}[x]$ does not exist, hence the required limit does not exist.

6. (a) $\lim_{x\to 1}\frac{\sqrt{f(x)}-1}{\sqrt{x}-1} \left(\frac{0}{0}\right)$ form using L' Hospital's rule

$$= \lim_{x\to 1}\frac{\frac{1}{2\sqrt{f(x)}}f'(x)}{1/2\sqrt{x}} = \frac{f'(1)}{\sqrt{f(1)}} = \frac{2}{1} = 2.$$

7. (b) **Let a is a rational number** other than 0, in [–5, 5], then

$f(a) = a$ and $\lim_{x\to a} f(x) = -a$

[As in the immediate neighbourhood of a rational number, we find irrational numbers]

$\therefore f(x)$ is not continuous at any rational number

**If a is irrational number**, then

$f(a) = -a$ and $\lim_{x\to a} f(x) = a$

$\therefore f(x)$ is not continuous at any irrational number clearly

$\lim_{x\to 0} f(x) = f(0) = 0$

$\therefore f(x)$ is continuous at $x = 0$

8. (d) $\because f''(x) - g''(x) = 0$

Integrating, $f'(x) - g'(x) = c$;

$\Rightarrow f'(1) - g'(1) = c \Rightarrow 4 - 2 = c \Rightarrow c = 2.$

$\therefore f'(x) - g'(x) = 2$;

Integrating, $f(x) - g(x) = 2x + c_1$

$\Rightarrow f(2) - g(2) = 4 + c_1 \Rightarrow 9 - 3 = 4 + c_1$;

$\Rightarrow c_1 = 2 \quad \therefore f(x) - g(x) = 2x + 2$

At $x = 3/2, f(x) - g(x) = 3 + 2 = 5.$

9. (c) $f(x+y) = f(x) \times f(y)$

Differentiate with respect to $x$, treating $y$ as constant

$f'(x+y) = f'(x)f(y)$

Putting $x = 0$ and $y = x$, we get $f'(x) = f'(0)f(x)$;

$\Rightarrow f'(5) = 3f(5) = 3 \times 2 = 6.$

10. (a) The given expression can be written as

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^4 - \lim_{n\to\infty}\frac{1}{n}.\lim_{n\to\infty}\frac{1}{n}\left(\frac{r}{n}\right)^3$$

$$= \int_0^1 x^4\,dx - \lim_{n\to\infty}\frac{1}{n}\times\int_0^1 x^3\,dx = \left[\frac{x^5}{5}\right]_0^1 - 0 = \frac{1}{5}$$

11. (d) $\lim_{x\to 0}\frac{\log(3+x)-\log(3-x)}{x} = K$ (by L'Hospital rule)

$$\lim_{x\to 0}\frac{\frac{1}{3+x}-\frac{-1}{3-x}}{1} = K \quad \therefore \frac{2}{3} = K$$

12. (d) $\lim_{x\to 0}\frac{\frac{d}{dx}\int_0^{x^2}\sec^2 t\,dt}{\frac{d}{dx}(x\sin x)} = \lim_{x\to 0}\frac{\sec^2 x^2.2x}{\sin x + x\cos x}$

(by L' Hospital rule)

$$\lim_{x\to 0}\frac{2\sec^2 x^2}{\left(\frac{\sin x}{x}+\cos x\right)} = \frac{2\times 1}{1+1} = 1$$

13. (b) $\lim_{x\to a}\frac{f(a)g'(x)-g(a)f'(x)}{g'(x)-f'(x)} = 4$

(By L' Hospital rule)

$$\lim_{x\to a}\frac{k\,g'(x)-k\,f'(x)}{g'(x)-f'(x)} = 4 \qquad \therefore k = 4.$$

**14. (d)** $\lim_{x\to\frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right).(1-\sin x)}{(\pi-2x)^3}$

*Let* $x=\frac{\pi}{2}+y; y\to 0$

$= \lim_{y\to 0} \frac{\tan\left(-\frac{y}{2}\right).(1-\cos y)}{(-2y)^3} = \lim_{y\to 0} \frac{-\tan\frac{y}{2} 2\sin^2\frac{y}{2}}{(-8).\frac{y^3}{8}.8}$

$= \lim_{y\to 0} \frac{1}{32}\frac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)}.\left[\frac{\sin y/2}{y/2}\right]^2 = \frac{1}{32}$

**15. (c)** $f(0)=0;\ f(x)=xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$

R.H.L. $\lim_{h\to 0}(0+h)e^{-2/h} = \lim_{h\to 0}\frac{h}{e^{2/h}} = 0$

L.H.L. $\lim_{h\to 0}(0-h)e^{-\left(\frac{1}{h}-\frac{1}{h}\right)} = 0$

therefore, $f(x)$ is continuous.

R.H.D $= \lim_{h\to 0}\frac{(0+h)e^{-\left(\frac{1}{h}+\frac{1}{h}\right)}-0}{h} = 0$

L.H.D. $= \lim_{h\to 0}\frac{(0-h)e^{-\left(\frac{1}{h}-\frac{1}{h}\right)}-0}{-h} = 1$

therefore, L.H.D. $\neq$ R.H.D.

$f(x)$ is not differentiable at $x=0$.

**16. (b)** We know that $\lim_{x\to\infty}(1+x)^{\frac{1}{x}} = e$

$\therefore \lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x} = e^2$

$\Rightarrow \lim_{x\to\infty}\left[\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{\left(\frac{1}{\frac{a}{x}+\frac{b}{x^2}}\right)}\right]^{2x\left(\frac{a}{x}+\frac{b}{x^2}\right)} = e^2$

$\Rightarrow e^{\lim_{x\to\infty} 2\left[a+\frac{b}{x}\right]} = e^2 \Rightarrow e^{2a}=e^2 \Rightarrow a=1$ and $b\in R$

**17. (c)** $f(x)=\frac{1-\tan x}{4x-\pi}$ is continuous in $\left[0,\frac{\pi}{2}\right]$

$\therefore f\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}} f(x) = \lim_{x\to\frac{\pi}{4}^+} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{4}+h\right)$

$= \lim_{h\to 0}\frac{1-\tan\left(\frac{\pi}{4}+h\right)}{4\left(\frac{\pi}{4}+h\right)-\pi}, h>0 = \lim_{h\to 0}\frac{1-\frac{1+\tan h}{1-\tan h}}{4h}$

$= \lim_{h\to 0}\frac{-2}{1-\tan h}.\frac{\tan h}{4h} = \frac{-2}{4} = -\frac{1}{2}$

**18. (d)** $\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}+\frac{3}{n^2}\sec^2\frac{9}{n^2}+....+\frac{1}{n}\sec^2 1\right]$

$= \lim_{n\to\infty}\frac{r}{n^2}\sec^2\frac{r^2}{n^2} = \lim_{n\to\infty}\frac{1}{n}.\frac{r}{n}\sec^2\frac{r^2}{n^2}$

$\Rightarrow$ Given limit is equal to value of integral

$$\int_0^1 x\sec^2 x^2 dx$$

or $\frac{1}{2}\int_0^1 2x\sec\ x^2dx = \frac{1}{2}\int_0^1 \sec^2 t dt$ [put $x^2=t$]

$= \frac{1}{2}(\tan t)_0^1 = \frac{1}{2}\tan 1$.

**19. (a)** Given limit $= \lim_{x\to\alpha}\frac{1-\cos a(x-\alpha)(x-\beta)}{(x-\alpha)^2}$

$= \lim_{x\to\alpha}\frac{2\sin^2\left(a\frac{(x-\alpha)(x-\beta)}{2}\right)}{(x-\alpha)^2}$

$= \lim_{x\to\alpha}\frac{2}{(x-\alpha)^2}\times\frac{\sin^2\left(a\frac{(x-\alpha)(x-\beta)}{2}\right)}{\frac{a^2(x-\alpha)^2(x-\beta)^2}{4}}\times\frac{a^2(x-\alpha)^2(x-\beta)^2}{4}$

$= \frac{a^2(\alpha-\beta)^2}{2}$.

**20. (c)** $f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$;

As function is differentiable so it is continuous as it is given that $\lim_{h\to 0}\frac{f(1+h)}{h} = 5$ and hence $f(1)=0$

Hence $f'(1) = \lim_{h\to 0}\frac{f(1+h)}{h} = 5$

**21. (a)** As $f(1)=-2$ & $f'(x) \geq 2\ \forall\ x\in[1,6]$

Applying Lagrange's mean value theorem

$\frac{f(6)-f(1)}{5} = f'(c) \geq 2 \Rightarrow f(6)\geq 10+f(1)$

$\Rightarrow f(6)\geq 10-2 \Rightarrow f(6)\geq 8$.

**22. (b)** $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

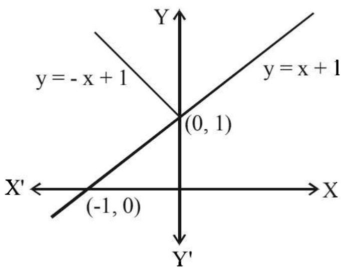
$|f'(x)| = \lim_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right| \leq \lim_{h\to 0}\left|\frac{(h)^2}{h}\right|$

$\Rightarrow |f'(x)|\leq 0 \Rightarrow f'(x)=0$

$\Rightarrow f(x)=$ constant

As $f(0)=0 \Rightarrow f(1)=0$

**23.** **(a)** $f(x)=\min\{x+1,|x|+1\} \Rightarrow f(x)=x+1 \ \forall\ x \in R$




Hence, $f(x)$ is differentiable everywhere for all $x \in R$.

**24.** **(b)** Given, $f(x)=\frac{1}{x}-\frac{2}{e^{2x}-1} \Rightarrow f(0)=\lim_{x\to 0}\frac{1}{x}-\frac{2}{e^{2x}-1}$

$$=\lim_{x\to 0}\frac{(e^{2x}-1)-2x}{x(e^{2x}-1)} \quad \left[\frac{0}{0}\text{ form}\right]$$

$\therefore$ using, L'Hospital rule

$$f(0)=\lim_{x\to 0}\frac{4e^{2x}}{2(xe^{2x}2+e^{2x}.1)+e^{2x}.2}$$

$$=\lim_{x\to 0}\frac{4e^{2x}}{4xe^{2x}+2e^{2x}+2e^{2x}} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to 0}\frac{4e^{2x}}{4(xe^{2x}+e^{2x})}=\frac{4.e^0}{4(0+e^0)}=1$$

**25.** **(c)** We have $f(x)=\begin{cases}(x-1)\sin\left(\frac{1}{x-1}\right), & if\ x\neq 1\\ 0 & , if\ x=1\end{cases}$

$$Rf'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{h\sin\frac{1}{h}-0}{h}=\lim_{h\to 0}\sin\frac{1}{h}=\text{a finite number}$$

Let this finite number be $l$

$$L\ f'(1)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$=\lim_{h\to 0}\frac{-h\sin\left(\frac{1}{-h}\right)}{-h}=\lim_{h\to 0}\sin\left(\frac{1}{-h}\right)$$

$$=-\lim_{h\to 0}\sin\left(\frac{1}{h}\right)=-(\text{a finite number})=-l$$

Thus $Rf'(1)\neq Lf'(1)$

$\therefore$ f is not differentiable at x = 1

Also, $f'(0)=\left.\sin\frac{1}{(x-1)}-\frac{x-1}{(x-1)^2}\cos\left(\frac{1}{x-1}\right)\right]_{x=0}$

$=-\sin 1+\cos 1 \ \therefore$ f is differentiable at x = 0

**26.** **(d)** f(x) is a positive increasing function.

$\therefore\ 0<f(x)<f(2x)<f(3x)$

$$\Rightarrow 0<1<\frac{f(2x)}{f(x)}<\frac{f(3x)}{f(x)}$$

$$\Rightarrow \lim_{x\to\infty}1\leq\lim_{x\to\infty}\frac{f(2x)}{f(x)}\leq\lim_{x\to\infty}\frac{f(3x)}{f(x)}$$

By Sandwich Theorem.

$$\Rightarrow \lim_{x\to\infty}\frac{f(2x)}{f(x)}=1$$

**27.** **(d)** $\lim_{x\to 2}\frac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}=\lim_{x\to 2}\frac{\sqrt{2}|\sin(x-2)|}{x-2}$

$$\text{L.H.L.}=\lim_{x\to 2^-}\frac{\sqrt{2}\sin(x-2)}{(x-2)}=-\sqrt{2}$$

$$\text{R.H.L.}=\lim_{x\to 2^+}\frac{\sqrt{2}\sin(x-2)}{(x-2)}=\sqrt{2}$$

Thus L.H.L. $\neq$ R.H.L.

Hence, $\lim_{x\to 2}\frac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}$ does not exist.

**28.** **(b)** $L.H.L=\lim_{x\to 0^-}f(x)=\lim_{h\to 0}\frac{\sin\{(p+1)(-h)\}-\sin(-h)}{-h}$

$$=\lim_{h\to 0}\frac{-\sin(p+1)h}{-h}+\frac{\sin(-h)}{-h}=p+1+1=p+2$$

$$R.H.L.=\lim_{x\to\sigma^+}f(x)=\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}$$

$$=\lim_{h\to 0}\frac{1}{(\sqrt{1+h}+1)}=\frac{1}{2}$$

and $f(0)=q\Rightarrow p=-\frac{3}{2},q=\frac{1}{2}$

**29.** **(d)** $\lim_{x\to 5}\frac{(f(x))^2-9}{\sqrt{|x-5|}}=0$

$$\lim_{x\to 5}[(f(x))^2-9]=0\Rightarrow\lim_{x\to 5}f(x)=3$$

**30. (a)** Let $f(x)=[x]\cos\left(\frac{2x-1}{2}\right)$

Doubtful points are $x=n, n\in I$

$$\text{L.H.L}=\lim_{x\to n^-}[x]\cos\left(\frac{2x-1}{2}\right)\pi$$

$$=(n-1)\cos\left(\frac{2n-1}{2}\right)\pi=0$$

($\because$ $[x]$ is the greatest integer function)

$$\text{R.H.L}=\lim_{x\to n^+}[x]\cos\left(\frac{2x-1}{2}\right)\pi=n\cos\left(\frac{2n-1}{2}\right)\pi=0$$

Now, value of the function at $x=n$ is $f(n)=0$

Since, L.H.L = R.H.L. = $f(n)$

$\therefore$ $f(x)=[x]\cos\left(\frac{2x-1}{2}\right)$ is continuous for every real $x$.

**31. (c)** $f(x)=|x-2|=\begin{cases}x-2, & x-2\ge 0\\ 2-x, & x-2\le 0\end{cases}$

$$=\begin{cases}x-2, & x\ge 2\\ 2-x, & x\le 2\end{cases}$$

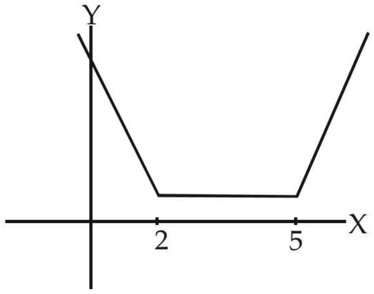
Similarly, $f(x)=|x-5|=\begin{cases}x-5, & x\ge 5\\ 5-x, & x\le 5\end{cases}$

$\therefore$ $f(x)=|x-2|+|x-5|=\{x-2+5-x=3,\ 2\le x\le 5\}$

Thus $f(x)=3,\ 2\le x\le 5$

$f'(x)=0,\ 2<x<5$

$f'(4)=0$ $\therefore$ statement 1 is true.




$\because$ $f(2)=0+|2-5|=3$ and $f(5)=|5-2|+0=3$

$\therefore$ statement-2 is also true and a correct explanation for statement 1.

**32. (d)** Multiply and divide by $x$ in the given expression, we get

$$\lim_{x\to 0}\frac{(1-\cos 2x)}{x^2}\frac{(3+\cos x)}{1}\cdot\frac{x}{\tan 4x}$$

$$=\lim_{x\to 0}\frac{2\sin^2 x}{x^2}\cdot\frac{3+\cos x}{1}\cdot\frac{x}{\tan 4x}$$

$$=2\lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\lim_{x\to 0}3+\cos x\cdot\lim_{x\to 0}\frac{x}{\tan 4x}$$

$$=2.4\ \frac{1}{4}\lim_{x\to 0}\frac{4x}{\tan 4x}=2.4.\frac{1}{4}=2$$

**33. (b)** Consider $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$

$$=\lim_{x\to 0}\frac{\sin\left[\pi(1-\sin^2 x)\right]}{x^2}$$

$$=\lim_{x\to 0}\sin\frac{(\pi-\pi\sin^2 x)}{x^2}\qquad[\because \sin(\pi-\theta)=\sin\theta]$$

$$=\lim_{x\to 0}\sin\frac{(\pi\sin^2 x)}{\pi\sin^2 x}\times\frac{\pi\sin^2 x}{x^2}$$

$$=\lim_{x\to 0}1\times\pi\left(\frac{\sin x}{x}\right)^2=\pi$$

**34. (a)** Multiply and divide by $x$ in the given expression, we get

$$\lim_{x\to 0}\frac{(1-\cos 2x)}{x^2}\frac{(3+\cos x)}{1}\cdot\frac{x}{\tan 4x}$$

$$=\lim_{x\to 0}\frac{2\sin^2 x}{x^2}\cdot\frac{3+\cos x}{1}\cdot\frac{x}{\tan 4x}$$

$$=2\lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\lim_{x\to 0}3+\cos x\cdot\lim_{x\to 0}\frac{x}{\tan 4x}$$

$$=2.4\ \frac{1}{4}\lim_{x\to 0}\frac{4x}{\tan 4x}=2.4.\frac{1}{4}=2$$

**35. (c)** Since g (x) is differentiable, it will be continuous at x = 3

$$\therefore \lim_{x\to 3^-}g(x)=\lim_{x\to 3^+}g(x)$$

$2k=3m+2$ ...(1)

Also g(x) is differentiable at x = 0

$$\therefore \lim_{x\to 3^-}g'(x)=\lim_{x\to 3^+}g'(x)$$

$$\frac{K}{2\sqrt{3+1}}=m$$

$k=4m$ ...(2)

Solving (1) and (2), we get

$$m=\frac{2}{5},\ k=\frac{8}{5}$$

$\therefore k+m=2$

**36. (d)** $g(x)=f(f(x))$

In the neighbourhood of $x=0$,

$f(x)=|\log 2-\sin x|=(\log 2-\sin x)$

$\therefore$ $g(x)=||\log 2-\sin|\log 2-\sin x||$

$=(\log 2-\sin(\log 2-\sin x))$

$\therefore$ g (x) is differentiable

and $g'(x)=-\cos(\log 2-\sin x)(-\cos x)$

$\Rightarrow g'(0)=\cos(\log 2)$

**37.** **(d)** $y = \lim_{n\to\infty}\left(\frac{(n+1)(n+2)...3n}{n^{2n}}\right)^{\frac{1}{n}}$

$$\ln y = \lim_{n\to\infty}\frac{1}{n}\ln\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)....\left(1+\frac{2n}{n}\right)$$

$$\ln y = \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+....+\ln\left(1+\frac{2n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\ln\left(1+\frac{r}{n}\right)$$

$$= \int_0^2 \ln(1+x)dx$$

Let $1 + x = t \Rightarrow dx = dt$

when $x = 0,\ t = 1$

$x = 2,\ t = 3$

$$\ln y = \int_1^3 \ln t\, dt = [t\ln t - t]_1^3 = \ln\left(\frac{3^3}{e^2}\right) = \ln\left(\frac{27}{e^2}\right)$$

$$\Rightarrow y = \frac{27}{e^2}$$

**38.** **(a)** $\ln P = \lim_{x\to 0^+}\frac{1}{2x}\ln(1+\tan^2\sqrt{x})$

$$\lim_{x\to 0^+}\frac{1}{x}\ln\left(\sec\sqrt{x}\right)$$

Applying L hospital's rule :

$$= \lim_{x\to 0^+}\frac{\sec\sqrt{x}\tan\sqrt{x}}{\sec\sqrt{x}\cdot 2\sqrt{x}}$$

$$= \lim_{x\to 0^+}\frac{\tan\sqrt{x}}{2\sqrt{x}}$$

$$= \frac{1}{2}$$