Appendix B
Relativistic Kinematics

In particle physics, most scattering interactions take place between particles whose speeds are comparable with the speed of light c. This is often true even in decays, particularly if light particles are emitted. The requirements of special relativity therefore cannot be ignored. In nuclear physics accurate predictions can also often only be obtained if relativistic effects are taken into account. In this appendix we review (usually without proof) some relativistic kinematical results and the use of invariants to simplify calculations.

B.1 Lorentz Transformations and Four-Vectors

Consider a particle of mass m in an inertial frame of reference S. Its co-ordinates are $(t, \mathbf{r}) \equiv (t, x, y, z)$ and its speed is $u = |\mathbf{u}|$, where $\mathbf{u}$ is its velocity. In a second inertial frame S' its co-ordinates are $(t', \mathbf{r}') \equiv (t', x', y', z')$ and its speed is $u' = |\mathbf{u}'|$ where $\mathbf{u}'$ is its velocity. If S and S' coincide at $t = 0$ and S' is moving with uniform speed v in the positive z-direction with respect to S, then the two sets of coordinates are related by the *Lorentz transformation*

$$\begin{aligned} x' &= x \\ y' &= y \\ z' &= \gamma(v)(z - vt) \\ t' &= \gamma(v)(t - vz/c^2) \end{aligned} \tag{B.1}$$

where $\gamma(v) = (1 - \beta^2)^{-\frac{1}{2}}$ is the *Lorentz factor* and $\beta \equiv v/c$. From the definition of velocity and using these transformations, the particle's speed in S' is related to its speed in S by

$$u' = \frac{u - v}{1 - uv/c^2} \tag{B.2}$$

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and hence

$$\gamma(u') \equiv [1 - (u'/c)^2]^{-\frac{1}{2}} = \gamma(u)\gamma(v)(1 - uv/c^2). \tag{B.3}$$

As $v \to 0$, the transformations in Equations (B.1) approach the Galilean transformations.

The most general Lorentz transformation has its simplest form in terms of *four-vectors*, whose general form is $a = (a_0, a_1, a_2, a_3) = (a_0, \mathbf{a})$. Then Equations (B.1) become

$$a'_0 = \gamma(a_0 - va_3/c); \quad a'_1 = a_1; \quad a'_2 = a_2; \quad a'_3 = \gamma(a_3 - va_0/c). \tag{B.4}$$

For example, the space-time four-vector is $x = (ct, \mathbf{x})$ and when used in Equations (B.4) reproduces Equations (B.1). The scalar product of two four-vectors a and b is defined as

$$ab \equiv a_0 b_0 - \mathbf{a} \cdot \mathbf{b} \tag{B.5}$$

and is an *invariant*, i.e. is the same in all inertial frames of references.

The basic four-vector in particle kinematics is the *four-momentum*, defined by

$$P \equiv mu, \tag{B.6}$$

where m is the *rest mass* and u is the *four-velocity*, whose components are $u = \gamma(v)(c,\ \mathbf{v})$, where $\mathbf{v}$ is the three-velocity and $v \equiv |\mathbf{v}|$. In terms of the *total energy E* (i.e. including the rest mass) and the three-momentum $\mathbf{p}$,

$$P = (E/c, \mathbf{p}). \tag{B.7}$$

Thus for two four-momenta P_1 and P_2 the invariant scalar product is

$$P_1 P_2 = E_1 E_2/c^2 - \mathbf{p}_1 . \mathbf{p}_2 \tag{B.8}$$

and for $P_1 = P_2 = P$,

$$P^2 = E^2/c^2 - \mathbf{p}^2. \tag{B.9}$$

However, from Equations (B.5) and (B.6) we have $u^2 = c^2$ and hence $P^2 = m^2c^2$, so combining this with Equation (B.9) gives

$$E^2 = \mathbf{p}^2 c^2 + m^2 c^4. \tag{B.10}$$

It follows that

$$E = \gamma(v) mc^2, \quad \mathbf{p} = \gamma(v) m\mathbf{v}, \quad \mathbf{v} = c^2\mathbf{p}/E. \tag{B.11}$$

The Lorentz transformations for energy and momentum follow from these definitions and Equations (B.4). Thus, in S' we have

$$E' = mc^2\gamma(u') = \gamma(v)(E - vp) \tag{B.12a}$$

and

$$p' = mu'\gamma(u') = \gamma(v)(p - vE/c^2), \tag{B.12b}$$

where $p = |\mathbf{p}|$ and $p' = |\mathbf{p}'|$. For a set of N non-interacting particles,

$$p'_z = \gamma(v)(p_z - vE/c^2); \quad p'_x = p_x; \quad p'_y = p_y; \tag{B.13a}$$

and

$$E' = \gamma(v)(E - vp_z), \tag{B.13b}$$

where

$$E = \sum_{i=1}^{N} E_i \quad \text{and} \quad \mathbf{p} = \sum_{i=1}^{N} \mathbf{p}_i. \tag{B.13c}$$

In the general case where the relative velocity $\mathbf{v}$ of the two frames is in an arbitrary direction, the transformations in Equations (B.12) become

$$\mathbf{p}' = \mathbf{p} + \gamma\mathbf{v}\left(\frac{\gamma\,\mathbf{v}\cdot\mathbf{p}}{\gamma + 1} - E\right)\frac{1}{c^2}, \quad E' = \gamma(E - \mathbf{v}\cdot\mathbf{p}). \tag{B.14}$$

B.2 Frames of Reference

The two most commonly used frames of reference for particle kinematics are the *laboratory system* (LS) and the *centre-of-mass system* (CMS). We will start by discussing these in the context of two-particle scattering. In the LS, a moving projectile a in a beam strikes a target particle b at rest, i.e.

$$P_a = (E_a/c, \mathbf{p}_a), \quad P_b = (m_b c, \mathbf{0}). \tag{B.15}$$

In the CMS, the three-momenta of the two particles a and b are equal and opposite, so that the total momentum is zero,[1] i.e.

$$P_a = (E_a/c, \mathbf{p}_a), \quad P_b = (E_b/c, \mathbf{p}_b), \tag{B.16a}$$

[1]Although 'centre-of-mass' system is the most frequently used name, some authors refer to this as the 'centre-of-momentum' system. Logically, a better name would be 'zero-momentum' frame.

with

$$\mathbf{p}_a + \mathbf{p}_b = \mathbf{0}. \tag{B.16b}$$

In a colliding beam accelerator, these two views become mixed. The colliding particles are both moving, but only if they have equal momenta and collide at zero crossing angle is the system identical to the centre-of-mass system.

The four-vectors of the initial-state particles in the two systems may be written (L = laboratory, T = target)

$$P_a = (E_{\mathrm{L}}/c, 0, 0, p_{\mathrm{L}}), \quad P_{\mathrm{T}} = (m_{\mathrm{T}}c, 0, 0, 0) \quad \mathrm{LS} \tag{B.17a}$$

with $E_{\mathrm{L}}^2 = m_{\mathrm{B}}^2 c^4 + p_{\mathrm{L}}^2 c^2$ (B = beam), and

$$P_a = (E_a/c, 0, 0, p), \quad P_b = (E_b/c, 0, 0, -p) \quad \mathrm{CMS} \tag{B.17b}$$

with $E_a^2 = m_{\mathrm{B}}^2 c^4 + p^2 c^2$ and $E_b^2 = m_{\mathrm{T}}^2 c^4 + p^2 c^2$.
The Lorentz transformations between them are

$$p = \gamma(p_{\mathrm{L}} - vE_{\mathrm{L}}/c^2), \quad E_a = \gamma(E_{\mathrm{L}} - vp_{\mathrm{L}}), \tag{B.18}$$

where

$$v = \frac{c^2 p_{\mathrm{L}}}{E_{\mathrm{L}} + m_{\mathrm{T}}c^2}, \quad \gamma = \frac{E_{\mathrm{L}} + m_{\mathrm{T}}c^2}{c^2\sqrt{s}}, \quad v\gamma = \frac{p_{\mathrm{L}}}{\sqrt{s}} \tag{B.19}$$

and s is the *invariant mass squared* of the system defined by

$$s \equiv (p_a + p_b)^2/c^2 = [(E_a + E_b)^2 - (\mathbf{p}_a c + \mathbf{p}_b c)^2]/c^4. \tag{B.20}$$

In particular, in the LS,

$$s = m_{\mathrm{T}}^2 + m_{\mathrm{B}}^2 + 2m_{\mathrm{T}}E_{\mathrm{L}}/c^2. \tag{B.21}$$

This result was used in Chapter 4 when discussing the relative merits of fixed-target and colliding beam accelerators.

Substituting Equations (B.19) into Equations (B.18) gives

$$p = \frac{p_{\mathrm{L}} m_{\mathrm{T}}}{\sqrt{s}}, \quad E_a = \frac{m_{\mathrm{B}}^2 c^2 + m_{\mathrm{T}}E_{\mathrm{L}}}{\sqrt{s}} \tag{B.22a}$$

and similarly for particle b:

$$p = \frac{p_{\mathrm{L}} m_{\mathrm{T}}}{\sqrt{s}}, \quad E_b = \frac{m_{\mathrm{T}}^2 c^2 + m_{\mathrm{T}}E_{\mathrm{L}}}{\sqrt{s}}. \tag{B.22b}$$

Finally we state, without proof, the transformation of scattering angles for the specific case of laboratory and centre-of-mass systems. Consider the general scattering reaction

$$B(E_{\mathrm{L}}, \mathbf{p}_{\mathrm{L}}) + T(m_{\mathrm{T}}^2, \mathbf{0}) \to P(E, \mathbf{q}) + \cdots\cdots, \tag{B.23}$$

where B is a beam particle incident on a target particle T at rest in the laboratory system and P is one of a number of possible particles in the final state. If $\mathbf{p}_{\mathrm{L}}$ is taken along the z-direction, then

$$\mathbf{p}_{\mathrm{L}} = (0, 0, p_{\mathrm{L}}) \quad \text{and} \quad \mathbf{q} = (0, q\sin\theta_{\mathrm{L}}, q\cos\theta_{\mathrm{L}}), \tag{B.24}$$

where θ_{L} is the scattering angle in the laboratory system, i.e. the angle between the beam direction and $\mathbf{q}$. In the CMS,

$$\mathbf{p}'_{\mathrm{B}} + \mathbf{p}'_{\mathrm{T}} = \mathbf{0}, \tag{B.25}$$

where $\mathbf{p}'_{\mathrm{B}}$ and $\mathbf{p}'_{T}$ are the CMS momenta of the beam and target, respectively. The relation between the scattering angle θ_{C} in this system and θ_{L} is

$$\tan\theta_{\mathrm{L}} = \frac{1}{\gamma(v)} \frac{q' \sin\theta_{\mathrm{C}}}{q' \cos\theta_{\mathrm{C}} + vE'/c^2}, \tag{B.26}$$

where

$$E' = m_P c^2 \gamma(u) \quad \text{and} \quad q' = m_P u \gamma(u) \tag{B.27}$$

and u is the magnitude of the velocity of P in the centre-of-mass frame.

It is instructive to consider the form of Equation (B.26) at high energies. From Equation (B.19) the velocity of the transformation is

$$v = p_{\mathrm{L}} c^2 [E_{\mathrm{L}} + m_{\mathrm{T}} c^2]^{-1}, \tag{B.28}$$

so at high energies where $E_{\mathrm{L}}^2 \approx p_{\mathrm{L}} c \gg m_{\mathrm{B}} c^2$, $m_{\mathrm{T}} c^2$, $v \approx c(1 - m_{\mathrm{T}} c/p_{\mathrm{L}}) \approx c$ and

$$\gamma(v) \approx \left(\frac{p_{\mathrm{L}}}{2m_{\mathrm{T}} c}\right)^{1/2}. \tag{B.29}$$

Substituting Equations (B.27), (B.28) and (B.29) into Equation (B.26) gives

$$\tan\theta_{\mathrm{L}} \approx \left(\frac{2m_{\mathrm{T}} c}{p_{\mathrm{L}}}\right)^{1/2} \cdot \frac{u \sin\theta_{\mathrm{C}}}{u\cos\theta_{\mathrm{C}} + c}. \tag{B.30}$$

Thus, unless $u \approx c$ and $\cos\theta_{\mathrm{C}} \approx -1$, the final-state particles will lie in a narrow cone about the beam direction in the laboratory system. Similarly, when a high-energy particle decays, its decay products will emerge predominantly at small angles to the initial beam direction.

B.3 Invariants

The transformations between laboratory and centre-of-mass systems for energy and momentum have been worked out explicitly above, but a more efficient way is to work with quantities that are invariants, i.e. have the same values in all inertial frames. We have already met one of these: s the invariant mass squared, defined in Equation (B.20). We will now find expressions for the energy and momentum in terms of invariants for both the LS and the CMS.

First, in the LS, from Equations (B.15), we have

$$\mathbf{p}_{\mathrm{B}} = \mathbf{0}, \quad E_{\mathrm{B}} = m_{\mathrm{B}} c. \tag{B.31}$$

However, from Equation (B.23),

$$s = m_{\mathrm{B}}^2 + m_{\mathrm{T}}^2 + 2m_{\mathrm{T}} E_{\mathrm{L}}/c^2 \tag{B.32}$$

i.e.

$$E_{\mathrm{L}} = \frac{(s - m_{\mathrm{T}}^2 - m_{\mathrm{B}}^2)c^2}{2m_{\mathrm{T}}} \tag{B.33}$$

and so

$$p_{\mathrm{L}}^2 = \frac{E_{\mathrm{L}}^2}{c^2} - m_{\mathrm{B}}^2 c^2 = \frac{(s - m_{\mathrm{B}}^2 - m_{\mathrm{T}}^2)^2 c^2 - 4m_{\mathrm{B}}^2 m_{\mathrm{T}}^2 c^2}{4m_{\mathrm{T}}^2}. \tag{B.34}$$

This can be written in the useful compact form

$$p_{\mathrm{L}} = \frac{c}{2m_{\mathrm{T}}} \lambda^{\frac{1}{2}}(s, m_{\mathrm{B}}^2, m_{\mathrm{T}}^2), \tag{B.35a}$$

where the *triangle function* λ is defined by

$$\lambda(x, y, z) \equiv (x - y - z)^2 - 4yz. \tag{B.35b}$$

This function is invariant under all permutations of its arguments and in particular Equation (B.35a) can be written in the form

$$p_{\mathrm{L}} = \frac{c}{2m_{\mathrm{T}}} \left\{ \left[s - (m_{\mathrm{T}} + m_{\mathrm{B}})^2 \right] \left[s - (m_{\mathrm{T}} - m_{\mathrm{B}})^2 \right] \right\}^{\frac{1}{2}}. \tag{B.36}$$

In a similar way it is straightforward to show that, in the CMS,

$$p = \frac{c}{2\sqrt{s}}\left\{\left[s-(m_{\mathrm{T}}+m_{\mathrm{B}})^2\right]\left[s-(m_{\mathrm{T}}-m_{\mathrm{B}})^2\right]\right\}^{\frac{1}{2}} \tag{B.37}$$

from which it follows that

$$E_a = \frac{(s+m_{\mathrm{B}}^2-m_{\mathrm{T}}^2)c^2}{2\sqrt{s}}, \quad E_b = \frac{(s-m_{\mathrm{B}}^2+m_{\mathrm{T}}^2)c^2}{2\sqrt{s}}. \tag{B.38}$$

The above formulae have many applications. For example, if we wish to produce particles with a certain mass M, the minimum laboratory energy of the beam particles is, from Equation (B.33),

$$E_{\mathrm{L}}(\mathrm{min}) = \frac{M^2c^2 - m_{\mathrm{B}}^2c^2 - m_{\mathrm{T}}^2c^2}{2m_{\mathrm{T}}}. \tag{B.39}$$

In the case of the decay of a particle A to a set of final-state particles $i = 1, 2, 3, \ldots, N$, i.e.

$$A \to 1 + 2 + 3 + \cdots + N, \tag{B.40}$$

the invariant mass W of the final-state particles is given by

$$W^2c^4 = \left(\sum_i E_i\right)^2 - \left(\sum_i \mathbf{p}_i c\right)^2 = E_A^2 - (\mathbf{p}_A c)^2 = M_A^2c^4. \tag{B.41}$$

Hence the mass of the decaying particle is equal to the invariant mass of its decay products. The latter can be measured if the particle is too short-lived for its mass to be measured directly.

Problems

B.1 The *Mandelstam variables s, t and u* are defined for the reaction $A + B \to C + D$ by

$$s = (p_A + p_B)^2/c^2, \quad t = (p_A - p_C)^2/c^2, \quad u = (p_A - p_D)^2/c^2,$$

where p_A etc. are the relevant energy-momentum four-vectors.

(a) Show that

$$s + t + u = \sum_{j=A,B,C,D} m_j^2.$$

(b) In the case of elastic scattering show that $t = -2p^2(1 - \cos\theta)/c^2$, where $p \equiv |\mathbf{p}|$, $\mathbf{p}$ is the centre-of-mass momentum of particle A and θ is its scattering angle in the CMS.

B.2 A pion travelling with speed $v \equiv |\mathbf{v}|$ in the laboratory decays via $\pi \to \mu + \nu$. If the neutrino emerges at right angles to $\mathbf{v}$, find an expression for the angle θ at which the muon emerges.

B.3 A pion at rest decays via $\pi \to \mu + \nu$. Find the speed of the muon in terms of the masses involved.

B.4 A neutral particle X^0 decays via $X^0 \to A^+ + B^-$. The momentum components of the final-state particles are measured to be (in GeV/c):

	p_x	p_y	p_z
A^+	−0.488	−0.018	2.109
B^-	−0.255	−0.050	0.486

Test the hypotheses that the decay is (a) $D^0 \to \pi^+ + K^-$ and (b) $\Lambda \to p + \pi^-$.

B.5 In a fixed-target e^-p scattering experiment, show that the squared four-momentum transfer is given by $Q^2 \approx 2E^2(1 - \cos\theta)/c^2$, where E is the total laboratory energy of the initial electron and θ is the laboratory scattering angle.

B.6 Calculate the minimum laboratory energy E_{min} of the initial proton for the production of antiprotons in a fixed-target experiment using the reaction $pp \to ppp\bar{p}$. If the protons are bound in nuclei, show that taking the internal motion of the nucleons into account leads to a smaller minimum energy given by

$$E'_{\text{min}} \approx (1 - p/m_P c)E_{\text{min}},$$

where p is the modulus of the average internal longitudinal momentum of a nucleon. Use a typical value of p to calculate E'_{min}.

B.7 A particle A decays at rest via $A \to B + C$. Find the total energy of B in terms of the three masses.

B.8 A meson M decays via $M \to \gamma\gamma$. Find an expression for the angle in the laboratory between the two momentum vectors of the photons in terms of the photon energies and the mass of M.

B.9 Pions and protons, both with momentum 2 GV/c, travel between two scintillation counters distance L m apart. What is the minimum value of L necessary to differentiate between the particles if the time-of-flight can be measured with an accuracy of 200 ps?

B.10 A photon is Compton scattered off a stationary electron through a scattering angle of 60° and its final energy is half its initial energy. Calculate the value of the initial energy in MeV.